CUT LOCUS AND MEDIAL AXIS IN GLOBAL SHAPE INTERROGATION AND REPRESENTATION

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Abstract

The cut locus C_A of a closed set A in the Euclidean space E is defined as the closure of the set containing all points p which have at least two shortest paths to A. We present a theorem stating that the complement of the cut locus i.e. $E\backslash(C_A \cup A)$ is the maximal open set in $(E\backslash A)$ where the distance function with respect to the set A is continuously differentiable. This theorem includes also the result that this distance function has a locally Lipschitz continuous gradient on $(E\backslash A)$. The medial axis of a solid D in E is defined as the union of all centers of all maximal discs which fit in this domain. We assume in the medial axis case that D is closed and that the boundary ∂D of D is a topological (not necessarily connected) hypersurface of E. Under these assumptions we prove that the medial axis of D equals that part of the cut locus of ∂D which is contained in D. We prove that the medial axis has the same homotopy type as its reference solid if the solid's boundary surface fulfills certain regularity requirements. We also show that the medial axis with its related distance function can be be used to reconstruct its reference solid. We prove that the cut locus of a solid's boundary is nowhere dense in the Euclidean space if the solid's boundary meets certain regularity requirements. We show that the cut locus concept offers a common frame work lucidly unifying different concepts such as Voronoi diagrams, medial axes and equidistantial point sets. In this context we prove that the equidistantial set of two disjoint point sets is a subset of the cut locus of the union of those two sets and that the Voronoi diagram of a discrete point set equals the cut locus of that point set. We present results which imply that a non-degenerate C^1-smooth rational B-spline surface patch which is free of self-intersections avoids its cut locus. This implies that for small enough offset distances such a spline patch has regular smooth offset surfaces which are diffeomorphic to the unit sphere. Any of those offset surfaces bounds a solid (which is homeomorphic to the unit ball) and this solid's medial axis is equal to the progenitor spline surface. The spline patch can be manufactured with a ball cutter whose center moves along the regular offset surface and where the radius of the ball cutter equals the offset distance.

Keywords : CAD, CAGD, CAM, Interrogation, Intersection, Finite Element Meshing

1 Introduction

The Medial Axis Transform in short (MAT) was introduced by Blum in [1] more than 20 years ago. Since then, a great deal of research has been done on the MAT, see the literature review in section 2. Initially the research performed on the MAT has mainly been from the vantage point of understanding how it can be useful for pattern recognition (see [2]). During the past five years the MAT concept has been employed in Computer Aided Design and Manufacture for:

- global shape interrogation
- global shape representation
- automated meshing algorithms

Although there exists extensive literature on the MAT which discusses mainly computational methods in a variety of practically relevant cases, basic global and even basic local aspects of the MAT concept are not sufficiently well understood. Here, for instance, the relations between the homotopy properties of an object and the homotopy properties of its MAT have not yet been systematically analyzed. Although it has been claimed occasionally (cf. eg. [2]) that the medial axis of a domain bounded by a simple closed curve is simply connected, there does not seem to exist any proof for this statement. Even in the planar case, there does not seem to exist any result discussing if the medial axis is in general connected. This is a severe gap because those topological relations often motivate the relevance of the medial axis for global shape interrogation and representation. Moreover, intuition frequently offers no immediate clue telling what conjectures are true. Therefore, in order to deduce correct results and construct proper proofs one has to utilize tools of topology and global differential geometry. Until now, the research activities performed in the whole MAT area have mainly focussed on computational techniques, and one misses a systematical foundational investigation of the concept as a whole. One of the main goals of this paper is to help fill this gap, and also to supply a systematical analysis of the above mentioned topological properties. In our effort to make a systematical analysis of the foundations of the MAT concept we investigate its relation to the concepts of cut loci, equidistantial sets, and Voronoi diagrams. We show that the cut locus concept offers a common frame work lucidly unifying different but related concepts such as Voronoi diagrams, equidistantial sets and medial axes. We want to point out that the distance function and its differentiability properties play a crucial role for many considerations in this paper.

There is one aspect which makes the MAT problem particularly interesting for the research in Computer Aided Geometric Design, namely the fact that it requires and integrates difficult intersection computations, offset computations and distance function computations. Therefore MAT computations are a challenging test bed for the most fundamental tools in geometric modeling.

We give now a summary of the main results of our paper:

The cut locus C_A of a closed set A in the n-dimensional Euclidean space E^n is defined as the closure of the set containing all points which have at least two shortest paths (minimal joins) to A. The medial axis $M(D)$ of a connected solid D in E^n is defined as the union of all centers of all maximal discs which fit in this solid. We assume in the medial axis case that D is closed and that the boundary ∂D of D is a topological (not necessarily connected) hypersurface of E^n. We prove in Theorem 1 that:

> Under these assumptions the medial axis of D equals that part of the cut locus of ∂D which is contained in D.

This theorem 1 as well as the two subsequent results illuminate that the cut locus concept offers a common framework unifying different concepts such as equidistantial sets, Voronoi diagrams, and medial axes. Namely we prove and discuss Theorem 6:

> The equidistantial set of two disjoint closed sets A, B is a subset of the cut locus $C_{A\cup B}$.

We also show and discuss Theorem 7:

> The Voronoi diagram of any discrete set P of points in R^n equals the cut locus C_P of the set P.

There exists a practically important characterization of the complement of the cut locus of any closed set A in E^n via differentiability properties of the distance function $d(A,x)$ which describes the distance of a variable point x to the set A. For this let $F\setminus G$ denote the set theoretic difference of any two sets F, G i.e. $F\setminus G$ is the subset of F obtained after removing all those elements of F which are also contained in G. Then by Theorem 2 we have:

> The set $E^n\setminus(C_A\cup A)$ is the maximal open subset of $E^n\setminus A$ where the distance function $d(A,x)$ is C^1-smooth. The gradient $\nabla d(A,x)$ is locally Lipschitz continuous on $E^n\setminus(C_A\cup A)$.

We present several basic topological results on medial axis and cut locus which answer open questions in this area. We show the *Topological Shape Theorem of the Medial Axis* (Theorem 8) which says:

> If ∂D is a C^2–smooth manifold or if ∂D is a one-dimensional, piece wise C^2–smooth manifold then the medial axis $M(D)$ is a deformation retract of D, hence $M(D)$ has the same homotopy type as D.

Consequences of this theorem are that $M(D)$ is path connected as D is path connected and $M(D)$ is simply connected if and only if D is simply connected. The local structure of the cut locus $C_{\partial D}$ is addressed by the next result Theorem 3D which holds under the same conditions for ∂D as in theorem 8:

> The cut locus of ∂D (and hence the medial axis $M(D)$) is nowhere dense in E^n.

This means $C_{\partial D}$ together with its limit points cannot contain any (arbitrarily small) n-dimensional disc. The preceding result does not hold if ∂D is only a C^1-smooth manifold with Lipschitz continuous first derivatives. However under those weaker regularity assumptions for ∂D we have Theorem 5 :

> The cut locus $C_{\partial D}$ of ∂D does not meet the set ∂D.

We even show the following stronger result (Theorem 4):

Let $f(u,v):[0,1]\times[0,1]\rightarrow R^3$ be a non-degenerate C^1-smooth representation of a surface S which is free of self-intersections. Non-degenerate means that the Jacobian of $f(u,v)$ has rank 2 everywhere. We assume also that the partial derivatives $\partial_u f(u,v), \partial_v f(u,v)$ are Lipschitz continuous. Under those assumptions the distance $d(S,C_S)$ between S and C_S is larger than some positive number $i(S)$.

In view of the above characterization of the cut locus via differentiability properties of the distance function, the last theorem proves that the distance function $d(S,x)$ is C^1-smooth in a neighborhood around S for all points with $0 < d(S,x) < i(S)$. This implies the practical property that offsets to S with offset distance smaller than $i(S)$ are non-degenerate C^1-smooth manifolds. It shows also that a C^1-smooth non-degenerate rational B-spline patch S which is free of self-intersections avoids its cut locus. This implies (Corollary 4.1) which says:

For every small enough offset distance the above spline patch S has a regular C^1-smooth offset surface which is diffeomorphic to the unit sphere. This offset surface bounds a solid (homeomorphic to the unit ball) whose medial axis is equal to the progenitor spline surface S.

In practical terms corollary 4.1 states that the spline patch S can be manufactured with a ball cutter whose center moves along a regular offset surface and where the radius of the ball cutter equals the offset distance.

The general problem of reconstructing the solid D by using its medial axis $M(D)$ is addressed in the subsequent result employing the (real valued) maximal disc radius function
$r : M(D) \rightarrow R$. This function assigns to any point $x \in M(D)$ the radius $r(x)$ of the maximal disc $K(x, r(x))$ contained in D. We show the *Reconstruction Theorem* (Theorem 9) which says:

If for a solid D the medial axis $M(D)$ and the maximal radius function $r : M(D) \rightarrow R$ are given then it is possible to reconstruct the solid D. Namely $D = \bigcup_{x \in M(D)} K(x, r(x))$.

This paper is structured as follows. In section 2 we give a survey of previous work on the medial axis. In section 3 we present definitions, characterizations and various various local results for Cut Locus, Medial Axis, equidistantial sets and Voronoi sets. In subsection 3.2 of section 3 we show that the cut locus avoids certain reference sets and we draw conclusions from this result among those that offset surfaces of a spline patch are C^1-smooth for sufficiently small offset distances. In subsection 3.3 we investigate the relation of the cut locus to equidistantial sets and Voronoi diagrams. We show that the cut locus concept offers a common framework unifying different concepts such as Voronoi diagrams, equidistantial sets and medial axes. We show that the equidistantial set of two disjoint sets is a subset of the cut locus of the union of those two sets. We also prove that a Voronoi diagram is the cut locus of a discrete point set In section 4 we present global results on the medial axis. We prove in subsection 4.1 that under appropriate assumptions for a solid's boundary the medial axis has the homotopy type of its enclosing solid. In subsection 4.2 we show that the medial axis can be used to reconstruct the engulfing solid. The appendix contains two lemmata. The first is used as a crucial part for the homotopy result in subsection 4.2. The second describes properties of cut locus points if the reference set is a closed surface being the union of planar facets.

2 Survey of Previous Work on the Medial Axis

The concept of the equidistantial point set with respect to two reference sets is basic for the concepts of cut locus, medial axis and Voronoi diagrams. The concept of the equidistantial point set is as old as geometry. Euclid used the concept of the equidistantial point set of two distinct points or straight lines in the plane. Apollonius defined the parabola as the equidistantial point set of a point and a straight line in the plane. The concept of equidistantial loci in the context of discrete point sets goes back at least as early as the work of Voronoi [43], his name being usually associated with the concept of a Voronoi diagram. The concept of the Cut Locus of a single point on a surface is due to Poincare [32], which he called in French "ligne de partage". However prior to Poincare the concept of the cut locus of a point on a surface occurs at least implicitly in Mangoldt's paper [22]. There has been a lot of work in Riemannian Geometry using the cut locus of a single point in particular for the investigation of geodesics and positively curved Riemannian manifolds, for an overview see e.g. [35], [17], [45] and the lists of references given there. The concept of the Medial Axis Transform (which is also called symmetric axis or skeleton) appears to have been introduced first by Blum in [1] as a method to describe and recognize biological shape, see also Blum's extensive article [2].

There exists a considerable body of literature on algorithms to compute the medial axis of a planar polygonal domain or of a planar domain bounded by circular arcs and polygons see e.g. Montanari [24], Preparata [33], Lee and Drysdale [20], Lee [21], Yap [46], Gursoy [8], Patrikalakis and Gursoy [30]. The amount of research done in the three dimensional case is smaller. Here we have the work of O'Rourke and Badler [28]. Motivated by work of Blum and Nagel [3] in the planar case, Nackman was the first to derive curvature relations between the curvature of the medial axis axis surface and the curvature of the boundary surface see Nackman [25] and Nackman and Pizer [26]. More recently, Hoffmann [12], [13] and Dutta and Hoffmann in [6] compute equidistantial curves and surfaces. Nackman and Srinivasan [27] investigate bisectors of linearly separable sets. Hoffmann and Vermeer [14] present systems of equations defining equidistantial curves and surfaces where they eliminate extraneous solutions in curve and surface operations.

The author introduced the concept of the cut locus for arbitrary closed sets in a Riemannian manifold with and without boundary [45]. Motivated by his work in [44] he could show that even under those very general assumptions and under the weak requirement of Lipschitz continuity for the Riemannian metric the cut locus can be characterized through differentiability properties of the distance function, cf. [45]. As a special case see also theorem 2 in this paper.

During the past five years there has been an increasing interest in the medial axis area by researchers involved in geometric modelling and computer aided design, analysis and manufacture. There are several reasons for this. First the medial axis appears to be useful for the extraction of gross features of a two or three dimensional solid cf. e.g. Rosenfeld [36], Patrikalakis and Gursoy in [29] and [30]. Further the medial axis appears to be an appropriate preprocessing tool for automated finite element mesh generation on topologically complicated two and three dimensional domains, cf. e.g. Srinivasan, Nackman, Tang, Meshkat in [42], Gursoy [8], Gursoy and Patrikalakis in [9]. This relevance as an appropriate preprocessing tool for topologically complicated domains is corroborated by the observation that numerical medial axis computations of complicated two dimensional solids yield objects which have the homotopy type of the enclosing domain e.g. the same number of holes, cf. Srinivasan, Nackman [41] and Gursoy [8]. Held [10] develops and applies the concept of equidistantial point sets and Medial Axes and Voronoi diagrams in numerical control 2.5 D machining applications. Held's book [10] as well as the thesis by Gursoy [8] provide extensive references in this general area and its applications.

3 Definitions, Characterizations and Local Results for Cut Locus, Medial Axis, Equidistantial Sets and Voronoi Sets

3.1 Review of some Concepts used in the Paper

To make the paper self-contained and more easily readable we review here some concepts from point set topology, differentiable manifolds and analysis which are used very often in this paper. We don't give the most general definitions of the concepts, but explain only the meaning within the scope of this paper. For more background on point set topology see e.g. Hu [15] or Kelley [16], for algebraic topology and differential topology see eg. Spanier [40], Massey [23] or Guillemin and Pollack [7], Hirsch

[11] respectively.

An open subset G of R^n is characterized by the property that for every point $x \in G$ there exists a positive number ε such that the disc $\{y \in R^n \,|\, |x-y| < \varepsilon\}$ is contained in G. The interval $(0,1) = \{s \in R \mid 0 < s < 1\}$ is an open subset of R^1.

A point q is a limit point of a set $C \subset R^n$ if there exists a sequence of points $x_n \in C$ converging to q. A set may not contain all its limit points eg. the point 0 is a limit point of the interval $(0,1)$ but 0 is not contained in $(0,1)$. A closed subset C of R^n is characterized by one of the two equivalent properties:
1) The set C includes all its limit points.
2) The complement $R^n \backslash C$ is an open subset of R^n.
The sets $\{s \in R^1 \mid 0 \le s\}$, $\{(x,y) \in R^2 \mid 0 \le x, 0 \le y\}$ are closed subsets of R^1, R^2, respectively.

A subset $B \subset R^n$ is called bounded if B is contained in some finite disc $\{y \in R^n \,|\, |0-y| < d\}$ with radius d. The sets $(0,1)$, $\{s \in R \mid 0 < s\}$ are bounded and unbounded subsets of R^1 respectively.

A subset K of R^n is compact if and only if K is closed and bounded. Hence the set $\{s \in R \mid 0 \le s \le 1\}$ is compact while both of the sets $\{s \in R \mid 0 \le s\}$, $\{s \in R \mid 0 \le s < 1\}$ are not compact. Compact sets have the property that continuous real valued functions attain a finite minimal and maximal value on them.

A subset D of S is dense in the set S if every point in S is a limit point of D. The rational numbers are a dense subset of the real numbers because every real number can be approximated by a sequence of rational numbers. A set $A \subset R^n$ is nowhere dense in R^n if D is not dense in some n-dimensional disc $\{x \in R^n \,|\, |x-q| < \varepsilon\}$. Let the set A be a subset of R^n. A function $f: A \rightarrow R^n$ is continuous in some point $q \in A$ if for any sequence of points $q_n \in A$ with limit point q the sequence of function values $f(q_n)$ has the limit point $f(q)$. Let A, B be subsets of R^n, R^m respectively. A function $f: A \rightarrow B$ is a homeomorphism if the function f is continuous and has a continuous inverse. Two subsets A, B of R^n, R^m respectively are called homeomorphic or are said to have the same homeomorphy type if there exists a homeomorphism $f: A \rightarrow B$.

An unbordered, k-dimensional topological submanifold S of R^n (with $0 \le k \le n$) is characterized by the property that for every point $q \in S$ there exists a positive number ε such that for the disc $K^o(q,\varepsilon) = \{x \in R^n \,|\, |x-q| < \varepsilon\}$ the intersection $K^o(q,\varepsilon) \cap S$ containing q (and being a neighborhood of q in S) is homeomorphic to R^k. A k-dimensional topological submanifold S with boundary ∂S is characterized by the properties that:
1) For every boundary point $p \in \partial S$ there exists a positive number δ such that the intersection $K^o(p,\varepsilon) \cap S$ containing p (and being a neighborhood of the boundary point p in S) is homeomorphic to the k-dimensional halfspace $H^k = \{(x_1, \ldots, x_k) \in R^k \mid x_1 \ge 0\}$.
2) The set $S \backslash \partial S$ is nonempty and for every every point $q \in S \backslash \partial S$ there exists a positive number δ such that the intersection $K^o(q,\delta) \cap S$ containing q (and being a neighborhood of the non-boundary point q in S) is homeomorphic to R^k.
The sets $O_1 = \{(x_1,x_2,x_3) \in R^3 \,|\, |x_1| < 1, x_2 = x_3 = 0\}$, $O_2 = \{(x_1,x_2,x_3) \in R^3 \,|\, |x_1^2 + x_2^2| < 1, x_3 = 0\}$, $O_3 = \{(x_1,x_2,x_3) \in R^3 \,|\, |x_1^2 + x_2^2 + x_3^2| < 1\}$ are one-, two-, three- dimensional submanifolds of R^3 respectively, and all three of those submanifolds have no boundary. The sets $B_1 = \{(x_1,x_2,x_3) \in R^3 \,|\, |x_1| \le 1, x_2 = x_3 = 0\}$, $B_2 = \{(x_1,x_2,x_3) \in R^3 \,|\, |x_1^2 + x_2^2| \le 1, x_3 = 0\}$, $B_3 = \{(x_1,x_2,x_3) \in R^3 \,|\, |x_1^2 + x_2^2 + x_3^2| \le 1\}$ define bordered[1] one-, two- and three dimensional submanifolds of R^3 respectively. Their boundaries are $\partial B_1 = \{(x_1,0,0) \in R^3 \,|\, |x_1| = 1\}$,

[1]We are using the terminology bordered manifold as a synonym to manifold with boundary.

$\partial B_2 = \{(x_1,x_2,0) \in R^3 \mid |x_1^2+x_2^2| = 1\}$, $\partial B_3 = \{(x_1,x_2,x_3) \in R^3 \mid |x_1^2+x_2^2+x_3^2| = 1\}$ where $\partial B_2, \partial B_3$ represent a unit circle and a unit sphere in R^3 respectively.

Let A be any subset of R^n. Any function $f: A \to R^m$ is Lipschitz continuous on A with some Lipschitz constant L if for all points $x,y \in A$ we have $|f(x)-f(y)| \le L|x-y|$. It is easily seen that a Lipschitz continuous function is continuous in all points of its domain of definition. However a continuous function need not be Lipschitz continuous, an example being the function $f(x) = +\sqrt{x}$ defined on the interval $[0,1] = \{0 \le x \le 1\}$. All C^0-smooth rational B-spline functions are Lipschitz continuous. A function f is locally Lipschitz continuous on a domain D if for every point $p \in D$ there exists a number ε such that the function f is Lipschitz continuous on $D \cap K^o(p,\varepsilon)$.

The notation C^k - smooth will refer to functions which have continuous partial derivatives of order k. The notation $C^{k,1}$ - smooth will refer to functions which have Lipschitz continuous derivatives of order k. The function $f: R \to R$ defined by $f(x) = 0$ for $x \le 0$ and $f(x) = x^2$ for $x \ge 0$ is $C^{1,1}$-smooth but not C^2 smooth. All rational B-spline functions (with simple knots) of degree k in each parameter are $C^{k,1}$-smooth.

A k-dimensional, C^r-smooth submanifold S of R^n is a k-dimensional, topological submanifold of R^n with the property that for every point $p \in R^n$ there exists a positive number ε such that:
There exists a homeomorphism $h: D = \{x \in R^k \mid |0-x| < 1\} \to S \cap K^o(p,\varepsilon)$ with $p \in h(D)$; the map h has continuous partial derivatives of k-th order on D and the Jacobian matrix $h'(x)$ has rank k every where on D. Any C^r-smooth k-dimensional submanifold S_1 of R^n can be locally represented by solutions of (n-k) (generally non-linear) equations described by $n-k$ C^r-smooth functions. This means for every point $x \in S_1$ there exists an open set U in R^n and a C^r-smooth function $e: U \to R^{n-k}$ whose differential has rank $n-k$ on all U and $x \in U \cap S_1 = e^{-1}(0)$. Using the implicit function theorem (cf. e.g. [5]) it is easily seen that for any open set $U \subset R^n$ and for any C^r-smooth function $e: U \to R^{n-k}$ whose Jacobian e' has rank $n-k$ on all U the preimage set $e^{-1}(0)$ defines a $n-k$ dimensional C^r-smooth submanifold of R^n.

We also need to explain smooth functions defined on submanifolds which are not open subsets of R^n. For this let S_1 be any C^k-smooth m-dimensional submanifold of R^n. A continuous map $f: S_1 \to R^w$ is C^r-smooth if for every point $x \in S_1$ there exists a positive number ε and and a C^k-smooth homeomorphisms $h: K(0,1) \to K(x,\varepsilon) \cap S_1$, $x \in K(x,\varepsilon) \cap S_1$ with the Jacobian $h'(z)$ of rank w on all $K(0,1)$ such that the composition map $f \circ h: K(0,1) \to R^w$ is C^r-smooth on all $K(0,1)$. The differential of map f has rank w at x if at the preimage point $z = h^{-1}(z)$ the Jacobian $(f \circ h)'(z)$ has rank w. Let S_1 be any C^k-smooth m-dimensional submanifold of R^n and let S_2 be any C^r- smooth m-dimensional submanifold of R^d then a map $f: S_1 \to S_2$ is a C^r-smooth diffeomorphism if f is a homeomorphism and if the map f as well as its inverse f^{-1} are both C^r-smooth. These conditions are already fulfilled if the map f is a C^r-smooth homeomorphism whose differential has rank m on all S_1. Two smooth submanifolds S_1, S_2 of R^n, R^m respectively are C^r- diffeomorphic if there exists a C^r-smooth diffeomorphism $f: S_1 \to S_2$. The mappings $\psi(x,0,1) = (x^3,1)$, $\phi(x,0,1) = (x,1)$ define homeomorphisms between the two C^∞-smooth submanifolds $S_1 = \{(x,0,1) \in R^3 \mid x \in R\}$, $S_2 = \{(x,1) \in R^2 \mid x \in R\}$ of R^3, R^2 respectively; here the map ϕ is a C^∞-smooth diffeomorphism, while ψ is not even a C^1-smooth diffeomorphism. Note that the inverse ψ^{-1} is continuous but not locally Lipschitz continuous, due to the fact that the Jacobian $\psi'(0,0) = 0$. Let S^1 denote the unit circle being a C^∞-smooth submanifold of R^2. Let $r(x,y), \gamma(x,y)$ be polar coordinates in R^2. The map $\beta: S^1 \to S^1$ with $\beta(x,y) = (cos(2\gamma(x,y)), sin(2\gamma(x,y)))$ is C^∞-smooth and its Jacobian has maximal rank on all S^1. This map β is locally invertible this means here that a mapping defined by restriction of β

to any sufficiently small subarc[2] of S^1 yields a diffeomorphism onto the image set of the small subarc. However β has not the global property to be a homeomorphism. Let $S_3 = \{(x,0) \in R^2 \mid x \in R\}$, $S_4 = \{(x, f(x)) \in R^2 \mid f(x) = 0 \quad \text{for} \quad x \le 0, f(x) = x^2 \quad \text{for } x \ge 0\}$. The map $\Omega(x) : S_3 \to S_4$ provides a C^1-smooth diffeomorphism between both submanifolds of R^2. However both submanifolds are not C^2-diffeomorphic submanifolds of R^2. Note also that the fact that a submanifold is diffeomorphic to some other submanifold does not say much on how complicated any of those submanifolds has been embedded in a Euclidean space. For instance a knotted curve K in R^3 is a submanifold of R^3 diffeomorphic to the unit circle in R^3, however the curve K may be embedded in a complicated way into the ambient space R^3. Note in this context that a diffeomorphism (or homeomorphism) between two submanifolds S_1, S_2 of R^n need not be extendable to a diffeomorphism (or homeomorphism) of R^n to itself. An example for this situation is provided by a closed knotted curve K in R^3. The curve K is diffeomorphic to the unit circle in R^3, however *no* homeomorphisms between K and the unit circle in R^3 can be extended to a homeomorphism of R^3 to itself, see e.g. Hirsch [11].

We shall use also one-dimensional piecewise smooth submanifolds of the Euclidean plane R^2. A piecewise possibly disconnected one-dimensional C^k–smooth submanifold S is a topological submanifold of R^2 with the subsequent additional property:
For every point $p \in S$ there exists a positive number ε and a homeomorphism[3] $h(t) : (-1,0] \cup [0,1) \to S \cap K^o(p,\varepsilon)$ such that $p \in h((-1,0] \cup [0,1))$ and each of the functions $h : (-1,0] \to R^2$, $h : (0,1] \to R^2$, is C^k-smooth and has non-zero first derivative on its respective domain of definition $(-1,0]$, $[0,1)$.
Note that, the two paths $h((-1,0])$, $h([0,1))$ will generally not have collinear tangents at the vertex point $h(0)$. Polygons which are free of self-intersections can be used to get one-dimensional piecewise C^∞-smooth submanifolds of R^2. Another example covered by the definition is given by the union of the two subsequent paths $\{(t,t^2) \in R^2 \mid 0 \le t < \infty\}$, $\{(t,0) \in R^2 \mid 0 \le t < \infty\}$.

3.2 Definitions, Characterizations and Local Properties of the Cut Locus and the Medial Axis

The MAT of a closed planar region B bounded by a curve has been defined by Blum to be the union of the centers of all maximal discs (which fit inside B) together with the radius function, defining the radius of a maximal disc for a point in M(B). Therefore, in the sense of Blum

Definition of the MAT: The MAT of a planar region B is a real valued function

$r: M(B) \to R$

together with its domain of definition M(B); the set M(B) $\subset$ B is called the medial axis or symmetric axis or skeleton of B. A point p $\in$ B is contained in M(B) if and only if there exists a closed disc

$K(p, r(p))$

with center p and radius r(p), which is not contained in a larger disc W with

$K(p, r(p)) \subset W \subset B$.

Blum defined the MAT concept initially for a domain in the Euclidean plane. We will generally assume that the set B is a bordered n-dimensional submanifold of the n-dimensional Euclidean space. For some of the results in this paper we need to make specific continuity requirement for the boundary ∂B like e.g.

[2] A subarc of length smaller than π is sufficiently short.

[3] We shall often use the notation $(-1,0]$, $[0,1)$ for the intervals $\{s \in R \mid -1 < s \le 0\}$, $\{s \in R \mid 0 \le s < 1\}$ respectively.

being a piecewise C^2-manifold.

Redefinition of the MAT: Note that we extend Blum's MAT definition in the following way:

- We include in the medial axis M(B) also all limit points of all centers of all maximal discs.
- We redefine the preceding function $r: M(B) \to R$ by $r(p) = d(p, \partial B)$ i.e. $r(p)$ is the distance of the the point p to the boundary ∂B.

This yields a well-posed definition of the function $r(p)$ also in case the point p is a limit point of centers of maximal discs in B. This redefinition yields a continuous function $r: M(B) \to R$ and Lemma 2 below will prove that this redefinition of $r(p)$ is consistent with the preceding one. Namely this holds by Lemma 2 because if a point p is a center of a maximal disc K in B then the radius of K equals the distance of p to the boundary ∂B.

We explain now why the redefinition of the function $r: M(B) \to R$ is important. For this note that in case the boundary ∂B is only a $C^{1,1}$-smooth manifold then a limit point p_o of centers of maximal discs need not be a center of a maximal disc in B. Hence for such a limit point p_o the function value $r(p_o)$ cannot be defined as the radius of the maximal disc in B with center p_o as p_o need not be center of a maximal disc. However we need to assign a value to $r(p_o)$ if we want to include limit points into the medial axis transform concept.

Example 1: We explain now an example of a planar domain with $C^{1,1}$– smooth boundary where a limit point of maximal disc centers is *not* a center of a maximal disc in the domain. For this purpose we define the function $f: R \to R$ by $f(x) = (1/2)x^4 sin(1/x)$ if $x \geq 0$ and $f(x) = 0$ for $x \leq 0$. The domain B is defined by all points above the graph of the function $f(x)$ i.e. the set $B = \{(x,y) \in R^2 \mid y \geq f(x)\}$. The function $f(x)$ is $C^{1,1}$-smooth. For $x > 0$ the first and second derivative of $f(x)$ are given by $f'(x) = 2x^3 sin(1/x) - (1/2)x^2 cos(1/x)$ and $f''(x) = 6x^2 sin(1/x) + (1/2) sin(1/x)$ respectively. The function $f(x)$ has infinitely many local minima on each interval between 0 and any positive number. Let x_m be such a minimum. Let Ra be a ray which starts at $(x_m, f(x_m))$. We assume that Ra is parallel to the y-axis and that Ra points into the domain B. The ray Ra contains a curvature center c_m which is located arbitrarily close to the axis $\{(x,y) \mid y = 1/2\}$ if x_m is sufficiently small; the point c_m is a curvature center respective the point $(x_m, f(x_m))$ on the curve $x \to (x, f(x))$. It can be shown that those curvature centers c_m are centers of maximal discs respective the domain B. This claim can be verified by elementary estimations[4].- With x_m converging to 0 the corresponding sequence of maximal disc centers has a limit point l on the y-axis precisely $l = (0, 1/2)$. This point l cannot be a center of a maximal disc in B because the (candidate) disc $K(l, 0.5)$ (with center l and radius 0.5) is subset of the larger disc $K((0,1),1)$ (with center (0,1) and radius 1) which is easily shown to be contained in B. Similarly if $K((0,1),1)$ is subset of B then no point in $\{(0,y) \mid 0 \leq y < 1\}$ can be center of a maximal disc in B. The claim that the disc $K((0,1),1)$ is subset of B follows from the subsequent inequalities which can be easily verified:

$$\text{For } 0 \leq x \leq 1 \text{ is } \quad 1 - \sqrt{1 - x^2} \geq (1/2)x^2 \geq (1/2)x^4 sin(1/x) \tag{1}$$

Therefore the two-dimensional bordered submanifold $B \subset R^2$ (with ∂B being $C^{1,1}$–smooth) contains centers of maximal discs with some limit point *not* being center of a maximal disc in B. This establishes the properties claimed for our example.

As we shall see later in theorem 1, the medial axis is a special subset of the cut locus concept studied in

[4]Note that the ray Ra cannot be a distance minimal path to ∂B after the ray has passed through c_m. Therefore the segment *seg* of Ra bounded by the two points c_m, $(x_m, f(x_m))$ must contain a non-extender point explained in the definition below. By lemma 1 below, a non-extender is a center of a maximal disc. Therefore the segment *seg* contains a center of a maximal disc. Those centers of maximal discs must have some limit point on the y-axis between the two values 0, 0.5 .

[45]. Therefore, we can successfully apply results from [45] in this context. For this we introduce the following notation:

Definition: A point $p \in R^n$ is called non-extender relative to the closed set A, if there exists a minimal join from A to p which cannot be extended as a minimal join beyond p.

Example: The midpoint of the unit circle S^1 is a non-extender relative to S^1 in the Euclidean plane R^2.

Using a simple estimation employing the triangle inequality it is easily seen that the preceding definition of a non-extender point yields immediately the subsequent corollary.

Corollary 1: If a point $q \in R^n$ is a non-extender with respect to some closed set $A \subset R^n$ then no minimal join from A to q can be extended distance minimally beyond q.

Using the concept of non-extender points we define next the cut locus with respect to a reference set.

Definition : The cut locus C_A of a closed set $A \subset R^n$ is then defined as the set of all non-extenders relative to A together with all limit points.

We want to give a result which relates the cut locus with the medial axis. For this purpose, we need to explain for what kind of sets B in R^n we want to define the medial axis. Note while we have defined the reference set A for the cut locus to be very general namely any closed set[5] we shall be more restrictive for the reference set B of the medial axis. Unless stated otherwise, let us from now on assume that B is always a closed bordered n-dimensional topological submanifold of R^n assume that the non-empty boundary ∂B of B is a $n-1$-dimensional topological manifold.

The preceding conditions imply

Proposition 1: The boundary ∂B separates B and its complement $R^n \backslash B$. This means if we join any point $p \in B$ with any point $q \in (R^n \backslash B)$ by a continuous path $c(t) : [0,1] \to R^n$

where $c(0) = p$, $c(1) = q$, then there exists a $t_o \in [0,1]$ such that $c(t_o) \in \partial B$.

Proof of Proposition 1 : We argue by contradiction. Therefore we assume that the whole path $c[0,1]$ does not meet the boundary ∂B. Hence $c[0,1]$ is contained in $R^n \backslash \partial B$. Thus $c[0,1] \subset (B \backslash \partial B) \cup (R^n \backslash B)$. Therefore the interval $[0,1]$ is represented by the subsequent union of two preimage sets $c^{-1}(B \backslash \partial B) \cup c^{-1}(R^n \backslash B)$. As $(B \backslash \partial B)$, $(R^n \backslash B)$ are both open sets in R^n their preimage sets $c^{-1}(B \backslash \partial B)$, $c^{-1}(R^n \backslash B)$ are open sets as well because the map $c(t)$ is continuous. Clearly those two preimage sets are also disjoint i.e. $c^{-1}(B \backslash \partial B) \cap c^{-1}(R^n \backslash B) = \varnothing$ because $(B \backslash \partial B) \cap (R^n \backslash B) = \varnothing$. The two preimage sets are both non-empty because $0 \in c^{-1}(B \backslash \partial B)$ and $1 \in c^{-1}(R^n \backslash B)$ as by assumption $c(0) \in (B \backslash \partial B)$ and $c(1) \in (R^n \backslash B)$. This means that the interval $[0,1]$ can be represented by the union of two open, disjoint, non-empty sets $c^{-1}(B \backslash \partial B)$, $c^{-1}(R^n \backslash B)$. This implies that the interval $[0,1]$ is disconnected (cf. eg. [15]), a contradiction. This proves proposition 1.

Under the above stated assumptions for B, we can conveniently characterize the medial axis as a subset of the cut locus. Namely we have:

Theorem 1: (Medial Axis as Interior Cut Locus of a Solid's Boundary)

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[5] A closed set may be completely disconnected and may have many components being isolated points, isolated curves and surface pieces.

Let B be a closed bordered n-dimensional topological manifold of the n-dimensional Euclidean space and assume that ∂B is a topological $n{-}1$-dimensional manifold. Then the medial axis M(B) equals the subset of the cut locus $C_{\partial B}$ which is contained in B, i.e. $M(B) = C_{\partial B} \cap B$.

In other words, the medial axis of a solid B is that subset of the solid's boundary cut locus which is contained in the solid. Theorem 1 is a consequence of the combination of the subsequent Lemmata 1 and 2.

Lemma 1: (A non-extender is a center of a maximal disc)
If ∂B is a topological $n{-}1$-dimensional manifold being boundary of a closed solid body B in R^n then a point $q \in B$ being a non-extender respective ∂B is a center of a maximal disc contained in B.

Proof of Lemma 1 : There exists a minimal join s_1 from ∂B to q. This segment s_1 is distance minimal from the boundary ∂B to q and s_1 and joins some boundary point $p_1 \in \partial B$ with q. Thus,

$$d(q,\partial B) = d(q,p_1) \tag{2}$$

By assumption of the lemma 1 s_1 cannot be extended distance minimally beyond q. We claim that

the disc $K(q, d(p_1, q))$ is a maximal disc contained in B. (3)

In order to show (3) we first prove

$$K(q,d(p_1,q)) \subset B \tag{4}$$

In order to prove (4) we argue by contradiction. Namely assume $K(q,d(p_1,q))$ contains a point $w \in R^n \backslash B$. Join q with w by an arc-length parametrized Euclidean segment c(t) with c(0)=q, c(d(q,w))=w. By proposition 1 the segment c(t) necessarily meets the boundary ∂B in a point $c(t_o)$. The point $c(t_o) \neq c(d(q,w)) = w$ as $w \in R^n \backslash B$ is not on the boundary ∂B. Therefore

$$d(q,\partial B) \le d(q,c(t_o)) < d(q,w) \le d(q,p_1) \tag{5}$$

a contradiction with (2). This proves (4). The next claim we want to establish is that:

$K(q,d(p_1,q))$ is a maximal disc contained in B. (6)

To prove (6) we have to show that:

$$K(q,d(p_1,q)) \text{ is not contained in any disc } K(\bar{q},r) \subset B \text{ with } r > d(p_1,q). \tag{7}$$

To prove (7) we argue by contradiction. Namely assume that (7) is not true. Then there would exist a disc

$$K(\bar{q},r) \subset B \text{ with } r > d(p_1,q) \text{ and } K(q,d(p_1,q)) \subset K(\bar{q},r) \tag{8}$$

We show now first that in this case

$$r = d(\bar{q},p_1) \tag{9}$$

Clearly $r \ge d(\bar{q},p_1)$ because otherwise (i.e. if $r < d(\bar{q},p_1)$) the point p_1 would not be contained in $K(\bar{q},r)$ and this would yield a contradiction with the assumption

$$K(q,d(p_1,q)) \subset K(\bar{q},r).$$

made in (8). Therefore in order to establish (9) it remains to show

$$r \leq d(\overline{q}, p_1) \tag{10}$$

In order to show (10) we need the subsequent assertion:

Any arbitrarily small disc $K(p_1,\varepsilon)$contains points of $R^n \setminus B$ (11)

The claim (11) holds because p_1 is in ∂B. To make the latter reasoning for (11) formally precise we derive now a contradiction from the negation of (11) which will prove (11). For this note if $K(p_1,\varepsilon) \subset B$ then $K^o(p_1, \varepsilon/2) = \{ x \in R^n \mid |x - p_1| < \varepsilon/2 \}$ would be a neighbourhood around p_1 in B. Now $K^o(p_1, \varepsilon/2)$ is homeomorphic to R^n and not homeomorphic to the halfspace $H^n = \{ (x^1,\ldots,x^n) \,/\, x^1 \geq 0 \}$[6]. However (if B is a bordered manifold then) a boundary point $p_1 \in \partial B$ cannot have a neighbourhood $U \subset B$ with U being homeomorphic to R^n. This yields a rigorous argument for (11).

Using (11) it is now easy to establish (10). Namely assuming $r > d(\overline{q}, p_1)$ we conclude that there exists a positive number ε such that:

$$K(p_1, \varepsilon) \subset K(\overline{q}, r) \tag{12}$$

Thus, by (11) $K(\overline{q}, r)$ must contain points of $R^n \setminus B$ a contradiction with the assumption $K(\overline{q}, r) \subset B$ in (8). This shows (10) and completes the argument for (9).

After this intermediate step we proceed now with the proof of (7). Denote with $S(\overline{q})$, $S(q)$ the spheres being the boundaries of the discs $K(\overline{q},r)$, $K(q,d(p_1,q))$ respectively.

Assume now that the center $\overline{q}$ of $K(\overline{q}, r)$ is not contained in the extension of the ray z which starts at p_1 and passes through q.[7] Then the two spheres $S(q)$, $S(\overline{q})$ either intersect transversally at p_1 or they have only the point p_1 in common. In both cases there exist points on $S(q) \subset K(q, d(p_1, q))$ which are not in $K(\overline{q}, r)$, hence a contradiction with the assumption $K(q, d(p_1, q)) \subset K(\overline{q}, r)$ in (8). Therefore $\overline{q}$ must be contained in z. Let the ray z be parameterized by arc length z(t) with $z(0)=p_1$. There must exist a number $\overline{r}$ such that $z(\overline{r}) = \overline{q}$. Clearly $\overline{r} = r$. We want to prove that

$$\overline{r} = d(p_1, q) \tag{13}$$

Now if $\overline{r} < d(p_1, q)$ then $K(\overline{q}, r)$ could not include all points of $K(q,d(p_1,q))$ a contradiction with (8). Therefore $\overline{r} \geq d(p_1, q)$. Thus, it remains to exclude the possibility that

$$\overline{r} > d(p_1, q) \tag{14}$$

For this we argue again by contradiction and we assume that (14) is true, hence there exists a positive number δ such that

$$\overline{r} = d(p_1, q) + \delta\,. \tag{15}$$

Now $K(z(\overline{r}), r) = K(\overline{q}, r) \subset B$. Therefore with considerations similar to the one proving (11) above we find that the open disc

$$K^o(z(d(p_1, q) + \delta), r) = \{ x \in R^n / | x - z(d(p_1, q) + \delta) \,| < d(p_1, q) + \delta \}$$

contains no points of the boundary ∂B. Thus

[6]It is a well known result from algebraic topology that R^n and H^n are not homeomorphic, see e.g. [39], [40]

[7]The ray z is an extension of the interior normal of the sphere S(q).

$$d(\partial B, z(d(p_1,q)+\delta)) \geq d(p_1,q)+\delta \tag{16}$$

Therefore the segment $z_{\tilde{q}} = \{ z(t)/0 \leq t \leq d(p_1,q)+\delta \}$ is distance minimal from $\tilde{q} = z(d(p_1,q)+\delta)$ to the boundary ∂B. This segment $z_{\tilde{q}}$ contains $q = z(d(p_1,q))$ as an interior point. Thus the minimal join $s_q = \{ z(t)/0 \leq t \leq d(p_1,q) \}$ going from from ∂B to q can be extended as a minimal join beyond q. This is a contradiction with the assumption stated in lemma 1 that q is a nonextender with respect to the boundary ∂B. Therefore it disproves (8) and shows (7). This proves (6) and completes the proof of lemma 1.

Lemma 2: Let B be a closed solid body in R^n with boundary ∂B a topological (n-1) dimensional manifold. Let $K(q,r)$, $r > 0$ be a maximal disc contained in B. Then the center q of this disc is a non-extender respective ∂B and the radius $r = d(q, \partial B)$.

Proof of Lemma 2 : The proof is performed in two steps. In the first step we prove that there exist boundary points nearest to q and that all those points are located on the boundary of the disc K(q,r), i.e. they all have distance r to q.
Therefore in the first part of the proof of step 1 we show that:

$$\text{There exists a point } p \in \partial B \text{ with } d(q,p) = d(q,\partial B). \tag{17}$$

The second claim in step 1 can be rephrased by the conclusion:

$$\text{If } q \in \partial B \text{ with } d(p,q) = d(p,\partial B) \text{ then } d(p,q) = r \tag{18}$$

In the second step of the proof of lemma 2 we shall show that the assumption q being an extender respective ∂B can be used to disprove the maximality condition in lemma 2. In other words we show if q is an extender respective ∂B then we can find a disc D contained in B where D contains also K(q,r) as a proper subset. Thus, step 2 will establish lemma 2.

We show now the claims of step 1. The distance function $x \rightarrow d(q,x)$ is continuous and the boundary ∂B is compact. Therefore the distance function attains its minimum in some boundary point $p \in \partial B$. This proves (17).
We show now (18). For this we first prove that

$$d(p,q) \geq r \tag{19}$$

Assume the contrary of (19) then there exists a point of ∂B in $K^o(q,r) = \{ y \in R^n / |y-q| < r \}$. This implies using the argument for the proof of (11) that there exist points of $R^n \setminus B$ in $K^o(q,r)$. This yields a contradiction with the assumption of the lemma that $K(q,r) \subset B$. This shows (19). Next we prove

$$d(p,q) \leq r \tag{20}$$

For this assume $d(p,q) > r$; then there exists a positive number ε such that K(q,r+ ε) contains no points of ∂B. This implies that

$$K(q,r+\varepsilon) \text{ contains no point } e \in R^n \setminus B \tag{21}$$

because otherwise by Proposition 1 the Euclidean segment joining $q \in B$ with $e \in R^n \setminus B$ would meet ∂B in K(q,r+ ε). This would yield a contradiction with the preceding statement that K(q,r+ ε) contains no points of ∂B. This shows (21). Now (21) implies that K(q,r+ ε) is contained in B. This is a contradiction with the assumption of the lemma that K(q,r) is a *maximal* disc contained B. Thus we disproved $d(p,q) > r$ and have shown (20). This completes the proof of (18) and establishes the claims contained in the first part of the lemma's proof.

We give now the argument for the second step of the lemma's proof. For this let c(t) be an arc length parametrized Euclidean ray which starts at the boundary point p described by (17) and passes through q, hence c(0)=p and c(r)=q. It follows from (18) that the segment c([0,r]) is a distance minimal join from ∂B to the point q. Assume now that q is an extender with respect to ∂B. Then there exists a positive number δ such that c([0,r+ δ) is a distance minimal join to ∂B. This implies obviously that

$$D^o = K^o(c(r+\delta), r+\delta) = \{ y \in R^n / |c(r+\delta) - y| < r + \delta \}$$
contains no points of ∂B (22)

for otherwise c([0,r+ δ]) could not be distance minimal to ∂B. Using the argument which proved (21) together with (22) one finds that

$$D = K(c(r+\delta), r+\delta) \text{ contains no points of } R^n \setminus B \quad (23)$$

Note if D would contain a point w of $R^n \setminus B$ then an arc-length parametrized segment g joining c(t+δ) with w would meet ∂B in an interior point x of g because $x \neq w$ as x is not in $R^n \setminus B$. Since the boundary point x is an interior point of g this point x must be in D^o a contradiction with (22). This consideration yields a formally complete argument for (23). Therefore D is contained in B. Also D obviously contains K(q,r). This yields a contradiction with the lemma's assumption that K(q,r) is a maximal disc contained in B and completes the proof of lemma 2.

Remark : Analyzing the preceding proof one finds that the key properties used in the arguments are :

- that the boundary ∂B separates the interior of the solid B from its complement $R^n \setminus B$;
- subsets of the boundary ∂B which are contained in any closed bounded disc are compact.

We used in our lemmata 1 and 2 that Both of those itemized properties will hold not only if B is compact but also in case the solid B is a unbounded, closed, bordered n-dimensional submanifold of R^n, with the boundary ∂B being an $n-1$-dimensional topological manifold which may even have infinitely many unbounded components.

Based on these considerations one can obviously define the concept of an exterior medial axis with respect to the solid B as the centers of all maximal discs which are contained in $\overline{(R^n \setminus B)} \cup \partial B$. Analogue to theorem 1 this exterior medial axis can now be characterized also as that subset of the cut locus $C_{\partial B}$ which is contained in $(R^n \setminus B) \cup \partial B$.

Next we give a series of results which explain mainly local properties (or the local nature) of the points in the cut locus (which agrees in B with the medial M(B) by Theorem 1). To simplify some of the statements in the results below, we introduce a name (notation) for a specific non-extender called pica.

Definition : A pica q with respect to a closed set A is a point which has at least two nearest neighbors on A, see Wolter [45].

The proofs of results in this paragraph as well as the proof of our global topological shape theorem in the next section makes use of the subsequent Theorem 2 which holds under very weak general assumptions. We state now a simplified (weakened) version of this result in [45]. In this version, we require the set A to be a closed, possibly disconnected, subset of R^n. Under these assumptions, we have:

Theorem 2: (Characterization of the Cut Locus of a Closed Set A in R^n)

- A) The picas with respect to A are dense in C_A. Hence the cut locus C_A consists of those points and their limit points.
- A') $R^n \setminus C_A$ is in R^n the maximal open set of points, which have a unique minimal join to A.
- B) The complement of the cut locus C_A, i.e. precisely $R^n \setminus (A \cup C_A)$ is the maximal

open set in $R^n \setminus (A \cup C_A)$ where the distance function $d(A, .)$[8] is C^1-smooth, and its gradient $\nabla\, d(A, .)$ is locally Lipschitz continuous on $R^n \setminus (A \cup C_A)$. At any point $q \in R^n \setminus (A \cup C_A)$ the gradient $\nabla d(A,q)$ equals the unit direction vector of the minimal join from the set A to q.

In order to illuminate the preceding statement A') we mention here:

Remark: That there exists always a unique minimal join from every point $p \in A$ to C_A does not hold in general if A is only a piecewise C^2-smooth submanifold of R^n. It holds however if A is a regular C^1-smooth submanifold of R^n. To illuminate the statement in the piecewise C^2-smooth case take a planar polygonal domain then it easy to construct examples where a concave vertex has more than one minimal join to the cut locus of the boundary polygon.

The next result describes local properties of points in the cut locus and also local aspects of its topological structure:

Theorem 3: Local Properties of Points in the Cut Locus Let A be a closed $n-1$-dimensional submanifold of R^n. In case $n > 2$ we assume the manifold A to be C^2-smooth. If $n = 2$ we only require A to be piecewise C^2-smooth. Under those assumptions the following statements hold

- A) A limit point of non-extenders with respect to A is a non-extender with respect to A. All points in the cut locus C_A are non-extenders respective the set A.
- B) In the piecewise linear boundary case, all non-extenders are picas. A limit of picas is here a pica[9].
- C) In the C^2-smooth boundary case, if a non-extender is not a pica, then it is a curvature center of the boundary A it may be both, e.g. the center of a circle.
- D) The set C_A is nowhere dense in R^n.

Proof of Theorem 3: The parts A), B), C) of theorem 3 are are shown in lemma A.1 contained in the appendix of this paper. It remains to prove part D).

Proof of Theorem 3 D) : Assume that the set C_A were some where dense in R^n. Then C_A being defined as a closed set would contain some solid n-dimensional disc $K(\mathbf{q}, r) = \{x \in R^n \,||\mathbf{x} - \mathbf{q}| \le r\},\ r > 0$. Obviously A being an $n-1$-dimensional submanifold of R^n cannot be dense in any n-dimensional disc. Therefore, we can find some n-dimensional disc $K(\mathbf{p}, w) = \{x \in R^n|\ |\mathbf{x} - \mathbf{p}| \le w\},\ r > w > 0$ with $K(\mathbf{p}, w) \subset K(\mathbf{q}, r)$ such that

$$K(\mathbf{p}, r) \cap A = \emptyset \tag{24}$$

There must exist a distance minimal segment $g(t)$ from the set A to the point $\mathbf{p}$. Let $g(t)$ be arc-length parameterized and assume that $g(d(A, \mathbf{p})) = \mathbf{p}$ with $d(A, \mathbf{p})$ being the distance of the point p to the set A. Then the point $g(d(A, \mathbf{p}) - w/2)$ being contained in $K(\mathbf{p}, w) \subset C_A$ must be a non-extender by theorem 3 A). This yields a contradiction with corollary 1 because the path $g(t)$ is distance minimal beyond $g(d(A, p) - w/2)$ up to the point p. This proves theorem 3 D) and completes the proof of theorem 3.

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[8] d(A,.) being the Euclidean distance function with respect to the closed set A.The point "." in the expression d(A,.) is a place holder for the variable of this function. Evaluating the function d(A,.) for a specific variable value ie. for a specific point p yields d(A,p) which is the distance of the point p to the set A.

[9] If this limit is on A we have a degenerate case, which we allow.

In order to illuminate the subtleties in the preceding theorem 3, we want to point out:

Remark: If we require the boundary ∂B above to be only C^1-smooth manifold (even with Lipschitz-continuous derivatives), then a limit of picas may be an extender. Thus here a limit of non-extenders may be an extender, cf. also example 1; moreover, here the picas (with respect to ∂B) may be dense in some open subregions of B, thus here the cut locus and hence the medial axis M(B) will be dense in some open sub-area of B. Note that also if dimension $n > 2$ and if the boundary ∂B is piecewise linear then statements A) and B) in theorem 3 are violated because a limit of picas may be a nonextender in this case, cf. also lemma A.2 in the appendix. In the general C^∞-smooth boundary case, e.g. in the plane with ∂B being a simple closed curve, the medial axis M(B) may have infinitely many end points which are caused by infinitely many curvature centers of ∂B; hence here M(B) may not be the union of finitely many arc pieces.

3.3 The Cut Locus Avoids Certain Reference Sets

There exists one important result which holds under very weak regularity assumptions. This result says that the cut locus stays away at least a certain positive distance from a set if that set fulfills certain regularity requirements. This result implies that the cut locus stays away at least a certain positive distance from a C^1-smooth rational spline patch defined over a rectangular domain. This holds if the surface patch is free of self-intersections and if the surface map has a Jacobian of rank 2 at all points. We shall actually prove a more general result which includes spline patches as a special case.

Theorem 4: Cut Locus avoids certain reference surface patches. Let $q(s,t): D = [0,1] \times [0,1] \rightarrow R^3$ be a regular C^1 - smooth surface S which is free of self-intersections. Regular means that the Jacobian $q' = (\partial_s q, \partial_t q)$ has rank 2 everywhere. We assume further that the partial derivatives of q(s,t) are Lipschitz continuous. Under those assumptions there exists a positive number λ such that the cut locus C_S stays away farther than distance λ from the surface S.

Note the requirement that the partial derivatives of q(s,t) are Lipschitz continuous is weaker than C^2 smoothness and it is already fulfilled if the surface is a C^1-smooth rational B-spline patch.

Remark: The requirement of Lipschitz continuity of the first partial derivatives can not be left out in theorem 4, it follows from [45], p. 65. that this Lipschitz continuity is also a necessary condition to prevent the cut locus from coming arbitrarily close to the surface S. It is easy to construct non-degenerate C^1-smooth planar curves which have their cut locus coming arbitrarily close. Namely define a planar curve $\{ (x(t), y(t)) / -1 \le t \le 1 \}$ by $x(t) = t$ and $y(t) = 0$ if $t \le 0$ otherwise $y(t) = t^{3/2}$. This yields a non-degenerate C^1-smooth curve which has infinitely large curvature at $(x(0), y(0)) = (0,0)$ and the cut locus of this curve approaches (and contains) the curve point (0,0).

We give now a proof of theorem 4. For this purpose we shall need the following

Lemma 3: Let D be a compact, convex set in R^n and assume that D is n-dimensional i.e. D contains an n-dimensional disc. Let m be any positive integer number and assume that the function $f(x): D \rightarrow R^{n+m}$ is C^1- smooth and regular i.e. $|f'(x)h| \neq 0$ if $h \neq 0$. We assume further that the Jacobian $f'(x)$ is Lipschitz continuous in the variable x. Under these assumptions there exist two positive numbers r_o, h_o such that for any unit vector N(x)

$$|f(x+h) - f(x) - rN(x)| > |r| \tag{25}$$

for all r with $0 < |r| < r_o$ if $|h| < h_o$ and if $f'(x)h$ is orthogonal on the unit vector N(x).

Proof of Lemma 3: In this proof we shall use a first order Taylor development of f(x+h) with a Lipschitz continuous remainder term. Namely representing f(x+h) by approximation with its Jacobian $f'(x)$ we get

$$f(x+h) = f(x) + f'(x)(h) + R(x,h)|h| \tag{26}$$

where $R(x,h)|h|$ is a remainder term and $f'(x)(h)$ means that the linear map $f'(x)$ is applied on the vector h c.f. e.g. Dieudonne [5].

We show next the Lipschitz continuity of the remainder term R(x,h) precisely we shall estimate the norm of R(x,h) by a product built by the norm of h multiplied with a constant number M, where M is independent of x. For this observe the Lipschitz continuity of the differential $f'(x)$ in the variable x means that there exists a number M such that

$$|f'(x+h) - f'(x)| \leq M|h| \tag{27}$$

if (x+h),x are points in D.

If the points x, (x+h) are in D then using (26) and (27) the remainder term fulfills

$$|R(x,h)| = \frac{|f(x+h) - f(x) - f'(x)(h)|}{|h|}$$

$$= \frac{|f(x+h) - f'(x)(h) - f(x+0) - f'(0)|}{|h|}$$

$$\leq \sup_{0 \leq s \leq 1} |\psi(s)| \frac{1}{|h|} \tag{28}$$

$$\leq \sup_{0 \leq s \leq 1} |f'(x+sh) - f'(x)| \tag{29}$$

$$\leq M|h| \tag{30}$$

if we define

$$\psi(s) = f(x+sh) - f'(x)(sh) \tag{31}$$

then (28) follows from a generalized mean value theorem see Dieudonne [5] as (31) implies

$$\psi'(s) = f'(x+sh)(h) - f'(x)(h) \tag{32}$$

Now (32) implies

$$|\psi'(s)| \leq |f'(x+sh) - f'(x)|\,|h| \tag{33}$$

and (33) yields (29) and (27) yields (30). In summary the remainder term for the first order Taylor development of the function f(x) fulfills

$$|R(x,h)| \leq M|h| \tag{34}$$

where the number M is independent of the point x in D.

We proceed now with the proof of lemma 3. For this inserting a first order Taylor development for f(x+h) yields

$$|f(x+h)-f(x)-rN(x)| = |f(x)+f'(x)(h)+R(x,h)|h|-f(x)-rN(x)|$$

$$\geq |f'(x)(h)-rN(x)|-|R(x,h)||h|$$

$$\geq \sqrt{e^2|h|^2+r^2}-M|h|^2 =: A \qquad (35)$$

where

$$e =: min\{|f'(x)(h)|/x \in D, |h|=1\}$$

[10]note to get (35) we use (34) and we exploit that (by assumption of the lemma 3) $f'(x)(h)$ is orthogonal on N(x). Applying now the mean-value theorem on the square root function (expression) in (35) we find that there exists a number $\xi \in (0,1)$ such that :

$$A = \frac{e^2|h|^2}{2\sqrt{r^2+\xi e^2|h|^2}} - M|h|^2 + r$$

$$\geq \frac{e^2|h|^2}{2\sqrt{r^2+e^2|h|^2}} - M|h|^2 + r.$$

Now choose two positive numbers r_o, h_o so small that

$$\frac{e^2}{2\sqrt{r_o{}^2+e^2|h_o|^2}} > M$$

then (25) obviously holds. This completes the proof of the lemma.

Proof of Theorem 4 :

We shall prove :

That there exists a number $\lambda > 0$ such that every minimal join emanating from S is distance minimal to S for a length λ. (36)

The proof of (36) follows from the two subsequent assertions namely assertion 1 and assertion 2.

Assertion 1: There exist two positive numbers δ, R such that the following holds:

Let x be any point in D and let $g_x(t)$ be any arc-length parametrized segment with $g_x(0) = q(x)$. Assume there exist two (arbitrarily small) positive numbers ω, η such that the segment $g_x[0,\eta]$ is distance minimal to the subset $q(U_\omega)$ of S where $U_\omega = \{y \in D / |x-y| \leq \omega\}$.

Then for all points

$y \in U_\delta(x) \setminus \{x\}$ we have $|q(y) - g_x(t)| < t$ if $t \leq R$.

This means a segment $g_x(t)$ which starts as a locally distance minimal join at any point q(x) is distance minimal to the (whole) subset $q(U_\delta)$ if the length of $g_x(t)$ is $\leq R$.

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[10]Note e exists because D is compact and $e > 0$ because here $f'(x)(h) \neq 0$ as $f'(x)$ has maximal rank.

Proving this assertion 1 is the difficult part of the theorem's proof. We shall give the proof of assertion 1 further down.

The other assertion used to complete the proof of theorem 4 is the following

> **Assertion 2:** For any positive number δ' there exists a number $\alpha(\delta')$ such that for any two points x,y in D with $|x-y| \geq \delta'$ we have $|q(x)-q(y)| \geq \alpha(\delta')$.

Proof of Assertion 2: Assertion 2 holds because the surface S is free of self intersections and because it is defined over a compact parameter domain D.

Namely define

$$\alpha(\delta') = min\{\, |q(x)-q(y)| \,/\, (x,y) \in D \times D,\ |x-y| \geq \delta' \}. \tag{37}$$

The set $D \times D$ is a compact set in R^4 and $\{(x,y) \in R^4 \,/\, |x-y| \geq \delta'\}$ is a closed set in R^4. Therefore the set $W = (D \times D) \cap B$ is a compact subset in R^4. The function $(x,y) \rightarrow a(x,y) := |q(x)-q(y)|$ is a continuous function on R^4. This function $a(x,y)$ is positive on W because $x \neq y$ and because the map $q(s,t)$ is free of self-intersections. The function $a(x,y)$ being a continuous function defined on a compact set W must attain its minimum which must be positive here. This shows that $\alpha(\delta') > 0$ and proves assertion 2.

Combining assertion 1 and assertion 2 we finish now the proof of theorem 4. This will prove the theorem 4 by using the still unproven assertion 1 which we will show further down.

Completing the proof of theorem 4 by using assertion 1 and assertion 2: Let $\delta, R > 0$ be the numbers described in assertion 1 and let $\alpha(\delta')$ be the number described in assertion 2. Then the claim of theorem 4 will hold if we define

$$\lambda = min\{\, \frac{1}{2}\alpha(\delta),\ R \,\}.$$

This means any minimal join $g_x(t)$ starting at any point q(x) in S remains distance minimal[11] to the surface S over the length λ. This holds because by assertion 2 no point q(y) in D with $|x-y| \leq \delta$ can have have a distance less than λ to the point $g_x(\lambda)$. Therefore at most a point q(y) with $|y-x| > \delta$ may have a distance smaller than λ to the point $g_x(\lambda)$. However this is impossible because by assertion 2 for points with $|x-y| \geq \delta$ is $|q(x)-q(y)| \geq \alpha(\delta)$. Thus if $|x-y| \geq \delta$ and $0 \leq t \leq \lambda$ then:

$$2\lambda \leq \alpha(\delta) \leq |q(y)-q(x)| \leq |q(y)-g_x(t)| + |q(x)-g_x(t)|$$

$$2\lambda \leq |q(y)-g_x(t)| + t$$

$$\lambda \leq |q(y)-g_x(t)|.$$

Thus for points y outside $U_\delta(x)$ a point $q(y)$ is not closer than distance t to the point $g_x(t)$ if $0 \leq t \leq \lambda$. This proves theorem 4 using the unproven assertion 1.

We finish now the proof of theorem 4 by giving a proof for assertion 1

Proof of Assertion 1: The Lemma 3 implies assertion 1 in most but not all cases where a minimal

[11] To specify our notation we say here that we assume that $g_x(t)$ is arc-length parametrized.

segment $g_x(t)$ starts on a surface patch S. (Note we assume that $g_x(t)$ is arc-length parametrized.)

It covers all the cases where the segments initial point $q(x)=g_x(0)$
is not on the boundary of the patch, because in such a case the
initial direction of the segment $g_x(t)$ must be normal to the patch S. (38)

The lemma covers even more cases. Namely if the minimal join $g_x(t)$ starts in the interior of one boundary edge b then it must be orthogonal to that boundary edge b. Here now the lemma 3 implies that $g_x(t)$ remains to be (locally) a minimal join to the boundary edge b. In other words in this situation lemma 3 shows that:

all points in $q(U_\delta(x)\cap b)$ are further from $g_x(t)$ than the point q(x) if
$t \le R$ and if we assume that R stands for the number r_o in lemma 3. (39)

Note assertion 1 is equivalent with the statement:

for all x in D the distance $d(q(U_\delta(x)), C_{U_\delta(x)}) \ge R$ (40)

As the picas are dense in the cut locus by Theorem 2, (40) is equivalent to the statement

for all x in D the set $q(U_\delta(x))$ has no picas
coming closer to it than distance R. (41)

The preceding conclusions so far drawn from lemma 3 show that $C_{q(U_\delta(x))}$ contains no pica p in distance closer than R to $q(U_\delta(x))$ such that one of the foot points of p[12] is either an

interior point of the patch (42)

or a vertex point of the patch (43)

Clearly the case (42) is excluded by the above statement (38) and (43) is excluded by the combination of (38) and (39). Namely if one foot point is a vertex point v then (under our nearness assumptions) the other foot point of the pica must either be an interior patch point or must be on a boundary edge containing the vertex v. Therefore the only remaining case which needs to be treated *i.e. shown to be impossible* is the one :

where a pica point p has two oblique minimal joins to S which have two foot
points q(x) and q(y) in two adjacent boundary curves and where $|x-y| \le \delta$. (44)

Indeed case (44) is actually the situation which allows the cut locus to come arbitrarily close to a boundary vertex in case the vertex is concave. Before we start a detailed discussion of the oblique pica case (44) we show now first that

any corner part of the patch S can be locally
approximated by a convex planar subset. (45)

Proof of (45): Let

$$L = (\partial_s q(0,0)\ \partial_t q(0,0))$$

be the differential related to the vertex point q(0,0) of the patch S. Let

$$Co(\varepsilon) = \{(s,t) \in [0,1] \times [0,1] \,/\, |(s,t)^t| < \varepsilon\}$$

Co(ε) is obviously a convex set and the linear map L (preserving convexity) will map Co(ε) onto a convex set $L(Co(\varepsilon)) \subseteq L(R^2)$.

[12]A foot point of p on $q(U_\delta(x))$ is defined as a point nearest of $q(U_\delta(x))$ to p.

The set L(Co(ε)) must be contained in a *proper* sector in the Euclidean plane with an opening angle $\omega < \pi$. This term proper holds because L(Co(ε)) cannot contain a straight line segment g passing unbroken through $L((0,0)^t)$ because otherwise we could find two vectors $x_1, x_2 \in Co(\varepsilon)$ such that $L(x_1)$, $L(x_2)$ would be collinear to g. This would yield a contradiction with the facts that x_1, x_2 are linear independent and that the linear map L having maximal rank preserves linear independence.

Exploiting the approximation properties of the differential L we find that $D_\varepsilon = q(Co(\varepsilon))$ is contained in a a set Ap(ε) which can be described as follows[13]

$$D_\varepsilon \subseteq \{ L(s,t) + \overline{R}(s,t) \ / \ L(s,t) \in L(Co(\varepsilon)), |\overline{R}(s,t)| \leq \frac{M}{\beta} |L(s,t)|^2 \}$$

[14]where

$$\beta = min\{ |L(s,t)|^2 / |(s,t)| = 1 \}$$

and M is defined by (27), (34).

Obviously for sufficiently small $\varepsilon > 0$ the set D_ε can be shown to be contained in a convex set say

$$D_\varepsilon \subseteq \{ L(s,t) + \overline{R}(s,t) \ / \ L(s,t) \in L(Co(\varepsilon)), |\overline{R}(s,t)| \leq \alpha(\varepsilon) M |L(s,t)|^2 \}$$

where α(ε) can be made arbitrarily small if ε is shrinking to zero. This completes the proof of (45).

We continue now the discussion of (44) that is we continue to show why (44) is not possible if the number δ in (44) is chosen sufficiently small. For this pick any point $p_o = q(s_o,0)$ on a boundary curve b_o where $b_o = \{q(s,0) / 0 < s \leq 1\}$. The surface normal $N(q(s_o,0)$ and the two tangent vectors $\partial_s q(s_o,0), \partial_t q(s_o,0)$ span the 3-space $R^3{}_{p_o}$ at $q(s_o,0)$. The plane spanned by $N(q(s_o,0))$, $\partial_s q(s_o,0)$ separates the 3-space $R^3{}_{p_o}$ into two half spaces. The vector $\partial_t q(s_o,0)$ points into the half space $H^+{}_{p_o}$ corresponding to the interior of the patch at p_o. The vector $-\partial_t q(s_o,0)$ points into the half space $H^-{}_{p_o}$ corresponding to the exterior of the patch at p_o. Let $v^+{}_{p_o}$ be any unit vector vector in $H^+{}_p$ and let

$$g(s) = \{p_o + sv^+{}_{p_o} / 0 \leq s \leq 1\}$$

be a segment starting at p_o and pointing into the direction $v^+{}_{p_o}$. Then :

for sufficiently small numbers s the orthogonal projection of g(s) onto S will be contained on the patch S in a neighborhood of the point p_o. (46)

Here (46) holds because the projection $p_T(v^+{}_{p_o})$ of $v^+{}_{p_o}$ into the tangent plane spanned by $\partial_s q(s_o,0), \partial_t q(s_o,0)$ is transversal to the boundary curve at p_o and points into the patch interior if $v^+{}_{p_o}$ is not collinear to the surface normal at p_o. (In case $v^+{}_{p_o}$ is collinear to the surface normal at p_o then (46) holds anyhow.) Using (46) it is not difficult to see that for arbitrarily small values of s there are points

[13]Moreover this set Ap(ε) yields also a quadratic approximation of D_ε

[14]For fixed given values (s,t) the vector $\overline{R}(s,t)$ attains all points in a disc of radius $\frac{M}{\beta}|L(s,t)|^2$.

x(s) on S such that $|x(s) - g(s)|$ is smaller than s.[15] Therefore g(s) cannot be (a locally) minimal join to S if the initial direction v_{p_o} is chosen from $H^{+}_{p_o}$ and if v_{p_o} is not collinear to the surface normal at p_o. Thus in order to have an oblique minimal segment g(s) at the boundary point p_o we must choose an initial direction $v^{-}_{p_o} \in H^{-}_{p_o}$. We can now assume that $v^{-}_{p_o}$ is not collinear to the surface normal at p_o because that case had already been settled in the preceding discussions essentially as a consequence of lemma 3. Let now $g(s) = \{p_o + sv^{-}_{p_o} \,/\, 0 \le s \le 1\}$.

Now if g(s) is locally distance minimal to S then :

$g'(0)$ must be orthogonal to the boundary edge b_o as
$p_o \neq q(0,0)$, hence $v^{-}_{p_o}$ is orthogonal to $\partial q(s_o, 0)$. (47)

Let $\tilde{g}(s)$ be an arc-length parametrized distance minimal segment emanating from the boundary edge b_1 adjacent to b_o i.e.

$$b_1 := \{q(0,t) \,/\, 0 < t < 1\}$$

The segment $\tilde{g}(s)$ is oblique to the boundary edge in a way analogue to g(s), i.e. $\tilde{g}'(0)$ points also into the corresponding (exterior) half space $H^{-}_{\tilde{g}(0)}$. We want to show that:

there exists a positive number R such that $\tilde{g}(s) \neq g(s)$ for all $s \le R$
if the initial points g(0), $\tilde{g}(0)$ are sufficiently close. (48)

Now let $g_T(s)$, $\tilde{g}_T(s)$ be projections of g(s), $\tilde{g}(s)$ into the tangent plane T spanned by $\partial_s q(0,0)$, $\partial_t q(0,0)$ at q(0,0). If g(s), $\tilde{g}(s)$ are supposed to intersect then also their projections. We are essentially interested in the case where g(0), $\tilde{g}(0)$ are located arbitrarily close to the vertex q(0,0). We have established above below (45) that $\partial_s q(0,0)$, $\partial_t q(0,0)$ build a convex vertex angle β smaller than π. For positive sufficiently small numbers s,t say $0 < s,t < \delta'_o$ the angle between $\partial_s q(s,0)$, $\partial_t q(0,t)$ comes arbitrarily close to β and is therefore also smaller than π as well. Using elementary planar geometry it can be shown that the segments $g[0,\infty)$, $\tilde{g}[0,\infty)$ will not intersect if the initial points g(0)=q(s,0), $\tilde{g}(0)$=q(0,t) are chosen such that $s,t < \delta'_o$. Therefore in order to have minimal joins which start oblique from the boundary edges $b_o \setminus \{q(0,0)\}$, $b_1 \setminus \{q(0,0)\}$ intersect one has to choose the initial points g(0)=q(s,0), $\tilde{g}(0)$=q(0,t) such that $s,t > \delta'_o$. This proves that (44) is not possible if δ is here smaller than δ'_o. The same considerations can be applied for the corresponding situations at the remaining three vertices. It is now obvious that for an appropriately small chosen δ the case (44) is impossible. This finishes the proof of assertion 1.

As we have now established assertion 1 we have also completed the proof of the theorem 4.

Analyzing the preceding proof of theorem 4 one finds that theorem 4 holds also in the more general case if the domain D is chosen to be any set in R^2 which has the property that there exists a C^1-diffeomorphism ϕ from D to a compact convex set in R^2 with the derivative of ϕ being Lipschitz continuous. The preceding theorem is useful in studying surface intersections, see Kriezis, Patrikalakis, Wolter [19] and Kriezis [18]. Another result being essentially a consequence of the preceding theorem 4 is the subsequent corollary.

Corollary 4.1: Using notations and assumptions of Theorem 4 then for any positive number $\varepsilon \le \lambda$:

A) The offset $O_\varepsilon(S) = \{x \in R^3 \,|\, d(x,S) = \varepsilon\}$ is a $C^{1,1}$-smooth manifold, diffeomorphic to the embedded

[15] This is obvious in case S agrees with its tangent plane at p_o. In the general case it follows because this tangent plane yields a first order approximation of the patch S in a neighborhood of the point p_o and because the difference between s and the distance of g(s) to the tangent plane at p_o is given by a positive linear function $\phi(s)$ in the variable s say $\phi(s) = m\,s$.

two-dimensional unit sphere and the offset region $OR_\varepsilon(S) = \{x \in R^n \mid d(x,S) \leq \varepsilon\}$ is a solid homeomorphic to the 3-dimensional unit disc $\{x \in R^3 \mid |x| \leq 1\}$.

B) The surface S is the medial axis of the solid $OR_\varepsilon(S)$.

Proof of Corollary 4.1 : Our proof of part A) will be sketchy and we will omit some detailed steps which are not difficult to carry out. Let $Q = \{(u,v,w) \in R^3 \mid w = 0,\ (u,v) \in [0,1] \times [0,1]\}$ be the unit square embedded in R^3. Let $OR_\varepsilon(Q) = \{y \in R^3 \mid d(y,Q) \leq \varepsilon\}$, $O_\varepsilon(Q) = \{y \in R^3 \mid d(y,Q) = \varepsilon\}$ be offset region and offset surface respectively for the progenitor set Q and offset distance ε. It is not difficult to show that $OR_\varepsilon(Q)$, $O_\varepsilon(Q)$ are homeomorphic to the closed three-dimensional unit disc and the two-dimensional unit sphere respectively with $O_\varepsilon(Q)$ being the boundary surface of the bordered manifold $OR_\varepsilon(Q)$. We prove part A) by defining a homeomorphism $\psi: OR_\varepsilon(Q) \to OR_\varepsilon(S)$. This homeomorphism ψ which also induces a homeomorphism between $O_\varepsilon(Q)$ and $O_\varepsilon(S)$ is constructed such that

ψ maps distance minimal segments between Q and $O_\varepsilon(Q)$
on distance minimal segments between S and $O_\varepsilon(Q)$. (49)

We give now a detailed description of the map ψ. For this we denote the parametric surface map representing S by $f(u,v): [0,1] \times [0,1] \to R^3$. The surface normal of S at a point $f(q) \in S$ is denoted by $N(q)$ and depends continuously on the variable point $q \in Q$. Let $e_1 = \{(u,v,w) \in Q \mid v = 0\}$, $e_2 = \{(u,v,w) \in Q \mid u = 1\}$, $e_3 = \{(u,v,w) \in Q \mid v = 1\}$, $e_4 = \{(u,v,w) \in Q \mid u = 0\}$ be the four edges of ∂Q. These edges can also be viewed as paths depending on the variables u or v respectively, in this context they are denoted by $e_1(u)$, $e_2(v)$, $e_3(u)$, $e_4(v)$. For any of those four edges e_i, $1 \leq i \leq 4$ we define the *exterior boundary normal n_i in the tangent plane of S* say at a point $e_1(u)$ by a unit tangent vector $n_1(u)$ of S at the point $e_1(u)$; the exterior boundary normal vector $n_1(u)$ must be chosen orthogonal on the tangent vector $\partial_u f(u,0)$ and the sign of $n_1(u)$ is determined by the condition that the angle between $-\partial_v f(u,0)$ and $n_1(u)$ must be smaller than $\pi/2$. Note that the line parallel to the tangent vector $Df(e'_i(u))$ (or $Df(e'_i(v))$ respectively) separates the related tangent plane into two half planes and n_i is chosen to point into the exterior half space which does not contain the "interior" tangent vector $\partial_v f(u,0)$, $-\partial_u f(1,v)$, $-\partial_v f(u,1)$, $\partial_u f(0,v)$ respectively for the cases $i = 1,2,3,4$. These considerations together with the condition that n_i must be orthogonal on the tangent vector $Df(e'_i(u))$ (or $Df(e'_i(v))$ respectively) give the complete definition of the exterior boundary normal n_i. To simplify the description of the map ψ we need also to introduce the subsequent definitions

If $u,v \in [0,1]$ then $\Delta u = \Delta v = 0$

If $u,v \notin (0,1)$ then

$$\Delta u = -u \text{ if } u \leq 0,\ \Delta u = u - 1 \text{ if } u \geq 1$$
$$\Delta v = -v \text{ if } v \leq 0,\ \Delta v = v - 1 \text{ if } v \geq 1 \tag{50}$$

With these notations we define now the map $\psi(q)$ with $q = (u,v,w)$. If $(u,v) \in [0,1] \times [0,1]$ then $\psi(q) = wN(u,v) + f(u,v)$.

If $v \leq 0$ and $u > 0$ and if $\sqrt{(\Delta u)^2 + (\Delta v)^2} > 0$ then

$$\psi(q) = f(u - \Delta u, v + \Delta v) + wN(u - \Delta u, v + \Delta v) +$$
$$\sqrt{(\Delta u)^2 + (\Delta v)^2}\ \frac{\Delta u\, n_1 + \Delta v\, n_2}{|\Delta u\, n_1 + \Delta v\, n_2|} . \tag{51}$$

On the other three rectangular strips around the boundary of the unit square Q the map $\psi(q)$ is defined analog to the definition given in (51). The map ψ is obviously continuous and elementary considerations show that the map ψ has property (49). It is not difficult to verify that the preimage under the map ψ of any shortest segment between S and $O_\varepsilon(S)$ is a shortest segment between Q and $O_\varepsilon(Q)$. All these considerations together show that $\psi: OR_\varepsilon(Q) \to OR_\varepsilon(S)$ is a continuous, injective map onto $OR_\varepsilon(S)$, where the injectivity follows because $\varepsilon < d(S, C_S)$ i.e. the distance of S to its cut locus is larger ε. This shows that ψ defined on compact set and being a continuous, injective map onto its image set is a homeomorphism[16]. This fact in conjunction with theorem 2 essentially imply part A of the corollary. Note the claim that $O_\varepsilon(S)$ is a $C^{1,1}$-smooth two-dimensional submanifold of R^3 follows with the implicit function theorem (c.f. [5]) from the fact that the distance function $d(S,.)$ is $C^{1,1}$*–smooth* with a non-zero gradient on $O_\varepsilon(S)$ which holds by theorem 2B because $d(S, C_S) > \varepsilon$. Finally the claim that $O_\varepsilon(S)$ is diffeomorphic to the the unit sphere S^2 follows because $O_\varepsilon(S)$ has the homeomorphy type of the unit sphere[17] and because smooth, compact two-dimensional homeomorphic manifolds are diffeomorphic cf. Hirsch [11].

Proof of corollary 4.1 B) : It has been established in the proof of part A) that the homeomorphism ψ maps disjoint shortest segments between $O_\varepsilon(Q)$ and Q on disjoint shortest segments between $O_\varepsilon(S)$ and S and that the inverse map of ψ maps disjoint shortest segments between $O_\varepsilon(S)$ and S onto disjoint shortest segments between $O_\varepsilon(Q)$ and Q. The homeomorphism ψ provides *a one to one correspondence between the intersection points of minimal joins* in both sets $OR_\varepsilon(Q)$ and $OR_\varepsilon(S)$; those intersection points are given in $OR_\varepsilon(Q)$ by the intersection of minimal joins between $O_\varepsilon(Q)$ and Q and in $OR_\varepsilon(S)$ by the intersection of minimal joins between $O_\varepsilon(S)$ and S. Clearly those intersection points of minimal joins are picas with respect to either one of two reference sets $O_\varepsilon(Q)$, $O_\varepsilon(S)$. Therefore and because Q is the set of picas in $OR_\varepsilon(Q)$ respective $O_\varepsilon(Q)$ it is obvious that the image set $\psi(Q) = S$ is the set of picas in $OR_\varepsilon(S)$ respective $O_\varepsilon(S)$. This proves part B) in view of theorem 1 and employing the fact that the picas are dense in the cut locus by theorem 2 A). This finishes the proof of corollary 4.1

In practical terms this corollary 4.1 states that any regular C^1-smooth regular spline surface patch which is free of self-intersections can be manufactured (modelled) with a ball cutter of radius ε where the center of the ball cutter moves along a compact $C^{1,1}$-smooth offset surface $O_\varepsilon(S)$ being diffeomorphic to the unit 2-sphere. This offset surface $O_\varepsilon(S)$ bounds a solid $OR_\varepsilon(S)$ whose medial axis equals the surface S. In other words, if the offset distance is smaller than the distance of the progenitor surface S to its cut locus then the progenitor surface is the medial axis of the offset region, see also figure 1 illustrating a surface S and a related offset surface $OR_\varepsilon(S)$. Another engineering application for the discussed offset surfaces and offset regions arises within the context of tolerancing where one wants to determine if a manufactured object fits within a specified tolerance region (offset region) of an ideal design surface, see Rossignac [37], Rossignac and Requicha [38] and Patrikalakis and Bardis [31].

Analyzing the preceding proof of theorem 4 one can derive another conclusion interesting enough to be called a theorem. Namely we have

Theorem 5: Cut Locus avoids compact unbordered submanifolds of R^n. Let A be any compact unbordered C^1-smooth submanifold of R^n. We assume that all local parametrizations of A have

[16]Note that it is a well known fact from point set topology that a continuous, injective map defined on a compact domain yields a homeomorphism onto the image set of the compact domain cf. e.g. [16], [15].

[17]Note that $O_\varepsilon(S)$ is homeomorphic to S^2 because $O_\varepsilon(S)$ is via ψ homeomorphic to $O_\varepsilon(Q)$ and because it is easy to construct a homeomorphism between S^2 and $O_\varepsilon(Q)$ as this construction may employ convexity properties of the solid $OR_\varepsilon(Q)$.

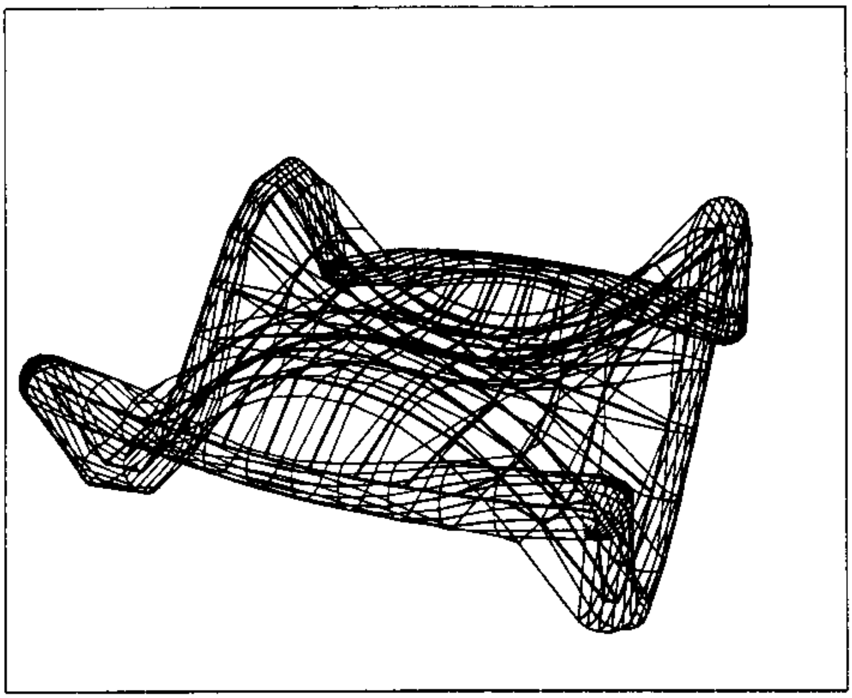

Figure 1: The Progenitor Surface S as Medial Axis of the Offset Region $OR_{\varepsilon}(S)$

locally Lipschitz-continuous first derivatives. Then there exists a positive number β such that the cut locus C_A stays away further than distance β from A.

Proof of Theorem 5 : This proof exploits essentially that lemma 3 is formulated for any function $f(x)$ defined on any convex solid in R^n and that the range of the function $f(x)$ is the space R^{n+m}, m any integer larger than zero. This lemma 3 proves case (38) of assertion 1 the only case needed if the reference set A is an unbordered C^1-smooth manifold. Exploiting also that A being a submanifold is free of self-intersections it easy to generalize assertion 2 to local parameterizations of a compact submanifold A of R^{n+m}. Applying these considerations together with the arguments used while *completing the preceding proof of theorem 4 using assertion 1 and assertion* 2 on a finite number of local parameterizations which cover A then employing compactness arguments it is not difficult to show that C_A cannot come arbitrarily close to A. This completes the proof of theorem 5.

3.4 The Relation of the Cut Locus to Equidistantial Sets and Voronoi Diagrams

We want to explain how the concept of cut locus is related to two other related concepts in computational geometry and geometric modelling. Those related concepts are the concept of a Voronoi diagram of a discrete point set and the concept of an equidistantial set (surface) or mid set of two disconnected sets. We hope that our subsequent results will help to clarify possible confusions in this area. It will turn out that the cut locus concept introduced by us offers a common framework unifying apparently different concepts such as Voronoi diagrams, equidistantial sets and medial axes.

Let A, B be closed and disjoint subsets of R^n. The disjointness condition means that

$$A \cap B = \varnothing. \tag{52}$$

The equidistantial set with respect to the pair of sets A, B is denoted by V(A,B) is defined by

$$V(A,B) = \{x \in R^n / d(A,x) = d(B,x)\} \tag{53}$$

Under these assumptions we have

Theorem 6: Equidistantial Sets as Subsets of the Cut Locus. The equidistantial set of two disjoint closed sets A, B is a subset of the cut locus $C_{A \cup B}$ of the union $A \cup B$ i.e. with the notation introduced above we have

$$V(A,B) \subseteq C_{A \cup B}$$

Proof : Let x be a point in V(A,B). Then by [45], p. 38 there exists a point x_A being nearest on A to x and there exists a point x_B being nearest on B to x. Because of (53) the two minimal joins seg[x_A,x], seg[x_B,x] are both distance minimal from x to $A \cup B$. Therefore and because $x_A \neq x_B$ as (52) the point x must be a pica respective $A \cup B$. Thus x is in $C_{A \cup B}$ which proves the theorem.

We want to point out that the relations between equidistantial sets and cut loci become much more complicated in case one removes the disjointness condition (52). To illuminate this we describe the following example. Consider two half circles S_1, S_2 the union of which builds the planar unit circle and we assume that $S_1 \cap S_2 = \{x_1 = (0,1), x_2 = (0,-1)\}$. In this situation $V(S_1,S_2)$ contains the whole y-axes while $C_{S_1 \cup S_2}$ contains only the point (0,0).

Another quite instructive example is the following one being a modification of the former example : Here S_1 is defined to be the circular arc $\{(x,y)/(x+0.75)^2 + y^2 = 1, x \leq 0\}$ and S_2 the circular arc defined by $\{(x,y)/(x-0.75)^2 + y^2 = 1, x \geq 0\}$. In this example S_1, S_2 intersect transversally while in the former example the intersection was tangential. Here now $V(S_1,S_2)$ equals the y-axis while the medial axis of $S_1 \cup S_2$ equals the segment $\{(x,y)/y = 0, |x| \leq 0.75\}$. The cut locus $C_{S_1 \cup S_2}$ contains the latter medial axes together with the set $\{(x,y)/\, x = 0, |y| \geq \sqrt{7/16}\}$.

In order to state our next theorem we need to review some definitions related to the concept of Voronoi diagrams. We follow here essentially Preparata and Shamos [34].

Let $P = \{p_i \in R^n | i \in I\}$ a set of discrete points in R^n, with the set I being used as a set of indices to distinguish the points in P. This set may even be infinite we assume however that the points in P do not have a cluster point. *In order to explain the concept of a Voronoi diagram we define first for every p_i in P* **the locus of proximity** **V(i)** *containing those points which are closer to (or at least not farer from) p_i than to any other point of $P \setminus \{p_i\}$. Clearly the set V(i) can be characterized as*

$$V(i) = \{x \in R^n /\, d(x,p_i) \leq d(x, P \setminus \{p_i\})\} \tag{54}$$

Obviously the set $V(i)$ can also be characterized by the equation

$$V(i) = \{x \in R^n / d(x,p_i) \leq d(x,p_j) \text{ for all } p_j \in P \setminus \{p_i\}\} \tag{55}$$

The set

$$H(i,j) = \{x \in R^n /\, d(x,p_i) \leq d(x,p_j)\} \tag{56}$$

defines a closed half space in R^n. The boundary of this half space is given by the plane containing all points which are equidistantial to the two points p_i, p_j. Or with the notation introduced above the boundary of $H(i,j)$ can be described also as the medial set $V(\{p_i\}, \{p_j\})$ with respect to the two point sets $\{p_i\}$, $\{p_j\}$ each of which containing a single point. With (56) and (55) we can obviously redefine $V(i)$ as an intersection of half spaces i.e.

$$V(i) = \bigcap_{i \neq j} H(i,j) \tag{57}$$

This redefinition of V(i) also shows that

V(i) being an intersection of convex sets is convex. (58)

Using concepts and notations introduced above in (54), (55) we give now the following definitions:

Definition : The boundary $\partial V(i)$ of the locus of proximity V(i) is the *Voronoi polygon (polytope)* respective the point p_i of the given set P.

It is obvious that a point in $\partial V(i)$ is contained in a boundary plane of some $H(i,j)$.

Definition : We call the union of all the polytopes $\partial V(i), p_i \in P$ is the *Voronoi diagram V(P)* respective the point set P in R^n i.e.

$$V(P) := \bigcup_{p_i \in P} \partial V(i) \tag{59}$$

We shall use the subsequent characterization of $\partial V(i)$ i.e we need that

$$\partial V(i) = \{ x \in R^n / \, d(x,p_i) = d(x, P \setminus \{p_i\}) \} \tag{60}$$

Proof of (60) : Let $x \in \partial V(i)$. Then in view of (57) there must exist a point $p_j \in P \setminus \{p_i\}$ such that x is contained in the boundary plane of $H(i,j)$. This boundary plane is equidistantial between between p_i and p_j hence $d(x,p_i) = d(x,p_j)$ for some $j \neq i$. Thus

$$d(x,p_i) = d(x,p_j) \geq d(x, P \setminus \{p_i\}) \tag{61}$$

The point x being contained in $\partial V(i)$ is also in V(i). Therefore (54) together with (61) imply $d(x,p_i) = d(x, P \setminus \{p_i\})$. Thus the point x must be contained in the set given by the right hand side of equation (60). This proves the inclusion "$\subset$" claimed by (60). It remains to show the converse inclusion "$\supset$" which is also claimed by (60). For this let x be a point contained in the set described by the right hand side of (60). Then by (54) the point x is contained in $V(i)$. Let p_j be a point nearest in $P \setminus \{p_i\}$ to x. Then x is in the boundary plane of $H(i,j)$. Thus the half space $H(i,j)$ cannot include any open n-dimensional disc D *containing* x. Therefore $V(i)$ being (by (57)) a subset of $H(i,j)$ cannot include such a disc D either. This proves that x cannot be an interior point of $V(i)$ and thus x must be a boundary point of $V(i)$. This shows the inclusion "$\supset$" and completes the proof of (60).

We give now our description of the Voronoi diagram by the cut locus i.e. we have the following result.

Theorem 7: The Voronoi Diagram as Cut Locus of a Discrete Point set. For any discrete set of points $P = \{ p_i \in R^n \mid i \in I \}$[18] is the related Voronoi diagram characterized by the relation

$$V(P) = C_P \tag{62}$$

Proof of theorem 7 : We show now (62). This means according to our definition of a Voronois diagram stated in (59) we have to prove

$$\bigcup_{p_i \in P} \partial V(i) = C_P \tag{63}$$

[18] The set I serves here as a set of indices used to distinguish the points in P.

For this we show first that a point x in $V(P)$ must also be contained in C_P. Let $x \in V(P)$. Then there exists a point $p_i \in P$ such that $x \in \partial V(i)$. Clearly for any $p_i \in P$ is $d(x,P) = min\{ d(x,p_i), d(x,P\backslash\{p_i\}) \}$. Thus using (60) we find that

$$d(x,P) = d(x,p_i) = d(x,P\backslash\{p_i\}) \tag{64}$$

Therefore and because $\{p_i\} \cap (P\backslash\{p_i\}) = \emptyset$ there exist two distinct distance minimal joins from x to P. One of those joins ends in $\{p_i\}$ the other ends in $P\backslash\{p_i\}$. Thus x is a pica respective P. Therefore x is in C_P. This proves the desired inclusion.

We show now the other inclusion claimed by (63). For this let x be a point in C_P. As by theorem 3 A) the picas are dense in C_P it is easily seen that x must be a pica respective P. Thus there exist at least two distinct minimal joins from x to P. Those two minimal joins end up in two distinct points p_i, p_j in P. Thus we have

$$d(x,P) = d(x,p_i) = d(x,p_j) \tag{65}$$

Now using that $p_j \in P\backslash\{p_i\}$ we get

$$d(x,P) \leq d(x,P\backslash\{p_i\}) \leq d(x,p_j) \tag{66}$$

The combination of (65) and (66) yields

$$d(x,p_i) = d(x,P\backslash\{p_i\}) \tag{67}$$

Therefore (60) implies that the point x is in $\partial V(i)$. This proves that x is in $V(P)$ and finishes the proof (63). This completes the proof of theorem 7.

4 Global Results on the Medial Axis

4.1 The Medial Axis has the Homotopy Type of its Reference Solid

The fundamental global topological shape relation between a solid B and its medial axis M(B) is stated in the following:

> **Theorem 8:** <u>**Global Topological Shape Theorem for the Medial Axis:**</u> Let B be a compact bordered n–dimensional submanifold of R^n.[19] Let us assume that ∂B is C^2-smooth submanifold if $B \subset R^n$; in case $B \subset R^2$ the weaker boundary regularity namely ∂B being piecewise C^2-smooth (possibly disconnected) submanifold is sufficient. Under these assumptions the medial axis M(B) is a deformation retract of B.

Proof of the Shape Theorem:

The proof of the global shape theorem consists in constructing a *retract*[20] :

$$R : B \backslash \partial B \rightarrow M(B) \backslash \partial B \tag{68}$$

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[19] This means in practical terms that B is a compact solid in R^n.

[20] See e.g. Massey [23] for the definition and discussion of a deformation retract.

and a *homotopy*

$$f(x,t) : (B\setminus\partial B) \times I \rightarrow B\setminus\partial B \quad \text{with } I = [0,1], \tag{69}$$

such that

$$f(x,0) = x, \ f(x,1) = R(x) \quad \text{for all } x \in B\setminus\partial B \tag{70}$$

and

$$f(y,t) = y \quad \text{for all } (y,t) \in (M(B) \times I). \tag{71}$$

The retract map R must be continuous and must satisfy

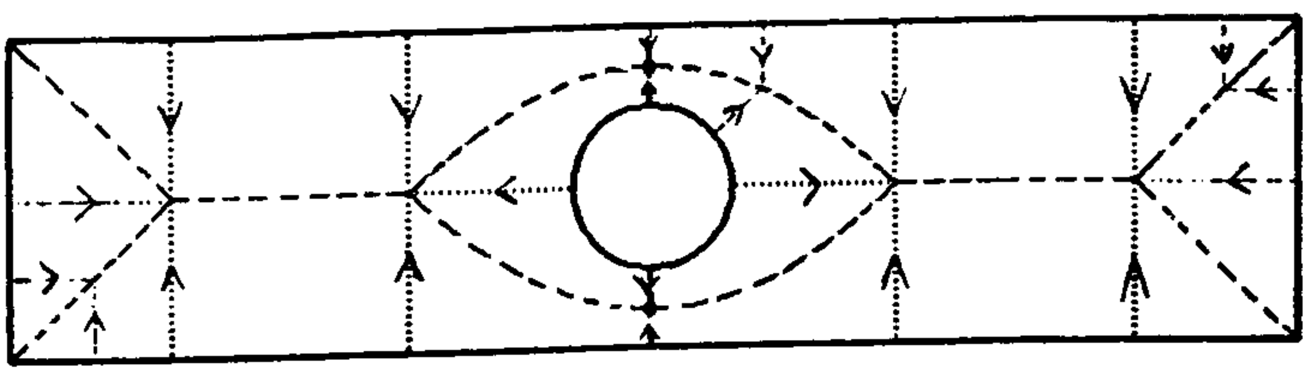
$$R(x) = x \quad \text{for all } x \in M(B)\setminus\partial B. \tag{72}$$

In order to construct the deformation retract we define the homotopy f(x,t) by

$f(x,t) := x + t\, d(x,\psi(x))\, \nabla d(\partial B,x)$
with $\nabla d(\partial B,x)$ being the gradient of the
distance function $x \rightarrow d(\partial B,x)$ at the point x
for $x \in M(B)$ we define $f(x,t) = x$; (73)

Here in (73)

$\psi(x)$ is defined to be the point where the extension
of the minimal join from ∂B to x meets $M(B) = C_{\partial B} \cap B$. (74)



The solid B in figure 2 is a rectangle with a circular hole. The curve drawn as – – – – – indicates the medial axis M(B). The arrows indicate the vector field $x \rightarrow \nabla d(\partial B,x)$. Any dotted curve ········ traces the orbit $f(x,t)$ of some point x during the homotopic deformation $t \rightarrow f(x,t)$, $t \in [0,1]$.

Figure 2: Deformation Retract

See also figure 2 illustrating the deformation defined by (73), (74). The proof for the continuity of the

map $\psi(x)$ makes use of part A) of theorem 3. We shall show the continuity of the map $\psi(x)$ later. To prove the continuity of $f(x,t)$ we need to exploit also part B) of theorem 2 which in view of theorem 1 guarantees the continuity of the gradient function
$x \to \nabla d(\partial B,x)$ on $B \setminus (\partial B \cup M(B))$.
Note the range of the homotopy $f(x,t)$ is indeed in $B \setminus \partial B$ for any $(x,t) \in (B \setminus \partial B) \times I$

i.e. $f(x,t) \in B \setminus \partial B$ (75)

because obviously $d(f(x,t), \partial B) \geq d(x,\partial B) > 0$ for all $t \in I$ and because (by Proposition 1) ∂B separates $B \setminus \partial B$ from $R^n \setminus B$. These two conditions imply (75). Namely assume there exists a point $x \in B \setminus \partial B$ with $f(x,t_1) \in (R^n \setminus B \cup \partial B)$. Then there would exist a number t^* with $0 < t^* < t_1$ such that $f(x,t^*) \in \partial B$, $d(f(x,t^*), \partial B) = 0$ a contradiction. The reason why we defined $f(x,t)$ on $B \setminus \partial B$ is that $\nabla d(\partial B,.)$ can generally not be extended continuously to the boundary ∂B if ∂B is not smooth.

We need to make the preceding proof formally complete. For this we we must show that:

the map $f(x,t)$ is well defined and continuous. (76)

We also need to verify that

$R(x)=x$ for $x \in M(B)$. (77)

In view of the definition of $R(x)$ in order to show (77) one needs to prove

$f(x,1)=x$ for $x \in M(B)$ (78)

To prove (76) and (78) we shall use that

the map $\psi(x)$ is :
well defined, (79)

continuous, (80)

and

$\psi(x)=x$ for $x \in M(B)$. (81)

We shall prove (79), (80), (81) later. Let us for the time being assume that those three claims are correct and and let us use them to establish (76) and (78). To do this we use also theorem 2B) and theorem 1. Namely by theorem 2B) the gradient of the function describing the distance to ∂B i.e. $\nabla d(\partial B,x)$ is continuous on $B \setminus (\partial B \cup C_{\partial B})$ and by theorem 1 we have $M(B) = C_{\partial B} \cap B$. Therefore

$\nabla d(\partial B,x)$ is continuous on $B \setminus (\partial B \cup M(B))$ (82)

Using (82) together with (80) and (79) it is obvious that the map $f(x,t)$ is continuous and also well defined if x is outside of $M(B)$. Thus to complete the proof of (76) it remains to show that

$f(x,t)$ is also well defined and continuous if x is in $M(B)$. (83)

Clearly by (81) we have $f(x,t)=x$ for x in $M(B)$. This shows (78). Let now be x_o be any point in $M(B)$, t_o any point in $[0,1]$ and let (x_n,t_n) be any sequence in $(B \setminus \partial B) \times [0,1]$ converging to (x_o,t_o). For proving the continuity of $f(x,t)$ for any point x in $M(B)$ we have to show that $f(x_n,t_n)$ converges to $f(x_o,t_o)$. Using (81) and (80) we find that the sequence $t_n d(x_n, \psi(x_n))$ is converging to 0. This together with the fact that the norm of the vectors $\nabla d(\partial B,x)$ is bounded by 1 proves that the sequence $t_n d(x_n, \psi(x_n)) \nabla d(\partial B, x_n)$ must converge to 0, hence $f(x_n,t_n)$ must converge to x_o. This proves that $f(x,t)$ is also continuous at any point x_o in $M(B)$ thus it completes the proof (83) and finishes the proof of (76).

It remains to show (79), (81) and (80). Clearly the claim of (81) can be viewed to be a consequence of the definition of $\psi(x)$. This proves (81). We have to show (79). For this we have to prove that for every point x in $B\backslash\partial B$ the definition given for the function ψ describes a unique point $\psi(x)$. Let x be any point in $B\backslash\partial B$. The case where x is in $M(B)$ has already been settled before. Let us therefore assume that x is not in $M(B)$. By theorem 2A ' we know that there exists a unique minimal join g_x from ∂B to x. This segment g_x has length larger than 0 because x is not on ∂B. Extending g_x beyond x the extension must eventually meet ∂B by Proposition 1 because x is in $B\backslash\partial B$. This means that the latter extension segment fails eventually to be distance minimal to ∂B. Thus the extension must meet $C_{\partial B}\cap B = M(B)$ before leaving $B\backslash\partial B$, say it meets $C_{\partial B}$ not closer than in distance $\delta > 0$ to ∂B. The extension segment say extended up to distance $\delta/2$ to the boundary is compact and contained in $B\backslash\partial B$. Denote this extension segment by ***seg***. The intersection of the compact set *seg* with the closed set $M(B)$ is compact, recall $M(B)$ was defined to be closed as it includes all limit points. is compact because $M(B)$ is closed. Therefore ***there exists*** a unique point nearest to x on the intersection of $M(B)$ with the extension segment *seg*. This proves (79).

It remains to show (80). We do this now. For this we show that:

If x_n is any sequence in $B\backslash\partial B$ is converging to any point x_o in $B\backslash\partial B$
then $\psi(x_n)$ converges to $\psi(x_o)$. (84)

To prove (84) let us discuss first the case that x_o is outside M(B) i.e.

$$\alpha = d(x_o, M(B)) > 0 \qquad (85)$$

The minimal join g_{x_o} from ∂B to x_o can by (79) be extended until it meets $M(B)$ in a point $\psi(x_o) \neq x_o$. The segment g_{x_o} starts in a boundary point b_o and g_{x_o} contains x_o as an interior point. Let g_{x_n} be the minimal join from ∂B to x_n. Then the segment sequence g_{x_n} must converge to the segment g_{x_o} because otherwise the point x_o would be a pica contradicting the assumption that x_o is not in $M(B)$.[21] Therefore the sequence of of segments defined to be the extensions of g_{x_n} until $\psi(x_n) \in M(B)$ has all its limit points in an extension of g_{x_o}. As $M(B)$ is closed any limit w of the sequence $\phi(x_n)$ must be contained in $M(B)$. Such a limit point w of $\psi(x_n)$ cannot be an interior point of the segment joining x_o with $\phi(x_o)$ as this segment (being the extension part of the minimal join from the boundary ∂B to x_o) does *not* meet $M(B)$ before it reaches $\psi(x_o)$. We want to show that

$$w = \psi(x_o) \qquad (86)$$

It remains to exclude the possibility that w is located on the extension of seg[b_o, $\psi(x_o)$] after the point $\psi(x_o)$. Assume the latter happens. The sequence of minimal boundary joins yields a subsequence converging to a minimal segment g_l from w to ∂B. This minimal segment would now include $\psi(x_o)$ as an interior point contradicting the assumption that $\psi(x_o) \in M(B)$ is a nonextender because all points in $M(B)$ are nonextenders by theorem 3 A) under the continuity assumptions stated above for ∂B in theorem 8. [22] This proves (86) for the case that x_o is outside of $M(B)$. Let us therefore discuss now the case that x_o is in $M(B)$. Again we have to prove (86). Let now x_n be a sequence converging to x_o,x_o a point in $M(B)$. Let $d_n = seg[b_n, \psi(x_n)]$ be the segments defined by extending the minimal join from the boundary ∂B to x_n up to the point $\psi(x_n)$; we assume here that b_n is the point where the segment d_n starts at the boundary. Let w be any cluster point of the sequence $\psi(x_n)$. We must prove (86). Assume that d_n denotes also the subsequence of segments whose end points $\psi(x_n)$ converge against w. The sequence d_n contains a

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[21] It is well known that any sequence of minimal joins contained in a compact set contains a subsequence converging against a minimal join, c.f. [4], this result is applied here and will be applied often in proofs without explicit reference.

[22] Note to establish the continuity of $\psi(x)$ we use at this point that all points in $M(B)$ are nonextenders. As we also use the property that $M(B)$ is closed we need in this proof sufficient conditions under which theorem 2 A) holds i.e. we need that a limit of nonextenders must be a nonextender itself.

subsequence which converges to a minimal join d_o from ∂B to w, c.f. [4], p. 20 or [45]. As all d_n contain the corresponding x_n the limit segment d_o must contain the limit point x_o of the sequence x_n. Note by definition of the map ψ we have $\psi(x_o) = x_o$ by (81) because x_o is now in the set $M(B)$ which contains only nonextenders. Therefore the segment d_o being a minimal join from the boundary to the point $w = lim\psi(x_n)$ contains the nonextender point $x_o = \psi(x_o)$. Clearly this is only possible if $w = \psi(x_o)$. This shows (86) in case x_o is in $M(B)\setminus\partial B$ and completes the proof (84), hence the proof of (80) is finished. Therefore the proof of Theorem 8 is now complete.

We now draw some conclusions from the fundamental shape theorem by applying standard results of homotopy theory cf. eg. [40]:

Corollary 8.1: Under the assumptions of Theorem 8 the medial axis $M(B)$ is path-connected because B is path connected and it has the same homotopy type as B; hence all homotopy groups of B and $M(B)$ agree, hence $M(B)$ is simply connected if B is simply connected.

Note that although the medial axis is connected under the assumptions stated in theorem 8 the cut locus is generally not connected as we explain in the subsequent

Remark : Even if ∂B is a C^∞-smooth simple closed planar curve bounding a topological disc B then the cut locus $C_{\partial B}$ is generally not connected unless B is convex. Moreover the cut locus $C_{\partial B}$ may even have arbitrarily many connected components in $R^2\setminus B$, each of which may start in a curvature center of the curve $C_{\partial B}$. Those components being unbounded will proceed to infinity.

4.2 The Reconstruction of a Solid by its Medial Axis

The preceding theorem explained the relations between the topological (global shape) structure of a bordered manifold B and its medial axis $M(B)$. Next, we are going to discuss how it is possible to reconstruct B via $M(B)$. Before that, note that the maximal disc radius function:

$$r{\cdot}M(B) \to R$$

which was defined by $r(x) := d(\partial B, x)$ is obviously a continuous function, because $d(A,.)$ is continuous for any closed set A in R^n; $d(A,.)$ is even Lipschitz continuous and its restriction to $M(B)$ is Lipschitz continuous as well.

The result of this section is the

Theorem 9: **<u>(Reconstruction Theorem:)</u>**
Assume we know the medial axis transform M(B), r: M(B)$\to$ R of a domain B, then we can reconstruct B. Namely, we have:

$$B = \bigcup_{x \in M(B)} K(x, r(x))$$

where the union is taken for all discs with center $x \in M(B)$ with $K(x,r(x)) = \{y \in R^n \,/\, |x-y| \le r(x)\}$.

Proof of the Reconstruction Theorem:

We want to prove that

$$B = \bigcup_{x \in M(B)} K(x,r(x)) \tag{87}$$

For this we show the following assertions

$$B \supset \bigcup_{x \in M(B)} K(x,r(x)) \tag{88}$$

$$B \subset \bigcup_{x \in M(B)} K(x,r(x)) \tag{89}$$

Clearly (87) is a consequence of (88) and (89).
Assertion (88) is true as

$$B \supset K(x,r(x)) \text{ for all points } x \in M(B) \tag{90}$$

We show (90). Namely by definition $r(x) = d(x, \partial B)$. Now in case $K(x,r(x))$ would contain any point $y \in R^n \setminus B$ then by proposition 1 the segment connecting x and y would contain a boundary point z with a distance smaller than $r(x)$ to x a contradiction. This proves (90).

In order to prove (89) we show that:

For every point $y \in B$ there exists a point x_o

$$\text{such that } y \in K(x_o, r(x_o)). \tag{91}$$

If here $y \in M(B)$ then the claim (91) is obviously true because $y \in K(y, r(y))$ even if $r(y)$ is zero. Therefore assume $y \notin M(B)$ thus

$$d(y, M(B)) > 0 \tag{92}$$

because $M(B)$ is a closed subset of R^n. Now as B is a manifold with boundary ∂B it is possible to approximate y with a sequence of points $y_n \in (B \setminus \partial B)$. For every point y_n in this sequence there exists a minimal join s_n to the boundary ∂B , see [45]. It is possible to extend any of these minimal joins s_n to get a minimal join $\bar{s}_n$ from the boundary ∂B to a point q_n in $M(B)$. Recall by theorem 1 is $M(B) = B \cap C_{\partial B}$. Therefore employing the definition of $C_{\partial B}$ any minimal join from the boundary ∂B to a point $b \in (B \setminus \partial B)$ can be extended as a minimal join α until it hits $M(B)$ in a point q. Thus α yields then also a minimal join from q to ∂B. We can choose a subsequence $\bar{s}_{n_k}$ of $\bar{s}_n$ which converges against a minimal join $\bar{s}$, see [45], Busemann.[23] The segment $\bar{s}$ is a minimal join from ∂B to a point in $M(B)$. Note the sequence of segments $\bar{s}_{n_k}$ contains a sequence of points y_{n_k} (being a subsequence of y_n) which converges against y. Therefore the limit segment $\bar{s}$ contains y. As all $\bar{s}_{n_k}$ meet $M(B)$ also the limit segment $\bar{s}$ meets $M(B)$ in some point. Let $x(y)$ be the point where the segment $\bar{s}$ meets the first time $M(B)$. The point $x(y)$ is not on the boundary ∂B because
$d(y, M(B)) > 0$ by (92); note that

$$d(x(y), \partial B) \geq d(x(y), y) \tag{93}$$

because $\bar{s}$ being a minimal join from ∂B to $x(y)$ contains y .

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[23]It is here necessary to choose a subsequence because there may exist several distinct minimal joins all being cluster points of the sequence s_n.

To finish the proof of (91) we choose in (91) $x_o = x(y)$. Now (93) and the definition of the maximal disc radius function $r(.)$ imply

$$K(x_o, d(x(y), y)) \subset K(x_o, d(x_o, \partial B)) = K(x_o, r(x(y))) \tag{94}$$

Therefore as $y \in K(x_o, d(x(y), y))$ we have $y \in K(x_0, r(x_0))$.
This proves (91) and finishes the proof of the reconstruction theorem.

5 Appendix

We supply here in the appendix several lemmata used by us in the proofs of major theorems in the preceding sections. Some of those lemmata may be considered to be of technical character while others may be of geometrical interest per se.

Lemma A.1: Let B be a compact solid in R^2 and assume that ∂B is piecewise C^2-smooth or let B be a compact solid in R^n and assume ∂B is C^2 - smooth. Then the following claims hold:

- A) A limit of picas respective ∂B is a non-extender respective ∂B. Specifically a limit of picas is a pica or a curvature center of ∂B; it may be both e.g. a center of a circle.
- B) A limit of non-extenders respective ∂B is a nonextender respective ∂B.
- C) A nonextender is either a pica or a curvature center respective ∂B. It may be both e.g. center of a circle. If a nonextender is not a pica then it must be a curvature center respective ∂B.
- D) If the boundary $\partial B \subset B \subset R^2$ is piecewise linear then every nonextender is a pica.

Proof of lemma A.1: We first prove lemma A.1 A),B),C) in case ∂B is a C^2-smooth hypersurface of R^n. In this case lemma A.1 A),B),C) are contained in theorem 5.3 of [45]. Indeed the latter theorem 5.3 covers the more general case where R^n can be replaced by any complete n-dimensional Riemannian manifold. Thus for the proof of lemma A.1 A),B),C) in case ∂B is a C^2-smooth hypersurface of R^n it is sufficient to refer to theorem 5.3 in [45].

Thus it remains to prove lemma A.1 A),B),C) in case ∂B is only piecewise C^2 here however employing the additional assumption that $R^2 \supset B \supset \partial B$. We first prove now part A) of lemma A1.1. The other parts B) and C) will further below be shown to be easy conclusions of part A).

Proof of lemma A.1 A): We want to prove that

a limit of picas respective ∂B is a nonextender respective ∂B (95)

We argue by contradiction and assume for this purpose that

there exists a sequence of picas q_n respective ∂B
whose limit is an extender respective ∂B (96)

Each q_n being a pica has at least two distinct nearest points p_{n1}, p_{n2} on ∂B. If now the sequence q_n converges against a point q_o being a pica then there is nothing more to prove because then the limit q_o is a nonextender. Let us therefore assume the case that q_o is not a pica. In that case the foot point sequences p_{n1}, p_{n2} converge against a (unique) point p_o being the foot point of q_o this foot point is characterized by the subsequent distance property

$$d(\partial B, q_o) = d(p_o, q_o) \tag{97}$$

We show now first that

the segment $seg[p_o, q_o]$ is normal on ∂B (98)

The point p_o must be contained in a boundary edge. This edge is represented by a path $b(t):[0,1] \to R^2$ being a regular C^2 parametrization, with b(0), b(1) being vertex points. This means each of the points b(0), b(1) is contained in an edge adjacent to b[0,1]. Now

if $p_o = b(t_o)$ is not a vertex point then it is easily seen that
the segment $seg[q_o, p_o]$ being a minimal join to b(0,1) must be normal on b(0,1). (99)

Let us therefore assume that p_o is a vertex point of b[0,1] say p_o=b(1). The sequences p_{n1}, p_{n2} converge against p_o. Therefore there exists a disc $K(p_o, \delta)$ which contains no other boundary vertex except p_o[24] and all p_{n1}, p_{n2} for n larger than a certain number N(δ) are contained in $K(p_o, \delta)$. For each given *n not* both points p_{n1}, p_{n2} can coincide with p_o. Thus we can assume that $p_{n1} \neq p_o$ for all $n \geq N(\delta)$[25]. Therefore p_{n1} must be contained either in $b(0,1) = \{b(t)/0 < t < 1\}$ or in the adjacent edge $c(0,1) = \{c(t)/0 < t < 1\}$ where b(1)=c(0). In any case

we find a sequence of points p_{n1} which is contained say in b(0,1)[26]
and p_{n1} converges toward p_o. (100)

Now

by conclusion (99) for $n \geq N(\delta)$
the segment $seg[q_n, p_{n1}]$ must be normal on b(0,1). (101)

As the normal vector of b[0,1] is continuous up to the boundary also the limit segment $seg[q_o, p_o]$ is normal on b[0,1] in b(1). This proves (98).

Note further down we shall make use of the property that every boundary edge can be viewed to be a subpart of an enclosing open regular C^2 smooth path. Thus say b[0,1] is subpath of a C^2 regular path $\bar{b}(-\varepsilon, 1+\varepsilon)$. This subpath property can be shown by extending the path $b[0,1]$ C^2-smooth and regular beyond the boundary points. We can define the extension $\bar{b}(t)$ of the path b(t) by:

$\bar{b}(t) = b(t)$ for $t \leq 1$

and for $t \geq 1$ by

$$\bar{b}(t) = b(1) + b'(1)(t-1) + (1/2)b''(1)(t-1)^2 \tag{102}$$

The extension beyond the point b(0) can be defined analogous.

It is easily seen that this extension $\bar{b}[-\varepsilon, 1+\varepsilon]$ is C^2-smooth, regular and
free of self intersections if ε is chosen sufficiently small. (103)

Let the segment $seg[q_o, p_o]$ be represented by an arc length parametrized path w(s) with $w(0)=p_o$ and $w(|p_o - q_o|)=q_o$. Now if the point q_o were a curvature center respective the foot point p_o and the arc b[1-ε,1] then we could show that q_o is a nonextender respective the boundary arc b[1-ε,1]. This means for any $\gamma > 0$ it is possible to construct a path starting in b[1-ε,1] and ending in $w(|p_o - q_o| + \gamma)$ and this path

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[24] This holds because the number of boundary vertices is finite.

[25] This can be achieved by swapping p_{n1} with p_{n2} as far as this is necessary.

[26] If necessary we swap the notations for the edges c[0,1] and b[0,1]

is shorter than $|p_o - q_o| + \gamma$. The latter claim follows e.g. from a more general result in [45] which gives an extension of a theorem of Jacobi. Therefore and because seg$[q_o,p_o]$ is a minimal join from ∂B to q_o:

the assumption that q_o is a curvature center respective the point
$p_o=b(1)$ and the arc $b[1-\varepsilon,1]$ implies that q_o is nonextender. (104)

Therefore our initial contradiction assumption (96) saying that q_o is an extender respective ∂B leads us to conclude that the point q_o is *not* a curvature center respective the arc $b[1-\varepsilon,1]$. Now if q_o is not a curvature center of b(1) then:

the normal map
$\phi(r,t) = b(t) + rN(b(t))$[27]
yields for sufficiently small numbers $\beta,\omega > 0$
a diffeomorphism
$\phi: U_b=[r_o-\beta,r_o+\beta] \times [1-\omega,1] \rightarrow D_b=\{\phi(r,t)/ (r,t) \in U_b\}$[28]
with $\phi(r_o,1) = q_o$ (105)

Now choosing some sufficiently small ρ then in view of (100) we can assume that all picas q_n in $K(q_o,\rho)$ have their foot point p_{n1} in $b(1-\lambda,1)$ and the points q_n must be in D_b if ρ is sufficiently small. Therefore and because of the diffeomorphy property (105) the other foot point p_{n2} of q_n must be in the adjacent boundary arc $c(0,1)$[29]. Next we show that p_{n2} cannot agree with $c(0)=b(1)=p_o$. This follows from a sublemma which we state now:

Sublemma A.1A': Let $f:[0,1]^n \rightarrow R^{n+1}$ be a regular C^2-smooth hypersurface patch. Denote the surface normal at f(x) by N(f(x)) and assume that for some x_o in $(0,1)^n$ and for some $r_o > 0$ the segment $\{ f(x_o) + rN(f(x_o)) \ / \ 0 \le r \le r_o \}$ does not contain a curvature center respective the point $f(x_o)$ on this surface patch. Then there exists a disc $K(x_o,\varepsilon)$ in R^n around x_o and an interval $(r_o-\delta, r_o+\delta)$ such that for all $(x,r) \in D_o = (K(x_o,\varepsilon) \times (r_o - \delta, r_o + \delta))$ the normal segments $g(x,r) = \{f(x) + sN(f(x)) \ / \ 0 \le s \le r\}$ are distance minimal to the subpatch $P_\varepsilon = \{f(x) \ / \ x \in K(x_o,\varepsilon)\}$. This implies that for any $(x,r) \in D_o$ any segment $\bar{g}$ joining the point $f(x)+rN(f(x))$ with P_ε is longer than $g(x,r)$ unless $\bar{g}$ agrees with $g(x,r)$[30].

A proof of this sublemma is not very difficult and can be given by exploiting the local diffeomorphy of the normal map onto the neighborhood of a point which is *not* a curvature center. This sublemma can also be viewed as a special case of a combination of two results saying that geodesics emanating normal from a C^2- smooth hypersurface are locally distance minimal up to their first focal point and that if y is not a focal point respective some submanifold S then a whole open neighborhood of y stays free of focal points respective S c.f. [45]. Therefore we don't give here a proof of this sublemma.

The sublemma implies in our situation that if *for sufficiently large indices n* the foot points p_{n1}, p_{n2} are both on $b[1,1-\lambda)$ then $p_{n1}=p_{n2}$. This yields a contradiction because $p_{n1} \neq p_{n2}$. This implies in our situation that for sufficiently large n the point p_{n2} is unequal to $q_o=c(0)$, hence p_{n2} is in the open interval $c(0,1)$.

[27] Here N(b(t)) denotes the normal vector of the curve b(t) at the foot point b(t).

[28] Note that this diffeomorphism is defined using the *restriction* of a diffeomorphism which is originally defined on a larger open set $U_{\bar{b}}=(r_o-\beta,r_o+\beta) \times (1-2\omega,1+2\omega)$ where $\phi(r,t)$ is now defined for $t \ge 1$ is now defined by using the extension $\bar{b}(t)$ described in (102).

[29] Note we use here that (105) guarantees that the normals emanating from $b(1-2\omega,1]$ do not intersect in D_b.

[30] This implication holds because of the following argument: First we observe that the sublemma implies with the interval $(r_o-\delta,r_o+\delta)$ being open that for any $(\bar{x},\bar{r}) \in D_o$ the point $q(\bar{x},\bar{r}) = f(\bar{x})+\bar{r}N(f(\bar{x}))$ is an extender with respect to P_ε. This excludes that there exists some other minimal join from $q(\bar{x},\bar{r})$ to P_ε besides $g(\bar{x},\bar{r})$.

Now recall q_n converges to q_o, therefore for sufficiently large numbers n the point q_n must either be an interior point of the topological disc D_b or q_n is on the segment $\{ \phi(r,1) / r_o \leq r \leq r_o+\beta \}$. Clearly for large enough n the foot point $p_{n2} \in c(0,1)$ is outside D_b and the segments $w_n = seg[q_n,p_{n2}]$ must meet the boundary of the topological disc D_b in some point z_n. Using that p_{n2} is in $c(0,1)$ and that w_n converges toward the segment $seg[b(1),q_o] = \{ \phi(r,1) / 0 \leq r \leq r_o \}$ it is not difficult to see that for large enough numbers n the the intersection point z_n must be located on the segment $\{ \phi(r,1) / 0 \leq r \leq r_o+\delta \}$. Using that the segment $seg[z_n,p_{n2}]$ is a minimal join to the boundary ∂B it is also not hard to prove that the segment $seg[b(1),z_n]$ cannot be extended as a minimal join to the boundary ∂B beyond the point z_n. Therefore and because z_n must converge to q_o with q_n it follows that the point q_o must be a nonextender respective ∂B. Thus we get a contradiction with our assumption that q_o is an extender. This shows that a limit of picas must be a nonextender and proves the first part of lemma A.1A).

It still remains to show that q_o *must be a curvature center* respective its foot point if it is not a pica. Let us assume that q_o is not a curvature center respective its foot point $b(1)$ on any of both adjacent arcs $b(0,1] = \{ b(s)/0 < s \leq 1 \}, c[0,1) = \{ c(s)/0 \leq s \leq 1 \}$ and let us derive a contradiction. Precisely we shall show that q_o is the first curvature center (on the segment seg[b(1), q_0]) respective the foot point $b(1)$ on at least one of the two arcs $b(0,1]$, $c[0,1)$. For this we need to return to the considerations in the preceding proof. The preceding proof used 3 assumptions

1) q_o is a limit of picas
2) q_o itself is not a pica
3) q_o is an extender

We still need assumption 1) and 2) for the proof of the second part of lemma A.1A). The only locations in the preceding proof where we used the assumption that q_o is an extender was (except at the very end) when we used it to conclude that q_o is not a curvature center respective p_o on b[0,1] and p_o on c[0,1]. In this proof we can now assume directly the non-curvature center property of q_o and we don't need the nonextender property. Recall the picas q_n are related to minimal joins (segments) $seg[p_{n1},q_n]$, $seg[p_{n2},q_{n2}]$ which converge to a *minimal* join being the segment $seg[\, b(1),q_o]$. Because of this minimal length property the open segment $seg[b(1),q_o)$ which does not include q_o cannot contain any curvature center respective the foot point q_o on any of the arcs $b(0,1]$, $c[1,0)$ by (104). Arguing by contradiction we assume now also that q_o is *not a curvature center respective* the point $b(1) = c(0)$ on *both arcs* $b(0,1]$, $c[0,1)$. Therefore analogue to (105) we can now describe a diffeomorphism $\psi(r,s) : U_c \rightarrow D_c$ employing the normal map with normals of the path $c(s)$. Note that here now $\psi(r_o,0) = q_o$ and also like in proof of (98) we get now $\{\psi(r,0)/0 \leq r \leq r_o \} = seg[b(1),q_o]$. In the proof above (with q_n converging to q_o) the segments $seg[p_{n2},q_n]$ being subparts of normals on the curve $c[0,1)$ were shown to intersect $\{ \psi(r,0)/ r_o - \beta \leq r \leq r_o + \beta \}$. This yields a contradiction with the assumption that $\psi : U_c \rightarrow D_c$ is a diffeomorphism. This proves that q_o must be a curvature center of its foot point respective at least one of the arcs $b(0,1]$,$c[0,1)$. This completes the proof of lemma A.1A).

Proof of Lemma A.1B): Let q_n be a sequence of nonextenders converging against a limit point q_o. We have to prove that q_o is a non-extender. By theorem 2A) every nonextender is limit of a sequence of picas. Therefore for every n we can find a pica $\overline{q}_n$ within distance 1/n to q_n. Together with the sequence q_n also the sequence of picas $\overline{q}_n$ is converging to q_o. Thus by lemma A.1A) the limit q_o is a nonextender. This proves lemma A.1B).

Proof of Lemma A.1C): By theorem 2A) every nonextender is a limit of picas. Lemma A.1A) states that a limit of picas has the properties claimed by lemma A.1C) for any nonextender. Therefore the combination of lemma A.1A) and theorem 2A) prove lemma A.1C).

Proof of Lemma A.1D): Lemma A.1D) is a special case of lemma A.2B). Therefore lemma A.1D) follows from lemma A.2 given below. This proves lemma A.1D) and completes the proof of lemma A.1.

We finally present a result which pertains to the practically important special case where the solid B is contained in R^3 and where ∂B is piecewise linear. This means the solid's boundary consists of planar facets with edges being straight line segments. The subsequent lemma A.2 characterizes nonextenders and it also describes properties of limit points of nonextenders.

Lemma A.2: Let B be a compact solid in R^3 and assume that ∂B is piecewise linear. Then the following statements hold :

- A) If a limit of picas is not a pica then its nearest point q on ∂B is a vertex point of ∂B i.e. q is contained in more than two boundary planes.
- B) Every nonextender respective ∂B is a pica.

Proof of Lemma A.2: Every boundary plane P_i has a unique interior normal N_i. The number of those normals is finite. Let $\xi > 0$ be the smallest angle built by any two distinct (interior) boundary normals of ∂B.

Proof of Lemma A.2 A: We first show part A) of lemma A.2. For this we show that:

If a limit of picas is not a pica then its foot point on ∂B is a vertex point i.e. the foot point is contained in more than two boundary planes. (106)

To prove (106) we assume that its negation is true and derive a contradiction. Therefore assume there exists a sequence of picas q_n converging to a non-pica q_0 and the foot point p_0 of q_0 is contained in at most two hyperplanes[31]. Clearly as q_o is not a pica

the minimal joins from q_n to the boundary must converge against the segment joining q_o with p_o. (107)

As p_o is not a vertex there exists a small disc $K(p_0,\delta)$ such that $K(p_0,\delta)$ meets at most two hyperplanes P_1, P_2 and there is no vertex in $K(p_0,\delta)$. It is obvious that $K(p_0,\delta)$ must meet at least two boundary planes with distinct normals because

the foot point p_0 of q_0 cannot be an interior point of a boundary plane piece P_1 with normal N_1. (108)

As otherwise (for sufficiently large numbers n) the minimal segments g_n joining q_n with ∂B are either parallel to N_1 or built an angle ang_n larger than some positive number κ with N_1 where N_1 is parallel to the segment g_o joining q_o with p_o. This would yield a contradiction with the assumption (107) because the fact that the q_n are picas together with (107) implies that the angles ang_n attain arbitrarily small positive values. This proves (108). Therefore we can now assume that $K(p_o,\delta)$ meets precisely two hyperplanes P_1, P_2 with normals N_1, N_2 respectively. Let γ be the angle built by the two normals N_1, N_2. As the limit of the picas q_n is not a pica and as the foot points p_{n1}, p_{n2} must converge against p_o there exists a disc $K(q_o,\varepsilon)$ and a disc $K(p_o,\eta)$ such that:

For all q_n in $K(q_o, \varepsilon)$ the foot points p_{n1}, p_{n2} are in $K(p_o, \eta)$ and all pairs of segments $seg[q_n,p_{n1}]$, $seg[q_n,p_{n2}]$ build an angle smaller than $\gamma/10$. (109)

It can also be arranged that ε in (109) can be chosen so small that :

The convex hull CO of $K(p_o,\eta) \cup K(q_o,\varepsilon)$ meets only the planes P_1, P_2. (110)

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[31]To simplify our notation we shall call a nearest boundary point of any point q the foot point of q.

Here (110) holds because seg$[p_o,q_o]\setminus\{p_o\}$ does not meet ∂B. Let us take any pica q_n in $K(q_o,\varepsilon)$. The point q_n has (at least) two distinct foot points in $K(p_o,\eta)$. At most one of the two segments can be normal on a boundary hyperplane because of the angle provision (109). Assume that say p_{n1} is an interior point of one of the two planes say of P_1[32] The other foot point p_{n2} can not be an interior point of P_2 because of the angle provision (109). Therefore $p_{n2} \in P_1 \cap P_2 \cap K(p_o,\eta)$. Thus

$$length(seg[q_n,p_{n2}]) = \sqrt{|q_n - p_{n2}|^2 + |p_{n1} - p_{n2}|^2} \quad > \quad length(seg[p_{n2},p_{n2}]) \tag{111}$$

a contradiction with the assumption that p_{n1}, p_{n2} are both foot points of q_n. These considerations imply that both points p_{n1}, p_{n2} must be edge points thus

$$\{p_{n1},p_{n2}\} \subset P_1 \cap P_2 \subset K(p_o,\eta).$$

Now

The segment seg$[p_{n1},p_{n2}]$ is contained in $D = P_1 \cap P_2 \subset \cap K(p_o,\eta)$ (112)

because D is convex as an intersection of convex sets. The planar triangle W with the vertices p_{n1}, p_{n2}, q_{n2} is contained in the convex set CO defined in (110). The triangle W has two edges seg$[q_n,p_{n1}]$, seg$[q_n,p_{n2}]$ of equal length. Clearly by (112) the mid point m of seg$[p_{n1},p_{n2}]$ is in ∂B. Therefore the segment seg$[m,q_n]$ yields a boundary join shorter than say seg$[q_n,p_{n1}]$, a contradiction. This shows that the foot point p_o must be vertex point i.e. p_o meets more than two boundary planes. This proves part A) of lemma A.2.

Remark: Actually we also proved above that if the segment angle of a pica is smaller than some positive number then the foot points of this pica must be located close to a vertex point. Moreover analyzing the preceding geometric considerations it is not difficult to derive an estimation for the distance of a pica 'oot point to the nearest boundary vertex. This estimation would incorporate the segment angle of the pica.

Proof of Lemma A.2 B: Using lemma A.2 A) we show now lemma A.2 B). That is we prove that a nonextender is necessarily a pica if ∂B is piecewise linear. For the proof we argue by contradiction. Namely we derive a contradiction from the negation of lemma A.2 B). For this purpose we assume that there exists a nonextender q_o which is not a pica. By theorem 2 the picas are dense in the set of nonextenders, thus q_o is limit of a sequence of picas q_n. By lemma A.2 A) the foot point p_o of q_o is a boundary vertex. As q_o is a nonextender respective ∂B we know that for any $\varepsilon > 0$ the extension of seg$[p_o,q_o]$ by length ε to a point q_ε (beyond q_o) is not a minimal join to the boundary. Therefore there exists a minimal join g_ε from q_ε to the boundary which meets ∂B in a point p_ε. The point p_ε is different from p_o as otherwise the extension of seg$[p_o,q_o]$ would be minimal join to the boundary. As q_o is not a pica the segment g_ε is converging towards seg$[p_o,q_o]$ and p_ε converges toward p_o if ε converges to 0. Since the number of boundary vertices is finite there exists a positive number δ such that $K(p_o,\delta)$ contains only the boundary vertex p_o. Every segment joining a point of $\partial B \cap K(p_o,\delta)$ with the vertex p_o is completely contained in $\partial B \cap K(p_o,\delta)$[33]. Now choose the ε for the extension of seg$[p_o,q_o]$ so small that the foot point p_ε of q_ε (defined above) is contained say in $\partial B \cap K(p_o,\delta/10)$. The segment seg$[p_o,p_\varepsilon]$ as well as its extension by length $\delta/3$ beyond p_ε are contained in $\partial B \cap K(p_o,\delta)$. Let p_e be the end point of

[32]If p_{n1} is an interior point of P_2 we swap the names of the two planes.

[33]This holds because $\partial B \cap K(p_o,\delta)$ is built by a finite union of planar pieces S_k each planar piece S_k being a sector bounded by two segments (starting at p_o) and a circular arc with radius δ. Now any point $p \in \partial B \cap K(p_o,\delta)$ must be contained in some S_k. As S_k is convex S_k contains seg$[p,p_o]$. Therefore $\partial B \cap K(p_o,\delta)$ being the union of the sectors S_k must contain seg$[p,p_o]$. This proves our claim.

this extension of $e=seg[p_o,p_\varepsilon]$. If e is not normal on $seg[p_o,p_\varepsilon$ then it is easily seen that e contains a point p_d such that $seg[p_d,q_\varepsilon]$ yields a shorter join to the boundary than the minimal join $seg[p_\varepsilon,q_\varepsilon]$ a contradiction. Thus $seg[p_d,q_\varepsilon]$ must be orthogonal on e. Now the points $p_o,p_\varepsilon,q_\varepsilon$ built a triangle with a rectangular angle at vertex p_ε. This triangle contains a segment g which joins q with e and g is parallel to $seg[p_d,q_\varepsilon]$. Clearly g is shorter than the minimal join $seg[q_o,p_o]$ unless g and $seg[q_o,p_o]$ agree. Thus g and $seg[q_o,p_o]$ must agree. However this not possible because the assumption q being a nonextender implied that p_o and p_ε are distinct c.f. above. Therefore we get a contradiction with our assumption of the proof of lemma A.2 B). This completes the proof of lemma A.2 B).

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References

[1] Blum H.
A Transformation for Extracting New Descriptors of Shape.
Models for the Perception of Speech and Visual Form :362-381, ed: Weinant Wathen-Dunn MIT Press, 1967.

[2] Blum H.
Biological Shape and Visual Science (Part I).
Journal of Theoretical Biology 38:205-287, 1973.

[3] Blum H., Nagel R. N.
Shape Description Using Weighted Symmetric Axis Features.
Pattern Recognition 10(3):167-180, 1978.

[4] Busemann, H.
The Geometry of Geodesics.
Academic Press: New York, 1985.

[5] Dieudonne, J.
Foundations of Modern Analysis.
Academic Press, New York, 1969.

[6] Dutta, D., Hoffmann, C. M.
A Geometric Investigation of the Skeleton of CSG Objects.
In B. Ravani (editor), *Proceedings of the 16th ASME Design Automation Conference: Advances in Design Automation, Computer Aided and Computational Design*, pages 67-75. New York: ASME, Chicago, IL, September, 1990.

[7] Guillemin, V. and Pollack, A.
Differential Topology.
Prentice-Hall, Englewood Cliffs, NJ, 1974.

[8] Gursoy, H. N.
Shape Interrogation by Medial Axis Transform for Automated Analysis.
PhD thesis, Massachusetts Institute of Technology, November, 1989.

[9] Gursoy, H. N. and Patrikalakis, N. M.
Automated Interrogation and Adaptive Subdivision of Shape Using Medial Axis Transform.
Advances in Engineering Software , 1992.
(To appear).

[10] Held, M.
On the Computational Geometry of Pocket Machining.
Springer-Verlag, Berlin, Germany, 1991.

[11] Hirsch, M. W.
Graduate Texts in Mathematics, 33: Differential Topology.
Springer-Verlag, New York, 1976.

[12] Hoffmann, C. M.
A Dimensionality Paradigm for Surface Interrogation.
Computer Aided Geometric Design 7:517-532, 1990.

[13] Hoffmann, C. M.
Computer Vision, Descriptive Geometry, and Classical Mechanics.
Technical Report CSD-TR-91-073, Computer Sciences Department, Purdue University, October, 1991.
To appear in Proceedings of the Eurographics workshop, Computer Graphics and Mathematics, October 1991, Genova, Italy.

[14] Hoffmann, C. M. and Vermeer, P.
Eliminating Extraneous Solutions in Curve and Surface Operations.
International Journal of Computational Geometry and Applications 1:47-66, 1991.

[15] Hu, S. T.
Elements of General Topology.
Holden-Day, San Francisco, 1964.

[16] Kelley, J. L.
General Topology.
D. Van Nostrand Company, Princeton, NJ, 1955.

[17] Kobayashi, S.
On Conjugate and Cut Loci.
Mathematical Association of America Studies in Mathematics.
Mathematical Association of America, 1967, pages 96-122.

[18] Kriezis, G. A.
Algorithms for Rational Spline Surface Intersections.
PhD thesis, Massachusetts Institute of Technology, March, 1990.

[19] Kriezis G. A., Patrikalakis N. M., and Wolter F. - E.,
Topological and Differential Equation Methods for Surface Intersections.
Computer Aided Design 24(1):41-55, January, 1992.

[20] Lee, D. T. and Drysdale, R. L.
Generalization of Voronoi Diagrams in the Plane.
SIAM Journal on Computing 10(1):73-87, February, 1981.

[21] Lee, D. T.
Medial Axis Transformation of a Planar Shape.
IEEE Transactions on Pattern Analysis and Machine Intelligence PAMI-4(4):363-369, July, 1982.

[22] Mangoldt, H. von.
Über diejenigen Punkte auf positiv gekrümmten Flachen, welche die Eigenschaft haben, dass die von ihnen ausgehenden geodätischen Linien nie aufhören, kürzeste Linien zu sein.
Journal für Reine und Angewandte Mathematik 91:23-52, 1881.

[23] Massey, W. S.
Algebraic Topology: An Introduction.
Springer Verlag, New York, 1989.

[24] Montanari U.
Continuous Skeletons from Digitized Images.
Journal of the Association for Computing Machinery 16(4):534-549, October, 1969.

[25] Nackman, L. R.
Curvature Relations in Three-Dimensional Symmetric Axes.
Computer Graphics and Image Processing 20:43-57, 1982.

[26] Nackman L. R. and Pizer S. M.
Three-Dimensional Shape Description Using the Symmetric Axis Transform I: Theory.
IEEE Transactions on Pattern Analysis and Machine Intelligence PAMI-7(2):187-202, March, 1985.

[27] Nackman, L. R. and Srinivasan, V.
Bisectors of Linearly Separable Sets.
Discrete and Computational Geometry 6:263-275, 1991.

[28] O'Rourke, J. and Badler N.
Decomposition of Three-Dimensional Objects into Spheres.
IEEE Transactions on Pattern Analysis and Machine Intelligence PAMI-1(3):295-305, July, 1979.

[29] Patrikalakis, N. M.
Shape Interrogation.
In C. Chryssostomidis (editor), *Proceedings of the 16th Annual MIT Sea Grant College Program Lecture and Seminar, Automation in the Design and Manufacture of Large Marine Systems*, pages 83-104. Cambridge, MA, October, 1988.
New York: Hemisphere Publishing, 1990.

[30] Patrikalakis N. M., Gursoy, H. N.
Shape Interrogation by Medial Axis Transform.
In B. Ravani (editor), *Proceedings of the 16th ASME Design Automation Conference: Advances in Design Automation, Computer Aided and Computational Design, Vol. I*, pages 77-88. ASME, Chicago, IL, September, 1990.

[31] Patrikalakis, N. M. and Bardis, L.
Localization of Rational B-Spline Surfaces.
Engineering with Computers 7(4):237-252, 1991.

[32] Poincare, H.
Sur les lignes geodesiques des surfaces convexes.
Transactions of the American Mathematical Society 5:237-274, 1905.

[33] Preparata, F. P.
The Medial Axis of a Simple Polygon.
Lecture Notes in Computer Science: Mathematical Foundations of Computer Science.
Springer-Verlag, 1977, pages 443-450.

[34] Preparata F. P. and Shamos M. I.
Computational Geometry: An Introduction.
Springer-Verlag, New York, 1985.

[35] Rauch, H.
Geodesics and Curvature in Differential Geometry in the Large.
Yeshiva University Press, New York, 1959.

[36] Rosenfeld, A.
Axial Representations of Shape.
Computer Vision, Graphics and Image Processing 33:156-173, 1986.

[37] Rossignac, J. R.
Blending and Offseting Solid Models.
PhD thesis, University of Rochester, July, 1985.
Production Automation Project Technical Memorandum No. 54.

[38] Rossignac, J. R. and Requicha, A. G.
Offsetting Operations in Solid Modelling.
Computer Aided Geometric Design 3(2):129-148, 1986.

[39] Rushing, T. B.
Topological Embeddings.
Academic Press, New York, 1973.

[40] Spanier, E.
Algebraic Topology.
McGraw-Hill, New York, 1966.

[41] Srinivasan V. and Nackman L. R.
Voronoi Diagram for Multiply Connect Polygonal Domains I: Algorithm.
IBM Journal of Research and Development 31(3):361-372, May, 1987.

[42] Srinivasan, V., Nackman, L. R., Tang, J.-M., and Meshkat, S. N.
Automatic Mesh Generation using the Symmetric Axis Transformation of Polygonal Domains.
Technical Report RC 16132, IBM, September, 1990.

[43] Voronoi, G.
Nouvelles applications des parametres continus a la theorie des formes quadratiques.
Journal fur Reine und Angewandte Mathematik 134:198-287, 1908.

[44] Wolter, F.-E.
Distance Function and Cut Loci on a Complete Riemannian Manifold.
Archiv der Mathematik 32:92-96, 1979.

[45] Wolter, F.-E.
Cut Loci in Bordered and Unbordered Riemannian Manifolds.
PhD thesis, Technical University of Berlin, Department of Mathematics, December, 1985.

[46] Yap, C.-K.
An O(n log n) Algorithm for the Voronoi Diagram of a Set of Simple Curve Segments.
Discrete Computational Geometry 2:365-393, 1987.