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**INSTABILITY OF THE FAR WAKE OF A STEADILY
ADVANCING SHIP**

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Abstract

The three-dimensional linear stability of the viscous wake far behind a ship at low Froude numbers is analyzed. It is assumed that the wake has reached a self-similar parallel flow state, around which the Euler equations are linearized. The stability problem is formulated as an eigenvalue problem for waves travelling parallel to the direction of the ship. It is found that the wake becomes unstable, and that the instability is of the convective type. The free-surface manifestation of the instability wave exhibits a characteristic staggered pattern of alternating hills and valleys, which is antisymmetric about the axis of the wake. For low Froude numbers, the frequency and phase-velocity of these gravity waves is determined by the characteristics of the shear flow in the wake.

1 Introduction

The purpose of this paper is to investigate the instability of the viscous wake of a ship, and its manifestation on the ocean surface (figure 1). The best known identifiable feature of the flow behind a ship is the Kelvin wave pattern. The Kelvin wave pattern has been extensively studied, because it constitutes the main source of ship resistance at high speeds. The viscous wake of the ship has received much less attention, since it is for most ships thin, and it has often been assumed that its influence on the wavemaking of the ship can be neglected, even though there is some important evidence for the contrary (Tatinclaux, 1970). In recent years, however, the interest in the viscous wake of the ship has acquired a whole new dimension, in connection with the problem of satellite surveillance of the ocean. As aerial pictures of the ocean reveal, the viscous wake of the ship persists at large distances behind the ship. In fact, owing possibly to the ship's disturbing the biological contents of the water near the ocean surface, viscous wakes of ships are often visible in satellite pictures long after any hydrodynamic effects of the ship have been dissipated. Thus, the viscous wake, even though thin, leaves a very persistent "trace" of the ship on the ocean surface, and offers an effective means of ship detection. In particular, identification of unsteady flow-patterns in the wake that are capable of perturbing the free surface can be very helpful in detecting the presence of ships in the ocean. A solution of the problem through direct simulation of the Navier-Stokes equations is still impossible, owing to the very high value of the Reynolds numbers of ships (typically 10^9). Recently, Swaan (1987) has performed computations of the steady high-Reynolds number flow past a ship using a parabolic approximation to the Navier-Stokes equations, and a K, ϵ modelling of turbulence. His study shows the structure of the average flow in the wake of the ship, but does not address the issue of the stability of the the wake.

In this paper an attempt is made to study the *unsteady* patterns in the far wake of a ship by looking at one specific important aspect of the problem, namely the interaction between the instability of the wake far behind the ship and the ocean surface. Peregrine (1971), in a notable paper, studied the diffraction of the ship-generated waves by the viscous wake, using an earlier theory developed by Longuet-Higgins and Stewart (1961) for the diffraction of water waves by non-uniform currents. This theory is valid for waves with wavelengths much shorter than the width of the wake; such short wavelengths are stable. The problem discussed here is the generation of gravity waves by the shear-flow type of instability of the wake, and is therefore valid for waves with wavelengths comparable with the width of the wake.

The problem is formulated as follows: In agreement with the recent computations of Swaan (1987), it is

assumed that, sufficiently far behind the ship, the average flow in the wake reaches a self-similar state. Then the stability of small perturbations around the self-similar state is formulated. The perturbation has the form of waves that propagate parallel to the axis of motion of the ship, and have an eigenfunction type of dependence in the other directions. An eigenvalue problem is obtained for the frequency which is solved numerically for the unstable modes. From the computed eigenvectors, the shape of the free-surface manifestations of the instability waves is determined.

2 Formulation of the problem

We consider the linear instability of the viscous wake of a three-dimensional floating object at large distances behind the object. Let x, y, z be a system of coordinates fixed on the object. The axis x is parallel to the oncoming flow, the axis z parallel and opposite to the direction of gravity, and y is perpendicular to the other two (see figure 1). For low Froude numbers, the steady waves generated from the ship are mainly of the transverse type. Consequently, the ocean surface lying above the viscous wake is free of steady waves and can be considered approximately flat. The average flow in the far wake can thus be approximated by half of the "double body" one, which far from the object tends to become axisymmetric. This implies that far behind the ship the average velocity becomes independent of the angle $\theta = \text{atan}(z/y)$ in the y, z plane, reaching asymptotically a self-similar state. Recently, Swaan(1987) computed the steady flow past a ship using parabolic Navier-Stokes equations. Swaan's results show that indeed the average-flow tends to become self-similar as assumed above; his computational results show good agreement with the experimental results of Mitra, Neu and Schetz (1985,1986). If $U(x,y,z)$ represents the average velocity in the x direction in the wake we have that $U = U(x,r)$, where $r = \sqrt{y^2+z^2}$. Furthermore, the wake varies slowly in the x direction so we can approximately set $U = U(r)$. We will study the stability of this flow to small perturbations. We assume that all velocities have been non-dimensionalized with respect to the free stream velocity U_∞ , the pressure with respect to ρU_∞^2 (ρ is the density), and all lengths with respect to the width b of the wake. Consistent with this non-dimensionalization the acceleration of gravity g will be replaced in the equations of motion by $1/F^2$, where F is the Froude number defined as $F = U_\infty/\sqrt{gb}$.

The linearized equations of motion about the time-averaged flow, written in the system of the cylindrical coordinates x, r, θ , are (Drazin & Reid, 1981):

$$\frac{\partial u_x}{\partial t} + U \frac{\partial u_x}{\partial x} + \frac{dU}{dr} u_r = -\frac{\partial p}{\partial x} \quad (1)$$

$$\frac{\partial u_\theta}{\partial t} + U \frac{\partial u_\theta}{\partial x} = -\frac{1}{r} \frac{\partial p}{\partial \theta} \quad (2)$$

$$\frac{\partial u_r}{\partial t} + U \frac{\partial u_r}{\partial x} = -\frac{\partial p}{\partial r} \quad (3)$$

The continuity equation is:

$$\frac{\partial u_x}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} = 0 \quad (4)$$

In equations (1) through (4), u_r, u_θ, u_x are respectively the components of the perturbation velocity along the directions r, θ, x ; p is the *dynamic* pressure, i.e. the pressure minus the hydrostatic one $-F^2 z$ (otherwise, the Froude number would appear as a constant forcing in the z -direction).

On the free surface we have the kinematic and the dynamic boundary conditions. Let $\eta(x, y, t)$ be the non-dimensional displacement of the free surface, and u_n the component of the velocity perpendicular to the free surface, considered positive when pointing *upwards*. Then on $\theta=0, -\pi$ the linearized kinematic boundary condition is :

$$\frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} = u_n \quad (5)$$

and the dynamic boundary condition can be written as :

$$p = F^{-2} \eta \quad (6)$$

Finally, for $r = \sqrt{y^2 + z^2} \rightarrow \infty$, $u_r, u_\theta, u_x, p, \eta$ tend to zero.

In order to obtain the dispersion relation of the flow, we set all quantities proportional to $e^{i(kx - \omega t)}$, where k, ω are the wavenumber and frequency; for notational simplicity we keep the same symbols p, u_r, u_θ, u_x for the transformed quantities. The dispersion relation is then given by the following set of equations:

$$i(kU - \omega) u_x + \frac{dU}{dr} u_r = -i k p \quad (7)$$

$$i(kU - \omega)u_\theta = -\frac{1}{r} \frac{\partial p}{\partial \theta} \quad (8)$$

$$i(kU - \omega)u_r = -\frac{\partial p}{\partial r} \quad (9)$$

$$iku_x + \frac{1}{r} \frac{\partial}{\partial r}(ru_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} = 0 \quad (10)$$

subject to the boundary conditions at the free surface, and at infinity. The former become:

$$i(kU - \omega)\eta = u_n \quad (11)$$

$$p = F^{-2}\eta \quad (12)$$

whereas the latter remain as they were, i.e. for $r = \sqrt{y^2 + z^2} \rightarrow \infty$, $u_r, u_\theta, u_x, p, \eta$ tend to zero.

Because of the boundary conditions on the free surface, we found it more convenient express the perturbation velocities in terms of the perturbation pressure. To this purpose, we multiply equation (7) by ik , and operate on (8) with $\partial/(r\partial\theta)$; then by adding the two and using (10), we obtain an equation relating p, u_r :

$$-i(kU - \omega) \frac{1}{r} \frac{\partial}{\partial r}(ru_r) + iku_r \frac{dU}{dr} = k^2 p - \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} \quad (13)$$

We now use equation (9) to eliminate u_r from (13) and obtain, after some term regrouping, the following partial differential equation for $p(r, \theta)$:

$$(kU - \omega) \left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} - k^2 p \right) - 2k \frac{dU}{dr} \frac{\partial p}{\partial r} = 0 \quad (14)$$

For the boundary conditions on the free surface, we note that on $\theta = 0$, $u_n = u_\theta$, whereas on $\theta = -\pi$, $u_n = -u_\theta$. In combined form, the relation between u_n, u_θ on $\theta = 0, -\pi$ can be written as follows:

$$u_\theta = u_n \cos(\theta) \quad (15)$$

By eliminating therefore η and u_θ the two boundary conditions at the free surface (11), (12) can be

combined into a single condition of the mixed type, as follows:

$$\cos(\theta)(kU - \omega)^2 F^2 p - \frac{1}{r} \frac{\partial p}{\partial \theta} = 0 \quad \text{on } \theta = 0, -\pi \quad (16)$$

Finally, we impose the condition that far from the wake the perturbation vanishes, i.e.

$$p(r, \theta) \rightarrow 0 \quad r \rightarrow \infty \quad (17)$$

Equations (14), (16), (17) define for any given k an eigenvalue problem for the frequency ω ; that is to say, they constitute the dispersion relation for gravity waves propagating above the wake of the ship.

3 Fourier Series Solution

We note that because of the linearity of the problem and the symmetry of the average flow $U(r)$ around the plane $y = 0$, it is possible to decompose any arbitrary disturbance into two parts: one in which the perturbation pressure is *anti-symmetric* around $y = 0$, which we will call Mode I, and one in which the pressure is *symmetric* around $y = 0$, which we will call Mode II (see figure 2). Thus, given that the free surface displacement is proportional to the value of the perturbation pressure there, Mode I disturbs the free surface in an antisymmetric manner around $y = 0$, whereas Mode II disturbs the free surface in a symmetric manner. The fact that the two modes are separable facilitates the numerical solution of the problem.

3.1 Mode I

We start with mode I, which from its definition satisfies the following anti-symmetry conditions:

$$p(r, -\pi) = -p(r, 0) \quad (18)$$

$$\frac{\partial p}{\partial \theta}(r, 0) = \frac{\partial p}{\partial \theta}(r, -\pi)$$

Since the initial flow is independent of the angle θ , it is convenient to expand $p(r, \theta)$ in a cosine-series in terms of the angle θ . In accordance with the antisymmetric character of Mode I, the Fourier series will contain odd-order coefficients only. Therefore we set:

$$p(r, \theta) = \sum_{n=1}^{\infty} p_n(r) \cos(n\theta) \quad (19)$$

where in (19) it is implied that the summation is carried over all *odd* n only; the same convention applies

for the rest of this section. The coefficients $p_n(r)$ of the Fourier series in (19) are given by:

$$p_n(r) = \frac{1}{\pi} \int_{-\pi}^0 p(r, \theta) \cos(n\theta) d\theta \quad (20)$$

In an infinite fluid direct substitution of (19) into (14) will yield an infinite set of *uncoupled* ordinary differential equations for $p_n(r)$, completely equivalent to those obtained for the perturbation velocity by Batchelor and Gill, 1962. In the problem considered here, however, owing to the mixed boundary condition at the free surface, double differentiation of the Fourier series with respect to θ yields a divergent Fourier series for $\partial^2 p / \partial \theta^2$. One way to overcome this difficulty, is to independently expand $\partial^2 p / \partial \theta^2$ in a cosine series in θ , and express the coefficients p_n in terms of the coefficients of the Fourier expansion of $\partial^2 p / \partial \theta^2$. This amounts to integrating twice the cosine-series expansion of $\partial^2 p / \partial \theta^2$, which is always possible, rather than differentiating twice p . Therefore we set:

$$\frac{\partial^2 p}{\partial \theta^2} = \sum_{n=1}^{\infty} a_n(r) \cos(n\theta) \quad (21)$$

where $a_n(r)$ are given by:

$$a_n(r) = \frac{1}{\pi} \int_{-\pi}^0 d\theta \frac{\partial^2 p}{\partial \theta^2} \cos(n\theta) \quad (22)$$

By substituting (19), (21) into (14) we obtain the following equations :

$$(kU - \omega) \left(\frac{d^2 p_n}{dr^2} + \frac{1}{r} \frac{dp_n}{dr} + \frac{1}{r^2} a_n - k^2 p_n \right) - 2k \frac{dU}{dr} \frac{dp_n}{dr} = 0 \quad (23)$$

We now express p_n in terms of a_n . This can be done by integrating by parts twice equation (22):

$$\begin{aligned} \int_{-\pi}^0 \frac{\partial^2 p}{\partial \theta^2} \cos(n\theta) d\theta &= \frac{\partial p}{\partial \theta} \cos(n\theta) \Big|_{-\pi}^0 + n p \sin(n\theta) \Big|_{-\pi}^0 - n^2 \int_{-\pi}^0 p \cos(n\theta) d\theta = \\ &= \cos(\theta) \cos(n\theta) F^2 r (Uk - \omega)^2 p(r, \theta) \Big|_{-\pi}^0 - n^2 \pi p_n(r) \end{aligned} \quad (24)$$

or, equivalently, by using the definition of $a_n(r)$ in (22) and the symmetry conditions (18), we have:

$$p_n(r) = -\frac{a_n(r)}{n^2} + \frac{2}{\pi} F^2 r (kU - \omega)^2 p(r, 0) \frac{1}{n^2} \quad (25)$$

From (19), we have that $p(r,0)$ is given by:

$$p(r,0) = \sum_{m=1}^{\infty} p_m(r) \quad (26)$$

Consequently, by substituting (25) into (26) we express $p(r,0)$ in terms of $a_n(r)$:

$$p(r,0) = \sum_{m=1}^{\infty} -\frac{a_m(r)}{m^2} + \frac{2}{\pi} F^2 r (kU - \omega)^2 p(r,0) Q \quad (27)$$

where Q in equation (27) is the following constant:

$$Q = \sum_{m=1}^{\infty} \frac{1}{m^2} = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) \quad (28)$$

We now can solve equation (27) for $p(r,0)$, which gives:

$$p(r,0) = -\frac{1}{1 - (2/\pi) r F^2 (kU - \omega)^2} \sum_{m=1}^{\infty} \frac{a_m}{m^2} \quad (29)$$

We substitute (29) into (25); this yields the expression of the coefficients p_n in terms of the coefficients $a_n(r)$:

$$p_n(r) = -\frac{a_n(r)}{n^2} - \frac{1}{n^2} \frac{(2/\pi) F^2 r (kU - \omega)^2}{1 - (2/\pi) F^2 r Q (kU - \omega)^2} \sum_{m=1}^{\infty} \frac{a_m}{m^2} \quad (30)$$

Substitution of (30) into (23) will give an infinite set of *coupled* homogeneous ordinary differential equations containing only $a_n(r)$, subject to the boundary conditions that $a_n(r)$ vanish as $r \rightarrow \infty$. Thus, for a given k , an eigenvalue problem is defined for the frequency ω . This approach, however, is not appropriate for the numerical solution of the problem, since the eigenvalue problem is non-linear with respect to the eigenvalue ω . Although the problem can, presumably, still be solved by shooting, performing shooting on a large set of simultaneous differential equations seems very difficult. Instead a different approach has been implemented, that leads to a set of equations depending linearly on ω . This is explained in section 4.

We now discuss the behaviour of the solution for $r \rightarrow \infty$. Assuming that $F > 0$, for $r \rightarrow \infty$, equation (30)

is reduced to:

$$p_n(r) = -\frac{a_n(r)}{n^2} - \frac{1}{n^2} \frac{1}{Q} \sum_{m=1}^{\infty} \frac{a_m}{m^2} \quad (31)$$

We substitute (31) into (23); then since $dU/dr (r \rightarrow \infty) \rightarrow 0$ we obtain:

$$\frac{1}{n^2} \left(\frac{d^2 a_n}{dr^2} + \frac{1}{r} \frac{da_n}{dr} - k^2 a_n \right) + \frac{1}{n^2} \left(\frac{d^2 S}{dr^2} + \frac{1}{r} \frac{dS}{dr} - k^2 S \right) = 0 \quad n = 1, 3, \dots \quad (32)$$

where in equation (32) S stands for:

$$S = \sum_{n=1}^{\infty} \frac{a_n}{n^2} \quad (33)$$

We now sum equations (32) over all n , and using equations defining Q, S (28), (33) we obtain:

$$\frac{d^2 S}{dr^2} + \frac{1}{r} \frac{dS}{dr} - k^2 S = 0 \quad (34)$$

We substitute (34) back into (32). This gives:

$$\frac{d^2 a_n}{dr^2} + \frac{1}{r} \frac{da_n}{dr} - k^2 a_n = 0 \quad n = 1, 3, \dots \quad (35)$$

From equation (35) we conclude that for $r \rightarrow \infty$, $a_n(r)$ behaves like a zeroth order Bessel function; the behaviour of such functions for large arguments is thus given by (Abramowitz and Stegun, 1970):

$$a_n(r) \sim \frac{1}{\sqrt{r}} \exp(-kr) \quad (36)$$

Thus the eigenmode decays exponentially with the distance from the mid-axis of the wake. Equation (36) is useful in the numerical solution of the problem as a truncation condition for the infinite domain.

3.2 Mode II

Mode II satisfies from its definition the following symmetry relations:

$$p(r,0) = p(r,-\pi) \quad (37)$$

$$\frac{\partial p}{\partial \theta}(r,0) = -\frac{\partial p}{\partial \theta}(r,-\pi)$$

The eigenvalue problem for Mode II can thus be formulated in exactly the same manner as for Mode I, with the difference that the Fourier series will now contain only the *even* order terms. We omit the details of the derivation, and simply report that only Mode I was found to be unstable.

4 Numerical solution

In this section we describe the numerical solution of the eigenvalue problem for the Mode I. We start by introducing the auxiliary functions $G(r)$, $H(r)$, defined as follows:

$$G(r) = 2F^2 r (kU - \omega)^2 p(r,0) \quad (38)$$

$$H(r) = 2Fr(kU - \omega) p(r,0) \quad (39)$$

From their definition in (38), (39) the two auxiliary functions G, H are related by:

$$G(r) = F(kU - \omega)H(r) \quad (40)$$

We multiply both sides of equation (27) by $2Fr(kU - \omega)$ and use the definitions of G, H in equations (38), (39) to obtain:

$$H(r) = 2Fr(kU - \omega) \sum_{m=1}^{\infty} \left(-\frac{a_m}{m^2} \right) + \frac{2Q}{\pi} Fr(kU - \omega)G(r) \quad (41)$$

Since we are solving for Mode I, the summation in (41) is carried out over odd m only. In terms of the new variables, equation (25) can be re-written as:

$$p_n(r) = -\frac{a_n}{n^2} + \frac{1}{\pi} \frac{G(r)}{n^2} \quad (42)$$

Then by substituting (42) into (23) we obtain:

$$(Uk - \omega) \left(\frac{d^2 a_n}{dr^2} + \frac{1}{r} \frac{da_n}{dr} - \left(k^2 + \frac{n^2}{r^2} \right) a_n - \frac{1}{\pi} \left(\frac{d^2 G}{dr^2} + \frac{1}{r} \frac{dG}{dr} - k^2 G \right) \right) - 2k \frac{dU}{dr} \left(\frac{da_n}{dr} - \frac{1}{\pi} \frac{dG}{dr} \right) = 0 \quad (43)$$

Equations (40), (41) and (43), together with the conditions that the unknown functions $H(r), G(r), a_n(r)$ $n = 1, 3, 5, \dots$ vanish for $r \rightarrow \infty$, define for any given k an eigenvalue problem that depends *linearly* on the eigenvalue ω . Consequently, if we truncate the domain to $0 < r < R$, and use finite-differences to approximate the derivatives in (43) at specified points $r_i, i = 1, 2, \dots, N$, the *discretized versions* of equations (40), (41) and (43) define a generalized algebraic eigenvalue problem for ω . The latter can be solved using standard algorithms.

We now assume that, for numerical computation purposes, the Fourier series is truncated after M terms, and that N points are used in the finite-difference grid. Then we form a compound eigenvector \mathbf{X} of order $(M+2) \times N$ as follows: The first N positions of the eigenvector are occupied by the values of $H(r)$ at the N discretization points $r_i, i = 1, 2, \dots, N$, the next N positions by the values of $G(r)$, the next N positions by the values of $a_1(r)$, and so on; finally, the last N positions are occupied by the values of $a_M(r)$ at the discretization points. Then the discretized equations can be combined in a single matrix equation of the form:

$$\mathbf{A} \cdot \mathbf{X} = \omega \mathbf{B} \cdot \mathbf{X} \quad (44)$$

where \mathbf{A}, \mathbf{B} are compound matrices of order $((M+2)N) \times ((M+2)N)$. In general, depending on the required value of M , the order of the eigenvalue problem can be quite high, and require enormous amounts of computation. For low values of the Froude number F , which are of interest here, the coupling between the coefficients a_n is weak (of order F^2), and the Fourier series converges after just a few terms. Furthermore, the decomposition into mode I and II proves quite helpful in that respect, since for the mode I, using M coefficients implies that the series has been truncated at the $(2M+1)$ -th term. As a result, for low F , the required value of M is low, and the order of the eigenvalue problems is such that can be handled with the standard Q-Z algorithms.

For the self-similar average flow, the following non-dimensional velocity distribution can be assumed:

$$U(r) = 1 - A \exp(-\alpha r^2) \quad (45)$$

For the calculations reported here, the values $A = 0.368$, $\alpha = 0.89$ were used. Those are the values for the self-similar profile measured by Ogata and Sato (1966), far behind an axisymmetric body in infinite fluid. We have thus assumed that the velocity profile behind this object being half-submerged will be half of that in an infinite fluid at the same Reynolds number. For low Froude numbers, the eigenvalues were

found to depend very weakly on the Froude number and on the number of Fourier modes used in the expansion.

Thus, for Froude numbers up to 0.5, which covers the range of interest of the present study, the computed frequency was found to be practically independent of the Froude number (to within one per cent). The mapping of the k -real axis into the complex ω -plane is shown in figure 3, for a Froude number equal to 0.5. For all Froude numbers up to 0.5, the most unstable wave was found for a wavenumber equal to 0.55, which gave a complex frequency (0.4525, 0.0170). This value is almost identical with the one occurring in infinite fluid. Therefore, for the problem at hand, the phase velocity of the *unstable* gravity waves above the wake is controlled by the shear flow characteristics. The free surface displacement A can easily be calculated from the computed values of $H(r)$ (which occupy the first N positions of the compound eigenvector \mathbf{x}) as follows:

$$A = F^2 p(r,0) = \frac{F H(r)}{2r(kU(r) - \omega)} \quad (46)$$

The variation of the free surface displacement generated by the most unstable wave-mode along an $x = \text{constant}$ plane as a function of the coordinate y is shown in figure 4. Only half of the displacement is shown, i.e. for $0 < y < \infty$. Far away from the wake, the elevation decays exponentially with the distance y . The vertical scale is arbitrary, since an eigenfunction is plotted. Again the dependence of the eigenmode on the Froude number was found very weak. The phase of the free surface elevation of the same eigenmode is shown in figure 5 (again only the part $0 < y < \infty$ is shown). The free surface elevation in the middle of the wake, where the fluid velocity is reduced, lags behind the elevation outside the wake, where the fluid velocity has its free-stream value. The fact that lines $x = \text{constant}$ on the free surface are not constant phase curves shows that the wavecrests of the eigenmodes will be curved. A three-dimensional computer made visualization of the free surface displacement is shown in figure 6, constructed on an IRIS machine. A length equal to two wavelengths of the disturbance is represented. The free surface elevation produced by the instability wave consists of two parallel series of alternating hills and valleys, in agreement with its antisymmetric character.

Finally the character of the wake instability was determined, i.e. whether it is of the absolute or the convective type. The procedure suggested in Triantafyllou et al. (1987) was followed, in which the complex wavenumber plane is mapped through the dispersion relation into the complex frequency plane; the pinch-point type of double roots are located from the local angle-doubling property of the map. If

these points lie in the upper half ω plane the instability is absolute; else it is convective. For this problem the instability was found to be convective for all low Froude numbers (including zero Froude number). This means that the wake acts as a *spatial amplifier*, i.e. under a time-harmonic forcing, spatially growing waves are produced.

5 Conclusions

The main outcome of this investigation is that the interaction between the instability of the wake and the ocean surface results in an antisymmetric pattern. This pattern consists of two parallel series of "hills" and "valleys", and is antisymmetric about the line of motion of the ship; the phase velocity of the wave is controlled by the characteristics of the shear flow in the wake. This "chessboard" type of pattern is certainly very characteristic, and it remains to be seen whether it is readily identifiable too. The fact that the instability of the wake is convective, implies that unstable wave groups will be convected with the mean flow and amplified in space. Possible excitation sources for the wake instability are ambient waves, or even the unsteadiness of the Kelvin wave pattern.

An important simplification in this study was offered by the self-similarity assumption. An interesting question then is how the above conclusions will be modified, if the self-similarity assumption is dropped. This problem models the stability of the wake close to the ship, where the average flow is not self-similar. In fact, in reality, owing to the presence of the propeller, a self-similar state might be reached very far behind the ship. Therefore this improved analysis will be pertinent to the part of the wake where the instability waves are excited before they propagate into the self-similar part of the wake. For a wake that is not self-similar, the average flow itself should be expanded in a Fourier series. This will render the analysis of section 3 considerably more complicated. Furthermore the numerical effort required will be higher, because the Fourier coefficients in the expansion will be strongly coupled, even for low Froude number (and thus M will be high). This extension is therefore far from trivial, and is planned as a future investigation.

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Figure Captions

Figure 1: Definition figure.

Figures 2a, 2b: Anti-symmetric (I) and symmetric mode (II) in the wake.

Figure 3: Highest Branch map of the k -real axis into the complex ω -plane.

Figure 4: Amplitude of the free-surface displacement as a function of y for the most-unstable wave.

Figure 5: Phase of the free-surface displacement as a function of y for the most-unstable wave.

Figure 6: Perspective view of the free-surface manifestation of the most-unstable wave.

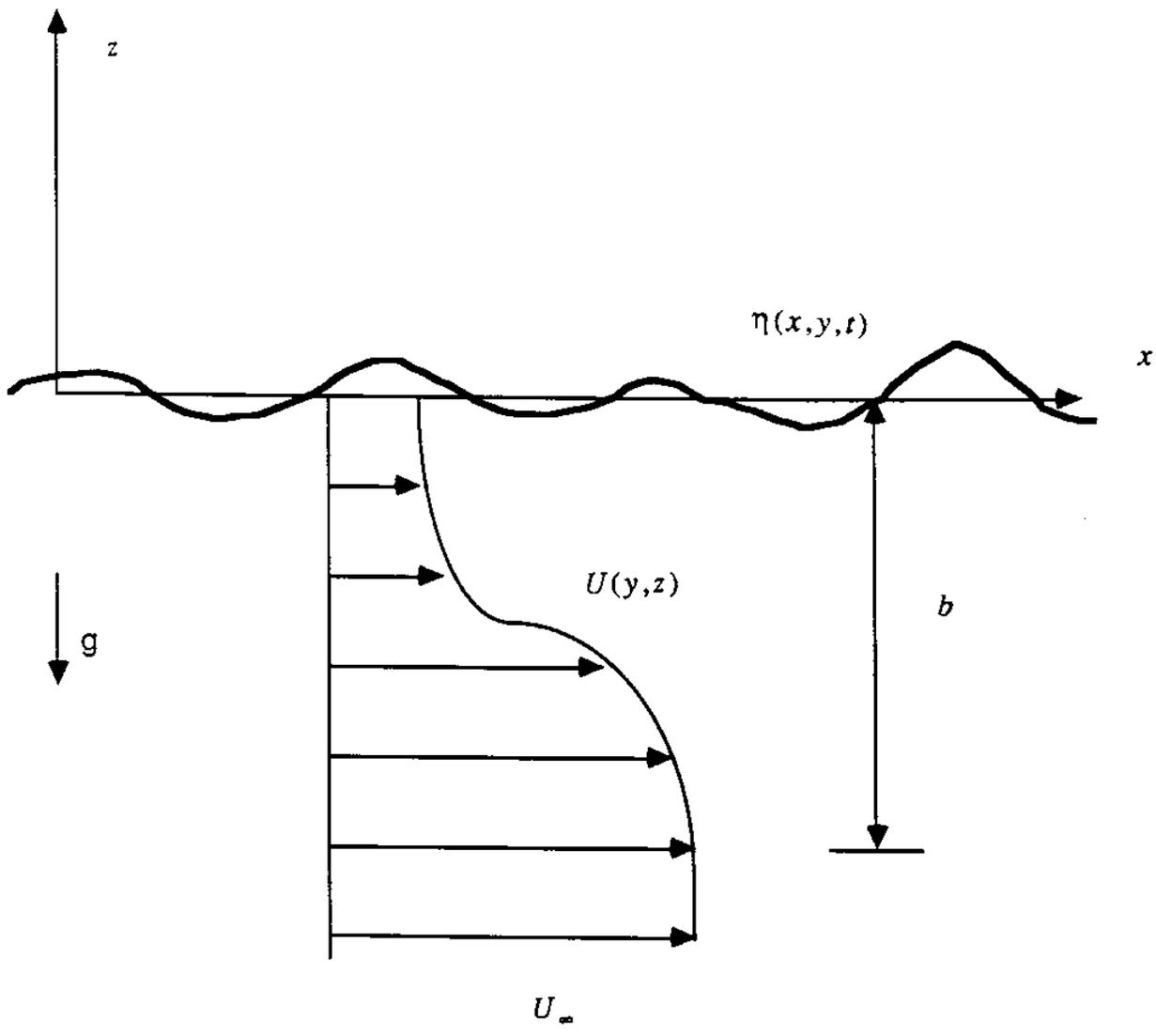
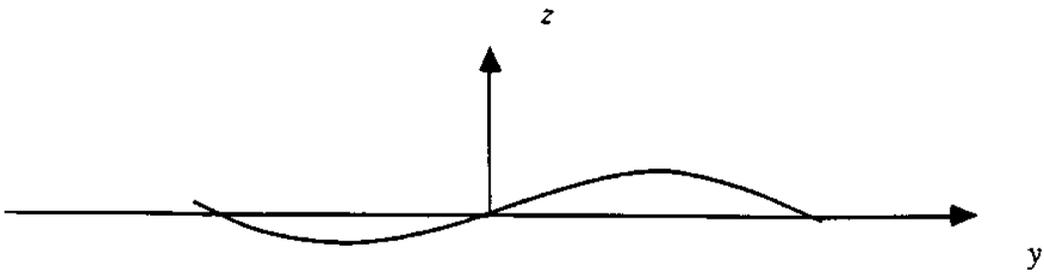
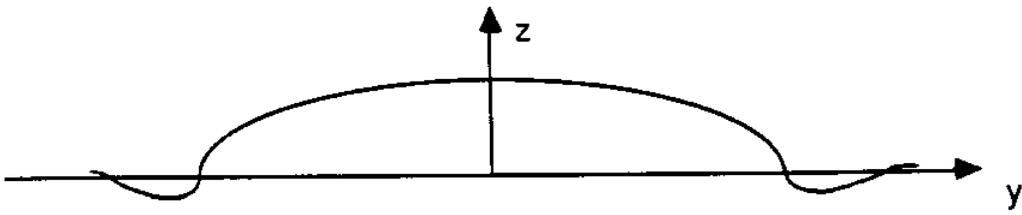


Figure 1

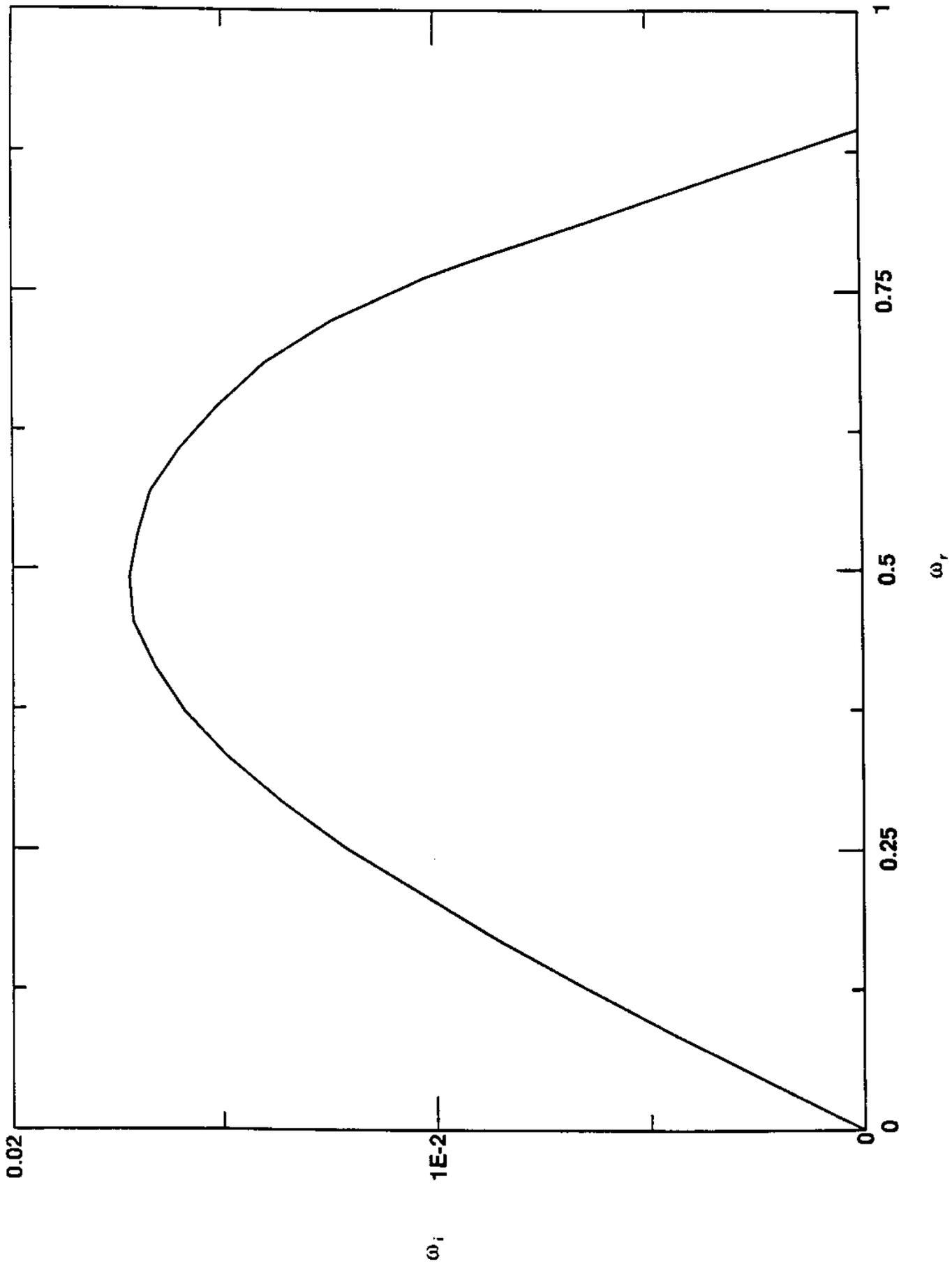


MODE I : $\eta(y) = -\eta(-y)$



MODE II : $\eta(y) = \eta(-y)$

Figure 2



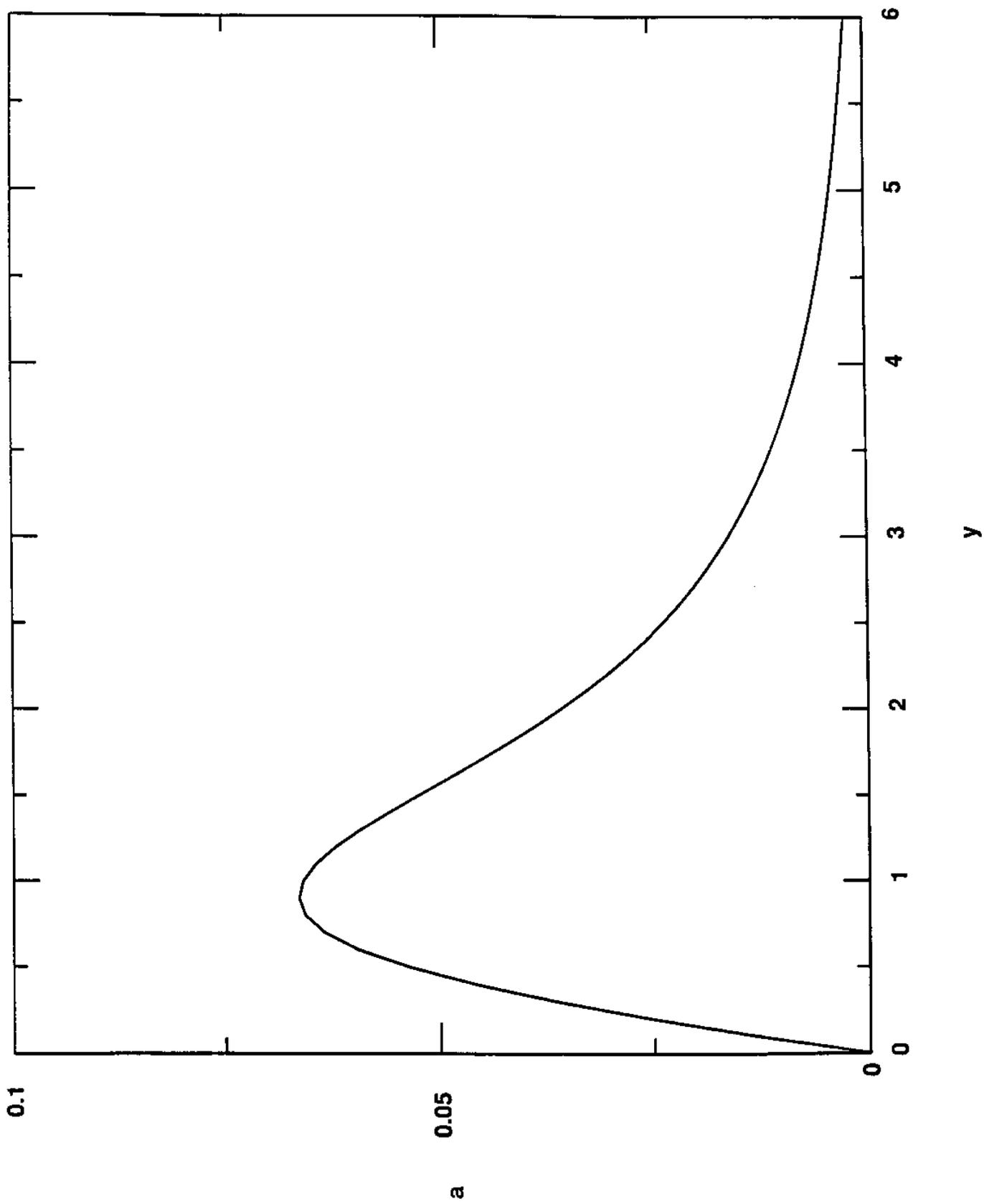


Figure 4

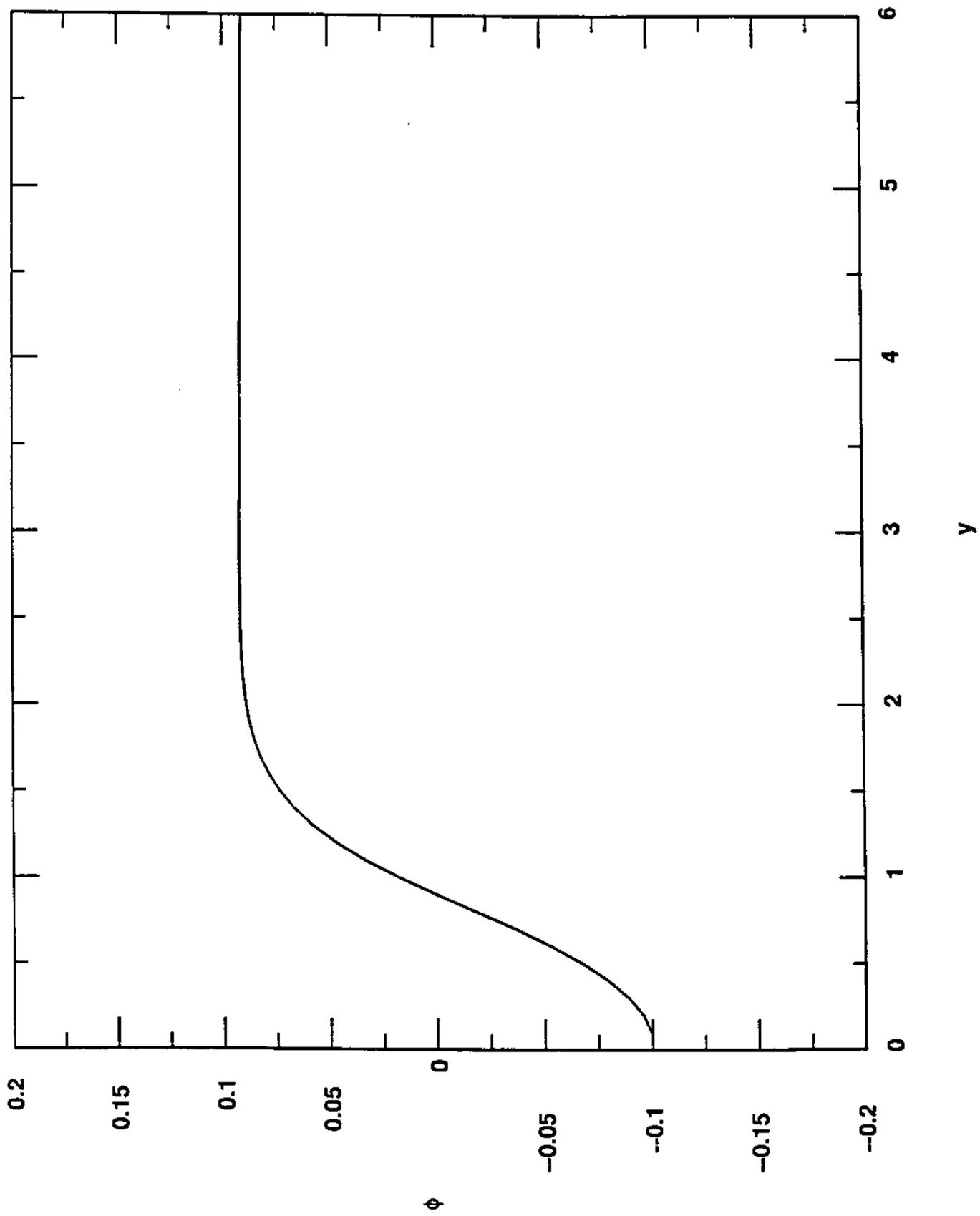


Figure 5

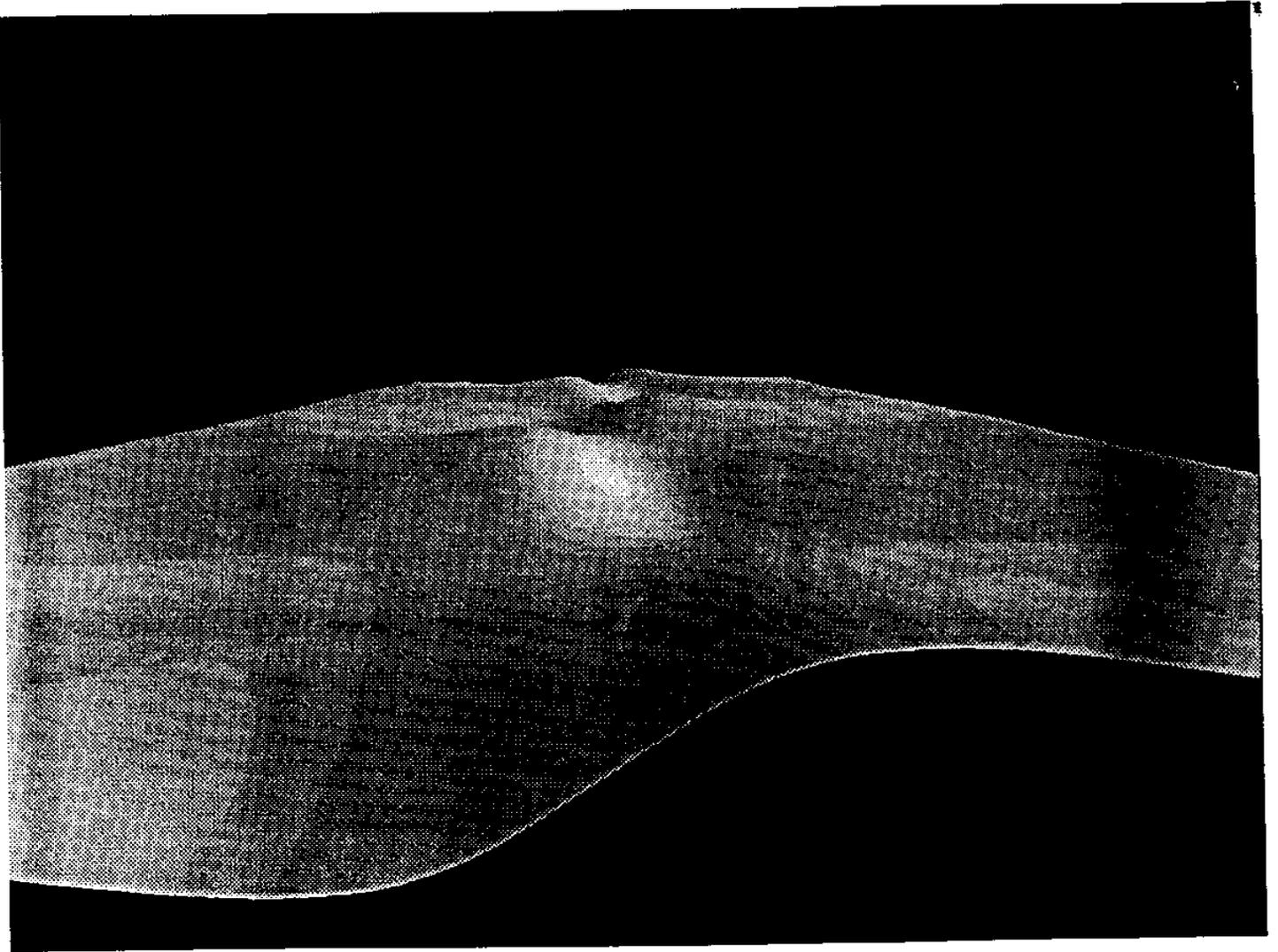


Figure 6