A SIMULATION PROCEDURE FOR VORTEX FLOW OVER AN OSCILLATING WING

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MITSG 94-7

Sea Grant College Program
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

Grant No. NA90AA-D-SG424

Project No: 90-RU-21

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A Simulation Procedure for Vortex Flow over an Oscillating Wing

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Abstract

A method for simulating hydrofoils performing large amplitude oscillations in a flow with regions of concentrated vorticity has been developed. The method is based on two–dimensional potentional flow and the theory of functions of a complex variable. The shape of the foil profile is obtained as a Joukowski transformation of a circle. There are three main components in the method: (1) Careful formulation of the velocity potentials for the three degrees of freedom and the point vortices. (2) A new way of releasing vortices at the trailing edge to model the vortex sheet that constitute the wake of the foil. (3) Closed form expressions for the force and moment on the foil, facilitating rapid and accurate calculation these and related quantities.

Acknowledgments

This work, project number 90–RU–21 was funded in part by the MIT Sea Grant College Program under a federal grant number NA90AA-D-SG424 from the National Sea Grant College Program, National Oceanic and Atmospheric Administration, U.S. Department of Commerce.

Additional financial support of ARPA, the Office of Naval Research, under contracts N00014-92-J-1726 and N00014-93-1-0587, is also gratefully acknowledged.

1 Introduction

The problem of a foil moving in a flow with regions of concentrated vorticity is of interest in several contexts. In aerodynamics, the vortex wake of one helicopter rotor blade impinges on the following one, resulting in pressure fluctuations and noise [Booth90, Fujinami86, Panaras87, Straus90]. An example of constructive interference is the so called Kasper foil, where the ability to trap a vortex in a stationary position above the foil results in enormous lift coefficients [Saffman77]. In biofluiddynamics, flow visualizations indicate that the flapping tail of a fish may recover some of the energy in vortices formed over the anterior parts of the body [Rosen59]. As a model of the last problem, consider a foil that manipulates the Karman vortex street behind a bluff body so as to de–energize it. Leaving little kinetic energy behind, the bluff body – foil system is in effect a streamlined entity, perhaps more streamlined than a conventional hull – propeller combination could ever be. This mechanism will also reduce the wake signature associated with a bluff body. An application could be to locate a vortex controlling foil behind submarine bridge, thereby improving stealthworthiness of the submarine.

This report describes a method for simulating such problems, assuming that the flow is two–dimensional and inviscid. These are drastic assumptions, but they enable simulations to be performed at a modest computational cost, which is is a great advantage when exploring the rather large parameter space associated with the applications mentioned above. The assumption of two–dimensional flow limits the scope of the method to cases with high aspect ratio foils and incoming vortices of high correlation length, although we expect the qualitative results to be valid even when this assumption is relaxed. The inviscid flow assumption restricts the method to cases with high Reynolds number (which is not serious) and absence of stall, or leading edge separation. Thus there is an implicit assumption of low local angles of attack in order to keep the flow attached to the foil, otherwise we should have to resort to computationally very intensive methods or experiments.

As a further simplification, the foil shape is taken to be a Joukowski profile, which is obtained through a simple transformation of a circular cylinder. This leads to a closed form description of the flow in the circle plane, where the vortex field is modeled with point vortices. Furthermore, the question of which face of the foil the wake is tangent to at the trailing edge [Basu78] becomes a moot point, as the trailing edge of a Joukowski foil is cusped.

In order to evaluate the performance of the foil (mainly the efficiency), it is necessary to calculate forces and moment experienced by the foil during the simulation. It is possible to numerically integrate the pressure on the foil at any time, but this is a slow process that requires many function calls involving all the vortices in the flow. Additional difficulties arise in cases where the foil has zero thickness and a sharp leading edge, which are important for verification. Therefore, a major part of this report is devoted to the derivation of closed form expressions for force and moment, using the fact that the foil shape is a Joukowski profile.

In this report, no examples of actual application of the method has been presented, these will be reported later in a Ph.D. thesis. The motivation for issuing this report as a separate item, is a hope that the methodology used will be of general interest in unsteady foil simulations.

2 Coordinate Systems

In the following we shall need 3 different plane coordinate systems to describe and solve for the flow over a wing profile in surge, heave and pitch. In complex notation, these are (1) $z_0 = x_0 + iy_0$, (2) $z = x + iy$ and (3) $z' = x' + iy'$. They are shown together in Figure 1.

(1) The z_0 plane is used to describe the mean flow and the motion of the foil. z_0 is fixed in the mean position of the foil, and an observer in this reference frame will see the foil oscillating in heave and pitch and

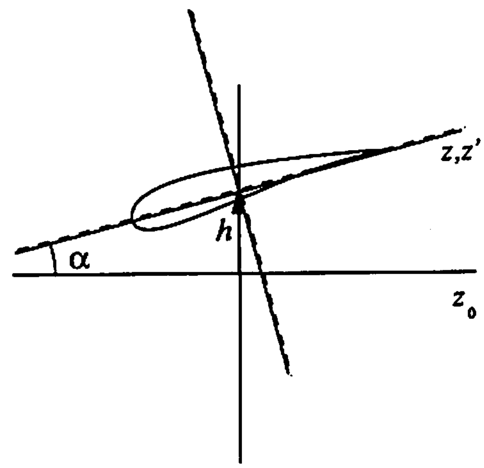

Figure 1: Definition of coordinate systems.

a free stream U_0 flowing from left to right. The motion of the foil has the following form:

$$\begin{aligned} h &= h_0 \cos(\omega t + \mu) - \alpha_0 b \sin(\omega t + \mu) \\ \alpha &= \alpha_0 \sin(\omega t + \mu) \end{aligned} \tag{1}$$

Here, h_0 and α_0 are heave and pitch amplitude respectively. The phasing between the two motions is determined by b, for motions of moderated pitch amplitude, b can also be thought of as the pitch point. ω, t and μ are frequency, time and phase, respectively. The freedom to choose the phase μ is convenient in certain cases, $\omega = 0$, $\mu = -\pi/2$ for instance, specifies an impulsively started foil. The above definitions (with $\mu = 0$) coincides with Lighthill's [Lighthill75] in the limit of small amplitudes. Other motions, like a foil moving in circles without pitching [Schmidt65], can be simulated with minor modifications.

(2) The z plane coincides with the foil at the instant under consideration, and is at rest in the fluid at infinity. An observer in this reference frame will see the foil moving in the x- and y-direction with velocities U and V respectively, sometimes denoted $W = U + iV$ for brevity. Furthermore, the foil is rotating counterclockwise about $z = 0$ with angular velocity Ω. This is the reference frame in which we solve the boundary value problem for the oscillating foil. The two coordinate systems are related by

$$z_0 = ih + ze^{i\alpha} \tag{2}$$

and the velocities are given by

$$\begin{aligned} W &= \left(-U_0 + i\dot{h}\right) e^{-i\alpha} \\ \Omega &= \dot{\alpha} \end{aligned} \tag{3}$$

(3) The z' plane is fixed in the foil at all times and coincides with z at any particular instant. An observer in the z' system sees the fluid moving over a stationary foil, the fluid motion being everywhere rotational if $\Omega \neq 0$. The fact that the foil is fixed in this frame, makes it suitable for deriving force and moment expressions from pressure integration.

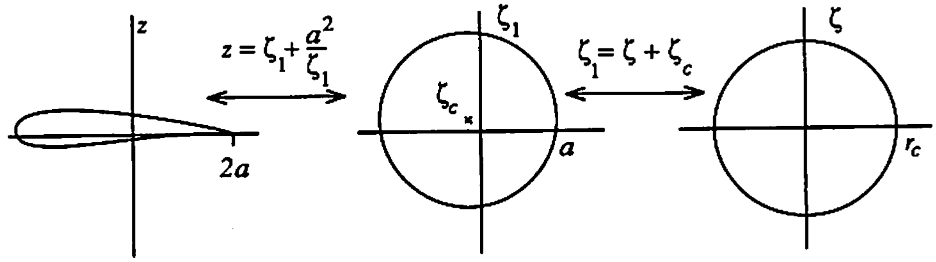

Figure 2: Mapping between Joukowski profile and circle.

3 Conformal mapping

As Figure 2 shows, a Joukowski foil shape can be obtained by mapping a circle of radius r_c in the ζ plane by:

$$z = F(\zeta) = \zeta + \zeta_c + \frac{a^2}{\zeta + \zeta_c} \tag{4}$$

This function is obtained in two steps. First, the foil is mapped to a circle in an intermediate plane, ζ_1, that has center at ζ_c and goes through the point a on the real axis. Thus, $r_c = |a - \zeta_c|$. This circle is mapped to the ζ plane by a simple translation. a and ζ_c are parameters which determine the size, thickness and camber of the foil. The inverse of the mapping function is given by:[1]

$$\zeta = F^{-1}(z) = \frac{1}{2}\left[z + \sqrt{z^2 - 4a^2}\right] - \zeta_c \tag{5}$$

Three quantities of interest are the area, center of area and polar moment of area for the Joukowski foil. The simplest way to find these is by using either of the following expressions for an arbitrary function of z and its complex conjugate, $f(z, \bar{z})$ over a region R, bounded by ∂R:

$$\int_{\partial R} f d\bar{z} = -2i \int\int_R \frac{\partial f}{\partial z} dA \ , \ \int_{\partial R} f dz = 2i \int\int_R \frac{\partial f}{\partial \bar{z}} dA \tag{6}$$

These are readily found by applying Stokes theorem in two dimensions to the real and imaginary parts of f [MilneThomson60]. Replacing ∂R by S, we can now write for the area:

$$A = \int\int_R dA = \frac{1}{2i}\int_S \bar{z} dz$$

Next, the variable of integration is changed to ζ and the integral is performed on the circle C:

$$A = \frac{1}{2i}\int_C \left(\bar{\zeta} + \bar{\zeta}_c + \frac{a^2}{\bar{\zeta} + \bar{\zeta}_c}\right) F'(\zeta) d\zeta$$

Here, the prime denotes derivative, $F'(\zeta) = dF/d\zeta$, not to be confused with the coordinate system fixed in the foil. Using the fact that $\bar{\zeta} = r_c^2/\zeta$ on C, we obtain an integrand that is analytic except for discrete poles:

$$A = \frac{1}{2i}\int_C \left(\frac{r_c^2}{\zeta} + \bar{\zeta}_c + \frac{a^2}{r_c^2/\zeta + \bar{\zeta}_c}\right)\left(1 - \frac{a^2}{(\zeta + \zeta_c)^2}\right) d\zeta$$

[1] When implementing a multivalued expression like this on a computer, it is important to choose the proper branch. A simple way is to use the expanded form: $\zeta = \frac{1}{2}\left[z + \sqrt{z - 2a}\sqrt{z + 2a}\right] - \zeta_c$.

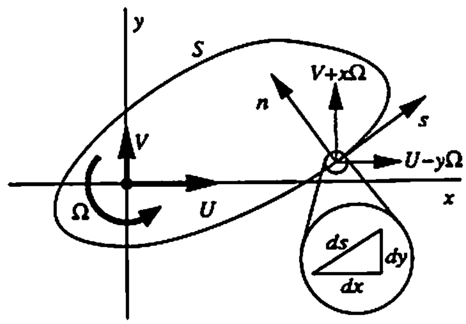

Figure 3: Definitions for boundary value problem.

The integrand is then expanded out to 6 terms that are evaluated by the residue theorem, and when they are added, we arrive at an expression for the area:

$$A = \pi r_c^2 \left[1 - \frac{a^4}{(r_c^2 - \delta^2)^2}\right] \tag{7}$$

where $\delta^2 = \zeta_c \overline{\zeta}_c$.

Similarly, we have for the center of area, z_c, and radius of gyration, r_g:

$$z_c A = \int\int_R z dA = -\frac{1}{4i}\int_S z^2 d\overline{z} = \pi r_c^2 \left[\zeta_c + \frac{a^6 \overline{\zeta}_c}{(r_c^2 - \delta^2)^3}\right] \tag{8}$$

$$r_g^2 A = \int\int_R z\overline{z} dA = -\frac{1}{4i}\int_S z^2 \overline{z} d\overline{z} = \frac{\pi}{2} r_c^2 \left[r_c^2 + 2\delta^2 - a^8 \frac{r_c^2 + 2\delta^2}{(r_c^2 - \delta^2)^4}\right] \tag{9}$$

We will need these quantities in the calculation of force and moment on the foil.

4 Boundary value problem for translating and rotating section

Consider an arbitrary profile S in the $z = x + iy$ plane, translating with velocity $W = U + iV$ and rotating with angular velocity Ω as shown in Figure 3. The velocity of a material particle at a point (x, y) on S, has a component v_n in the $\mathbf{n}$ direction. The boundary condition of no flow through S, can be written:

$$v_n = -\frac{\partial \psi}{\partial s} = -\frac{\partial \psi}{\partial x}\frac{dx}{ds} - \frac{\partial \psi}{\partial y}\frac{dy}{ds} \tag{10}$$

where ψ is the stream function of the flow. From the kinematics of the body it can be seen that:

$$v_n = (V + \Omega x)\frac{dy}{ds} - (U - \Omega y)\frac{dx}{ds}. \tag{11}$$

For these expressions to be consistent for arbitrary slopes dy/dx, the stream function must satisfy

$$\frac{\partial \psi}{\partial x} = -V - \Omega x \quad , \quad \frac{\partial \psi}{\partial y} = U - \Omega y \quad , \text{ on } S$$

i.e.

$$\psi = Uy - Vx - \frac{1}{2}\Omega(x^2 + y^2) \quad , \text{ on } S \tag{12}$$

We shall see later how vortices can be added to the flow without violating this condition.

Since the flow is assumed irrotational and incompressible, a complex potential can be formed from the velocity potential and the stream function in the usual manner:

$$w(z) = \phi(x, y) + i\psi(x, y)$$

The conditions on w is that it must be analytic exterior to S (except at the sharp trailing edge) and have a vanishing derivative as $z \to \infty$. The form of the boundary condition (12) indicates that the solution is a linear superposition of three unit velocity potentials;

$$w = Uw_1 + Vw_2 + \Omega w_3$$

which must satisfy

$$\mathrm{Im}\{w_1\}_S = \mathrm{Im}\{z\}_S\ ,\quad \mathrm{Im}\{w_2\}_S = \mathrm{Im}\{-iz\}_S\ ,\quad \mathrm{Im}\{w_3\}_S = \mathrm{Im}\left\{-\frac{i}{2}z\bar{z}\right\}_S \tag{13}$$

The subscript on the curly brackets refers to the contour at which the relationships must hold.

Since the value of ψ must be the same at corresponding points in the physical and the map plane, the mapping (4) provides a way of solving these boundary value problems in the ζ plane. For example, the boundary condition for w_1 may be stated as;

$$\mathrm{Im}\{w_1\}_C = \mathrm{Im}\left\{\zeta + \zeta_c + \frac{a^2}{\zeta + \zeta_c}\right\}_C \tag{14}$$

We are looking for a solution that is analytic outside C. One might suggest that $\zeta + \zeta_c + a^2/(\zeta + \zeta_c)$ is in fact the solution for w_1. This would be acceptable, except for the first term, which violates the condition at infinity. However, we have the fact that:

$$\mathrm{Im}\{\zeta\}_C = \mathrm{Im}\left\{-\frac{r_c^2}{\zeta}\right\}_C$$

Thus, by replacing the ζ term, the correct solution can be constructed:

$$w_1 = -\frac{r_c^2}{\zeta} + \zeta_c + \frac{a^2}{\zeta + \zeta_c} \tag{15}$$

The solution could be written explicitly in the physical coordinate through (5), but that would be an unnecessary complication. For example, if we need to know the fluid velocity at a point z, we apply the chain rule:

$$u - iv = \frac{dw}{dz} = \frac{dw}{d\zeta}\frac{d\zeta}{dz} = \left[\frac{dw}{d\zeta}\frac{1}{F'(\zeta)}\right]_{\zeta = F^{-1}(z)}$$

The reason for keeping the constant ζ_c in w_1, is that the boundary condition (12) is assumed to hold without any additional constants in the calculation of the moment on the profile.

In the same way, w_2 is found to be:

$$w_2 = -i\frac{r_c^2}{\zeta} - i\zeta_c - i\frac{a^2}{\zeta + \zeta_c} \tag{16}$$

For w_3, which is somewhat more complicated, we use the intermediate variable ζ_1:

$$
\begin{aligned}
\operatorname{Im}\{w_3\}_C &= \operatorname{Im}\left\{-\frac{i}{2}\left(\zeta_1+\frac{a^2}{\zeta_1}\right)\left(\overline{\zeta}_1+\frac{a^2}{\overline{\zeta}_1}\right)\right\}_C \\
&= \operatorname{Im}\left\{-\frac{i}{2}\left[\zeta_1\overline{\zeta}_1+a^2\left(\frac{\overline{\zeta}_1}{\zeta_1}+\frac{\zeta_1}{\overline{\zeta}_1}\right)+\frac{a^4}{\zeta_1\overline{\zeta}_1}\right]\right\}_C \\
&= \operatorname{Im}\left\{-\frac{i}{2}\left[\zeta_1\overline{\zeta}_1+2a^2\frac{\overline{\zeta}_1}{\zeta_1}+\frac{a^4}{\zeta_1\overline{\zeta}_1}\right]\right\}_C \\
&= \operatorname{Im}\left\{-\frac{i}{2}\left[\zeta\overline{\zeta}+\overline{\zeta}_c\zeta+\zeta_c\overline{\zeta}+\delta^2+2a^2\frac{\overline{\zeta}+\overline{\zeta}_c}{\zeta+\zeta_c}+\frac{a^4}{\zeta\overline{\zeta}+\overline{\zeta}_c\zeta+\zeta_c\overline{\zeta}+\delta^2}\right]\right\}_C
\end{aligned}
\qquad (17)
$$

Now,

$$\zeta\overline{\zeta}=r_c^2 \quad \text{on } C \quad \text{and} \quad \operatorname{Im}\{i\overline{\zeta}_c\zeta\}=\operatorname{Im}\{i\zeta_c\overline{\zeta}\},$$

so that

$$
\begin{aligned}
\operatorname{Im}\{w_3\}_C &= \operatorname{Im}\left\{-\frac{i}{2}\left[r_c^2+2\zeta_c\overline{\zeta}+\delta^2+2a^2\frac{\overline{\zeta}+\overline{\zeta}_c}{\zeta+\zeta_c}+\frac{a^4}{r_c^2+\overline{\zeta}_c\zeta+\zeta_c\overline{\zeta}+\delta^2}\right]\right\}_C \\
&= \operatorname{Im}\left\{-\frac{i}{2}\left[r_c^2+2\zeta_c\frac{r_c^2}{\zeta}+\delta^2+2a^2\frac{r_c^2/\zeta+\overline{\zeta}_c}{\zeta+\zeta_c}+\frac{a^4}{r_c^2+\overline{\zeta}_c\zeta+\zeta_c r_c^2/\zeta+\delta^2}\right]\right\}_C
\end{aligned}
\qquad (18)
$$

The only thing preventing the use of the expression inside the curly brackets as a solution for w_3 is the last term, which contains a dipole singularity at $\zeta=-r_c^2/\overline{\zeta}_c$. This singularity must be neutralized by adding another dipole of opposite strength, according to the circle theorem of two-dimensional potential flow [Batchelor67, MilneThomson60]. It goes as follows: Consider a flow with complex potential $f(z)$, whose singularities are all at a distance greater than r_c away from the origin. A flow with the same singularities and far field behavior, but internally bounded by a circle of radius r_c centered at the origin, has the potential:

$$w(z)=f(z)+\overline{f}\left(\frac{r_c^2}{z}\right)$$

It is straightforward to verify that this potential is purely real on the circle, $|z|=r_c$, making the circle streamline. Now, a dipole of moment q, say, can be put at $\zeta=-r_c^2/\overline{\zeta}_c$ without affecting the boundary condition (12):

$$\frac{q}{\zeta+r_c^2/\overline{\zeta}_c}-\frac{\overline{q}}{r_c^2/\zeta+r_c^2/\overline{\zeta}_c}$$

Here,

$$q=-\frac{ia^4r_c^2}{(r_c^2-\delta^2)\overline{\zeta}_c}$$

We add these terms inside the curcy brackets of (18), and when the terms are combined, the correct solution for w_3 is obtained:

$$w_3=-i\left[\frac{\zeta_c r_c^2}{\zeta}+a^2\frac{r_c^2/\zeta+\overline{\zeta}_c}{\zeta+\zeta_c}+\frac{a^4\zeta_c}{(\zeta+\zeta_c)(r_c^2+\delta^2)}+\frac{a^4+r_c^4-\delta^4}{2(r_c^2-\delta^2)}\right] \qquad (19)$$

Figure 4 shows the streamlines associated with the three unit potentials in the case of a profile with $a=0.5$, $\zeta_c=-0.05+i0.1$.

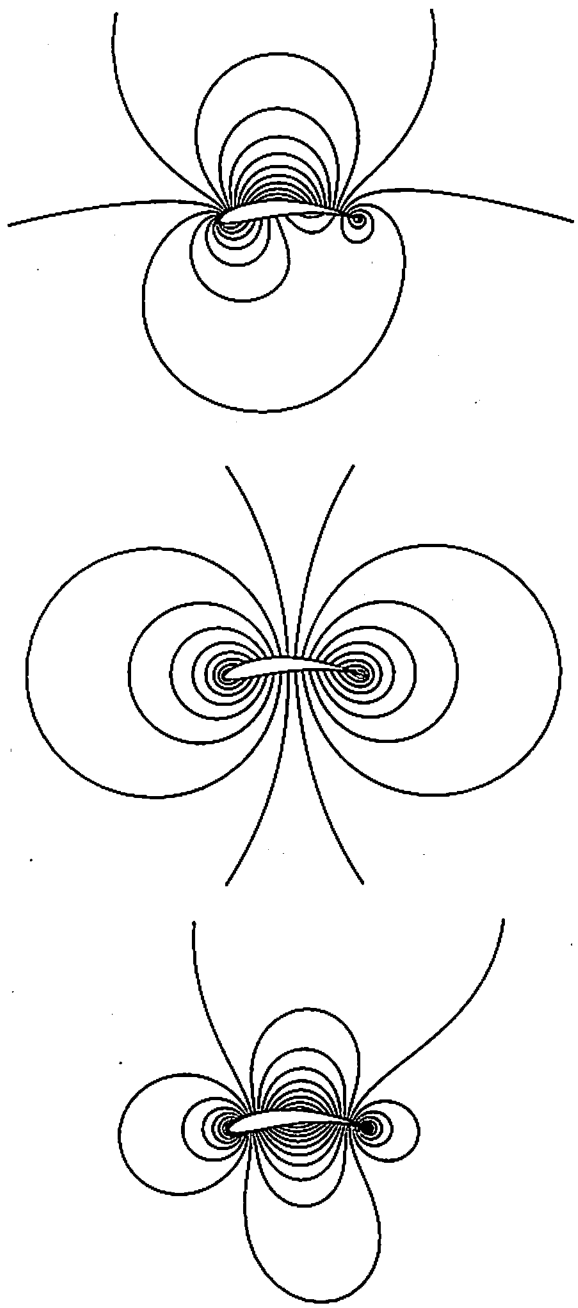

Figure 4: Streamlines of the unit velocity potentials. From top to bottom: w_1, w_2 and w_3.

5 Vortex potentials

An arbitrary amount of circulation around the foil is provided by a central vortex potential in the ζ plane:

$$\gamma_c w_4 = \gamma_c i \log \frac{\zeta}{r_c} \tag{20}$$

w_4 is purely real on C and S, so this term can be added without affecting the boundary condition (12).

Finally, a free vortex at some point z_k, say, is introduced by adding a vortex at the corresponding point in the ζ plane, $\zeta_k = F^{-1}(z_k)$. According to the circle theorem, this vortex must have an oppositely signed image at the *inverse point*, $r_c^2/\overline{\zeta}_k$ and a same sign image at $\zeta = 0$. However, the latter can be absorbed in $\gamma_c w_4$, and is not considered a part of the free vortex potential. Instead, the value of γ_c is determined on physical grounds when a new vortex is introduced in the flow. Any number of vortices of different strengths are incorporated in the flow:

$$\sum_k \gamma_k w_5(\zeta;\zeta_k)$$

where

$$w_5 = i \log \left(-\frac{r_c}{\zeta_k} \frac{\zeta - \zeta_k}{\zeta - r_c^2/\overline{\zeta}_k} \right) \tag{21}$$

As with the central vortex potential, w_5 is written such that it is purely real on S, and (12) holds without any additional constant. In addition, this form of w_5 has the desirable property that for $|\zeta| = \mathcal{O}(r_c)$,

$$w_5 \longrightarrow -i \log \frac{\zeta}{r_c} \quad \text{as} \quad \zeta_k \longrightarrow \infty$$

Thus, as a vortex is removed far away from the circle, its image moves toward the center, giving rise to a potential af the same form as w_4.

A Taylor expansion of w_5 about ζ_k, verifies that it has the correct singular behavior of a vortex at z_k in the physical plane. Note that the strength of the vortices differ from what many researchers use by a factor -2π. For instance, the total counterclockwise circulation about the foil caused by the free vortex images and the central vortex, is:

$$\Gamma = 2\pi \left(\sum_k \gamma_k - \gamma_c \right) \tag{22}$$

The flow in the z plane is now completely described by:

$$w = U w_1(\zeta) + V w_2(\zeta) + \Omega w_3(\zeta) + \gamma_c w_4(\zeta) + \sum_k \gamma_k w_5(\zeta;\zeta_k)$$

and the mapping (4).

6 Vortex convection and vortex shedding algorithm

The vortices convect with velocities given by *Routh's rule* [Clements73, Sarpkaya88, Sheen86];

$$\frac{d\overline{z}_k}{dt} = \frac{d}{dz}\left[w(z) - i\gamma_k \log(z - z_k)\right]_{z=z_k} = \frac{1}{F'(\zeta_k)} \frac{d}{d\zeta}\left[w(\zeta) - i\gamma_k \log(\zeta - \zeta_k)\right]_{\zeta=\zeta_k} - \frac{i\gamma_k}{2} \frac{F''(\zeta_k)}{[F'(\zeta_k)]^2} \tag{23}$$

In the second step the chain rule has been applied, and the last term is a correction due to the difference between $\log(z - z_k)$ and $\log(\zeta - \zeta_k)$. This expression is used to step forward the simulation in a second

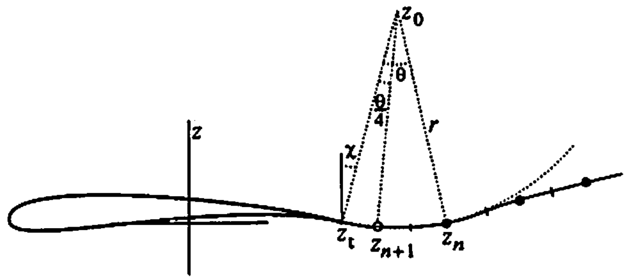

Figure 5: Vortex shedding algorithm.

order Runge-Kutta scheme. The derivative of the terms in the square bracket can be expressed in closed form. It will consist of contributions from the three unit potentials and all the vortex potentials except for the one due to vortex k itself. Thus, as the number of vortices N grows, this $\mathcal{O}(N^2)$ effort quickly becomes prohibitively slow. A method based on multipole expansions [Greengard87, Carrier88, Korsmeyer90] that reduces this effort to $\mathcal{O}(N)$ has been implemented, greatly reducing the computational effort when the number of vortices becomes large.

An algorithm that releases vortices into the flow to satisfy the Kutta condition of smooth flow at the trailing edge, will complete the simulation. A certain controversy exists regarding the validity range of the Kutta condition [McCroskey82], but this question will not be addressed here. We assume that the parameters of foil motion and inflow conditions are such that smooth flow over the trailing edge is a good model.

In general, there is no obvious way to introduce vortices near the trailing edge in order to satisfy the Kutta condition. Matters are simplified, however, by the fact that the trailing edge of the Joukowski foil is cusped, and that the wake must leave the trailing edge parallel to it. A vortex shedding algorithm based on an idea from [Sarpkaya75] is illustrated in Figure 5. The idea is that the location of a new vortex can be determined from an interpolation between the trailing edge and one or more previously shed vortices. Each vortex in the wake (solid dots) represents a segment (between tic marks) of a continuous vortex sheet. When it is time to shed a new vortex, a circular arc tangent to the cusped trailing edge, is fitted through the location of the previously shed vortex, z_n (the trailing edge angle χ is given by the conformal mapping). The center of the arc is denoted z_0 and the angle $z_t z_0 z_n$ is θ. The new vortex is put at z_{n+1} such that the angle $z_t z_0 z_{n+1}$ is $\theta/4$. Consequently, the new vortex will sit approximately in the middle of the segment it is supposed to represent. The following relations hold:

$$\begin{aligned} z_0 - z_n &= re^{i(\frac{\pi}{2}-\chi+\theta)} \\ z_0 - z_t &= re^{i(\frac{\pi}{2}-\chi)} \\ z_0 - z_{n+1} &= re^{i(\frac{\pi}{2}-\chi+\theta/4)} \end{aligned}$$

From these expressions, z_{n+1} can be found in terms of θ:

$$\begin{aligned} \frac{z_{n+1}-z_t}{z_n-z_t} &= \frac{1-e^{i\theta/4}}{1-e^{i\theta}} \\ &= \frac{1-e^{i\theta/4}}{(1-e^{i\theta/4})(1+e^{i\theta/4})(1+e^{i\theta/2})} \\ &= \frac{1}{1+e^{i\theta/4}+e^{i\theta/2}+e^{i3\theta/4}} \end{aligned} \tag{24}$$

and θ is determined from:

$$\frac{(z_n-z_t)^2}{|z_n-z_t|^2} = -e^{-i2\chi}\frac{1-e^{i\theta}}{1-e^{-i\theta}} = e^{-i2\chi}e^{i\theta} \tag{25}$$

Note that these expressions are well behaved at $\theta = 0$.

The strength of the new vortex is given by the requirement that $dw/d\zeta = 0$ at the point which maps onto the trailing edge, $\zeta = a - \zeta_c$. This is necessary for the flow to be smooth at the trailing edge in the physical plane and constitutes the Kutta condition in the simulation:

$$
\begin{aligned}
\frac{dw}{d\zeta} &= U\left[\frac{r_c^2}{\zeta^2} - \frac{a^2}{(\zeta+\zeta_c)^2}\right] + iV\left[\frac{r_c^2}{\zeta^2} + \frac{a^2}{(\zeta+\zeta_c)^2}\right] \\
&\quad + i\Omega\left[\frac{\zeta_c r_c^2}{\zeta^2} + a^2 r_c^2 \frac{2\zeta+\zeta_c}{\zeta^2(\zeta+\zeta_c)^2} + a^2\frac{\bar{\zeta}_c}{(\zeta+\zeta_c)^2} - \frac{a^4\zeta_c}{(\zeta+\zeta_c)^2(r_c^2-\delta^2)}\right] \\
&\quad + i\frac{\gamma_c}{\zeta} + \sum_{k=1}^{n+1} i\gamma_k\left[\frac{1}{\zeta-\zeta_k} - \frac{1}{\zeta - r_c^2/\bar{\zeta}_k}\right] \\
&= 0\,,\ \text{at}\ \zeta = a - \zeta_c
\end{aligned}
\tag{26}
$$

Denoting $a - \zeta_c = r_c e^{i\theta_c}$, $\zeta_k = r_k e^{i\theta_k}$ and $\zeta_c = \delta_R + i\delta_I$, we can solve for γ_{n+1}, and write the answer in terms of real quantities only:

$$
\begin{aligned}
\gamma_{n+1} &= -\frac{r_c^2 - 2r_c r_{n+1}\cos(\theta_c+\theta_{n+1}) + r_{n+1}^2}{r_c^2 - r_{n+1}^2} \\
&\quad \left[2r_c(U\sin\theta_c + V\cos\theta_c) + 2\Omega\left(a^2 - a\delta_R - \delta^2\frac{a-\delta_R}{a-2\delta_R}\right)\right. \\
&\quad \left. + \gamma_c + \sum_{k=1}^{n}\frac{\gamma_k(r_c^2 - r_k^2)}{r_c^2 - 2r_c r_k\cos(\theta_c+\theta_k) + r_k^2}\right]
\end{aligned}
\tag{27}
$$

In cases where the foil starts to move from an initial state of rest and the fluid is otherwise undisturbed, the total circulation around the foil and its wake must remain zero by virtue of Kelvin's circulation theorem. This implies that $\gamma_c = 0$ at all times. In the general case, vortices other than those shed at the trailing edge may be introduced in the flow at any time. Then, γ_c must change so as to preserve the circulation around the foil alone. In fact, due to the way the vortex potentials are defined, the total effect on w of such a vortex vanishes as the point where it is introduced goes to infinity.

We can test this algorithm versus linear theory for a flat plate impulsively pitched to $\alpha_0 = 0.01$. This is a good test, because the the step function response obtained will contain the frequency response for all frequencies. The effect of the finite angle is vanishing for the steady case, we assume this is true also here. Figure 6 shows that the vortex shedding algorithm described above yields a foil circulation in close agreement with linear theory, even for fairly large time steps. Time has been nondimensionalized by c/U_0, where c is the half chord length.

7 Pressure in a rotating and translating coordinate frame

In order to develop expressions for force and moment, it is useful to express the pressure in terms of coordinates that are fixed in the foil, z' in Figure 1. In the z coordinate system, which is an inertia frame, the pressure is given by

$$p = -\frac{\partial\phi}{\partial t} - \frac{1}{2}(u^2 + v^2)$$

where u and v are flow velocity components. Here, as in the rest of this report, fluid density is assumed to be unity. The z' frame moves with velocities $W = U + iV$ and Ω, as given by (3), and the flow velocities

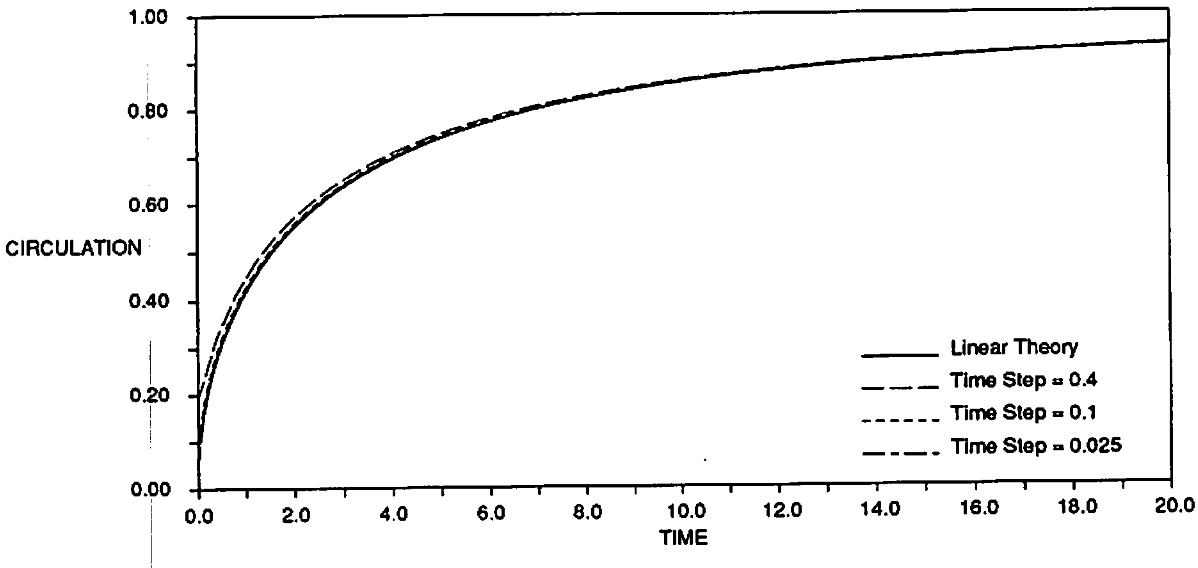

Figure 6: Circulation around an impulsively started flat plate normalized by its steady state value.

seen by an observer in this reference frame are:

$$u' = u - U + \Omega y \ , \ \ v' = v - V - \Omega x$$

For later reference we note that

$$u' + iv' = \overline{\left(\frac{dw}{dz}\right)} - W - i\Omega z \tag{28}$$

Furthermore, the time derivative must be adjusted for spatial variation when the moving coordinate system is used:

$$\frac{\partial}{\partial t} \leftarrow \frac{\partial}{\partial t} - (U - \Omega y)\frac{\partial}{\partial x} - (V + \Omega x)\frac{\partial}{\partial y}$$

Thus, in the foil fixed coordinate system the pressure is given by:

$$p = -\frac{\partial \phi}{\partial t} - \frac{1}{2}(u'^2 + v'^2) + \frac{1}{2}(W + i\Omega z)(\overline{W} - i\Omega \overline{z}) \tag{29}$$

The first term should be though of as the rate of change of the potential that describes the flow in the z frame as seen by an observer in the z' frame.

The utility of expressing pressure this way is that the profile S, over which the pressure must be integrated, is independent of time. Thus, the time derivative in the first term may be taken outside the integral sign.

Figure 7 shows the difference in pressure, calculated by the above expression, between the lower and upper face of a flat plate in impulsively started motion. We note that the simulation results agree well with linear theory, except at the trailing edge where our method fails to predict zero pressure loading. This is a result of discretizing the wake into point vortices, because they cannot represent a flow that is discontinuous across the trailing edge. If the velocity on the upper and lower side of the trailing edge is denoted u_+ and u_- respectively, it is easy to show that

$$\frac{d\Gamma}{dt} = \frac{1}{2}(u_-^2 - u_+^2)$$

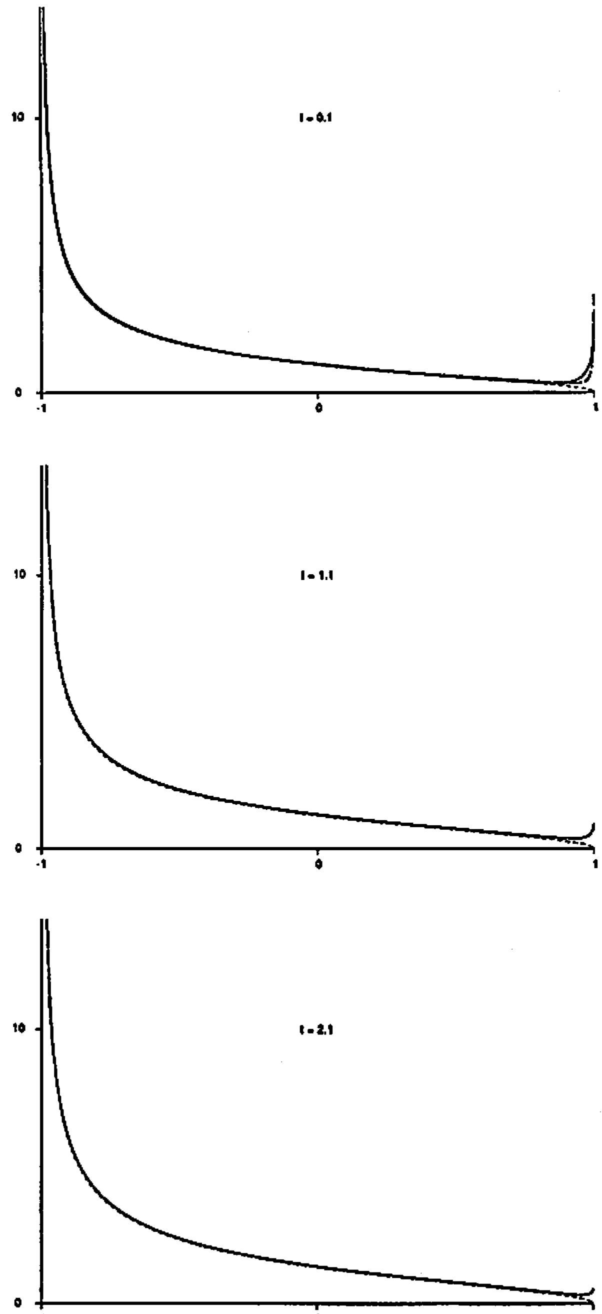

Figure 7: Pressure distribution at three different times for an impulsively started foil. Solid line: $\Delta t = 0.1$, Dashed Line: $\Delta t = 0.02$, Dotted line: linear theory.

is necessary for zero pressure loading [Basu78]. From linear theory we know that $d\Gamma/dt \sim t^{-1/2}$ as $t \to 0$, i.e. the velocity discontinuity is arbitrarily large at $t = 0$. Therefore, it is not surprising that the error in the simulation is largest initially. Furthermore we note that the discrepancy is confined to a smaller region when the time step is reduced and the trailing edge vortices are more packed together.

8 Force

Representing the forces per unit span X and Y along the x and y axes as one complex number, it is easily seen that

$$X + iY = i\int_S p dz \tag{30}$$

Using (29), we obtain:

$$\begin{aligned} X + iY &= -i\frac{d}{dt}\int_S \phi dz - \frac{i}{2}\int_S (u'^2 + v'^2)dz + \frac{i}{2}\int_S (W + i\Omega z)(\overline{W} - i\Omega\overline{z})dz \\ &= -i\frac{d}{dt}\left[\int_S w dz + iWA - \Omega A z_c\right] - \frac{i}{2}\int_S \overline{\left[\frac{dw}{dz} - \overline{W} + i\Omega\overline{z}\right]^2} dz + i\Omega A(W + i\Omega z_c) \\ &= -i\frac{d}{dt}\int_S w dz + \dot{W}A + i\dot{\Omega}Az_c - \frac{i}{2}\int_S \overline{\left[\left(\frac{dw}{dz}\right)^2 + 2\frac{dw}{dz}(-\overline{W} + i\Omega\overline{z})\right]} dz - i\Omega A(W + i\Omega z_c) \\ &= -i\frac{d}{dt}\int_S w dz + \dot{W}A + i\dot{\Omega}Az_c - \frac{i}{2}\int_S \overline{\left(\frac{dw}{dz}\right)^2} dz - \Omega\int_S z dw + iW\Gamma + i\Omega A(W + i\Omega z_c) \end{aligned} \tag{31}$$

where A, z_c and Γ are given by (7), (8) and (22), respectively. To obtain these expressions, we have made use of (6), (28), the fact that the flow is parallel to S:

$$(u'^2 + v'^2)dz = (u' + iv')(u' - iv')dz = (u' + iv')^2 d\overline{z}$$

and the boundary condition (12):

$$\phi = w - i\psi = w - \frac{1}{2}(\overline{W}z - W\overline{z} - i\Omega z\overline{z}), \text{on} S$$

More details are given in [MilneThomson60].

The three integrals in (31) will be referred to as (I), (II) and (III) respectively. They can be evaluated by replacing the contour of integration by one that encloses all singularities and branch cuts, as well as S, and is closed at a large distance from the foil. This method has been used by [Graham80] and [Sarpkaya75] for flow over stationary objects. It is an extension of Lagally's theorem, which is valid for vortices that are being held fixed in a steady flow. The contour of integration is shown in Figure 8 for the case of a single free vortex. We have denoted the vortex position z'_k as a reminder that its time derivative must be taken as seen by an observer moving with the foil,

$$z'_k = z_k = F(\zeta_k) \text{ , but } \frac{dz'_k}{dt} = \frac{dz_k}{dt} - W - i\Omega z_k = \frac{d}{dt}F(\zeta_k)$$

In the region within the entire contour, all the integrands are analytic, so the sum of the integrals evaluated over the different parts of the contour must be zero. This yields expressions for the integrals over S in terms of the other contributions, which are easier to evaluate:

$$\int_S = \int_{S_\infty} - \int_{S_v} - \int_{S_b}$$

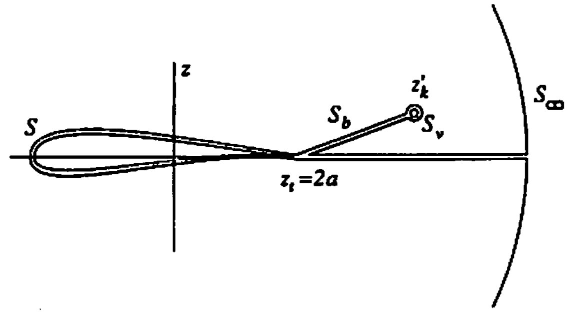

Figure 8: Integration contour for force.

- the direction of integration being counterclockwise. S_b encloses the branch cut between the vortex and its image, passing through the trailing edge. The integrals in (31) can be evaluated as follows:

(I) In the first integral, w_4 is not suited to integration on the contour in Figure 8, and will be evaluated later directly on S. Omitting w_4, the first integral gets a contribution from the cut S_b, where w makes a jump $2\pi\gamma_k$ from the upper to the lower bank, and from S_∞. The following expression is then true for arbitrary profiles:

$$\int_S w dz = \int_{S_\infty} w dz - 2\pi\gamma_k(z_k' - z_t)$$

where z_t is the location of the trailing edge. Limiting ourselves to the previously defined Joukowski profile, we can readily find the far field behavior of w and identify the terms that go like $1/z$. By the residue theorem we then find the contribution from S_∞:

$$\int_{S_\infty} w dz = 2\pi\left[iU\left(a^2 - r_c^2\right) + V\left(a^2 + r_c^2\right) + \Omega\left(r_c^2\zeta_c + a^2\overline{\zeta}_c - \frac{a^4\zeta_c}{r_c^2 - \delta^2}\right) + \gamma_k\left(\zeta_k - \frac{r_c^2}{\overline{\zeta}_k}\right)\right]$$

Then we must add the contribution from $\gamma_c w_4$ by integrating by parts on S:

$$\int_S w_4 dz = [w_4 z]_-^+ - \int_S \frac{dw_4}{dz} z dz = -2\pi z_{cut} - \int_C F(\zeta) dw_4 = -2\pi(z_{cut} - \zeta_c)$$

The square bracket with limits means that the difference should be taken at opposite sides of the branch cut associated with the logarithmic potential, with the upper and lower limits corresponding to a counterclockwise direction of integration. The value of the integral depends on where this branch cut is chosen to be. The choice is made based on the notion that when an external vortex is introduced, the effect on the total potential integrated around the foil must vanish as the vortex is introduced further away. This requires that the $z_{cut} = 2a$, i.e. the branch cut for w_4 must pass through the trailing edge just like for the free vortices. Now,

$$\begin{aligned}\int_S w dz \;&=\; 2\pi\left[iU\left(a^2 - r_c^2\right) + V\left(a^2 + r_c^2\right) + \Omega\left(r_c^2\zeta_c + a^2\overline{\zeta}_c - \frac{a^4\zeta_c}{r_c^2 - \delta^2}\right)\right.\\ &\left. -\gamma_c(2a - \zeta_c) + \gamma_k\left(\zeta_k - \frac{r_c^2}{\overline{\zeta}_k} - z_k' + 2a\right)\right] \qquad (32)\end{aligned}$$

(II) The second integral gets no contributions from the cut S_b where the integrand is continuous. Neither is there a contribution from S_∞, as $w = \mathcal{O}(\log z)$ in the far field (this is in contrast to steady flow past a stationary wing which gets $U\Gamma$ from this term). The only contribution is from S_v, this term is writing the potential with the singularity at z_k separated out:

$$w(z) = f_k(z) + i\gamma_k \log(z - z_k)$$

where $f_k(z)$ is analytic in the neighborhood of z_k. Then

$$\int_S \left(\frac{dw}{dz}\right)^2 dz = -\int_{S_v} \left(f_k'(z) + \frac{i\gamma_k}{z - z_k}\right)^2 dz = 4\pi\gamma_k f_k'(z_k)$$

where again, the prime denotes z derivative. From (23) it is seen that $\overline{f_k'(z_k)}$ is the convection velocity of the vortex, as seen in the z reference frame, i.e.

$$\overline{\int_S \left(\frac{dw}{dz}\right)^2 dz} = 4\pi\gamma_k \frac{dz_k}{dt} = 4\pi\gamma_k \left(\frac{dz_k'}{dt} + W + i\Omega z_k'\right) \tag{33}$$

This expression is valid for an arbitrary profile.

(III) In the last integral, $\gamma_c w_4$ also requires special attention. Apart from this term, we have

$$\int_S z dw = \int_S \frac{dw}{dz} z dz$$

which gets contributions from S_∞ and S_v. The latter is easily found, and we have the general expression:

$$\int_S z dw = \int_{S_\infty} \frac{dw}{dz} z dz + 2\pi\gamma_k z_k'$$

For the Joukowski profile,

$$\int_{S_\infty} z dw = 2\pi \left[iU\left(r_c^2 - a^2\right) - V\left(r_c^2 + a^2\right) - \Omega\left(r_c^2\zeta_c + a^2\overline{\zeta}_c - \frac{a^4\zeta_c}{r_c^2 - \delta^2}\right) + \gamma_k\left(\frac{r_c^2}{\overline{\zeta}_k} - \zeta_k\right)\right]$$

We saw above that

$$\int_S z dw_4 = -2\pi\zeta_c$$

Thus,

$$\int_S z dw = 2\pi \left[iU\left(r_c^2 - a^2\right) - V\left(r_c^2 + a^2\right) - \Omega\left(r_c^2\zeta_c + a^2\overline{\zeta}_c - \frac{a^4\zeta_c}{r_c^2 - \delta^2}\right) - \gamma_c\zeta_c + \gamma_k\left(\frac{r_c^2}{\overline{\zeta}_k} - \zeta_k + z_k'\right)\right]$$

We have now calculated all the necessary integrals for a closed form force expression. In cases with $\gamma_c = 0$, the following expression may be used to evaluate the force on an arbitrary profile in the presence of point vortices, requiring only the far field behavior of the velocity potential:

$$\begin{aligned} X + iY &= -i\frac{d}{dt}\left[\int_{S_\infty} w dz - 2\pi\sum_k \gamma_k(z_k' - z_t)\right] + \dot{W}A + i\dot{\Omega}Az_c - 2\pi i\sum_k \gamma_k \frac{dz_k'}{dt} \\ &\quad - \int_{S_\infty} \frac{dw}{dz} z dz + i\Omega A(W + i\Omega z_c) \end{aligned} \tag{34}$$

We have extended the result to an arbitrary number of vortices simply by summing over k. Two terms cancelled due to (22). In cases where the number of vortices around the profile remain constant, the time derivative may be taken inside the first summation and cancellation with the second sum is obtained. Here we want to use the above expression when vortices are released every time step into the flow, and we leave the time derivative outside the sum to take into account the changing number of vortices.

For the Joukowski foil, we use the results (I) - (III), and collect terms to obtain:

$$\begin{aligned} X+iY &= -\dot{U}[2\pi(r_c^2-a^2)-A] \\ &\quad -i\dot{V}[2\pi(r_c^2+a^2)-A] \\ &\quad -i\dot{\Omega}\left[2\pi\left(r_c^2\zeta_c+a^2\overline{\zeta}_c-\frac{a^4\zeta_c}{r_c^2-\delta^2}\right)-Az_c\right] \\ &\quad -iU\Omega\left[2\pi(r_c^2-a^2)-A\right] \\ &\quad +V\Omega\left[2\pi(r_c^2+a^2)-A\right] \\ &\quad +\Omega^2\left[2\pi\left(r_c^2\zeta_c+a^2\overline{\zeta}_c-\frac{a^4\zeta_c}{r_c^2-\delta^2}\right)-z_cA\right] \\ &\quad +iW\Gamma \\ &\quad -i\frac{d}{dt}\left[-2\pi\gamma_c(2a-\zeta_c)+2\pi\sum_k\gamma_k\left(2a-\zeta_c-\frac{r_c^2}{\overline{\zeta}_k}-\frac{a^2}{\zeta_k+\zeta_c}\right)\right] \\ &\quad -2\pi i\sum_k\gamma_k\frac{dz_k}{dt} \\ &\quad -\Omega\left[-2\pi\gamma_c\zeta_c+2\pi\sum_k\gamma_k\left(\frac{r_c^2}{\overline{\zeta}_k}+\zeta_c+\frac{a^2}{\zeta_k+\zeta_c}\right)\right] \end{aligned} \tag{35}$$

The conventional added mass notation, m_{ij} [Newman77], and the relation (22) provides a compact expression for the forces:

$$\begin{aligned} X+iY &= -\dot{U}(m_{11}+im_{21})-\dot{V}(m_{12}+im_{22})-\dot{\Omega}(m_{16}+im_{26}) \\ &\quad +\Omega U(m_{21}-im_{11})+\Omega V(m_{22}-im_{12})+\Omega^2(m_{26}-im_{16}) \\ &\quad +i(W+i\Omega\zeta_c)\Gamma+i\frac{d}{dt}[2\pi\mathcal{Z}_1-\Gamma(2a-\zeta_c)]-2\pi i\mathcal{Z}_2-2\pi\Omega\mathcal{Z}_1 \end{aligned} \tag{36}$$

where the added mass coefficients have been identified as:

$$m_{11} = \pi r_c^2-2\pi a^2+\pi r_c^2\frac{a^4}{(r_c^2-\delta^2)^2} \tag{37}$$

$$m_{12} = 0 \tag{38}$$

$$m_{21} = 0 \tag{39}$$

$$m_{22} = \pi r_c^2+2\pi a^2+\pi r_c^2\frac{a^4}{(r_c^2-\delta^2)^2} \tag{40}$$

$$m_{16} = Im\left\{-\pi r_c^2\zeta_c-2\pi a^2\overline{\zeta}_c+2\pi\frac{a^4\zeta_c}{r_c^2-\delta^2}+\pi r_c^2\frac{a^6\overline{\zeta}_c}{(r_c^2-\delta^2)^3}\right\} \tag{41}$$

$$m_{26} = Re\left\{\pi r_c^2\zeta_c+2\pi a^2\overline{\zeta}_c-2\pi\frac{a^4\zeta_c}{r_c^2-\delta^2}-\pi r_c^2\frac{a^6\overline{\zeta}_c}{(r_c^2-\delta^2)^3}\right\} \tag{42}$$

and

$$\mathcal{Z}_1 = \sum_k\gamma_k\left(\frac{r_c^2}{\overline{\zeta}_k}+\frac{a^2}{\zeta_k+\zeta_c}\right) \tag{43}$$

$$\mathcal{Z}_2 = \sum_k\gamma_k\frac{dz_k}{dt} \tag{44}$$

The added mass coefficients have been verified versus Sedov's [Sedov65] expressions, in the special case of a symmetric profile. Note that there is no added mass coupling between surge and heave even for a cambered Joukowski foil.

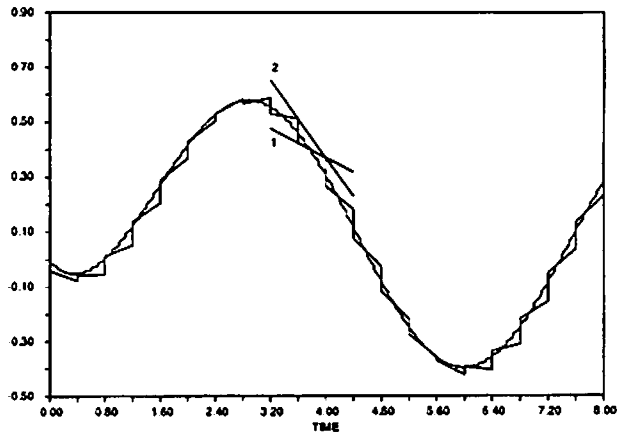

Figure 9: Typical time history of sum in brackets in equation (36) for two different time steps, $\Delta t = 0.4$ and $\Delta t = 0.1$.

The vortex convection velocities in $\mathcal{Z}_2$, given by (23), are expensive to calculate. However, these velocities are needed to perform the simulation in the first place, so there is no extra effort required for the force calculation.

Figure 9 shows a typical time history for the expression in square brackets in (36), whose time derivative we need to calculate forces. At the beginning of every time step, a new vortex is released and this sum makes a jump, which can be found to be of order $\Delta t^{3/2}$. It now becomes clear why it is important to take the growing number of vortices into account when the time derivative is taken. If we fail to do this, we effectively calculate the slope as marked by "1" in Figure 9, whereas the correct way consistently uses the values at either the beginning or end of each time step, marked "2". Method 1 will still converge as the time step is reduced, but only at a rate $\sqrt{\Delta t}$.

Using method 2 with a central difference gives the time derivative at intermediate times, $t = (n+1/2)\,\Delta t$. The values for Γ, $\mathcal{Z}_1$, and $\mathcal{Z}_2$ are available at times that are integer multiples of the time step in the simulation, $t = n\,\Delta t$. To form the force we must then linearly interpolate Γ, $\mathcal{Z}_1$, and $\mathcal{Z}_2$ to $t = (n+1/2)\,\Delta t$. The added mass forces can be exactly evaluated for any t, and are added. This is a more precise calculation than taking the central difference over two time steps and adding at $t = n\,\Delta t$.

Figure 10 shows how this calculation compares for an impulsively started flat plate at angle of attack, α_0. Lift and drag has been nondimensionalized by $2\pi U^2 c\alpha_0$ and $U^2 c\alpha_0^2$, respectively. We note that the results converge everywhere except at $t = 0$, as could be expected from the analysis of the pressure distribution.

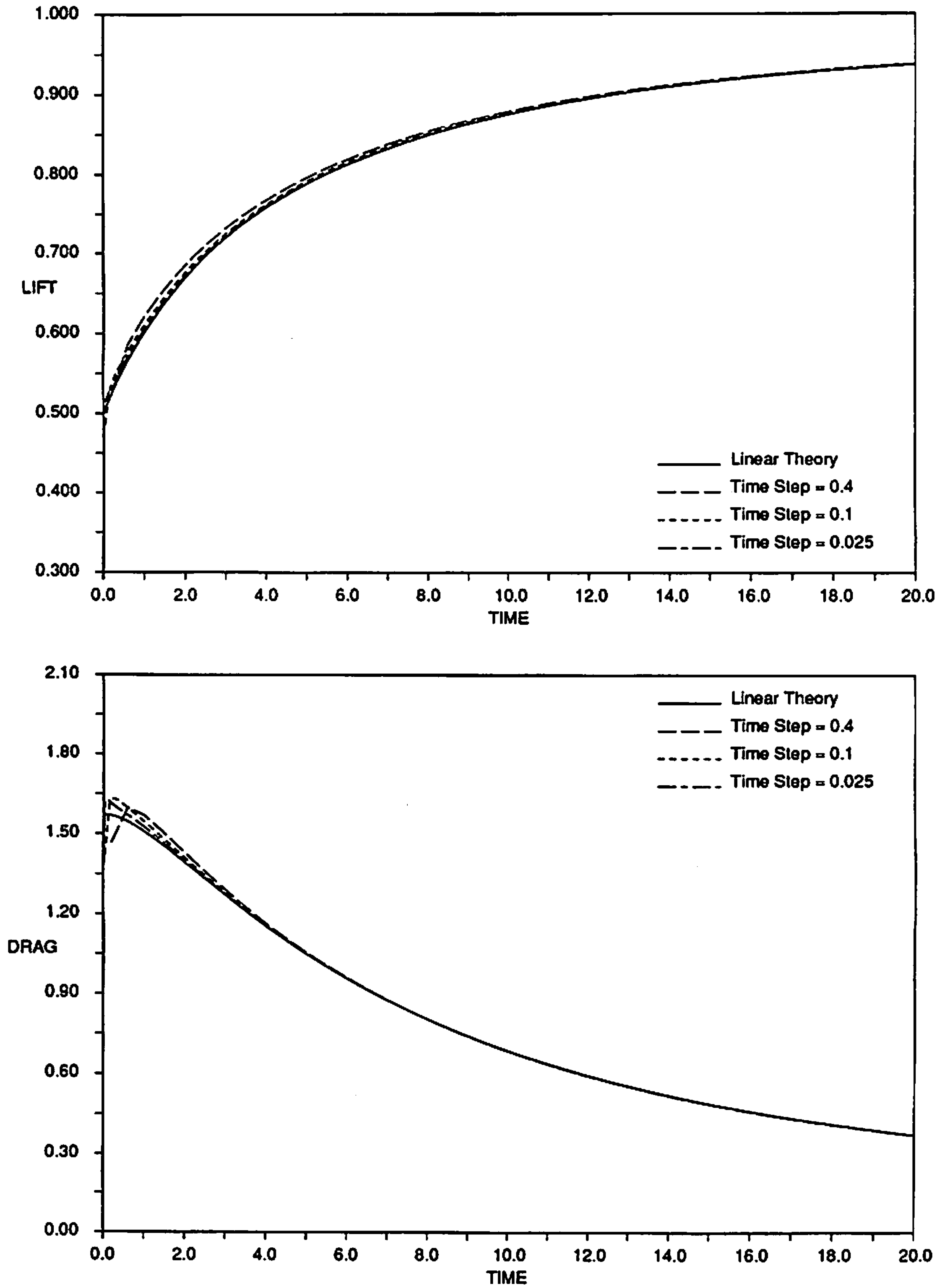

Figure 10: Forces on an impulsively started flat plate.

9 Moment

The counterclockwise moment on the foil about the point $z = 0$ is seen to be

$$M = Re\left\{\int_S pz d\overline{z}\right\} \tag{45}$$

Using the same technique as in the force calculation, we have:

$$\begin{aligned}
M &= Re\left\{-\frac{d}{dt}\int_S \phi z d\overline{z} - \frac{1}{2}\int_S (u'^2 + v'^2) z d\overline{z} + \frac{1}{2}\int_S |W + i\Omega z|^2 z d\overline{z}\right\} \\
&= Re\left\{-\frac{d}{dt}\left[\int_S wz d\overline{z} + 3i\overline{W} z_c A + 2\Omega r_g^2 A\right] - \frac{1}{2}\int_S \left[\frac{dw}{dz} - \overline{W} + i\Omega\overline{z}\right]^2 z dz + \Omega\overline{W} z_c A\right\} \\
&= Re\left\{-\frac{d}{dt}\int_S wz d\overline{z} - 3i\dot{\overline{W}} z_c A - 2\dot{\Omega} r_g^2 A - \frac{1}{2}\int_S \left(\frac{dw}{dz}\right)^2 z dz + \overline{W}\int_S z dw + \Omega\overline{W} z_c A\right\} \qquad (46)
\end{aligned}$$

where again, we denote the integrals (I), (II) and (III) respectively. Note that additive constants in w will give a contribution to (I), so it is important to formulate all the potentials w_i so that the boundary condition (12) holds. The first integrand is not analytic, and the contour decomposition that we used in the force calculation is not applicable here, no general expression equivalent to (34) exists. Instead, the mapping function will be substituted for z and the integral evaluated on the map circle C, where we may use the fact that $\overline{\zeta} = r_c^2/\zeta$. But first, (I) is further broken down into 5 parts:

$$\int_S wz d\overline{z} = U\int_S w_1 z d\overline{z} + V\int_S w_2 z d\overline{z} + \Omega\int_S w_3 z d\overline{z} + \gamma_c\int_S w_4 z d\overline{z} + \gamma_k\int_S w_5 z d\overline{z}$$

We shall refer to these intgrals as (I a) through (I e).

(I a)

$$\begin{aligned}
\int_S w_1 z d\overline{z} &= \int_C w_1 F(\zeta)\overline{F'(\zeta) d\zeta} \\
&= \int_C \left(-\frac{r_c^2}{\zeta} + \zeta_c + \frac{a^2}{\zeta + \zeta_c}\right)\left(\zeta + \zeta_c + \frac{a^2}{\zeta + \zeta_c}\right)\left(1 - \frac{a^2}{\overline{\zeta + \zeta_c}}\right) d\overline{\zeta} \\
&= \int_C \left(-\frac{r_c^2}{\zeta} + \zeta_c + \frac{a^2}{\zeta + \zeta_c}\right)\left(\zeta + \zeta_c + \frac{a^2}{\zeta + \zeta_c}\right)\left(-\frac{r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2 + \overline{\zeta}_c\zeta)^2}\right) d\zeta \\
&= 2\pi i\left[-a^2\zeta_c + a^4\overline{\zeta}_c\left(\frac{1}{r_c^2 - \delta^2} + \frac{r_c^2}{(r_c^2 - \delta^2)^2}\right) - 2\frac{a^6 r_c^2}{(r_c^2 - \delta^2)^3}\overline{\zeta}_c - r_c^2\zeta_c + \frac{a^4 r_c^2}{(r_c^2 - \delta^2)^2}\zeta_c\right] \qquad (47)
\end{aligned}$$

In the last step, the integrand was expanded out in 12 terms that were individually evaluated by the residue theorem.

(I b)

$$\int_S w_2 z d\overline{z} = 2\pi\left[a^2\zeta_c - a^4\overline{\zeta}_c\left(\frac{1}{r_c^2 - \delta^2} + \frac{r_c^2}{(r_c^2 - \delta^2)^2}\right) - 2\frac{a^6 r_c^2}{(r_c^2 - \delta^2)^3}\overline{\zeta}_c - r_c^2\zeta_c + \frac{a^4 r_c^2}{(r_c^2 - \delta^2)^2}\zeta_c\right] \tag{48}$$

(I c)

$$\int_S w_3 z d\overline{z} = 2\pi\left[a^4 + a^2\zeta_c^2 + a^4\frac{\delta^4 - 2r_c^2\delta^2}{(r_c^2 - \delta^2)^2} + \frac{a^6\overline{\zeta}_c^2}{(r_c^2 - \delta^2)^2} + \frac{2a^8 r_c^2\delta^2}{(r_c^2 - \delta^2)^4} - r_c^2\frac{a^4 + r_c^4 - \delta^4}{2(r_c^2 - \delta^2)} + a^4 r_c^2\frac{a^4 + r_c^4 - \delta^4}{(r_c^2 - \delta^2)^3}\right] \tag{49}$$

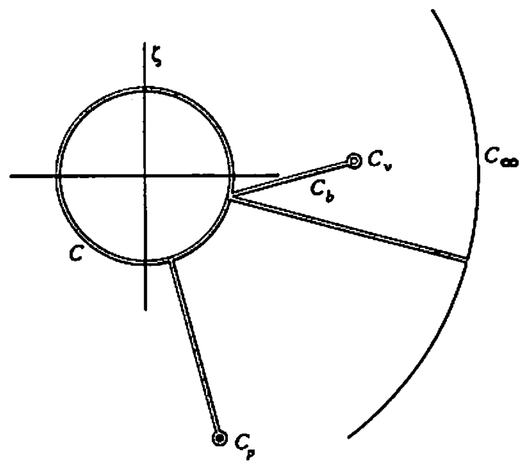

Figure 11: Integration contour for moment.

(I d) is difficult to evaluate, and in fact there is no need to. By arguing that a vortex introduced far away should not affect the integral (I), we can set (I d)$= -\lim_{\zeta_k \to \infty}$(I e).

(I e) Expanding out, this integral can be written:

$$\int_C i\log\frac{-r_c(\zeta-\zeta_k)}{\zeta_k(\zeta-r_c^2/\bar{\zeta}_k)}\left[-\frac{r_c^2}{\zeta}-\frac{r_c^2\zeta_c}{\zeta^2}-\frac{a^2r_c^2}{\zeta^2(\zeta+\zeta_c)}+\frac{a^2r_c^2\zeta}{(r_c^2+\bar{\zeta}_c\zeta)^2}+\frac{a^2r_c^2\zeta_c}{(r_c^2+\bar{\zeta}_c\zeta)^2}+\frac{a^4r_c^2}{(\zeta+\zeta_c)(r_c^2+\bar{\zeta}_c\zeta)^2}\right]d\zeta$$

Let $g(\zeta)$ denote the integrand. A partial fraction expansion of the 3rd and 6th term in the square brackets yields, after collecting terms:

$$g(\zeta) = i\log\frac{-r_c(\zeta-\zeta_k)}{\zeta_k(\zeta-r_c^2/\bar{\zeta}_k)}\left[\frac{\mathcal{A}}{\zeta}+\frac{\mathcal{B}}{\zeta^2}+\frac{\mathcal{C}}{\zeta+\zeta_c}+\frac{\mathcal{D}}{\zeta+r_c^2/\bar{\zeta}_c}+\frac{\mathcal{E}+\mathcal{F}\zeta}{(\zeta+r_c^2/\bar{\zeta}_c)^2}\right]$$

where

$$\begin{aligned}
\mathcal{A} &= -r_c^2+r_c^2\frac{a^2}{\zeta_c^2}\\
\mathcal{B} &= -r_c^2\zeta_c-r_c^2\frac{a^2}{\zeta_c}\\
\mathcal{C} &= -\frac{r_c^2a^2}{\zeta_c^2}+\frac{r_c^2a^4}{(r_c^2-\delta^2)^2}\\
\mathcal{D} &= -\frac{r_c^2a^4}{(r_c^2-\delta^2)^2}\\
\mathcal{E} &= -\frac{r_c^2a^4}{(r_c^2-\delta^2)\bar{\zeta}_c}+\frac{r_c^2a^2\zeta_c}{\bar{\zeta}_c^2}\\
\mathcal{F} &= \frac{r_c^2a^2}{\bar{\zeta}_c^2}
\end{aligned}$$

It is now apparent that g has a pole at $\zeta = -r_c^2/\bar{\zeta}_c$, in addition to the singularities inside C and at ζ_k. The integration contour in the ζ plane is shown in Figure 11. The branch cut between the vortex and its image runs inside C_b, which intersects C at the point that maps onto the trailing edge, $\zeta = a - \zeta_c$. C_p surrounds the pole at $-r_c/\overline{\zeta}_c$, and the contour is closed at a large distance by C_∞. We have

$$\int_C = \int_{C_\infty} - \int_{C_p} - \int_{C_b}$$

The vortex at ζ_k has a logarithmic singularity, which is too weak to give a contribution as C_v shrinks.

The terms of g that go like $1/\zeta$ in the far field are easily identified, and we get the following contribution from C_∞:

$$\int_{C_\infty} g(\zeta)d\zeta = -2\pi(\mathcal{A}+\mathcal{C}+\mathcal{D}+\mathcal{F})\log\frac{-r_c}{\zeta_k}$$

The contribution from C_p is found from the residue of a first and second order pole:

$$\begin{aligned}\int_{C_p} g(\zeta)d\zeta &= 2\pi i\left\{\mathcal{D}i\log\frac{-r_c(\zeta-\zeta_k)}{\zeta_k(\zeta-r_c^2/\overline{\zeta}_k)} + \frac{d}{d\zeta}\left[(\mathcal{E}+\mathcal{F}\zeta)i\log\frac{-r_c(\zeta-\zeta_k)}{\zeta_k(\zeta-r_c^2/\overline{\zeta}_k)}\right]\right\}_{\zeta=-r_c^2/\overline{\zeta}_c} \\ &= -2\pi\left[(\mathcal{D}+\mathcal{F})\log\frac{-r_c(r_c^2/\overline{\zeta}_c+\zeta_k)}{\zeta_k(r_c^2/\overline{\zeta}_c+r_c^2/\overline{\zeta}_k)} + (\mathcal{E}-\mathcal{F}r_c^2/\overline{\zeta}_c)\left(-\frac{1}{r_c^2/\overline{\zeta}_c+\zeta_k}+\frac{1}{r_c^2/\overline{\zeta}_c+r_c^2/\overline{\zeta}_k}\right)\right]\end{aligned}$$

The contribution from C_b can be evaluated using the fact that the logarithmic term in $g(\zeta)$ makes a jump of 2π from the upper to the lower bank of C_b. Thus

$$\int_{C_b} g(\zeta)d\zeta = \int_{a-\zeta_c}^{\zeta_k} 2\pi\left[\frac{\mathcal{A}}{\zeta}+\frac{\mathcal{B}}{\zeta^2}+\frac{\mathcal{C}}{\zeta+\zeta_c}+\frac{\mathcal{D}}{\zeta+r_c^2/\overline{\zeta}_c}+\frac{\mathcal{E}+\mathcal{F}\zeta}{(\zeta+r_c^2/\overline{\zeta}_c)^2}\right]d\zeta$$

This is a sum of elementary line integrals in ζ, running from $a-\zeta_c$ to ζ_k. The last term is conveniently evaluated by integrating by parts, and we have:

$$\begin{aligned}\int_{C_b} g(\zeta)d\zeta &= 2\pi\mathcal{A}\log\frac{\zeta_k}{a-\zeta_c} - 2\pi\mathcal{B}\left(\frac{1}{\zeta_k}-\frac{1}{a-\zeta_c}\right) + 2\pi\mathcal{C}\log\frac{\zeta_c+\zeta_k}{a} \\ &+ 2\pi(\mathcal{D}+\mathcal{F})\log\frac{r_c^2/\overline{\zeta}_c+\zeta_k}{a-\zeta_c+r_c^2/\overline{\zeta}_c} - 2\pi\frac{\mathcal{E}+\mathcal{F}\zeta_k}{\zeta_k+r_c^2/\overline{\zeta}_c} + 2\pi\frac{\mathcal{E}+\mathcal{F}(a-\zeta_c)}{a-\zeta_c+r_c^2/\overline{\zeta}_c}\end{aligned}$$

Then, all the terms can be collected to find the integral (Ie):

$$\begin{aligned}\int_S w_5 z d\overline{z} &= 2\pi\mathcal{A}\log\frac{\zeta_c-a}{r_c} + 2\pi\mathcal{B}\left(\frac{1}{\zeta_k}-\frac{1}{a-\zeta_c}\right) - 2\pi\mathcal{C}\log\frac{-r_c(\zeta_c+\zeta_k)}{a\zeta_k} \\ &+ 2\pi\mathcal{D}\log\frac{a-\zeta_c+r_c^2/\overline{\zeta}_c}{r_c^2/\overline{\zeta}_c+r_c^2/\overline{\zeta}_k} + 2\pi\mathcal{E}\left(\frac{1}{r_c^2/\overline{\zeta}_c+r_c^2/\overline{\zeta}_k}-\frac{1}{a-\zeta_c+r_c^2/\overline{\zeta}_c}\right) \\ &+ 2\pi\mathcal{F}\left(\log\frac{a-\zeta_c+r_c^2/\overline{\zeta}_c}{r_c^2/\overline{\zeta}_c+r_c^2/\overline{\zeta}_k}+1-\frac{\overline{\zeta}_k}{\overline{\zeta}_k+\overline{\zeta}_c}-\frac{a-\zeta_c}{a-\zeta_c+r_c^2/\overline{\zeta}_c}\right)\end{aligned} \tag{50}$$

Upon taking the real part, the logarithmic terms cancel. This should come as no surprise, since the logarithm is a multivalued function. Now, employing the relation

$$r_c^2 = (a-\zeta_c)(a-\overline{\zeta}_c)$$

we get

$$Re\left\{\int_S w_5 z d\overline{z}\right\} = 2\pi Re\left\{\mathcal{B}\left(\frac{1}{\zeta_k}-\frac{1}{a-\zeta_c}\right)+\mathcal{E}\left(\frac{1}{r_c^2/\overline{\zeta}_c+r_c^2/\overline{\zeta}_k}-\frac{\overline{\zeta}_c}{a(a-\zeta_c)}\right)+\mathcal{F}\left(\frac{\overline{\zeta}_c}{\overline{\zeta}_k+\overline{\zeta}_c}-\frac{\overline{\zeta}_c}{a}\right)\right\}$$

$\mathcal{B}, \mathcal{E}, \mathcal{F}$ blow up as $\zeta_c \to 0$, so this expression is not suitable for the special case of a flat plate. With this in mind, we can substitute and contract terms until we reach the following result:

$$Re\left\{\int_S w_5 z d\overline{z}\right\} = 2\pi Re\left\{\frac{-r_c^2\zeta_c}{\zeta_k} - \frac{a^2 r_c^2}{\zeta_k(\zeta_k+\zeta_c)} + a^2 - \delta^2 - \frac{a^4\zeta_c}{(\delta^2 - r_c^2)(\overline{\zeta}_k + \overline{\zeta}_c)} - \frac{a^2\overline{\zeta}_c}{\zeta_k + \zeta_c} + a\zeta_c + \frac{a^3\zeta_c}{\delta^2 - r_c^2}\right\} \tag{51}$$

(I d) Letting $\zeta_k \to \infty$ in the above expression yields the integral (I d):

$$Re\left\{\int_S w_4 z d\overline{z}\right\} = -2\pi Re\left\{a^2 - \delta^2 + a\zeta_c + \frac{a^3\zeta_c}{\delta^2 - r_c^2}\right\} \tag{52}$$

(II) The second integral in (46) can be evaluated by separating out the singular part of w at ζ_k the same way as in the case of force. The result is:

$$\int_S \left(\frac{dw}{dz}\right)^2 z dz = 4\pi\gamma_k \frac{d\overline{z}_k}{dt} z_k \tag{53}$$

(III) The third integral was found in the force calculation:

$$\int_S \frac{dw}{dz} z dz = 2\pi\left[iU(r_c^2 - a^2) - V(r_c^2 + a^2) - \Omega\left(r_c^2\zeta_c + a^2\overline{\zeta}_c - \frac{a^4}{r_c^2 - \delta^2}\zeta_c\right) + \gamma_c\zeta_c + \gamma_k\left(\frac{r_c^2}{\overline{\zeta}_k} - \zeta_k + z_k'\right)\right] \tag{54}$$

The moment can now be written in closed form:

$$\begin{aligned}
M \;=\; & -\dot{U}\pi\left[2a^2 - r_c^2 + 2\frac{a^4}{r_c^2 - \delta^2} - \frac{r_c^2 a^6}{(r_c^2 - \delta^2)^3}\right]\delta_I \\
& - \dot{V}\pi\left[2a^2 + r_c^2 - 2\frac{a^4}{r_c^2 - \delta^2} - \frac{r_c^2 a^6}{(r_c^2 - \delta^2)^3}\right]\delta_R \\
& - \dot{\Omega}\pi\left[2a^4 + 2a^2(\delta_R^2 - \delta_I^2) + 2a^4\frac{\delta^4 + a^2(\delta_R^2 - \delta_I^2) - 2r_c^2\delta^2}{(r_c^2 - \delta^2)^2}\right. \\
& \qquad \left. + a^8 r_c^2\frac{2\delta^2 - r_c^2}{(r_c^2 - \delta^2)^4} - \frac{r_c^2 a^4}{r_c^2 - \delta^2} + r_c^2\delta^2 + a^4 r_c^2\frac{a^4 + r_c^4 - \delta^4}{(r_c^2 - \delta^2)^3}\right] \\
& - U\Omega\pi\left[2a^2 + r_c^2 - 2\frac{a^4}{r_c^2 - \delta^2} - \frac{r_c^2 a^6}{(r_c^2 - \delta^2)^3}\right]\delta_R \\
& + V\Omega\pi\left[2a^2 - r_c^2 + 2\frac{a^4}{r_c^2 - \delta^2} - \frac{r_c^2 a^6}{(r_c^2 - \delta^2)^3}\right]\delta_I \\
& + UV2\pi\left[r_c^2 + \frac{r_c^2 a^4}{(r_c^2 - \delta^2)^2}\right] \\
& + 2\pi\dot{\gamma}_c Re\left\{a^2 - \delta^2 + a\zeta_c + \frac{a^3\zeta_c}{\delta^2 - r_c^2}\right\} \\
& - 2\pi\frac{d}{dt}\sum_k \gamma_k Re\left\{\frac{-r_c^2\zeta_c}{\zeta_k} - \frac{a^2 r_c^2}{\zeta_k(\zeta_k + \zeta_c)} + a^2 - \delta^2 - \frac{a^4\zeta_c}{(\delta^2 - r_c^2)(\overline{\zeta}_k + \overline{\zeta}_c)} - \frac{a^2\overline{\zeta}_c}{\zeta_k + \zeta_c} + a\zeta_c + \frac{a^3\zeta_c}{\delta^2 - r_c^2}\right\} \\
& - 2\pi\sum_k \gamma_k Re\left\{\frac{d\overline{z}_k}{dt} z_k\right\} \\
& + 2\pi Re\left\{\overline{W}\left[\gamma_c\zeta_c + \sum_k \gamma_k\left(\frac{r_c^2}{\overline{\zeta}_k} - \zeta_k + z_k'\right)\right]\right\}
\end{aligned} \tag{55}$$

In a more compact form:

$$\begin{aligned} M &= -\dot{U}m_{61} - \dot{V}m_{62} - \dot{\Omega}m_{66} - U^2 m_{21} + V^2 m_{12} \\ &\quad + UV(m_{11} - m_{22}) - U\Omega m_{26} + V\Omega m_{16} \\ &\quad + \frac{d}{dt}\left[2\pi\mathcal{S}_1 - \Gamma\left(a^2 - \delta^2 + a\zeta_c + \frac{a^3\zeta_c}{\delta^2 - r_c^2}\right)\right] - 2\pi\mathcal{S}_2 + Re\{\overline{W}(2\pi\mathcal{Z}_3 - \Gamma\zeta_c)\} \end{aligned} \tag{56}$$

where

$$m_{61} = m_{16} \tag{57}$$

$$m_{62} = m_{26} \tag{58}$$

$$\begin{aligned} m_{66} &= \pi\left[2a^4 + 2a^2(\delta_R^2 - \delta_I^2) + 2a^4\frac{\delta^4 + a^2(\delta_R^2 - \delta_I^2) - 2r_c^2\delta^2}{(r_c^2 - \delta^2)^2}\right. \\ &\quad \left. + a^8 r_c^2\frac{2\delta^2 - r_c^2}{(r_c^2 - \delta^2)^4} - \frac{r_c^2 a^4}{r_c^2 - \delta^2} + r_c^2\delta^2 + a^4 r_c^2\frac{a^4 + r_c^4 - \delta^4}{(r_c^2 - \delta^2)^3}\right] \end{aligned} \tag{59}$$

$$\mathcal{S}_1 = \sum_k \gamma_k Re\left\{\frac{r_c^2\zeta_c}{\zeta_k} + \frac{a^2 r_c^2}{\zeta_k(\zeta_k + \zeta_c)} + \frac{a^4\zeta_c}{(\delta^2 - r_c^2)(\overline{\zeta}_k + \overline{\zeta}_c)} + \frac{a^2\overline{\zeta}_c}{\zeta_k + \zeta_c}\right\} \tag{60}$$

$$\mathcal{S}_2 = 2\pi\sum_k \gamma_k Re\left\{\frac{d\overline{z}_k}{dt} z_k\right\} \tag{61}$$

$$\mathcal{Z}_3 = 2\pi\sum_k \gamma_k\left(\frac{r_c^2}{\overline{\zeta}_k} + \frac{a^2}{\zeta_k + \zeta_c}\right) \tag{62}$$

Figure 12 shows that this calculation compares nicely with linear theory for an impulsively started flat plate at a small angle of attack.

In addition to comparisons with linear theory for a flat plate, the algorithm has also been verified versus direct numerical integration of the pressure. The results agree to whatever accuracy we are able to evaluate the integrals.

10 Comments

By evaluating the integrals in the general expressions provided by Milne-Thomson, [MilneThomson60] we have obtained formulae (36) and (56) for force and moment experienced by a Joukowski foil in a flow with point vortices. The principal differences from previous work can be summarized as follows:

- The profile shape is a general Joukowski foil, compared to a flat plate in [Sarpkaya75], and a general power series transform in [Graham80].
- The profile is allowed to perform arbitrary motion in a fluid at rest. This is more general than flow over a stationary profile, which is kinematically equivalent in the case of translation, but precludes treatment of rotation.
- The influence of a growing number of vortices have been retained in the time derivatives, removing an $\mathcal{O}(\sqrt{\Delta t})$ error, where Δt is the time step of the simulation.
- To the author's knowledge, no closed form expression has been published for the moment, even in the special case of a flat plate.

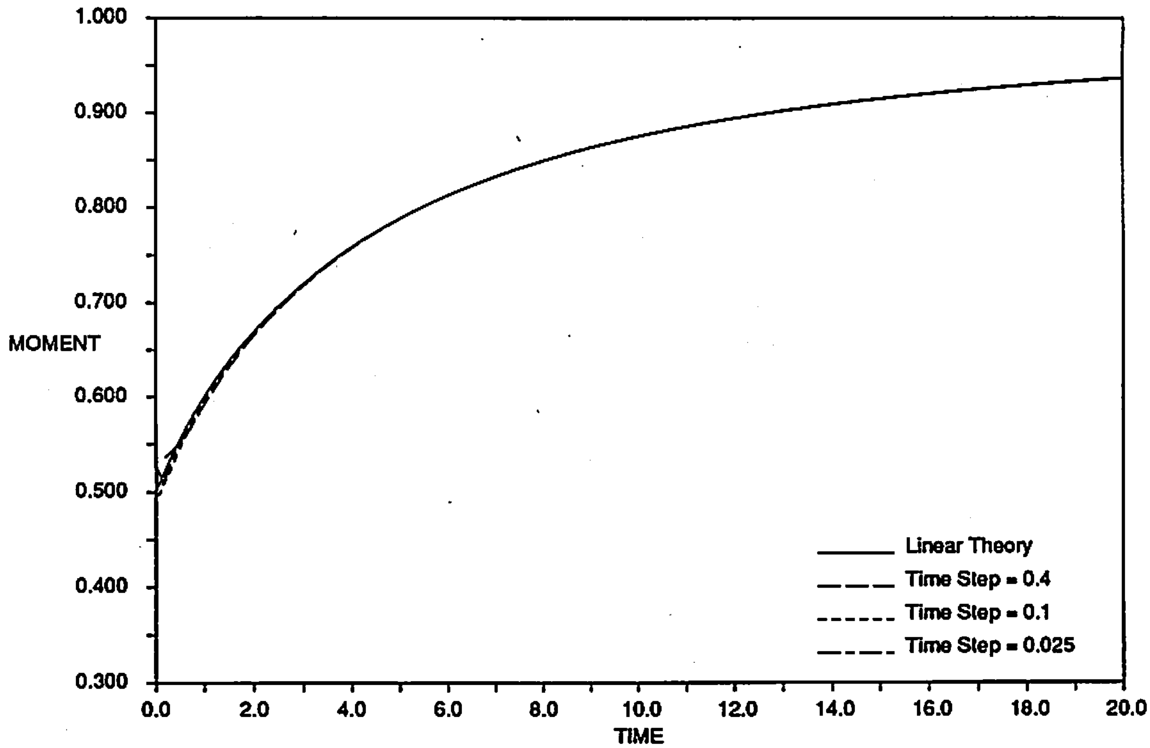

Figure 12: Moment on an impulsively started flat plate normalized by its steady state value, $\pi U^2 c^2 \alpha_0$.

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