MIT-T-86-001 C2

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Mooring Dynamics for Offshore Applications

Part I Theory

MIT Sea Grant
College Program

Massachusetts
Institute of Technology
Cambridge, MA 02139

MITSG 86-1
January 1986

MIT-T-SG-001 C2

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Index No. NA81AA-D00069
Project No. R/O-5

MITSG 86-1
January 1986

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Acknowledgments

This research was supported by the MIT Sea Grant Program through the National Oceanic and Atmospheric Administration in the US Department of Commerce and by NAVSEA and a consortium of companies, including Conoco, Exxon, Gulf, HydroNorsk, Mobil and Shell.

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Nomenclature

A_o	Area of the cable in unstretched condition
C_{Dt}	Tangential drag coefficient
C_{Dn}	Normal drag coefficient
C_i	Modal generalized displacement
D_o	Diameter of the cable in unstretched condition
D_w	Waterdepth
E	Young's modulus
F	Force per unit length
H	Horizontal Tension
H_*	Horizontal Tension projected on the chord ($H/\cos\phi_a$)
[H(s,r)]	Matrix influence function
L	Length of the unstretched cable
M	Mass plus the added mass per unit unstretched length
Q	Quantity governing slow dynamics $Q(\sigma)=-h(\phi_{o\sigma}^2 - M\omega^2L^2/(EA))$
T	Tension
T_e	Effective tension
T_o	Effective static tension
T_1	Effective dynamic tension

U	Current velocity
S	Impedance transfer function
b	Sag of the cable
b_d	Damping constant $0.5\ \rho_w C_D D_o$
$\vec{b}$	Bi-normal vector
c_{tr}	Transverse wavespeed
c_{el}	Elastic wavespeed
c_H	Transverse wavespeed projected on the horizontal direction
e	Strain
e_o	Static strain
e_1	Dynamic strain
g	Gravity constant
h	Ratio of the mass and the mass plus the added mass
k	Wavenumber
$\vec{k}$	Vertical unit vector
l	Length of the span of the cable
m	Mass per unit unstretched length
m_a	Added mass per unit unstretched length
$\vec{n}$	Normal unit vector
p	Tangential displacement [except in chapter 1 where it is the stretched Lagrangian coordinate along the cable]

q	Normal displacement
r	Bi-normal displacement
s	Unstretched Lagrangian coordinate along the cable
t	Time
$\vec{v}$	Velocity
$\vec{v}_r$	Relative velocity in the water
w_o	Net weight per unit unstretched length
A_i	Airy function
B_i	Bairy function
J_o	Zeroth order Bessel function of the first kind
Y_o	Zeroth order Bessel function of the second kind
W_o	Parabolic cylinder function
$\vec{\Omega}\ (\Omega_1,0,\Omega_3)$	Darboux vector
$\vec{\Omega}_o(\Omega_{1o},0,\Omega_{3o})$	Static component of the Darboux vector
$\vec{\Omega}_1(\Omega_{11},0,\Omega_{31})$	Dynamic component of the Darboux vector
α	Ratio of the total weight and the horizontal tension (W_oL/H) [except in chapter 2, where it is the angle between the current velocity and the (x,y) plane)
α_*	Ratio of the total weight and the projected horizontal tension (W_oL/H_*)
β	Projected elastic strain ($H_*/(EA)$)
ϵ	Small perturbation quantity

η	Non-dimensional normal displacement
θ	Euler angle around vertical axis
θ_o	Static component of θ
θ_1	Dynamic component of θ
λ	Ratio of the elastic stiffness and the catenary stiffness for horizontal cables [except in 3.6 where λ is a non-dimensional frequency parameter]
λ_*	Ratio of the elastic stiffness and the catenary stiffness for inclined cables
ξ	Non-dimensional tangential displacement
ρ	Density of the cable material [except in chapter 1 where it is the curvature]
ρ_w	Density of the water
σ	Non-dimensional unstretched length parameter
τ	Geometric stiffness parameter [except in chapter 1 where it is the torsion]
ϕ	Angle between the cable and the horizontal
ϕ_o	Static component of the ϕ
ϕ_1	Dynamic component of the ϕ
ϕ_a	Angle of the chord of the cable and the horizontal
ψ	Euler angle around the tangential direction of the cable
ψ_o	Static component of ψ

ψ_1 Dynamic component of ψ

ω Circular frequency

$\vec{\omega}$ Rotation vector

Introduction

This is the final report of a research effort conducted at the Ocean Engineering Department at M.I.T., supported jointly by the Sea Grant Program, NAVSEA and a consortium of offshore companies consisting of

Conoco Inc.

Exxon Production Research Company

Gulf Oil Exploration and Production Company

Norsk-Hydro

Mobil Research and Development Corporation

Shell Development Company

The purpose of the research was to study the dynamics of a single mooring line under conditions expected in a marine application. The discovery of some basic linear dynamic properties of cables was the principal reason for this study, which investigated the effect of the principal non-linearities on linear predictions. The equivalent linearization of nonlinear terms was also addressed as a means for reducing the computational complexity for assessing cable dynamic properties,

The interest in this project stems from the fact that deep water applications for mooring systems have been proposed for implementation in the near future, or have already been made (up to more than 2000 ft) in some instances.

The principal difference in deep water applications from previous mooring installations is that the natural frequencies of the lines lie within the wave spectrum and dynamic amplification is expected. At the same time an increase in pretension

is required due to the increased self-weight of the line as the line length increases, this reducing the available margin for dynamic effects.

The principal non-linearity for vibrating cables in water is the fluid drag, with the exception of vortex-induced oscillations. This was confirmed in this study, although other parameters enter such as the elastic stiffness-to-catenary stiffness ratio, which affect the dynamic performance also. An outline of the numerical results found for specific applications illustrates the effects of the principal parameters (see part II).

The effect and linearization of the cable bottom interaction is a novel issue for cable dynamic studies, although most lines used for mooring offshore structures exhibit such an interaction. This study addressed the problem with the purpose of gaining insight into the principal mechanism of interaction and a linearization procedure has been derived. A method to couple the dynamics of several mooring lines was also developed, based on the principle of dynamic impedance.

The problem of snap loading for a taut horizontal cable was addressed. Such a development for a curved cable is novel, because it accounts for all the properties recently discovered for cable dynamics. Traveling stress waves can be properly simulated with the new method. The numerical applications can be found in part II of the report.

The authors wish to acknowledge the support of the office of Sea Grant, NAVSEA and of the participating companies named in the beginning of the introduction. They are also indebted to the representatives of NAVSEA and the offshore companies who discussed the progress of the report, offered suggestions and shared their experience and information that was available to them: Mr. G. Prentice of NAVSEA; Drs. Pejaver and W.C.Kan of Exxon; Messrs. P.Erb, R.Vermeir and P.Wybro of Conoco; Dr. C.G.Caracostis of Shell Development

Company; Mr. P.O.Verlo of Norsk Hydro; Mr. T.Torsoy of Mobil Research and Development Corporation; and Mr. C.Barton of Gulf Oil Exploration and Production Company.

Chapter 1

THREE DIMENSIONAL EQUATIONS OF CABLE DYNAMICS

1.1 Introduction

Three dimensional cable dynamics can be studied in several coordinate systems. Critescu used a Cartesian coordinate system to study the motions of extensible cables [Critescu 67]. The fundamental dynamic properties of cables are better analysed in the so called natural coordinates of a cable, which are fixed on the cable. The cable coordinate system is, therefore, varying both in time and space. Another advantage of this system is that the fluid forces can be described easily. In chapter 3, we will show that this description leads to analytical solutions of the linearised cable dynamic equations.

A derivation of the three dimensional equations in natural coordinates has been done by Lenskii [Lenskii 78]. Cannon and Genin [Cannon 72a] derive the equations directly in terms of the velocities and angular rotations. Barr [Barr 74] obtained equations of motion in two dimensions, starting from a more general formulation, which includes bending effects. A study on the treatment of the hydrodynamic forces on cables can be found in [Breslin 74] and [Goodman 76].

1.2 Kinematics in Three Dimensions

We consider the cable idealized as a single curvilinear line. We define a certain material point on the cable as the origin and a certain direction as positive along the cable. For example, we will usually fix the origin at the lower end of a marine cable and the positive direction will be from the lower towards the upper cable end.

The cable is made of an elastic material, so it is extensible. As a result, the distance between two material points will vary depending on the state of stretching. Let s denote the unstretched distance of a material point from the origin and p(s,t) the stretched distance of the same point at a certain time, t. Both s and p are Langrangian coordinates of the material point.

The cable configuration, i.e. the shape of the idealized line is a continuous function of time and of the coordinates s (or p). Each material point can be described by its distance from the origin of a cartesian system (x, y, z), i.e.

$$x(s,t)$$

$$y(s,t)$$

$$z(s,t)$$

In order to account properly for the fluid forces, though, we will employ a different description system, which introduces a certain degree of complexity, so it is worthwhile establishing a few basic properties, which we can recall systematically in subsequent sections.

The system $\vec{t}$, $\vec{n}$, $\vec{b}$: We define a tangential unit vector $\vec{t}$, at a certain

point A of the cable configuration. The vector AC, where C is another point on the cable, has a limiting position as C tends towards A, which is the tangent direction (provided the cable configuration does not form an angle at the point A, in which case there are two tangents, one from the left and one from the right).

Next we define a normal unit vector $\vec{n}$ at point A: We pick a point B to the left of A and a point C to the right of A (both B and C lie on the cable configuration). We form two planes, one perpendicular to AC and passing through the middle point of AC, and a similar plane for AB. The two planes cross along a line whose shortest distance from A is AM. The limit position of AM, as B and C tend to A, define the normal direction (see figure 1-1), which is perpendicular to the tangential direction. The limit distance AM denoted by ρ is called the radius of curvature.

Finally, we define the binormal unit vector $\vec{b}$ such that the system of vectors $(\vec{t}, \vec{n}, \vec{b})$ is orthogonal and right-handed.

Since the configuration changes both with time and along the cable length, all three vectors are functions of t and p (or s), i.e.:

$$\vec{t}\ (t,p)$$

$$\vec{n}\ (t,p)$$

$$\vec{b}\ (t,p)$$

Strain e. A segment of the cable with unstretched length δs has at a certain time length δp. We define the longitudinal strain e as:

$$e = \lim_{\delta s \to 0} \left(\frac{\delta p - \delta s}{\delta s} \right) = \frac{dp}{ds} - 1 \qquad (1.1)$$

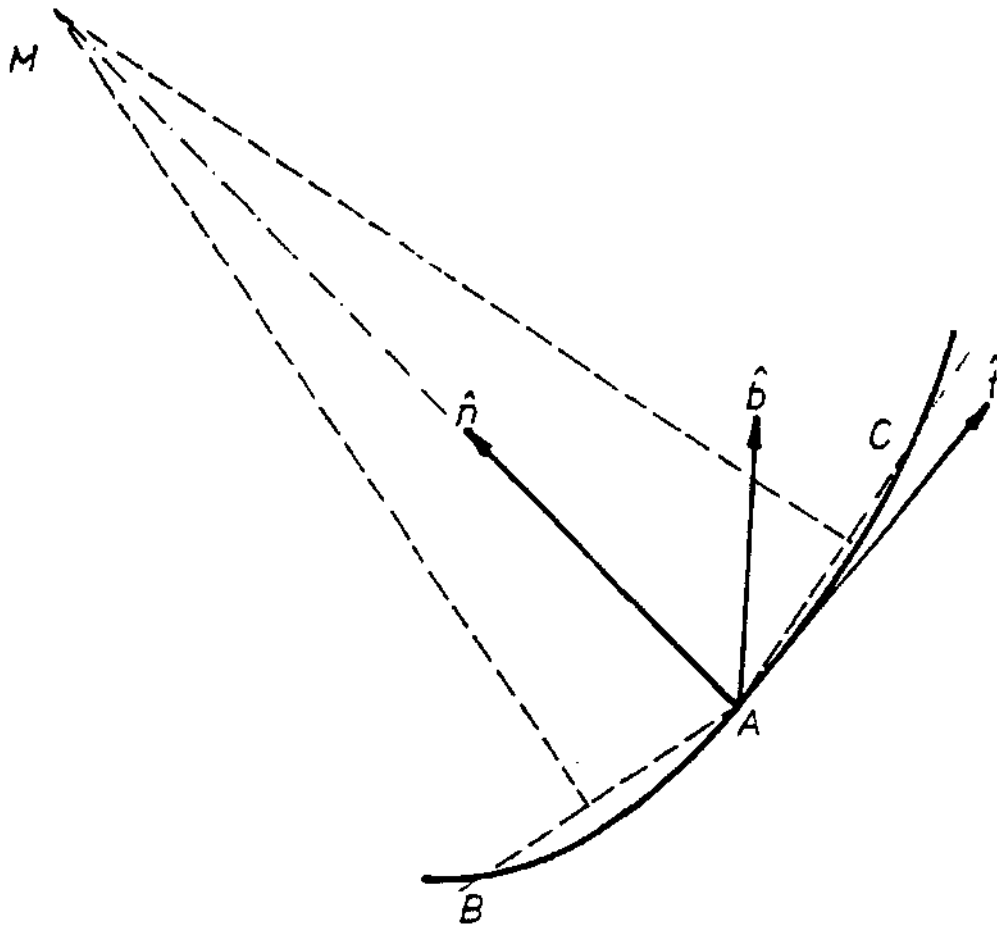

Figure 1-1: Definition of the Natural Coordinate System

As a result of stretching, the cross sectional area of the cable changes also, something that will be discussed later.

Changes in space: The position of $\vec{t}$ varies along p according to the Frenet relations, [Hildebrand 49]:

$$\frac{\partial \vec{t}}{\partial p} = \frac{1}{\rho} \vec{n}$$

$$\frac{\partial \vec{b}}{\partial p} = -\frac{1}{\tau} \vec{n} \qquad (1.2)$$

$$\frac{\partial \vec{n}}{\partial p} = \frac{1}{\tau} \vec{b} - \frac{1}{\rho} \vec{t}$$

where ρ is the radius of curvature and τ the radius of torsion. The notation

$$\frac{\partial \vec{t}}{\partial p}$$

indicates the vector which is obtained by subtracting the unit vector $\vec{t}(t,p)$ from the unit vector $\vec{t}\ (t,p+dp)$, divided by dp, as $dp \rightarrow 0$.

We rewrite the above relations in terms of the unstretched coordinate s, using the relation $dp = (1+e)ds$:

$$\frac{\partial \vec{t}}{\partial s} = \frac{1+e}{\rho} \vec{n}$$

$$\frac{\partial \vec{b}}{\partial s} = -\frac{1+e}{\tau} \vec{n} \qquad (1.3)$$

$$\frac{\partial \vec{n}}{\partial s} = \frac{1+e}{\tau} \vec{b} - \frac{1+e}{\rho} \vec{t}$$

We redefine ρ_s as $\rho/(1+e)$, and τ_s as $\tau/(1+e)$. These are the curvature and the torsion in the unstretched coordinate s. The subscript s will be omitted in the sequel, while ρ and τ wil denote the unstretched quantities.

Next we define the Darboux vector $\vec{\Omega}$:

$$\vec{\Omega} = \frac{1}{\tau}\vec{t} + \frac{1}{\rho}\vec{b} = (\Omega_1, \Omega_2, \Omega_3) \tag{1.4}$$

$$\Omega_1 = \frac{1}{\tau}$$

$$\Omega_2 = 0$$

$$\Omega_3 = \frac{1}{\rho}$$

which is a mathematical fabrication to facilitate operations involving spatial derivatives in the $(\vec{t}, \vec{n}, \vec{b})$ system. Let $\vec{F}$ (t,s) denote a vector which varies along the length of the cable and let its coordinates be F_1, F_2, F_3, i.e.

$$\vec{F} = F_1\vec{t} + F_2\vec{n} + F_3\vec{b} \tag{1.5}$$

Then the derivative of $\vec{F}$ in the $(\vec{t}, \vec{n}, \vec{b})$ system denoted as

$$\left[\frac{D\vec{F}}{Ds}\right]_{(\vec{t}, \vec{n}, \vec{b})}$$

becomes quite complex because in addition to the change of F_1, F_2, F_3 we must account for the rotation of the system $(\vec{t}, \vec{n}, \vec{b})$ along the cable length. To simplify the notation, we denote:

$$\left[\frac{D\vec{F}}{Ds}\right]_{(\vec{t}, \vec{n}, \vec{b})} = \frac{D\vec{F}}{Ds} \tag{1.6}$$

We will then prove that

$$\frac{D\vec{F}}{Ds} = \frac{\partial \vec{F}}{\partial s} + \vec{\Omega} \times \vec{F} \tag{1.7}$$

where

$$\frac{\partial \vec{F}}{\partial s} = \left[\frac{\partial F_1}{\partial s}, \frac{\partial F_2}{\partial s}, \frac{\partial F_3}{\partial s} \right]$$

Proof:

$$\frac{D\vec{F}}{Ds} = \frac{\partial \vec{F}}{\partial s} + F_1 \frac{\partial \vec{t}}{\partial s} + F_2 \frac{\partial \vec{n}}{\partial s} + F_3 \frac{\partial \vec{b}}{\partial s}$$

Using the Frenet relations (1.2) we find that

$$\frac{D\vec{F}}{Ds} = \frac{\partial \vec{F}}{\partial s} - \vec{t} \frac{F_2}{\rho} - \vec{n} \left[\frac{1}{\tau} F_3 - \frac{1}{\rho} F_1 \right] + \vec{b} \frac{F_2}{\tau}$$

$$= \frac{\partial \vec{F}}{\partial s} + \vec{\Omega} \times \vec{F}$$

Changes in time. For the same reason that changes in space become complicated in the $(\vec{t}, \vec{n}, \vec{b})$ system, changes in time must also be reformulated. We simply denote:

$$\left[\frac{D\vec{F}}{Dt} \right]_{(\vec{t}, \vec{n}, \vec{b})} = \frac{D\vec{F}}{Dt}$$

The equivalent of the Darboux vector is now the rotation vector of the classical dynamics:

$$\vec{\omega} = \omega_1 \vec{t} + \omega_2 \vec{n} + \omega_3 \vec{b}$$

and we obtain from dynamics that [Crandall 56]:

$$\frac{D\vec{F}}{Dt} = \frac{\partial \vec{F}}{\partial t} + \vec{\omega} \times \vec{F} \tag{1.8}$$

where

$$\frac{\partial \vec{F}}{\partial t} = \left[\frac{\partial F_1}{\partial t}, \frac{\partial F_2}{\partial t}, \frac{\partial F_3}{\partial t} \right]$$

1.3 Cable Dynamics

We define a velocity of a point on a the cable as:

$$\vec{v} = v_1\vec{t} + v_2\vec{n} + v_3\vec{b} \tag{1.9}$$

Newton's law can be written for an element with unstretched length ds and stretched length dp as:

$$m_o \frac{D\vec{v}}{Dt} ds = \sum_{i=0}^{n} \vec{F}_i dp \tag{1.10}$$

where m_o is the mass per unit length in the instretched coordinate and $\vec{F}$ the force per unit length on the cable.

$$m_o \frac{D\vec{v}}{Dt} = \sum_{i=0}^{n} \vec{F}_i(1+e) \tag{1.11}$$

The equation is rewritten using (1.8):

$$m_o \left[\frac{\partial \vec{v}}{\partial t} + \vec{\omega} \times \vec{v} \right] = \sum_{i=0}^{n} \vec{F}_i(1+e) \tag{1.12}$$

explicitly this is written as;

$$m_o \left[\frac{\partial v_1}{\partial t} - v_2\omega_3 + v_3\omega_2\right] = \sum_{i=0}^{n} F_{ti}(1+e)$$

$$m_o \left[\frac{\partial v_2}{\partial t} - v_3\omega_1 + v_1\omega_3\right] = \sum_{i=0}^{n} F_{ni}(1+e)$$

$$m_o \left[\frac{\partial v_3}{\partial t} - v_1\omega_2 + v_2\omega_1\right] = \sum_{i=0}^{n} F_{bi}(1+e)$$

Equation (1.12) is the equation of motion for a cable expressed in the natural coordinate system.

1.4 Compatibility Relations

Compatibility relations can be formulated directly in terms of displacements, or they can be formulated in terms of velocities. The two formulations are completely equivalent, and both can be used to solve the problem.

We will derive the compatibility relations using displacements. An alternative derivation to find the compatibility relations in terms of velocities can be found in appendix A.

We define $\vec{r}$ as the vector from the origin of a fixed coordinate system to a point on the cable. A reference state is defined as the position of the cable at some arbitrary, fixed time t_o. The vector to the same material point at that time is denoted by $\vec{r}_o$. (see figure 1-2)

Expressing the spatial changes using the Darboux vector of the reference state, we find:

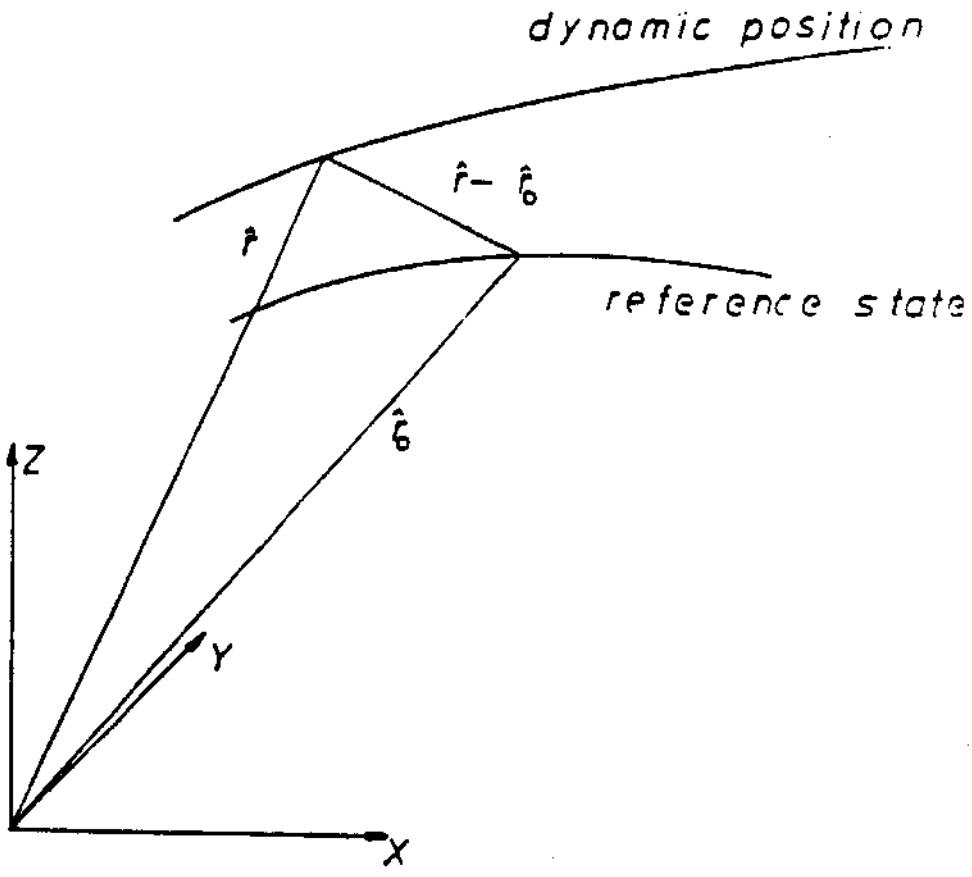

Figure 1-2: Derivation of the Compatibility Relations

$$\frac{D(\vec{r} - \vec{r}_o)}{Ds} = \frac{\partial(\vec{r} - \vec{r}_o)}{\partial s}_{(\vec{t}_o,\vec{n}_o,\vec{b}_o)} + \vec{\Omega}_o \times (\vec{r} - \vec{r}_o)$$

$$\frac{D\vec{r}}{Ds} - \frac{D\vec{r}_o}{Ds} = \frac{\partial(\vec{r} - \vec{r}_o)}{\partial s}_{(\vec{t}_o,\vec{n}_o,\vec{b}_o)} + \vec{\Omega}_o \times (r - r_o) \tag{1.13}$$

According to the definition of the tangential vector [Hildebrand 49]:

$$\vec{t}(t,p) = \frac{D\vec{r}}{Dp}(t,p)$$

$$= \frac{1}{(1+e)} \frac{D\vec{r}}{Ds} \tag{1.14}$$

Therefore, we obtain:

$$(1+e)\vec{t} - (1+e_o)\vec{t}_o = \frac{\partial(\vec{r} - \vec{r}_o)}{\partial s}_{(\vec{t}_o,\vec{n}_o,\vec{b}_o)} + \vec{\Omega}_o \times (\vec{r} - \vec{r}_o) \tag{1.15}$$

(1.15) is the compatibility relations in terms of displacements. To obtain the compatibility relations in terms of velocities, (1.15) is rewritten as:

$$(1+e)\vec{t} - (1+e_o)\vec{t}_o = \frac{D(\vec{r} - \vec{r}_o)}{Ds} \tag{1.16}$$

Taking the time derivative :

$$\frac{D}{Dt}\left[(1+e)\vec{t}\right] = \frac{D\vec{v}}{Ds}$$

$$\frac{D\vec{v}}{Ds} = \frac{\partial e}{\partial t}\vec{t} + (1+e)(\vec{\omega} \times \vec{t}) \tag{1.17}$$

These are the compatibility relations in terms of velocities. In this case, no reference system is involved. Explicity this can be written as:

$$\frac{\partial v_1}{\partial s} - v_2\Omega_3 = \frac{\partial e}{\partial t}$$

$$\frac{\partial v_2}{\partial s} + v_1\Omega_3 - v_3\Omega_1 = (1+e)\omega_3 \tag{1.18}$$

$$\frac{\partial v_3}{\partial s} + v_2\Omega_1 = -(1+e)\omega_2$$

1.5 Relation between the Rotation and the Darboux Vectors

The rotation and the Darboux vectors are related. This can be shown as follows. Equality of the mixed derivatives can be expressed as:

$$\frac{D}{Dt}\left[\frac{D\vec{v}}{Ds}\right] = \frac{D}{Ds}\left[\frac{D\vec{v}}{Dt}\right] \tag{1.19}$$

When expressing (1.19) in terms of the Darboux vector and the rotation vectors, using the formulas for triple vector products, the following equality is obained:

$$\frac{\partial\vec{\Omega}}{\partial t} = \frac{\partial\vec{\omega}}{\partial s} + \vec{\Omega}\times\vec{\omega} \tag{1.20}$$

or explicity

$$\frac{\partial\Omega_1}{\partial t} = \frac{\partial\omega_1}{\partial s} - \Omega_3\omega_2$$

$$0 = \frac{\partial\omega_2}{\partial s} - \Omega_1\omega_3 + \Omega_3\omega_1$$

$$\frac{\partial\Omega_3}{\partial t} = \frac{\partial\omega_3}{\partial s} + \Omega_1\omega_2$$

1.6 Forces Acting on the Cable

The forces acting on the cable are (a) the tension, (b) its weight, (c) external forces. The external forces include fluid related forces such as hydrostatic forces, drag forces and inertia forces.

1.6.1 Weight and Buoyancy forces

As shown in figure 1-3 the segment is in contact with the fluid only at its sides, so that the hydrostatic force is always perpendicular to the cable configuration (i.e., in the $\vec{t}$, $\vec{n}$ plane). It is very convenient to add and subtract the "missing" hydrostatic forces from a cable element (as shown in figure 1-3). Then, by lumping together all the hydrostatic forces pointing towards the interior of the cable element, we obtain Archimede's force in the vertical direction.

$$B = \rho_w \cdot A \cdot dp$$

where A is the (stretched) cable sectional area, while we lump together the tension T (pointing always in the tangential direction towards the exterior of the cable element) and the hydrostatic force acting in the same direction, to create the effective tension T_e.

$$T_e = T + p \cdot A$$

The introduction of the effective tension causes the equations in water to have the same form as the equations in air, except that the weight must be replaced by the net weight and the tension by the effective tension.

If m_o denotes the unstretched mass of the cable per unit length, and m the corresponding stretched quantity, then conservation of mass implies:

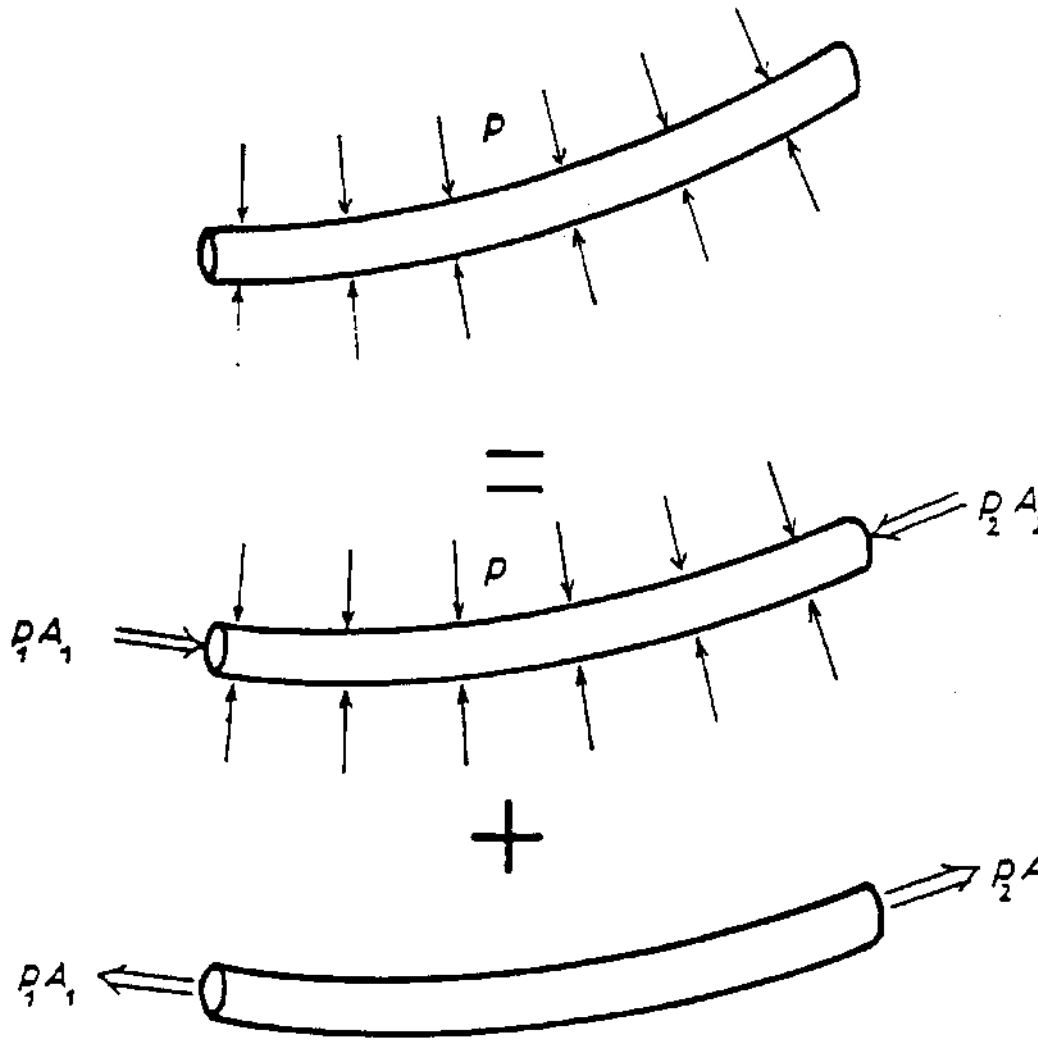

Figure 1-3: Effective Tension

$$m_o \cdot ds = m \cdot dp$$

Since we managed to artificially create a buoyancy force by introducing the effective tension, we can now use the net weight of the cable per unit (stretched) length in water,

$$m \cdot g - \rho_w \cdot g \cdot A$$

We assume for simplicity, following Breslin [Breslin 74] that the volume per unit length of the cable remains unchanged (which implies that Poisson's ratio ν is equal to 1/2) to obtain the net weight per unstretched unit length in water w_o as

$$w_o = g \cdot (m - \rho_w \cdot A_o)$$

where A_o is the unstretched cable sectional area. The corresponding quantity per stretched unit length, w_a, is related to w_o by the relation (based on $\nu = 1/2$):

$$w_o \cdot ds = w_a \cdot dp$$

Therefore

$$\vec{F}_W \cdot dp = - w_o \cdot \vec{k} \cdot ds$$

where $\vec{k}$ is the unit vector in positive direction in a fixed reference system.

1.6.2 Fluid Hydrodynamic Forces

In addition to the hydrostatic force, the cable is subject to a fluid force, which includes a fluid inertia component (related to the added mass m_a) and a viscosity related component (drag force). If a Morison type of loading is used, then the fluid forces are decomposed simply in an added mass force and

a velocity drag force.

The subject of the added mass for cables has attracted some attention in the literature, especially when a current is present. Lighthill [Lighthill 60] showed that for the transverse oscillations of a horizontal slender body whose longitudinal axis is parallel to the current, the added mass force can be approximated as:

$$F_A = - \frac{D}{Dt} \left[m_a \frac{DW}{Dt} \right]$$

where W is the vertical displacement, m_a the added mass per unit length of an infinite cylinder with the same cross section as the local section of the body, and D/DT denotes the substantial derivative, i.e. if U is the current speed:

$$F_A = - \left(\frac{\partial}{\partial t} - U \cdot \frac{\partial}{\partial x} \right) \left\{ m_a(x) \cdot \left(\frac{\partial}{\partial t} - U \cdot \frac{\partial}{\partial x} \right) \cdot W(x,t) \right\} \qquad (1.21)$$

The direction of F_A is vertical. Lighthill showed that this approximation is good, provided the wave length to diameter ratio is larger than 5, and the amplitude of oscillation is small. An extension to a curved inclined configuration must account for possible interactions among the various cable sections and, primarily, for separation effects. Breslin [Breslin 74] used potential theory to derive an expression for the added mass force for a cable, but his final expression, obtained by a strip theory approach is in error and does not reduce to Lighthill's expression. What is actually missing is a double material derivative as shown in (1.21). Breslin's expression contains a material and a regular derivative. In Lenskii [Lenskii 78] an expression to include stretching effects was derived.

It is the author's opinion that separation effects are predominant. As

shown in Allen and Perkins [Allen 51] any inclination above 3° - 5° causes expression (1.21) to fail. It is well known also that vortex shedding has a profound effect on added mass, so that any corrections for current or stretching effects within the frame of the investigators above may not be necessary. Until conclusive experiments are conducted therefore, and with the exception of cables towed along their axis, it is suggested in the present study to use an added mass force per unit length in the direction normal to the cable.

$$\vec{F}_A \, dp = - \, m_a \, \frac{\partial \vec{v}_n}{\partial t} \cdot ds \qquad (1.22)$$

where m_a the two-dimensional mass per unit length of an infinite cylinder with the same cross section as the cable, and under identical flow conditions, and $\vec{v}_n$ the normal velocity between the cylinder and the fluid particles. The value of m_a of course is difficult to find and only partial information is available as in Ramberg and Griffin [Ramberg 77].

To obtain the drag force, we use the separation principle. The drag force on an inclined cylinder is separated into a normal drag component, proportional to the square of the normal relative velocity and a tangential frictional drag component, proportional to the square of the tangential relative velocity. This force can be easily decomposed into a normal, tangential and binormal drag components [Breslin 74].

$$\vec{F}_t dp = -\frac{1}{2}\rho_w C_{Dt}(Re) D_o v_{rt} |v_{rt}| \vec{t}(1+e/2) ds$$

$$\vec{F}_n dp = -\frac{1}{2}\rho_w C_{Dn}(Re) D_o v_{rn} \left[v_{rn}^2 + v_{rb}^2\right]^{\frac{1}{2}} \vec{n}(1+e/2)ds \quad (1.23)$$

$$\vec{F}_b dp = -\frac{1}{2}\rho_w C_{Db}(Re) D_o v_{rb} \left[v_{rb}^2 + v_{rn}^2\right]^{\frac{1}{2}} \vec{b}(1+e/2)ds$$

where: D_o : the diameter of the cable
v_{rt}, v_{rn}, v_{rb} : the component of the relative velocity

Unsteady fluid forces, such as due to waves, vortex shedding and galloping are not considered in the present work.

1.6.3 Tension Force

The force on an element with unstretched length can be written as:

$$\frac{d}{ds}(T_e \vec{t})\, ds = \left[\frac{\partial T_e}{\partial s}\vec{t} + T_e \frac{d\vec{t}}{ds}\right]ds$$

$$= \left[\frac{\partial T_e}{\partial s}\vec{t} + \frac{T_e}{\rho}\vec{n}\right]ds \quad (1.24)$$

1.7 Equation of Motion

Using (1.22), (1.23) and (1.24), the equation of motion (1.10) can be rewritten as:

$$m_o \left[\frac{\partial \vec{v}}{\partial t} + \vec{\omega}\times\vec{v}\right] = \frac{\partial T_e}{\partial s}\vec{t} + \frac{T_e}{\rho}\vec{n}$$

$$- w_o\vec{k} - \left(m_{an}\frac{\partial v_n}{\partial t}\vec{n} + m_{ab}\frac{\partial v_b}{\partial t}\vec{b}\right)$$

$$+ \left(F_t\vec{t} + F_n\vec{n} + F_b\vec{b}\right)\frac{\partial p}{\partial s} \qquad (1.25)$$

The above equation is the complete equation of motion expressed in its natural coordinate system.

1.8 Governing Equations

The complete set of governing equations is obtained by combining (1.17). (1.20) and (1.25).

$$m_o \left[\frac{\partial \vec{v}}{\partial t} + \vec{\omega}\times\vec{v}\right] = \frac{\partial T_e}{\partial s}\vec{t} + T_e\Omega_3\vec{n}$$

$$- w_o\vec{k} - m_{an}\frac{\partial v_n}{\partial t}\vec{n} - m_{ab}\frac{\partial v_b}{\partial t}\vec{b}$$

$$+ \left(F_t\vec{t} + F_n\vec{n} + F_b\vec{b}\right)(1+e)$$

$$\frac{\partial \vec{v}}{\partial s} + \vec{\Omega}\times\vec{v} = \frac{\partial e}{\partial t}\vec{t} + (1+e)\vec{\omega}\times\vec{t} \qquad (1.26)$$

$$\frac{\partial \vec{\Omega}}{\partial t} = \frac{\partial \vec{\omega}}{\partial s} + \vec{\Omega}\times\vec{\omega}$$

This has to be supplemented with a tension-strain relation.

$$T_e = f(e\ ,\ \frac{\partial e}{\partial t}\ ,\ s) \tag{1.27}$$

The above is a set of 10 equations with 10 variables: v_1, v_2, v_3, ω_1, ω_2, ω_3, Ω_1, Ω_3, T_e, e . Given appropriate boundary conditions, a solution can be obtained. In the sequel, we will refer to m_o (the mass per unit unstretched length) as m, dropping the subscript o for convenience.

1.9 Euler Angles

The governing equations are best expressed in terms of Euler angles. Let ϕ, θ, ψ be the Euler angles defining the position of the $\vec{t}$, $\vec{n}$, $\vec{b}$ system relative to the (x,y,z) system. (See figure 1-4)

First we perform a rotation around the y axis by θ, then a rotation around the z_1 axis by ϕ. The x_2 axis is now the new tangential direction and the (y_2,x_2) plane is a vertical plane. Finally we perform a rotation ψ around the x_2 axis, so that finally x_3 corresponds to $\vec{t}$, y_3 corresponds to $\vec{n}$ and z_3 corresponds to $\vec{b}$. It is important to note that the Euler Angles are not unique when the cable element is completely vertical and when this case occurs new angles need to be defined.

It is possible to solve the equations without performing the rotation ψ. The (y_2,x_2) plane is then vertical, while this eliminates one variable ψ out of the equations. The expressions which were derived for the Darboux vector and also for the force component of the tension are not valid, however, and must be reformulated. For more details, see [Cannon 72a], [Cannon 72b] and [Firebaugh 72].

To investigate the properties of three dimensional dynamics, it is

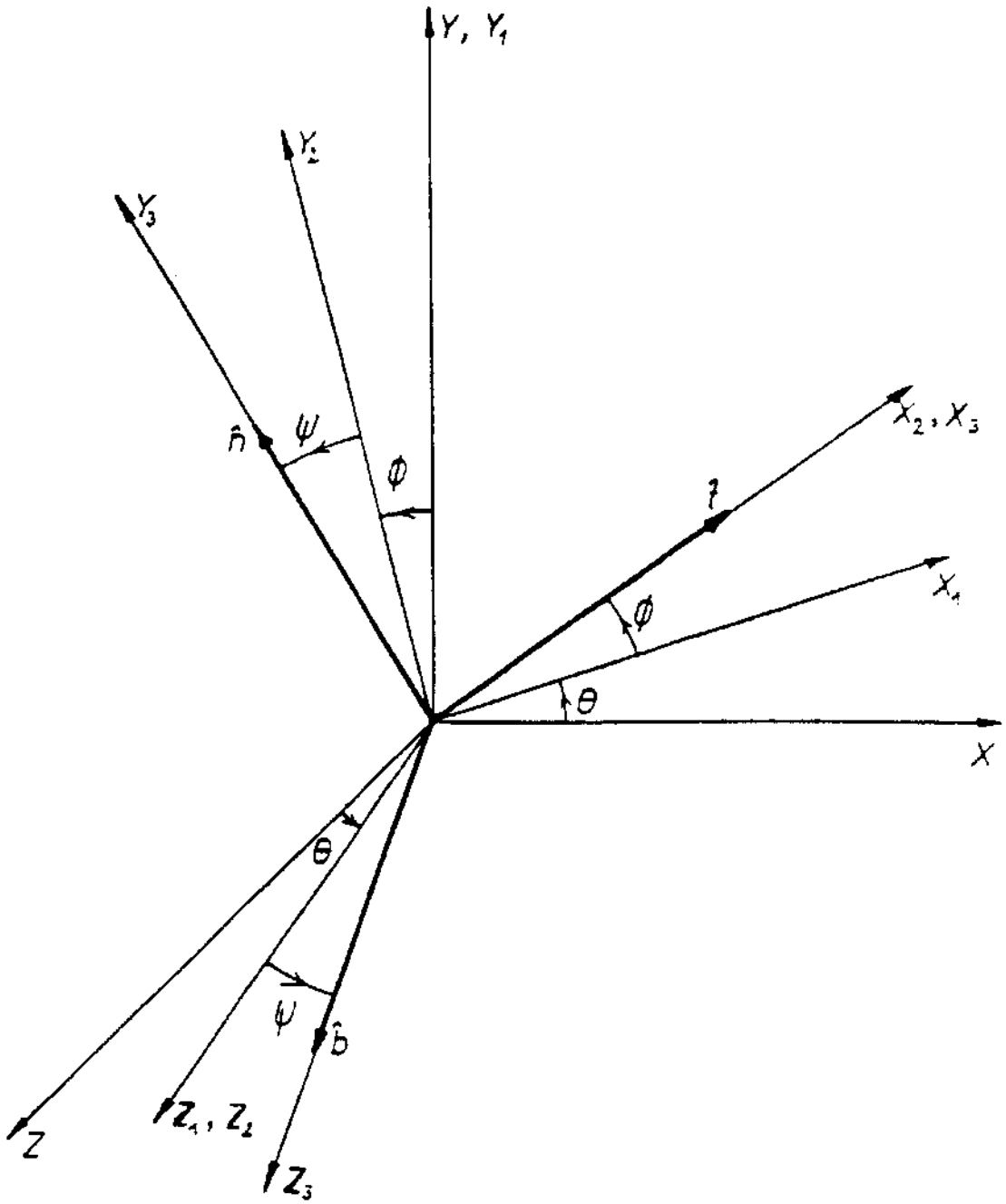

Figure 1-4: Definition of Euler Angles

preferable to use the natural coordinates. The transformation between the two coordinate systems is given by:

$$
\begin{aligned}
x &= \cos\theta \cdot \cos\phi \; x_o + \sin\phi \; y_o - \sin\theta \cdot \cos\phi \; z_o \\
y &= (\sin\psi \cdot \sin\theta - \cos\psi \cdot \sin\phi \cdot \cos\theta) \; x_o \\
&\quad + \cos\psi \cdot \cos\phi \; y_o + (\sin\psi \cdot \cos\theta + \cos\psi \cdot \sin\phi \cdot \sin\theta) \; z_o \\
z &= (\cos\psi \cdot \sin\theta + \sin\psi \cdot \sin\phi \cdot \cos\theta) \; x_o \\
&\quad - \sin\psi \cdot \cos\phi \; y_o + (\cos\psi \cdot \cos\theta - \sin\psi \cdot \sin\phi \cdot \sin\theta) \; z_o
\end{aligned}
\tag{1.28}
$$

where x_o, y_o, z_o are the coordinates in the fixed coordinate system and x,y,z are the coordinates in the natural coordinate system.

We can express the vertical unit vector in terms of the natural vector, using the transpose of (1.28)

$$
\vec{k} = \sin\phi \; \vec{t} + \cos\psi \cdot \cos\phi \; \vec{n} - \cos\phi \cdot \sin\psi \; \vec{b} \tag{1.29}
$$

The equations of motion are obtained as:

$$
m \left[\frac{\partial v_1}{\partial t} + \omega_2 \, v_3 - \omega_3 \, v_2 \right] = \frac{\partial T_e}{\partial s} + F_t(1+e) - w_o \cdot \sin\phi
$$

$$
(m+m_{ab}) \frac{\partial v_2}{\partial t} + m(\omega_3 v_1 - \omega_1 v_3) = \Omega_3 \; T_e + F_n(1+e) - w_o \cos\phi \; \cos\psi \tag{1.30}
$$

$$
(m+m_{ab}) \frac{\partial v_3}{\partial t} + m(\omega_1 v_2 - \omega_2 v_1) = F_b(1+e) + w_o \cos\phi \cdot \sin\psi
$$

The rotation vector expressed in Euler angles gives:

$$\vec{\omega}\, dt = (\sin\phi\ \delta\theta + \delta\psi)\ \vec{t} + (\cos\phi \cdot \cos\psi\ \delta\theta + \sin\psi\ \delta\phi)\ \vec{n} + (-\cos\phi \cdot \sin\psi\ \delta\theta + \cos\psi\ \delta\phi)\ \vec{b} \tag{1.31}$$

In component form we obtain:

$$\omega_1 = \sin\phi \cdot \frac{\partial\theta}{\partial t} + \frac{\partial\psi}{\partial t}$$

$$\omega_2 = \cos\psi \cdot \cos\phi \cdot \frac{\partial\theta}{\partial t} + \sin\psi \cdot \frac{\partial\phi}{\partial t} \tag{1.32}$$

$$\omega_3 = -\sin\psi \cdot \cos\phi \cdot \frac{\partial\theta}{\partial t} + \cos\psi \cdot \frac{\partial\phi}{\partial t}$$

The Darboux vector expressed in terms of Euler angles gives, using Frenet's formulas:

$$\frac{1}{\rho} = \frac{\partial\phi}{\partial s} \cdot \frac{1}{\cos\psi}$$

$$\frac{1}{\tau} = \frac{\partial\psi}{\partial s} - \tan\psi \cdot \tan\phi \cdot \frac{\partial\phi}{\partial s} \tag{1.33}$$

$$0 = \frac{\partial\theta}{\partial s} + \frac{\partial\phi}{\partial s} \cdot \tan\psi \cdot \frac{1}{\cos\phi}$$

1.10 Governing Equations using the Euler Angles

If (1.32) and (1.33) are substituted in the equations of motion and the compatibility relations, the governing equations in terms of Euler angles are obtained.

$$m\left[\frac{\partial v_1}{\partial t} + (\cos\phi \cdot \cos\psi \cdot \frac{\partial\theta}{\partial t} + \sin\psi \cdot \frac{\partial\phi}{\partial t})\, v_3 - (-\cos\phi \cdot \sin\psi \cdot \frac{\partial\theta}{\partial t} + \cos\psi \cdot \frac{\partial\phi}{\partial t})\, v_2\right] = \frac{\partial T_e}{\partial s} + F_t(1+e) - w_o\sin\phi$$

$$(m + m_{an})\frac{\partial v_2}{\partial t} - m(\sin\phi \cdot \frac{\partial\theta}{\partial t} + \frac{\partial\psi}{\partial t})\, v_3 + m(-\cos\phi \cdot \sin\psi \cdot \frac{\partial\theta}{\partial t} + \cos\psi \cdot \frac{\partial\phi}{\partial t})\, v_1 = \frac{\partial\phi}{\partial s} \cdot \frac{1}{\cos\psi} T_e + F_n(1+e) - w_o\cos\phi \cdot \cos\psi$$

$$(m + m_{ab})\frac{\partial v_3}{\partial t} + m(\sin\phi \cdot \frac{\partial\theta}{\partial t} + \frac{\partial\psi}{\partial t})\, v_2 - m(\cos\phi \cdot \cos\psi \cdot \frac{\partial\theta}{\partial t} + \sin\psi \cdot \frac{\partial\phi}{\partial t})\, v_1 = F_b(1+e) + w_o\cos\phi \cdot \sin\psi \qquad (1.34)$$

$$\frac{\partial v_1}{\partial s} - \frac{v_2}{\cos\psi}\frac{\partial\phi}{\partial s} = \frac{\partial e}{\partial t}$$

$$\frac{\partial v_2}{\partial s} - \left[\frac{\partial\psi}{\partial s} - \tan\psi \cdot \tan\phi \cdot \frac{\partial\phi}{\partial s}\right] v_3 + \frac{v_1}{\cos\psi}\frac{\partial\phi}{\partial s} = (1+e)\left[-\cos\phi \cdot \sin\psi \cdot \frac{\partial\theta}{\partial t} + \cos\psi \cdot \frac{\partial\phi}{\partial t}\right]$$

$$\frac{\partial v_3}{\partial s} + \left[\frac{\partial \psi}{\partial s} \cdot \tan\psi \cdot \tan\phi \cdot \frac{\partial \phi}{\partial s}\right] v_2$$

$$= -(1+e)\left[\cos\phi \cdot \cos\psi \cdot \frac{\partial \theta}{\partial t} + \sin\psi \cdot \frac{\partial \phi}{\partial t}\right]$$

$$T_e = f(e)$$

These constitute a set of seven equations with seven unknows. (1.34) are the non-linear governing equations in terms of Euler angles.

1.11 Two Dimensional Non-Linear Governing Equations

The governing equations (1.26) are simplified significantly when the motions are planar. (see figure 1-5)

$$\Omega_1 = \Omega_2 = 0$$

$$\omega_1 = \omega_2 = 0$$

$$v_3 = 0$$

Therefore the equations can be written explicitly as:

$$m\left[\frac{\partial v_1}{\partial t} - v_2\omega_3\right] = \frac{\partial T_e}{\partial s} - w_o \sin\phi + F_{dt}(1+e)$$

$$m\left[\frac{\partial v_2}{\partial t} + v_1\omega_3\right] = T_e \frac{\partial \phi}{\partial s} - w_o \cos\phi + F_{dn}(1+e) - m_a \frac{\partial v_2}{\partial t}$$

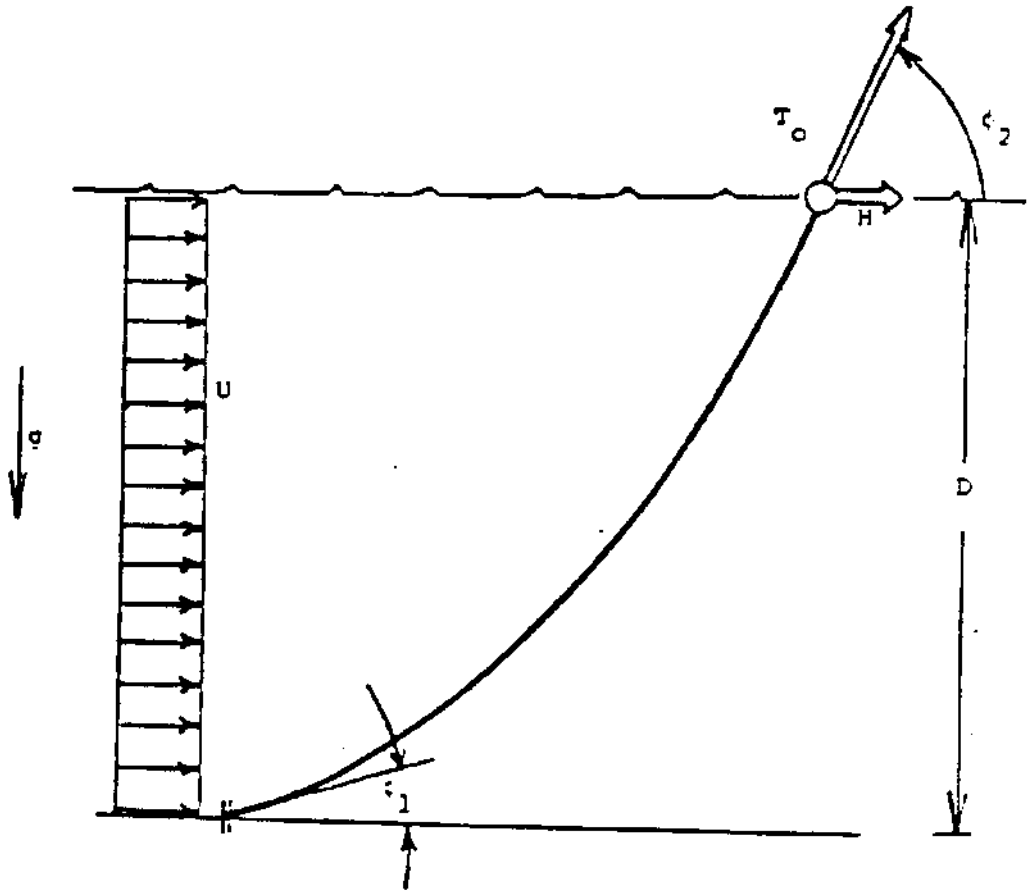

Figure 1-5: Two dimensional Cable Dynamics

$$\frac{\partial v_1}{\partial s} - \frac{v_2}{\rho} = \frac{\partial e}{\partial t} \qquad (1.35)$$

$$\frac{\partial v_2}{\partial s} + \frac{v_1}{\rho} = (1+e)\omega_3$$

$$\frac{\partial 1/\rho}{\partial t} = \frac{\partial \omega_3}{\partial s}$$

where: $F_{dt}(1+e) = \frac{1}{2}\rho_w C_{Dt} D_o (U\cos\phi - v_{rt}) |U\cos\phi - v_{rt}| (1+e/2)$

$$F_{dn}(1+e) = -\frac{1}{2}\rho_w C_{Dn} D_o (U\sin\phi + v_{rn}) |U\sin\phi + v_{rn}| (1+e/2)$$

U = current

In the two dimensional case we have:

$$\frac{1}{\rho} = \phi_s$$

$$\omega_3 = \phi_t$$

Therefore we obtain the final form of the non-linear, two dimensional equations expressed in natural coordinates:

$$m\left[\frac{\partial v_1}{\partial t} - v_2\phi_t\right] = \frac{\partial T_e}{\partial s} - w_o\sin\phi + F_{dt}(1+e)$$

$$m\left[\frac{\partial v_2}{\partial t} + v_1\phi_t\right] = T_e\frac{\partial \phi}{\partial s} - w_o\cos\phi + F_{dn}(1+e) - m_a\frac{\partial v_2}{\partial t}$$

$$\frac{\partial v_1}{\partial s} - \phi_s v_2 = \frac{\partial e}{\partial t} \qquad (1.36)$$

$$\frac{\partial v_2}{\partial s} + \phi_s v_1 = (1+e)\phi_t$$

$$T_e = f(e , \frac{\partial e}{\partial t} , s)$$

(5 equations with 5 unknowns)

1.12 Investigation of the Characteristics of the Governing Equations

The non-linear cable equations can be classified mathematically as a quasi-linear first order system, when the constitutive relation is of the form:

$$T_e = f(e)$$

In the case of a Kelvin visco-elastic damping model, the system is not quasi-linear and the investigation of the properties of the governing equations becomes very difficult. When restricted to the constitutive relation above, the system can be written as[1]:

$$\frac{\partial v_i}{\partial t} + a_{ij} \frac{\partial v_j}{\partial x} + b_i = 0 \qquad (1.37)$$

To investigate whether characteristic lines $x = \xi(t)$ exist, or equivalently whether the system can be written in terms of total derivatives only, linear combinations of the set of equations (1.37) are formed [Whitham 74].

$$l_i \frac{\partial v_i}{\partial t} + l_i a_{ij} \frac{\partial v_j}{\partial x} + b_i l_i = 0 \qquad (1.38)$$

---

[1]The tensor summation convention is used.

We want to rewrite these equations using total derivatives along the characteristic curves.

$$l_i \frac{dv_i}{dt} + b_i l_i = 0 \qquad \text{on} \quad \frac{d\xi}{dt} = c \tag{1.39}$$

This is only possible when:

$$l_i a_{ij} = l_j c$$

This is the condition to determine a characteristic line. To be able to determine completely the system, we must have n different characteristics, so that the condition:

$$|a_{ij} - c| = 0$$

must provide n real eigenvalues.

We restrict our analysis to the two dimensional case. See [Critescu 67] for a derivation in three dimensions and in a Cartesian coordinate system. The quasi-linear form of the non-linear, two dimensional equations is given in the next page.

After some manipulation, the characteristics can be found as:

$$c_{el} = \pm \left[\frac{1}{m} \cdot \frac{\partial f}{\partial e} \right]^{\frac{1}{2}}$$
$$c_{tr} = \pm \left[\frac{T}{M(1+e)} \right]^{\frac{1}{2}} \tag{1.40}$$

Four real and distinct eigenfrequencies are obtained, so the system is hyperbolic. Critescu obtained the same result for three dimensional shapes. The resulting waves can be classified as elastic waves and transverse waves. If the system is considered inextensible, the elastic wave speed goes to infinity and the system can be classified as hyperbolic-parabolic. For a linear-tension

$$\begin{bmatrix} \frac{\partial v_t}{\partial t} \\ \frac{\partial v_n}{\partial t} \\ \frac{\partial \phi}{\partial t} \\ \frac{\partial e}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & \frac{v_n}{1+e} & \frac{v_n v_t}{1+e} & \frac{\partial f}{\partial e}\frac{1}{m} & \frac{\partial v_t}{\partial s} \\ 0 & -\frac{hv_t}{1+e} & \frac{T}{M}\frac{hv_t^2}{1+e} & 0 & \frac{\partial v_n}{\partial s} \\ 0 & \frac{1}{1+e} & \frac{v_t}{1+e} & 0 & \frac{\partial \phi}{\partial s} \\ 1 & 0 & -v_n & 0 & \frac{\partial e}{\partial s} \end{bmatrix} + B$$

where:

$$B = \begin{bmatrix} -\frac{w_o}{m}\sin\phi + \frac{F_{dt}}{m}(1+e) + \frac{1}{m}\frac{\partial f}{\partial s} \\ -\frac{w_o}{M}\cos\phi + \frac{F_{dn}(1+e)}{M} \\ 0 \\ 0 \end{bmatrix}$$

$(M = m + m_a)$

Table 1-I: Quasi-linear Form of the Two Dimensional Equations of Motion

strain relation the elastic wavespeed can be rewritten as:

$$c_{el} = \left[\frac{E}{\rho} \right]^{\frac{1}{2}}$$

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Chapter 2

STATICS AND LINEARISED CABLE DYNAMICS

2.1 Introduction

Static solutions can be obtained from the governing equations derived in chapter 1 by retaining only the time independent terms. For a cable hanging solely under its own weight, the results are the well known catenary equations [Irvine 81]. An important simplification of the dynamic problem is obtained when the solution is assumed to consist of a static part and small oscillations around this mean position. The governing equations can then be separated in non-linear static and linearised dynamic equations.

The static problem, including hydrodynamic loading, has been investigated extensively. For a review see [Casarella 70]. The linearised cable dynamic equations have been derived by Goodman and Breslin [Breslin 74] for two dimensions. Linearised dynamic equations for the case of a towed configuration can be found in [Firebaugh 72].

2.2 Static Equations in Three Dimensions

By setting all dynamic quantities equal to zero in (1.30) and (1.33) we obtain the following set of static equations, with a subscript o to denote static quantities:

$$\frac{\partial T_{eo}}{\partial s} + F_{to}(1 + e_o) \cdot w_o \cdot \sin\phi_o = 0$$

$$\Omega_{30} \cdot T_{eo} + F_{no}(1 + e_o) \cdot w_o \cdot \cos\phi_o \cdot \cos\psi_o = 0$$

$$F_{bo}(1 + e_o) + w_o \cdot \cos\phi_o \cdot \sin\psi_o = 0$$

$$\Omega_{30} = \frac{1}{\cos\psi_o} \cdot \frac{\partial \phi_o}{\partial s} \tag{2.1}$$

$$\Omega_{10} = \frac{\partial \psi_o}{\partial s} \cdot \tan\psi_o \cdot \tan\phi_o \cdot \frac{\partial \phi_o}{\partial s}$$

$$\frac{\partial \theta_o}{\partial s} + \frac{\partial \phi_o}{\partial s} \cdot \tan\psi_o \cdot \frac{1}{\cos\phi_o} = 0$$

where F_{to} , F_{no} , F_{bo} are the components in the ($\vec{t}_o$, $\vec{n}_o$, $\vec{b}_o$) system of the hydrodynamic drag forces. The external forces can be expressed in terms of the Euler angles as:

$$F_{to}(1 + e_o) = -\frac{1}{2} \rho_w C_{dt} D_o v_{tr} \mid v_{tr} \mid (1+e_o/2)$$

$$F_{no}(1 + e_o) = -\frac{1}{2} \rho_w C_{dn} D_o (v_{nr}^2 + v_{br}^2)^{\frac{1}{2}} v_{nr}(1+e_o/2)$$

$$F_{bo}(1 + e_o) = -\frac{1}{2} \rho_w C_{db} D_o (v_{nr}^2 + v_{br}^2)^{\frac{1}{2}} v_{br}(1+e_o/2)$$

where v_{tr} , v_{nr} , v_{br} are, in this case, the components in the ($\vec{t}_o$, $\vec{n}_o$, $\vec{b}_o$) system of the current velocity, which can be expressed as:

$$v_{tr} = - U \cdot \cos(\theta_o - \alpha) \cdot \cos\phi_o$$

$$v_{nr} = - U \left[\sin(\theta_o - \alpha) \cdot \sin\psi - \cos(\theta_o - \alpha) \cdot \sin\phi_o \cdot \cos\psi_o \right]$$

$$v_{nb} = - U \left[\sin(\theta_o - \alpha) \cdot \cos\psi + \cos(\theta_o - \alpha) \cdot \sin\phi_o \cdot \sin\psi_o \right]$$

where α is the angle between the current velocity U and the X axis in the horizontal plane (see figure 2-1).

The above equations can be simplified as:

$$\begin{aligned}
&\frac{\partial T_{eo}}{\partial s} + F_{to}(1 + e_o) - w_o \cdot \sin\phi_o = 0 \\
&\frac{T_{eo}}{\cos\psi_o} \cdot \frac{\partial \phi_o}{\partial s} + F_{no}(1 + e_o) - w_o \cdot \cos\psi_o \cdot \cos\phi_o = 0 \\
&F_{bo}(1 + e_o) + w_o \cdot \sin\psi_o \cdot \cos\phi_o = 0 \\
&\frac{T_{eo} \cos\phi_o}{\sin\psi_o} \cdot \frac{\partial \theta_o}{\partial s} - F_{no}(1 + e_o) + w_o \cdot \cos\psi_o \cdot \cos\phi_o = 0
\end{aligned} \tag{2.2}$$

The following three geometric relations must be added:

$$\begin{aligned}
&\frac{\partial x}{\partial s_o} = (1+e_o) \cos\theta_o \cdot \cos\phi_o \\
&\frac{\partial y}{\partial s_o} = (1+e_o) \cdot \sin\phi_o \\
&\frac{\partial z}{\partial s_o} = - (1+e_o) \sin\theta_o \cdot \cos\phi_o
\end{aligned} \tag{2.3}$$

together with a tension-strain relation:

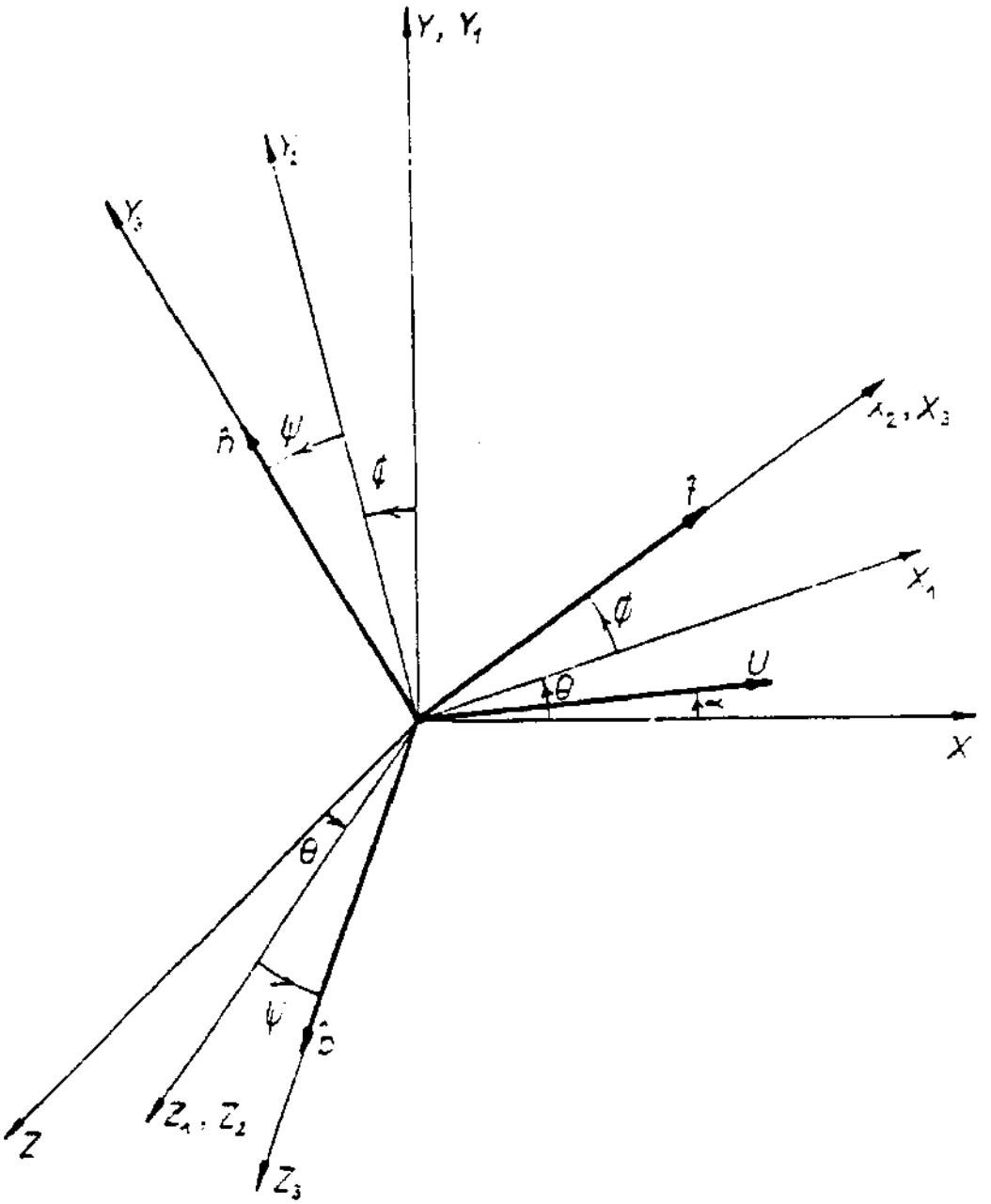

Figure 2-1: Definition of the Current in Euler Angles

$$T_{eo} = E \cdot A \cdot e_o$$

Equations (2.2) and (2.3) constitute the complete set of governing equations for static analysis. The non-linear ordinary differential equations must be supplemented with appropriate initial conditions for numerical integration. In most mooring applications a horizontal force or tension is applied at the top, in a prescribed direction. The prescribed water depth D_w is an integral constraint:

$$D_w = \int_o^l (1+e_o) \sin\phi_o \, ds \tag{2.4}$$

An iterative, shooting method, using the angle ϕ_o, can be used to satisfy this constraint.

In two dimensions the static equations can be written, with $\psi_o = 0$ and $\theta_o = 0$, as[2]:

$$\begin{aligned} \frac{\partial T_{eo}}{\partial s} &= w_o \sin\phi_o - F_{to}(1 + e_o) \\ T_{eo} \frac{\partial \phi_o}{\partial s} &= w_o \cos\phi_o - F_{no}(1 + e_o) \end{aligned} \tag{2.5}$$

with: $F_{to}(1 + e_o) = + 0.5\rho_w C_{Dt} D_o U \cos\phi_o \mid U\cos\phi_o \mid (1+e_o/2)$

$$F_{no}(1 + e_o) = - 0.5\rho_w C_{Dn} D_o U \sin\phi_o \mid U\sin\phi_o \mid (1+e_o/2)$$

The static equations (2.5) are accompanied by the following two geometric relations:

---

[2]The subscript $_o$ indicates static variables.

$$\frac{\partial x}{\partial s} = (1+e_o)\cos\phi_o$$
$$\frac{\partial y}{\partial s} = (1+e_o)\sin\phi_o \qquad (2.6)$$

and a tension-strain relation as in the three-dimensional case.

2.3 Linear Dynamic Equations

Next we derive the equations governing the dynamics of small deviations from the static configuration, i.e. we set:

$$T_e = T_o + T_1$$
$$\phi = \phi_o + \phi_1$$
$$\theta = \theta_o + \theta_1$$
$$\psi = \psi_o + \psi_1$$
$$\Omega_1 = \Omega_{10} + \Omega_{11}$$
$$\Omega_3 = \Omega_{30} + \Omega_{31}$$

The dynamic part[3], T_1 etc., is small compared with the static part, T_o etc. Similarly, the velocities (v_1, v_2, v_3) and rotations (ω_1, ω_2, ω_3) are assumed to be small when properly non-dimensionalized. To linearize the equation of motions we project the equations on the static reference system. Then we proceed to subtract the static equations from the dynamic equations and to simplify the remaining terms by retaining only first order combinations in the dynamic quantities. We obtain, after a lengthy linearisation process,

---

[3]The subscript for the effective tension will be dropped, for convenience, in the sequel.

the equations:

$$m \cdot \frac{\partial v_1}{\partial t} = \frac{\partial T_1}{\partial s} + F_{t1l}$$

$$+ \Omega_{30}\, T_o \cdot (\sin\psi_o \cdot \cos\phi_o \cdot \theta_1 - \cos\psi_o \cdot \phi_1)$$

$$M_2 \cdot \frac{\partial v_2}{\partial t} = \Omega_{30} \cdot T_1 + \Omega_{31} \cdot T_o + F_{n1l}$$

$$+ \frac{\partial T_o}{\partial s} (- \sin\psi_o \cdot \cos\phi_o \cdot \theta_1 + \cos\psi_o \cdot \phi_1) \qquad (2.7)$$

$$M_3 \cdot \frac{\partial v_3}{\partial t} = F_{b1l} + \frac{\partial T_o}{\partial s} (- \cos\psi_o \cdot \cos\phi_o \cdot \theta_1 - \sin\psi_o \cdot \phi_1)$$

$$+ \Omega_{30}\, T_o \cdot (\sin\phi_o \cdot \theta_1 + \psi_1)$$

where $M_2 = m+m_{an}$, $M_3 = m+m_{ab}$ and F_{t1l} , F_{n1l} , F_{b1l} denote the linearisation of the hydrodynamic drag forces $F_t(1+e)$, $F_n(1+e)$, $F_b(1+e)$ projected on the static reference system minus the drag forces on the static shape. (See chapter 6) Similarly, we proceed to linearise the compatibility relations.

$$\frac{\partial v_1}{\partial s} - v_2 \cdot \Omega_{30} = \frac{\partial e_1}{\partial t} = \frac{1}{EA} \cdot \frac{\partial T_1}{\partial t}$$

$$\frac{\partial v_2}{\partial s} + v_1 \cdot \Omega_{30} - v_3 \cdot \Omega_{10} = \omega_3 (1+e_o) \qquad (2.8)$$

$$\frac{\partial v_3}{\partial s} + v_2 \cdot \Omega_{10} = - \omega_2 (1+e_o)$$

The linearisation of the relation between the rotation and the Darboux vectors gives:

$$\frac{\partial \Omega_{11}}{\partial t} = \frac{\partial \omega_1}{\partial s} - \omega_2 \cdot \Omega_{30}$$

$$0 = \frac{\partial \omega_2}{\partial s} + \omega_1 \cdot \Omega_{30} - \omega_3 \cdot \Omega_{10} \tag{2.9}$$

$$\frac{\partial \Omega_{33}}{\partial t} = \frac{\partial \omega_3}{\partial s} + \omega_2 \cdot \Omega_{10}$$

The Darboux vector expressed in Euler angles gives:

$$\Omega_{31} \cdot \cos\psi_o - \Omega_{30} \cdot \sin\psi_o \cdot \psi_1 = \frac{\partial \phi_1}{\partial s}$$

$$\Omega_{11} = \frac{\partial \psi_1}{\partial s} - \left(\frac{\tan\phi_o}{\cos^2 \psi_o} \cdot \psi_1 + \frac{\tan\psi_o}{\cos^2 \phi_o} \cdot \phi_1 \right) \frac{\partial \phi_o}{\partial s} - \tan\psi_o \cdot \tan\phi_o \frac{\partial \phi_1}{\partial s} \tag{2.10}$$

$$0 = \frac{\partial \theta_1}{\partial s} + \frac{\partial \sigma_1}{\partial s} \cdot \tan\psi_o \cdot \frac{1}{\cos\phi_o} + \frac{\partial \phi_o}{\partial s} \cdot \frac{1}{\cos^2\psi_0} \cdot \psi_1 \cdot \frac{1}{\cos\phi_o} + \frac{\partial \phi_o}{\partial s} \tan\psi_o \cdot \frac{\tan\phi_o}{\cos\phi_o} \cdot \phi_1$$

The rotation vector, expressed in Euler angles, gives:

$$\omega_1 = \sin\phi_o \cdot \frac{\partial\theta_1}{\partial t} + \frac{\partial\psi_1}{\partial t}$$

$$\omega_2 = \cos\psi_o \cdot \cos\phi_o \cdot \frac{\partial\theta_1}{\partial t} + \sin\psi_o \cdot \frac{\partial\phi_1}{\partial t} \tag{2.11}$$

$$\omega_3 = -\sin\psi_o \cdot \cos\phi_o \cdot \frac{\partial\theta_1}{\partial t} + \cos\psi_o \cdot \frac{\partial\phi_1}{\partial t}$$

Next we introduce small dynamic displacements (p,q,r) along the static $(\vec{t}_o\ ,\ \vec{n}_o\ ,\ \vec{b}_o)$ directions respectively. To first order the velocities are given as:

$$v_1 = \frac{\partial p}{\partial t} + (\text{H.O.T.})$$

$$v_2 = \frac{\partial q}{\partial t} + (\text{H.O.T.}) \tag{2.12}$$

$$v_3 = \frac{\partial r}{\partial t} + (\text{H.O.T.})$$

We should point out here that relations (2.12) will be modified substantially in the non-linear case, because p, q, r are defined in the direction of the static vectors $\vec{t}_o\ ,\ \vec{n}_o\ ,\ \vec{b}_o$ while the velocities are defined along the dynamic vectors $\vec{t}\ ,\ \vec{n}\ ,\ \vec{b}$.

Next, we observe that all dynamic equations, except for (2.7), can be reduced by one time derivative if we use equations (2.11) to eliminate $(\omega_1\ ,\ \omega_2\ ,\ \omega_3)$. Thus we obtain for the equations of motion:

$$m \cdot \frac{\partial^2 p}{\partial t^2} = \frac{\partial T_1}{\partial s} + F_{t11}$$

$$+ \Omega_{30}\, T_o \cdot (\sin\psi_o \cdot \cos\phi_o \cdot \theta_1 - \cos\psi_o \cdot \phi_1)$$

$$M_2 \cdot \frac{\partial^2 q}{\partial t^2} = \Omega_{30} \cdot T_1 + \Omega_{31} \cdot T_o + F_{n11}$$

$$+ \frac{\partial T_o}{\partial s} (-\sin\psi_o \cdot \cos\phi_o \cdot \theta_1 + \cos\psi_o \cdot \phi_1) \qquad (2.13)$$

$$M_3 \cdot \frac{\partial^2 r}{\partial t^2} = F_{b11} + \frac{\partial T_o}{\partial s} (-\cos\psi_o \cdot \cos\phi_o \cdot \theta_1 - \sin\psi_o \cdot \phi_1)$$

$$+ \Omega_{30}\, T_o \cdot (\sin\phi_o \cdot \theta_1 + \psi_1)$$

The compatibility relations become:

$$\frac{\partial p}{\partial s} - q \cdot \Omega_{30} = \frac{T_1}{E \cdot A}$$

$$\frac{\partial q}{\partial s} + p \cdot \Omega_{30} - r \cdot \Omega_{10} = (-\sin\psi_o \cdot \cos\phi_o \cdot \theta_1 + \cos\psi_o \cdot \phi_1)(1+e_o) \qquad (2.14)$$

$$\frac{\partial r}{\partial s} + q \cdot \Omega_{10} = -(\cos\psi_o \cdot \cos\phi_o \cdot \theta_1 + \sin\psi_o \cdot \phi_1)(1+e_o)$$

where the linearised component of the Darboux vector satisfies relation (2.10).

We have as dynamic variables the quantities p, q, r, ϕ_1, θ_1, ψ_1, Ω_{11}, Ω_{31} and T_1, which must satisfy a set of nine differential equations (2.10), (2.13) and (2.14). The above equations are the complete, linear, three-dimensional equations of motion of a cable about an arbitrary, three-dimensional, static

configuration.

2.4 Three-dimensional Linear Dynamics of a Cable with Two-Dimensional Static Configuration

In the particular case of a cable with a two-dimensional static configuration, the governing equations can be set in a simple form, which can provide a number of important solutions.

When the static configuration is planar then:

$$\psi_o = 0$$

$$\theta_o = \text{const } (= 0 \text{ for convenience})$$

$$\Omega_{10} = 0$$

and equations (2.13) through (2.14) can be simplified as follows:

$$m \cdot \frac{\partial^2 p}{\partial t^2} = \frac{\partial T_1}{\partial s} - T_o \frac{\partial \phi_o}{\partial s} \phi_1 + F_{t11}$$

$$M_2 \cdot \frac{\partial^2 q}{\partial t^2} = \frac{d\phi_o}{ds} T_1 + \frac{\partial \phi_1}{\partial t} T_o + \frac{\partial T_o}{\partial s} \phi_1 + F_{n11} \qquad (2.15)$$

$$M_3 \cdot \frac{\partial^2 r}{\partial t^2} = - \frac{\partial T_o}{\partial s} \cos\phi_o \theta_1 + T_o \frac{\partial \phi_o}{\partial s} (\sin\phi_o \theta_1 + \psi_1) + F_{b11}$$

where we used that in this case

$$\Omega_{30} = \frac{d\phi_o}{ds}$$

$$\Omega_{31} = \frac{\partial \phi_1}{\partial s}$$

with the compatibility relations:

$$\frac{\partial p}{\partial s} - q\,\frac{\partial \phi_o}{\partial s} = \frac{T_1}{E \cdot A}$$

$$\frac{\partial q}{\partial s} + p\,\frac{\partial \phi_o}{\partial s} = \phi_1\,(1+e_o) \qquad (2.16)$$

$$\frac{\partial r}{\partial s} = -\cos\phi_o \cdot \theta_1\,(1+e_o)$$

The equations for the in-plane dynamics (p,q), therefore, including the compatibility relations, are completely decoupled from the equation for the out-of-plane motion r, for the case of a static two-dimensional configuration:

$$m \cdot \frac{\partial^2 p}{\partial t^2} = \frac{\partial T_1}{\partial s} - T_o \cdot \frac{\partial \phi_o}{\partial s} \cdot \phi_1 + F_{t11}$$

$$M_2 \cdot \frac{\partial^2 q}{\partial t^2} = \frac{d\phi_o}{ds} \cdot T_1 + \frac{\partial \phi_1}{\partial s} \cdot T_o + \frac{\partial T_o}{\partial s} \cdot \phi_1 + F_{n11} \qquad (2.17)$$

$$\frac{\partial p}{\partial s} - q \cdot \frac{d\phi_o}{ds} = \frac{T_1}{E \cdot A}$$

$$\frac{\partial q}{\partial s} + p \cdot \frac{d\phi_o}{ds} = \phi_1\,(1+e_o)$$

The solution method for the above equations, for the case of zero current, will be the focus of chapter 3.

2.5 Out-of-Plane Dynamics

From equation (2.10) by setting $\psi_o = 0$, we obtain:

$$\psi_1 = - \left[\frac{\partial \phi_o}{\partial s} \right]^{-1} \cos\phi_o \cdot \frac{\partial \theta_1}{\partial s}$$

Using the third compatiblity relation of (2.16), the equation of motion in the out-of-plane direction becomes:

$$M_3 \cdot \frac{\partial^2 r}{\partial t^2} = \frac{\partial}{\partial s} \left[T_o \cdot \frac{1}{1+e_o} \cdot \frac{\partial r}{\partial s} \right] + F_{b11} \qquad (2.18)$$

which is a string equation with variable tension. The out-of-plane dynamics are completely uncoupled within linear theory, therefore, and their governing equation can be brought to the simple form of (2.18).

2.6 Linearisation of External Forces

The linearisation of the external drag forces is a difficult problem. Not only are the hydrodynamic forces difficult to linearise due to their quadratic form, but they depend also on the local inclination angle. Therefore, their linearisation will provide additional angle dependent terms, which should be included in the equations. (See chapter 6)

2.7 References

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Chapter 3

SOLUTIONS FOR LINEAR DYNAMICS

3.1 Introduction

In this chapter, we will describe in detail the solution for the linearised equations of a uniform, single span cable, with a two dimensional static configuration. To highlight the interesting dynamic properties of cables, we will restrict ourselves to the case where the gravity force is dominant over the static current force, although the methodology can be extended to include a current. This assumption will also permit direct comparison with previous work on linearised cable dynamics.

The emphasis was put on analytical solutions and the physical explanation of the dynamic characteristics. This will hopely lead to a better understanding and improved designs of cable systems which are subject to dynamic loads.

Our primary interest will be in the calculation of the eigenmodes and eigenvalues, which can be used, within the frame of linear systems analysis, to solve any loading problem of the cable.

3.2 Governing Equations

The governing equations were derived in chapter 2. The in-plane and out-of-plane dynamics are uncoupled and can be written, under the assumptions described above, as:

In-plane motion:

$$m \frac{\partial^2 p}{\partial t^2} = \frac{\partial T_1}{\partial s} - w_o \cos\phi_o \, \phi_1$$

$$M \frac{\partial^2 q}{\partial t^2} = \frac{\partial \phi_o}{\partial s} T_1 + \frac{\partial \phi_1}{\partial s} T_o + w_o \sin\phi_o \, \phi_1 \qquad (3.1)$$

$$\frac{\partial p}{\partial s} - q \frac{\partial \phi_o}{\partial s} = e_1$$

$$\frac{\partial q}{\partial s} + p \frac{\partial \phi_o}{\partial s} = \phi_1 (1+e_o)$$

Out-of-plane motion:

$$M \frac{\partial^2 r}{\partial t^2} = \frac{\partial}{\partial s} \left[\left(\frac{\partial \phi_o}{\partial s} \right)^{-1} w_o \frac{\cos\phi_o}{1+e_o} \frac{\partial r}{\partial s} \right] \qquad (3.2)$$

The out-of-plane equation is a simple taut string equation with variable tension along the length.

3.3 Strings

The taut wire is the first cable to attract attention, because it was used for musical instruments. Pythagoras in the 6th century BC and Aristotle in the 3rd century BC knew quantatively the relation between frequency, tension and length of a taut chord. Galileo (1564 - 1642) in 1638, and the monk M. Mersenne (1588 - 1648) in 1636, published qualitative relations based on experimental measurements. R. Hooke (1635 - 1703) and J. Sauveur (1653 - 1716) published also experimental measurements of taut wire frequencies and obeservations of modes for the various harmonics.

B. Taylor (1685 - 1731) in 1713 published the first dynamic solution of the transverse cable dynamics by assuming a response shape. Daniel Bernoulli (1700 - 1782) published in 1738 theorems of oscillations of hanging chains and in 1755 his superposition principle of several harmonics for the taut wire, which was opposed, surprisingly, both by D'Alembert and Euler and remained controversial until 1822, when Fourier illustrated such superpositions. D'Alembert (1736 - 1813) was the first to derive the partial differential equation governing the small amplitude transverse motion of a taut wire and J.L. Lagrange (1736 -1813) solved fully the problem by considering the string as consisting of many (n) interconnected mass particles and then taking the limit as $n \rightarrow \infty$.

Euler (1707 - 1783) derived the equation of a hanging chain and then obtained a series solution and an estimate of the 3 first natural frequencies. S.D. Poisson (1781 - 1840) derived the equation of a cable element subject to an external force in 1820, and used it to derive a final solution to the problems of the hanging chain and of the taut wire.

In the case of a taut wire with zero gravity force, the static angle ϕ_o and the curvature $d\phi_o/ds$ are zero, so we obtain (see figure 3-1):

$$
\begin{aligned}
m \frac{\partial^2 p}{\partial t^2} &= \frac{\partial T_1}{\partial s} \\
M \frac{\partial^2 q}{\partial t^2} &= T_o \frac{\partial \phi_1}{\partial s} \\
\frac{\partial p}{\partial s} &= e_1 \\
\frac{\partial q}{\partial s} &= \phi_1 (1+e_o)
\end{aligned}
\qquad (3.3)
$$

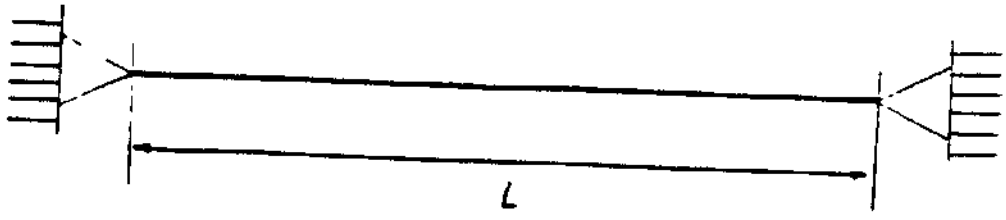


Figure 3-1: Taut String

with $T_1 = E \cdot A_o \cdot e_1$

Assuming a taut wire in air, we obtain:

$$\rho A_o \frac{\partial^2 q}{\partial t^2} = \frac{T_o}{1+e_o} \cdot \frac{\partial^2 q}{\partial s^2}$$

$$\rho \frac{\partial^2 p}{\partial t^2} = E \frac{\partial^2 p}{\partial s^2} \qquad (3.4)$$

Equations (3.4) are the equations for the transverse and for the longitudinal (elastic) dynamics. The two equations are uncoupled and the first equation gives the impression that elasticity plays no role in the transverse dynamics. This is of course erroneous, because an inelastic wire simply can not vibrate (it is geometrically impossible to create waves on an inelastic straight line, because it can not accept even the infinitesimal stretch, which is required to adjust the increased length of a non-straight configuration). So elasticity is not important quantitatively, but very important qualitatively, and

any solution that assumes the elastic stiffness as infinite is bound to fail to reproduce the taut wire results.

The natural frequencies for a taut wire of length L are obtained by taking the Fourier transform of (3.4).[4]

$$
\begin{aligned}
&- \rho A_o \, \omega^2 \, q = \frac{T_o}{1+e_o} \frac{d^2 q}{ds^2} \\
&- \rho \, \omega^2 \, p = E \frac{d^2 p}{ds^2}
\end{aligned}
\qquad (3.5)
$$

with: $q(0) = q(L) = 0$

$p(0) = p(L) = 0$

and since the general solution is:

$$q(s) = c_1 \sin(k_1 s) + c_2 \cos(k_1 s)$$

$$p(s) = c_3 \sin(k_2 s) + c_4 \cos(k_2 s)$$

with: $$k_1 = \frac{\omega}{\left[T/\rho A_o(1+e_o)\right]^{1/2}}$$

$$k_2 = \frac{\omega}{\left[E/\rho\right]^{1/2}}$$

---

[4]When an equation is written in the sequel in the frequency domain, the dynamic variables represent the complex amplitude of the dynamic quantities.

we obtain:

$$\omega_n = \frac{n\pi}{L}\left[\frac{T}{\rho A^{\circ}(1+e_o)}\right]^{1/2} \qquad \text{transverse, } n=1,2,3...$$

$$\omega_m = \frac{m\pi}{L}\left[\frac{E}{\rho}\right]^{1/2} \qquad \text{elastic, } m=1,2,3...$$

3.4 Hanging Chains

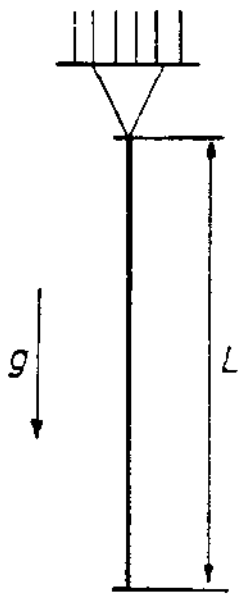


Figure 3-2: Hanging Chain

The freely hanging cable in air (see figure 3-2) has also been studied

extensively, first by, Poisson who derived the governing equations. Within the present framework, we have $\phi_o = \pi/2$, $\frac{d\phi_o}{ds} = 0$, $T_o = w_o s$:

$$m \frac{\partial^2 q}{\partial t^2} = \frac{\partial}{\partial s} \left[\frac{w_o s}{1+e_o} \frac{\partial q}{\partial s} \right]$$

$$\rho \frac{\partial^2 p}{\partial t^2} = E \frac{\partial^2 p}{\partial s^2} \tag{3.6}$$

The same elastic solution is found, therefore, in the tangential direction. In the normal direction, after making the assumption that the cable extension caused by its own weight is negligible, we obtain in air the equation:

$$- \omega^2 q = \frac{d}{ds} \left[gs \frac{dq}{ds} \right] \tag{3.7}$$

and the solution is:

$$q(s) = c_1 J_o \left(2\omega \left[\frac{s}{g} \right]^{1/2} \right) + c_2 Y_o \left(2\omega \left[\frac{s}{g} \right]^{1/2} \right) \tag{3.8}$$

where J_o is the Bessel function of the first kind and order zero, and Y_o the Bessel function of the second kind and order zero. In order for the solution to remain finite at $s = 0$, the second term must vanish since Y_o is infinite at $s = 0$, so the natural frequencies are obtained simply as roots of

$$J_o \left(2\omega \left[\frac{L}{g} \right]^{1/2} \right) = 0 \tag{3.9}$$

The first five roots can be found as [Hildebrand 49]

n	1	2	3	4	5
$\omega\left[L/g\right]^{1/2}$	1.204	2.760	4.327	5.896	7.465

In the case of a vertical chain with non zero lower end tension, the

equation for the transverse dynamics becomes, after neglecting the effect of the static strain:

$$m \frac{\partial^2 q}{\partial t^2} = \frac{\partial}{\partial s} \left[(T_o + w_o s) \frac{\partial q}{\partial s} \right] \tag{3.10}$$

The static tension is lineary varying from the bottom to the top. To find the eigenfrequencies, we have to solve the following eigenvalue problem:

$$- \omega^2 q = \frac{d}{ds} \left[(\frac{T_o}{m} + gs) \frac{dq}{ds} \right] \tag{3.11}$$

with boundary condition $q(0) = q(L) = 0$

The solution of known problem is also expressible in terms of zeroth order Bessel functions:

If we denote: $\gamma = \dfrac{2(m)^{1/2}}{w_o}$

then: $q(s) = c_1 J_o \left[\gamma\omega (w_o s + T_o)^{1/2} \right]$

$$+ c_2 y_o \left[\gamma\omega (w_o s + T_o)^{1/2} \right] \tag{3.12}$$

3.5 WKB Solution for Strings with Variable Tension

The transverse equation for strings with variable tension is given by:

$$M(s) \frac{\partial^2 q}{\partial t^2} = \frac{\partial}{\partial s} \left[T(s) \frac{\partial q}{\partial s} \right] \tag{3.13}$$

Taking the Fourier Transform and non-dimensionalising the length variable $\sigma = s/L$, we obtain:

$$\frac{d}{d\sigma}\left[T(\sigma) \frac{dq}{d\sigma} \right] + L^2 M(\sigma) \omega^2 q = 0 \qquad (3.14)$$

This equation is suitable to be solved using asymptotic analysis, if we assume that:

- $T(\sigma)$ and $M(\sigma)$ are slowly varying (compared with the dynamic solution) functions of σ

- or the non-dimensional frequency $\bar{\omega}^2$ is large. $\bar{\omega}^2 = M\omega^2 L^2/T$

It is important to note that if one or both of the above conditions are met, the asymptotic analysis will be valid. We perform a transformation of variables suggested by Nayfeh [Nayfeh 73]

$$\xi = \left[T \right]^{1/2} q$$

After substitution and expansion of the derivatives, we obtain:

$$\ddot{\xi} + \left[-\frac{1}{2}\frac{\ddot{T}}{T} + \frac{1}{4}\frac{\dot{T}^2}{T^2} + \frac{\omega^2 L^2 M}{T} \right] \xi = 0 \qquad (3.15)$$

If the conditions formulated above are met, we can try to find a WKB[5] expansion of the form:

$$\xi = e^{\omega g_0 + g_1 + g_2/\omega + g_3/\omega^2} \qquad (3.16)$$

where the coefficients are determined from substitution in (3.15). The leading order approximation is:

$$\xi = \frac{1}{(M/T)^{1/4}} \exp\{ \pm i\omega L \int_0^{\sigma} (M/T)^{1/2} d\sigma \} \qquad (3.17)$$

---

[5]Expansion named after Wentzel, Kramers and Brillouin

So the following solution for the transverse displacement is obtained:

$$q = \frac{1}{(MT)^{1/4}} \exp\{ \pm i\omega L \int_0^{\sigma} (M/T)^{1/2}\, d\sigma\} \tag{3.18}$$

(for a higher order approximation, see appendix B.)

Example: WKB solution for a string with linearly varying tension.

The solution is given as:

$$q = \frac{1}{\left[M(w_o L\sigma + T_o)\right]^{1/4}} \exp\left(\pm i\, 2\omega(M^{1/2}/w_o)\left[(w_o L\sigma + T_o)^{1/2} - T_o^{1/2}\right] \right) \tag{3.19}$$

where σ is the non-dimensional length.

This example is used in part II, to find the accuracy of the WKB method versus the exact solution. The above solution is increasingly more accurate as the tension variation per wavelength becomes smaller.

The solution breaks down when the lower end tension becomes zero as can be seen in (3.19). The governing equation has a singular turning point. The solution was first formulated by Langer and can be found in [Nayfeh 73] as follows:

We assume an expansion for the tension as:

$$T = a\sigma + b\sigma^2$$

The solution can be written as:

$$q(s) = \frac{(W)^{1/2}}{(TM)^{1/4}} C_o(\omega\, L\, W) \tag{3.20}$$

where: $W = \int_0^{\sigma} \frac{d\sigma}{(T/M)^{1/2}}$

C_o are the cylinder functions of zeroth order (J_o , Y_o). In the case of a

freely hanging chain, the solution (3.20) is identical with the exact solution. The WKB solution can be found from (3.20) or from the exact solution (3.8) using the large argument asymptotic approximation for the Bessel functions.

It is interesting to note that for constant tension, the WKB solution is the exact solution, and for linear varying tension, the singular turning point analysis gives the exact solution. In most cable analyses the tension variation between top and bottom is not very large, so that a regular WKB solution seems more than adequate to solve equations of the type (3.13)

Although the WKB method is not valid for a first order pole of the equation (3.15), the WKB method remains valid fore a second order pole of (3.15). This means that for static tension of quadratic dependence in the space coordinate, the WKB solution remains valid, even close to the singularity. This relatively unknown feature of the WKB solution is particular important in very slack cables, where the tension at the point of lowest tension varies quadratically.

This can be proved as follows: we consider a quadratic tension variation $T = b\sigma^2$. The first order WKB solution is obtained from:

$$\ddot{\xi} + \frac{\omega^2 L^2 M}{b\sigma^2} \cdot \xi = 0 \tag{3.21}$$

The solution is:

$$\xi = \frac{1}{(M/T)^{1/4}} \exp \{ \pm i\omega L \int_0^{\sigma} [(M/T)^{1/2}] ds \} \tag{3.22}$$

Expansion of (3.22) for small values of σ, assuming $\omega L(M/b)^{1/2}$ large, gives:

$$\xi \sim \sigma \exp \{1/2 \pm i\omega L (M/b)^{1/2}\} \tag{3.23}$$

If we perform local analysis of (3.21) directly by assuming a solution x^{ν} near $\sigma = 0$, we obtain again as solution:

$$\xi \sim \sigma \exp \{1/2 \pm i\omega L (M/b)^{1/2}\} \tag{3.24}$$

i.e., the WKB method has the correct asymptotic behavior for small σ.

3.6 Inextensible Cables

Next we consider the dynamic behavior of a uniform inextensible chain in the plane of its equilibrium configuration. The oscillations are assumed to be small, so that the linearised equations can be used. The governing equations (3.1) can now be written as:

$$\begin{aligned} m \frac{\partial^2 p}{\partial t^2} &= \frac{\partial T_1}{\partial s} - w_o \cos\phi_o \, \phi_1 \\ M \frac{\partial^2 q}{\partial t^2} &= \frac{\partial \phi_o}{\partial s} T_1 + \frac{\partial \phi_1}{\partial s} T_o + w_o \sin\phi_o \, \phi_1 \\ \frac{\partial p}{\partial s} - q \frac{\partial \phi_o}{\partial s} &= 0 \\ \frac{\partial q}{\partial s} + p \frac{\partial \phi_o}{\partial s} &= \phi_1 \end{aligned} \tag{3.25}$$

where m and M are assumed to be constant.

The previous equations were obtained from (3.1) using the assumption that the strain is zero. Note, however, that the dynamic tension is not equal to zero, i.e. a motion of the cable can still generate tension, although it does not generate strain. We write the displacements p and q in non-dimensional

form as ξ , η . The variable σ now denotes the non-dimensional length. Taking the Fourier transform of (3.25):

$$
\begin{aligned}
&- mL^2\,\omega^2\,\xi = \frac{dT_1}{d\sigma} - w_o\,L\cos\phi_o\,\phi_1 \\
&- ML^2\,\omega^2\,\eta = T_1\frac{d\phi_o}{d\sigma} + T_o\frac{d\phi_1}{d\sigma} + w_o\,L\sin\phi_o\,\phi_1 \\
&\frac{d\xi}{d\sigma} - \eta\frac{d\phi_o}{d\sigma} = 0 \\
&\frac{d\eta}{d\sigma} + \xi\frac{d\phi_o}{d\sigma} = \phi_1
\end{aligned}
\qquad (3.26)
$$

The study of the eigenfrequencies and modes of an inextensible chain hanging between two points at the same level was initiated by Rohrs in 1851 [Rohrs 51]. Routh studied the exact solution for a chain hanging in the form of a cycloid [Routh 55]. He obtained his cycloidal shape by taking a cable with non-uniform mass. For this case, the dynamic equations can be solved exactly. For the small sag case his results are reduced to the results of Rohrs who used a priori the small sag assumption for his analysis.

Pugsley [Pugsley 49] derived semi-empirical formulas to predict the eigenfrequencies for a uniform chain. An approximate solution for the linearised chain problem was derived by Saxon and Cahn [Saxon 53]. Their results, based on a second order perturbation method, predict accurately the eigenfrequencies and eigenmodes for relatively flat chains, or for higher order modes. Goodey [Goodey 61] arrived at similar results starting from the intrinsic form of the chain problem.

This analysis will follow closely the one used by Saxon and Cahn. A

more careful perturbation analysis of the problem reveals however that a revised first order perturbation scheme is able to predict the eigenfrequencies and eigenmodes far beyond the validity of the second order solution derived by Saxon and Cahn.

The effect of added mass is directly included in the derivation. The derived solutions can be used equally well to predict eigenmodes and eigenfrequencies of inclined inextensible chains, or to calculate the response to an imposed excitation at one end.

3.6.1 Statics

Some basic equations from the static analysis are needed in our derivation. For a catenary differential element equilibrium of the forces gives: (The following relations are valid for extensible cables as well, if the weight is the only external force.)

$$T_o \frac{d\phi_o}{d\sigma} = w_o L \cos\phi_o$$

$$\frac{dT_o}{d\sigma} = w_o L \sin\phi_o \qquad (3.27)$$

Then the following basic static relations are obtained from (3.27):

$$T_o = \frac{H}{\cos\phi_o}$$

$$\frac{d\phi_o}{d\sigma} = \alpha \cos^2\phi_o \qquad (3.28)$$

H is the horizontal tension in the chain and α ($=w_oL/H$) is the ratio of the total weight of the catenary in the appropriate fluid and the horizontal

tension.

The ratio α and the value of the static angle at some specific point along the cable define completely the static shape of the chain hanging under its own weight. As it will be seen in the sequel, they define also, together with the added mass, the dynamic behavior. The following useful relation exists between the top angle ϕ_t, the bottom angle ϕ_b and α:

$$\tan\phi_t = \tan\phi_b + \alpha$$

For a chain which is tangential to the bottom:

$$\tan\phi_t = \alpha$$

while for the case of a horizontal chain:

$$\phi_t = -\phi_b$$

so:

$$2\ \tan\phi_b = \alpha.$$

3.6.2 Horizontal, Small Sag, Inextensible Cable

Rohrs derived the dynamic solution for a shallow sag, inextensible cable, which we will re-derive here to illustrate the various concepts involved.

The case of an inextensible cable, hanging between two points of equal height is considered. A Lagrangian coordinate σ, with the origin at the center of the cable is introduced. (See figure 3-3). When the cable is assumed to be shallow (sag over span less than 1/8) the following approximations are valid.

$$T_o = \frac{H}{\cos\phi_o} \cong H$$

$$\frac{d\phi_o}{d\sigma} = \alpha\ \cos^2\phi_o \cong \alpha$$

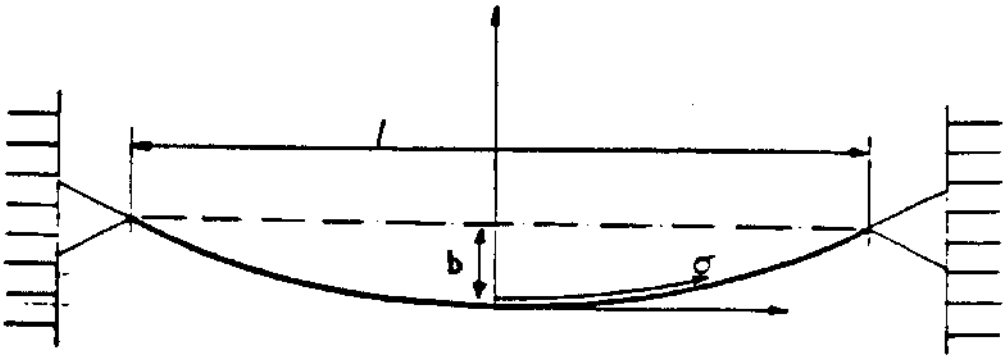


Figure 3-3: Shallow Sag, Horizontal Cable

l: Span of the cable
L: Length of the cable
H: Horizontal tension
b: sag

$$\phi_o = \alpha\ \sigma \tag{3.29}$$

$$\frac{b}{L} = \frac{\alpha}{8}$$

$$l = L\left[\ 1 - \frac{1}{24}\ (\alpha^2\ )\right]$$

where: b = sag
l = span
$\alpha = w_o L/H$
H = horizontal tension
L = length of the cable

From the above equations, we note that for shallow sag cables:

$$\alpha = \frac{w_o L}{H} << 1$$

Rohrs derived solutions for the eigenfrequencies and mode shapes. Starting with the general inextensible governing equations:

$$-\ mL^2\ \omega^2\ \xi = \frac{dT_1}{d\sigma} - w_o\ L\ \cos\phi_o\ \phi_1$$

$$-\ ML^2\ \omega^2\ \eta = T_1\ \frac{d\phi_o}{d\sigma} + T_o\ \frac{d\phi_1}{d\sigma} + w_o\ L\ \sin\phi_o\ \phi_1 \tag{3.30}$$

$$\frac{d\xi}{d\sigma} - \eta\ \frac{d\phi_o}{d\sigma} = 0$$

$$\frac{d\eta}{d\sigma} + \xi\ \frac{d\phi_o}{d\sigma} = \phi_1$$

Note that the non-dimensional natural frequencies will be of the same order of magnitude as those of a taut string with equivalent mass and average tension. This means that:

$$\omega_n^2 \frac{ML^2}{H} = O\ (n^2\ \pi^2\ ) \qquad\qquad n=1,2,3...$$

is a number much larger than 1 even for the first eigenfrequencies. Therefore we can conclude that:

$$\omega_n^2 \frac{ML^2}{H} >> \frac{w_o L}{H} \tag{3.31}$$

From the third relation of (3.30), we note that:

$$\frac{d\xi}{d\sigma} = \eta \frac{d\phi_o}{d\sigma} \tag{3.32}$$

so that the rate of change for ξ is small, of order α. Our main interest is in motions for which the boundary conditions are fixed (or at least at one end is fixed). If we assume that η is of the same order as ϵ, then we can conclude that:

$$\xi = O\ (\epsilon^2\ )$$

Using the last relation of (3.25), the order of ϕ_1 can be determined as:

$$\phi_1 = O\ (\epsilon)$$

We obtain from the equation for tangential dynamics, using (3.29):

$$T_1 = T_{1o} + \int_0^\sigma (-\ m\omega^2\ L^2\ \xi + w_o L \phi_1\ )\ ds \tag{3.33}$$

where T_{1o} is an integration constant.

This can be substituted in the relation for the normal dynamics, using (3.29):

$$M\omega^2\ L^2\ \eta = \alpha \left[\ T_{1o} + \int_0^\sigma (-m\omega^2\ L^2\ \xi + w_o L \phi_1\ )\ ds\ \right]$$

$$+ H \frac{d\phi_1}{d\sigma} + w_o L \alpha \sigma \phi_1 \tag{3.34}$$

To leading order:

$$- M\omega^2 L^2 \eta = \alpha T_{1o} + H \frac{d^2\eta}{ds^2} + O(\epsilon^2) \tag{3.35}$$

which gives as solution:

$$\eta(\sigma) = c_1 \sin(kL\sigma) + c_2 \cos(kL\sigma) - \frac{\alpha T_{1o}}{M\omega^2 L^2} \tag{3.36}$$

with: $k = \omega \left[\frac{M}{H} \right]^{1/2}$

To find the eigenfrequencies, we impose fixed boundary conditions:

$$\eta(-\frac{1}{2}) = \eta(\frac{1}{2}) = 0$$

Anti-symmetric modes of (3.36) are only possible if:

$$c_2 = 0$$

$$T_{1o} = 0$$

$$\frac{kL}{2} = n\pi \tag{3.37}$$

These are the odd modes with respect to the middle point of the cable and are completely equivalent to the corresponding string modes. Note also that no additional tension is generated.

The even, or symmetric modes (in the transverse natural displacement) are obtained as:

$$\eta(\sigma) = \frac{\alpha T_{1o}}{M\omega^2 L^2} \cdot \left[\frac{\cos kL\sigma}{\cos kL/2} - 1 \right] \tag{3.38}$$

Apparently no eigenfrequencies requirement can be derived from (3.38).

However, we have also to consider the boundary conditions in the tangential direction. Expressing the tangential motion as a function of the transverse solution, gives:

$$\xi(\sigma) = \frac{a^2\, T_{1o}}{M\omega^2\, L^2} \cdot \left[\frac{1}{kL} \cdot \frac{\sin kL\sigma}{\cos kL/2} - \sigma \right] + C \qquad (3.39)$$

The boundary conditions in this case are

$$\xi(1/2) = \xi(-1/2) = 0$$

They can only be satisfied if the integral $C = 0$ and:

$$\tan(kL/2) = kL/2$$

It is interesting to note that in this case a dynamic tension is generated in the cable. The first symmetric eigenfrequency is located at 2.86 π, compared to π for a string. This is due to the fact that the first symmetric mode of a string is possible only if stretching is allowed, so for an inelastic chain the geometric compatibility relations are modifying significantly the modal shapes.

Table 3-1 gives the eigenfrequencies for a string and an inextensible cable. Figure 3-4 shows a comparison between their mode shapes. Hopefully this example illustrates the fundamental difference between cables and strings. We will now discuss the general asymptotic solution for inextensible cables, without making the simplifying small sag assumption.

3.6.3 Derivation of the Governing Equation in the form of a Fourth Order Differential Equation

The linearised equations of motion of an inextensible chain can be written

Anti-symmetric Modes

	n	1	2	3	4
Chain	$\bar{\omega}_n$	2π	4π	6π	8π
String					

Symmetric Modes

	n	1	2	3	4
Chain	$\bar{\omega}_n$	2.86π	4.92π	6.94π	8.95π
String		π	3π	5π	7π

$$\text{where: } \bar{\omega}_n = \omega_n \, L \left[\frac{M}{H} \right]^{1/2}$$

Table 3-I: Comparison between the Natural Frequencies of a String and of an Inextensible Small Sag Cable

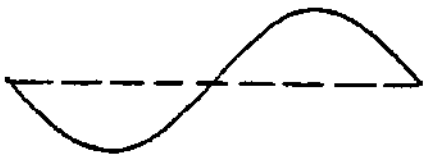
String | Inextensible Cable

Symmetric

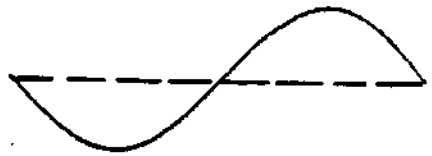
$\bar{\omega} = \pi$

does not exist

Anti-Symmetric

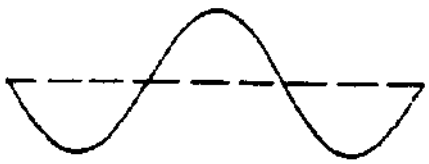


$\bar{\omega} = 2\pi$



$\bar{\omega} = 2\pi$

Symmetric



$\bar{\omega} = 3\pi$

$\bar{\omega} = 2.86\pi$

Figure 3-4: Comparison Mode Shapes of a String and of an Inextensible, Small Sag Cable

as:

$$- m\,\omega^2\,\xi\,L^2 = \frac{dT_1}{d\sigma} - w_o\,L\cos\phi_o\,\phi_1$$

$$- M\,\omega^2\,\eta\,L^2 = T_1\,\frac{d\phi_o}{d\sigma} + T_o\,\frac{d\phi_1}{d\sigma} + w_o\,L\sin\phi_o\,\phi_1 \qquad (3.40)$$

By defining a new dynamic angle $\phi_1^* = \phi_1/\alpha$ and a new non-dimensional dynamic tension $T_1^* = T_1/w_oL$, and by using the static relations (3.28), and the relation $d\phi_o/ds = \alpha\cos^2\phi_o$ to change the independent variable from σ to ϕ_o, the linearised equations can be written as:

$$- \frac{m\omega^2\,H}{w_o^2}\,\xi = \cos^2\frac{dT_1^*}{d\phi_o} - \cos\phi_o\,\phi_1^*$$

$$- \frac{M\omega^2\,H}{w_o^2}\,\eta = \cos^2\phi_o\,T_1^* + \cos\phi_o\,\frac{d\phi_1^*}{d\phi_o} + \sin\phi_o\,\phi_1^* \qquad (3.41)$$

We introduce two new parameters: h (=m/M), which is the ratio of the mass over the mass plus added mass, and λ^2 (=$M\omega^2\,H/w_o^2$), which is the non-dimensional frequency parameter.

After dropping the superscripts, the equations of motion can be written as:

$$- h\,\lambda^2\,\xi = \cos^2\phi_o\,\frac{dT_1}{d\phi_o} - \cos\phi_o\,\phi_1$$

$$- \lambda^2\,\eta = \cos^2\phi_o\,T_1 + \cos\phi_o\,\frac{d\phi_1}{d\phi_o} + \sin\phi_o\,\phi_1 \qquad (3.42)$$

The non-dimensional frequency parameter λ is fundamental for the perturbation analysis. It can be written as:

$$\lambda = \frac{\omega\,c_H}{h\,g\,(1-\rho fluid/\rho mat)} \qquad (3.43)$$

The parameter λ is large for high frequencies and for large horizontal wave speeds (c_H).

The compatibility relations are given by:

$$\frac{d\xi}{d\sigma} - \eta \frac{d\phi_o}{d\sigma} = 0$$

$$\frac{d\eta}{d\sigma} + \xi \frac{d\phi_o}{d\sigma} = \phi_1^* \, a \tag{3.44}$$

These equations can be rewritten (dropping the superscript) as:

$$\frac{d\xi}{d\phi_o} = \eta$$

$$\frac{d\eta}{d\phi_o} + \xi = \frac{\phi_1}{\cos^2\phi_o} \tag{3.45}$$

Equations (3.42) and (3.45) are the constitutive equations for the dynamics of an inextensible chain. A change of independent variable has been performed. The static angle ϕ_o is now the independent variable instead of the length scale. This results in a simplification of the equations, so that η , ϕ_1 , T_1 can be eliminated and a fourth order differential equation in ξ is obtained:

$$\cos\phi_o \, \xi^{IV} - 2 \sin\phi_o \, \xi^{III} + \left[\cos\phi_o + \frac{\lambda^2}{\cos^2\phi_o} \right] \xi^{II}$$

$$- 2 \left[\sin\phi_o - \frac{\sin\phi_o \lambda^2}{\cos^3\phi_o} \right] \xi^{I} - \frac{h \, \lambda^2}{\cos^2\phi_o} \xi = 0 \tag{3.46}$$

The superscripts denote differentiation with respect to ϕ_o. If $h = 1$, the original equation derived by Saxon and Cahn is recovered [Saxon 53].

The prescribed boundary conditions for mooring line applications are

normally:

$$\xi(\phi_1) = 0 \qquad \xi(\phi_2) = \text{given}$$

$$\eta(\phi_1) = \xi_{\phi_o}(\phi_1) = 0 \qquad \eta(\phi_2) = \xi_{\phi_o}(\phi_2) = \text{given}$$

Other boundary conditions can also be imposed. The eigenmodes and eigenfrequencies are found by imposing homogeous boundary conditions at both ends and searching for non-zero solutions.

3.6.4 Self-Adjoint Form

The constitutive equation is in self-adjoint form and the fourth order differential equation can be written as:

$$\left[\cos\phi_o \, \xi^{II} \right]^{II} + \left[\left(2\cos\phi_o + \frac{\lambda^2}{\cos^2\phi_o} \right)\xi^{I} \right]^{I} - \frac{h\lambda^2}{\cos^2\phi_o}\,\xi = 0 \qquad (3.47)$$

The above equation reduces to (3.46) by expanding the derivatives. Surprisingly, this fact seems to have escaped the attention of previous researchers.

The above equation describes completely the linear dynamic behavior of an inextensible cable under its own weight. The four governing equations can be reduced to a single fourth order differential equation, only for inextensible cables. The independent variable in the above equation is the static angle ϕ_o.

3.6.5 Orthogonality Condition

Using the self-adjoint form of the inextensible cable equation, the following very important property for the eigenmodes can be proven (see appendix C).

$$\int_{\phi_{bot}}^{\phi_{top}} \left[\frac{\xi_i^I \, \xi_j^I}{\cos^2\phi_o} + \frac{b \, \xi_i \, \xi_j}{\cos^2\phi_o} \right] d\phi_o = 0 \qquad \text{for } i \neq j \tag{3.48}$$

where ξ_i , ξ_j are the eigenmodes in the tangential direction with fixed boundaries. We can rewrite the compatibility condition, using the Lagrangian coordinate as the independent variable as, in the form:

$$\int_0^1 (M \, \eta_i \, \eta_j + m \, \xi_i \, \xi_j) \, d\sigma = 0 \qquad \text{for } i \neq j \tag{3.49}$$

where:

η_i , η_j the components of the eigenmodes in the transverse direction

ξ_i , ξ_j the components of the eigenmodes in the tangential direction

3.6.6 Derivation of Asymptotic Solutions

The solutions are achieved by postulating two different types of motion when λ is large. One is wave-like and fast varying compared to the coefficients of the equation. Physically it represents transverse waves in the chain. The other type of solution is slowly varying in space and represents instant adjustement of the chain to a disturbance, so as to preserve constant length (inextensible chain).

3.6.6.1 The Fast Solution

A solution in WKB form is proposed:

$$\xi = \exp \{ \lambda \int f_1 + \int g_1 + \frac{1}{\lambda} \int h_1 + \frac{1}{\lambda^2} \int i_1 + \ldots \} \tag{3.50}$$

The novelty of the approach consists of substituting (3.50) directly in (3.46) without making any simplifying assumption a priori.

The results were obtained by using MACSYMA[6]:

$$O(\lambda^4) \qquad f_1(\phi_o) = \pm \frac{i}{\cos^{3/2}\phi_o}$$

$$f_1^2(\phi_o) = 0 \tag{3.51}$$

The second solution indicates that two slow solutions must exist and will be analysed separately.

$$O(\lambda^3) \qquad g_1(\phi_o) = -\frac{7}{4}\frac{\sin\phi_o}{\cos\phi_o} \tag{3.52}$$

$$O(\lambda^2) \qquad h_1(\phi_o) = \mp i \cos^{3/2}\phi_o \left[\frac{28 - 16h}{32} - \frac{29}{32}\tan^2\phi_o \right] \tag{3.53}$$

Results for $O(\lambda)$ and $O(1)$ were also calculated but for reasons to be given later, the usefulness of these results seems limited.

The fast solution can be written as:

$$\xi_f = \cos^{7/4}\phi_o \begin{matrix}\cos \\ \sin\end{matrix} (\lambda f + h/\lambda) \tag{3.54}$$

where: $$f = \int_{\phi_b}^{\phi_o} \frac{1}{\cos^{3/2}\phi_o}\, d\phi_o$$

$$h = -\int_{\phi_b}^{\phi_o} \cos^{3/2}\phi_o \left[\frac{28 - 16h}{32} - \frac{29}{32}\tan^2\phi_o \right] d\phi_o$$

For the boundary value problem considered, the solution for the normal displacement is also required. It is given by:

---

[6]MACSYMA is a computer program used for performing symbolic and numerical mathematical manipulations. It is developed by the Mathlab Group of the M.I.T. Laboratory of Computer Science

$$\eta_f = -\ 7/4\ \cos^{3/4}\phi_o\ \sin\phi_o\ \begin{matrix}\cos\\ \sin\end{matrix}\ (\ \lambda f + h/\lambda)$$

$$\mp\ \cos^{7/4}\phi_o\ (\lambda \dot{f} + \dot{h}/\lambda\ )\ \begin{matrix}\sin\\ \cos\end{matrix}\ (\ \lambda f + h/\lambda) \qquad (3.55)$$

The solution (3.54) and (3.55) is, to leading order, equivalent with the Saxon and Cahn solution. However the second order terms are different from the ones found by Saxon and Cahn.

Even more important, to obtain the normal displacement correct to O(1), a term obtained directly from the leading order solution in (3.54), appears in (3.55). This term is not appearing in the Saxon and Cahn solution. The correct solution to O(1) is therefore given by:

$$\xi_f = \cos^{7/4}\phi_o\ \begin{matrix}\cos\\ \sin\end{matrix}\ (\lambda f) \qquad (3.56)$$

$$\eta_f = -\ 7/4\ \cos^{3/4}\phi_o\ \sin\phi_o\ \begin{matrix}\cos\\ \sin\end{matrix}\ (\lambda f) \mp \cos^{7/4}\phi_o\ \lambda \dot{f}\ \begin{matrix}\sin\\ \cos\end{matrix}\ (\lambda f) \qquad (3.57)$$

3.6.6.2 The Slow Solution

The slow solution is found from (3.46) by assuming λ large, when the equation becomes:

$$\frac{\lambda^2}{\cos^2\phi_o}\ \xi^{II} + 2\ \frac{\sin\phi_o}{\cos^3\phi_o}\ \lambda^2\ \xi^{I} - \frac{h\ \lambda^2}{\cos^2\phi_o}\ \xi = O(1) \qquad (3.58)$$

Simplification gives:

$$\xi^{II} + 2\tan\phi_o\ \xi^{I} - h\xi = O(1/\lambda^2) \qquad (3.59)$$

This equation is independent of the frequency. The solutions are [Triantafyllou 82a]:

$$\xi_{s1} = (1-h^2) + h^2\sin\phi_o + \frac{h}{2}(h-1)\cos^2\phi_o + O(h^3) \tag{3.60}$$

$$\xi_{s2} = (1-h)\cos^3\phi_o + 3(3+h)\left[(\phi_o - \frac{\pi}{2})\sin\phi_o + \cos\phi_o\right] + O(h^3) \tag{3.61}$$

$$\eta_{s1} = \left[h^2 - h(h-1)\sin\phi_o\right]\cos\phi_o + O(h^3) \tag{3.62}$$

$$\eta_{s2} = -3\cos\phi_o\left[(1-h)\cos\phi_o\sin\phi_o - (3+h)(\phi_o - \frac{\pi}{2})\right] + O(h^3) \tag{3.63}$$

3.6.6.3 Total Solution

The total solution to (3.46) is given by linear combination of (3.56) (3.57) and (3.60) - (3.63).

$$\begin{aligned}\xi &= A\xi_{f1} + B\xi_{f2} + C\xi_{s1} + D\xi_{s2}\\ \eta &= A\eta_{f1} + B\eta_{f2} + C\eta_{s1} + D\eta_{s2}\end{aligned} \tag{3.64}$$

The eigenvalues can be found, by finding the values of λ for which:

$$\text{Det}\begin{bmatrix} \eta_{f1}(\phi_1) & \eta_{f2}(\phi_1) & \eta_{s1}(\phi_1) & \eta_{s2}(\phi_1) \\ \xi_{f1}(\phi_1) & \xi_{f2}(\phi_1) & \xi_{s1}(\phi_1) & \xi_{s2}(\phi_1) \\ \eta_{f1}(\phi_2) & \eta_{f2}(\phi_2) & \eta_{s1}(\phi_2) & \eta_{s2}(\phi_2) \\ \xi_{f1}(\phi_2) & \xi_{f2}(\phi_2) & \xi_{s1}(\phi_2) & \xi_{s2}(\phi_2) \end{bmatrix} = 0 \tag{3.65}$$

3.7 Verification of the Solution

We will verify the results by calculating the eigenfrequencies of an inextensible chain hanging between two points at the same level. Many experimental and calculated results have been published. [Pugsley 49], [Saxon

53], [Goodey 61]

3.7.1 Eigenfrequencies

The solution is considerably simplified by noting that one slow and one fast solution are symmetric and the other anti-symmetric. The eigenfrequency equation (3.65) can be separated, therefore, in one for even modes and one for odd modes.

3.7.1.1 Odd Normal Modes

The requirement becomes[7]:

$$\frac{\eta_{f1}(\phi_o)}{\xi_{f1}(\phi_o)} = \frac{\eta_{s1}(\phi_o)}{\xi_{s1}(\phi_o)}$$

$$\tan(\lambda f+h/\lambda) = -\frac{1}{\lambda[1 + h/(\lambda^2 f)]}\left[\frac{\phi_o\cos^{5/2}\phi_o}{\cos\phi_o + \phi_o\sin\phi_o} + 7/4\sin\phi_o\cos^{1/2}\phi_o\right] \quad (3.66)$$

3.7.1.2 Even Normal Modes

The requirement becomes

$$\frac{\eta_{f2}(\phi_o)}{\xi_{f2}(\phi_o)} = \frac{\eta_{s2}(\phi_o)}{\xi_{s2}(\phi_o)}$$

$$\cot(\lambda f+h/\lambda) = \frac{1}{\lambda[1+h/(\lambda^2 f)]}\left[\frac{\cos^{5/2}\phi_o}{\sin\phi_o} + 7/4\sin\phi_o\cos^{1/2}\phi_o\right] \quad (3.67)$$

---

[7] To make use of the symmetry, the origin is selected in the middle of the cable. The lower limit of the integrals in f and h is therefore 0. The angle ϕ_0 is the static angle at the end point.

3.7.2 Consistency with Previous Results

For large λ, the eigenfrequencies calculated by (3.66) and (3.67) give the same results as derived by Saxon and Cahn [Saxon 53] and Goodey [Goodey 61].

3.7.2.1 Odd Normal Modes

Equation (3.66) can be rewritten for high order modes as:

$$\lambda = \frac{n\pi}{f}\left[\,1 - \frac{f(h + k_1)}{n^2\,\pi^2}\,\right] + O\left[\,\left(\frac{1}{n^4}\right)\,\right] \tag{3.68}$$

where: $$(h + k_1) = -\frac{1}{32}\int_0^{\phi_o}(12 - 29\tan^2\phi_o)\cos^{3/2}\phi_o d\phi_o + \frac{\phi_o\cos^{5/2}\phi_o}{\phi_o\sin\phi_o + \cos\phi_o} + \frac{7}{4}\sin\phi_o\cos^{1/2}\phi_o$$

which is identical to the formula derived by Saxon and Cahn, and by Goodey.

3.7.2.2 Even Normal Modes

Equation (3.67) can be rewritten for high order modes as:

$$\lambda = \frac{(n + 0.5)\pi}{f}\left[\,1 - \frac{f(h + k_2)}{(n + 0.5)^2\pi^2} + O\left(\frac{1}{n^4}\right)\,\right] \tag{3.69}$$

where: $$(h + k_2) = -\frac{1}{32}\int_0^{\phi_o}(12 - 29\tan^2\phi_o)\cos^{3/2}\phi_o d\phi_o + \frac{\cos^{5/2}\phi_o}{\sin\phi_o} + \frac{7}{4}\sin\phi_o\cos^{1/2}\phi_o$$

which is also identical to the result of Saxon and Cahn, and Goodey.

3.7.3 Comparison of the Perturbation Results with the Numerical Solutions

We will now compare numerically the results obtained by the new perturbation theory, with those of Saxon and Cahn and those of the finite difference scheme. The figures on which the results are plotted have the same format as the one used by Goodey (see figure 3-5). The horizontal axis is the static angle at the top. The frequency is non-dimensionalised with respect to the natural frequency of a pendulum with length equal to the chain span:

$$f \left[\frac{b}{g} \right]^{1/2} \tag{3.70}$$

where: f = frequency
b = sag at midspan

The relation between the non-dimensional parameter λ and the new non-dimensional frequency parameter is obtained directly from statics:

$$f \left[\frac{b}{g} \right]^{1/2} = 0.5 \left[\frac{1 - \cos\phi_{top}}{\cos\phi_{top}} \right]^{1/2} \frac{\lambda}{\pi} \tag{3.71}$$

The results are shown in the figures 3-6 and 3-7 and are significant, because:

- They clearly demonstrate the improved accuracy of the new first order theory compared to the first order theory derived by Saxon and Cahn.

- The new second order theory and that of Saxon and Cahn seem to agree very well up to a certain point, when λ becomes small and they break down.

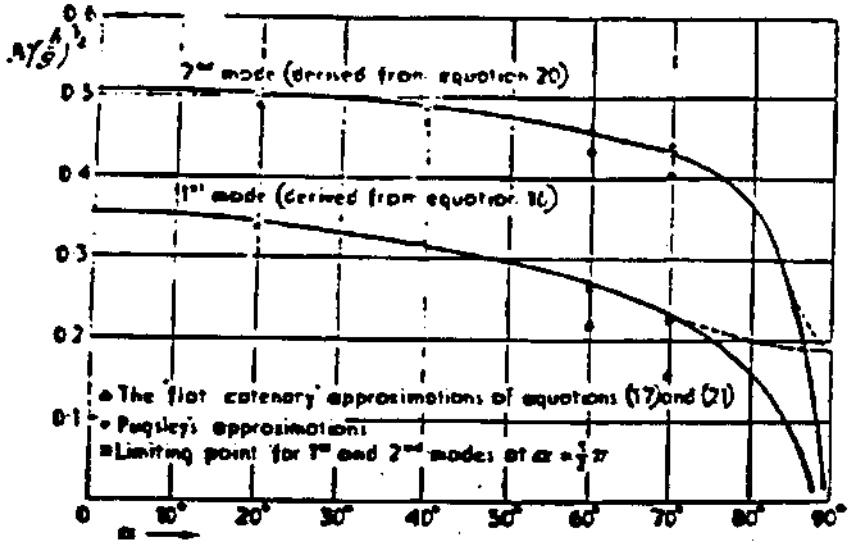


Figure 3-5: Eigenfrequencies of a Catenary [Goodey 61]

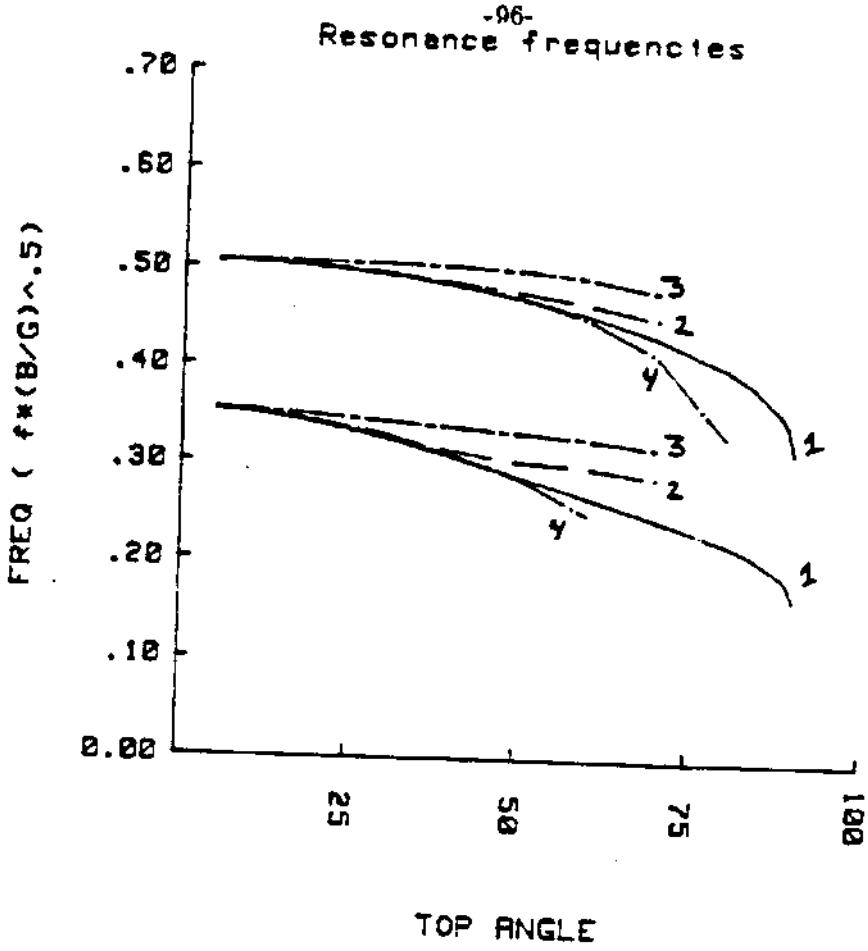


Figure 3-6: Comparison between the Present Perturbation Solution and the Solution by Saxon and Cahn

1. New perturbation theory (first order)

2. New perturbation theory (second order)

3. Saxon and Cahn (first order)

4. Saxon and Cahn (second order)

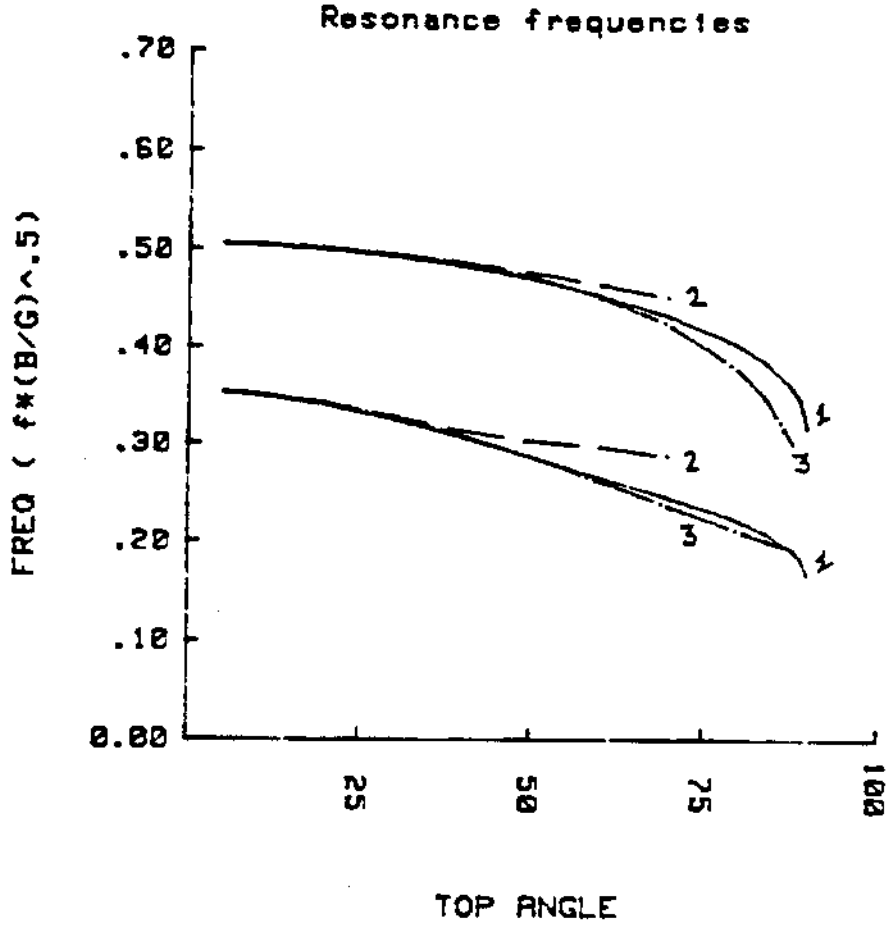


Figure 3-7: Comparison between the Perturbation Solution and the Finite Difference Solution

1. New perturbation theory (first order)

2. New perturbation theory (second order)

3. Numerical solution (explicit centered differences, 100 gridpoints. see section 3.9)

- It is quite interesting that the new first order theory gives good results even for λ small, which is outside of the validity range of the assumptions used. No breakdown of the solution is observed.

- There are some erroneous roots of very low frequency appearing in the first order Saxon and Cahn solution and to a lesser extent in the second order perturbation solution. (Not shown on figure.) The new first order solution does not have this problem.

- The first order solution provides valid results even for top angles very close to the limiting value of 90^{o}.

3.7.4 Nearly Vertically Hanging Cables

The limit, as the top angle approaches 90^{o}, exists for the first order perturbation theory.

For odd modes:

$$\tan(4\pi\bar{f}) = -\frac{1}{2\pi\bar{f}}\frac{7}{4} \rightarrow \tan(2\omega) = -\frac{1}{\omega}\frac{7}{4} \tag{3.72}$$

For even modes:

$$\operatorname{cotan}(4\pi\bar{f}) = \frac{1}{2\pi\bar{f}}\frac{7}{4} \rightarrow \operatorname{cotan}(2\omega) = \frac{1}{\omega}\frac{7}{4} \tag{3.73}$$

$$\text{with: } \bar{f} = f\left[\frac{b}{g}\right]^{1/2} \rightarrow \omega = 2\pi f\left[\frac{b}{g}\right]^{1/2}$$

The roots for ω can be found in table 3-II. The odd modes approach the correct values for higher frequencies. The approximation seems to be slightly better than the WKB approximation of the same problem (see table 3-II).

new theory (in the limit)		direct solution simple vertically hanging cable attached at one point	
odd modes	even modes	Bessel solution (exact)	WKB first order
1.057		1.204	$\pi/2$
	1.996	-	-
2.868		2.760	π
	3.706	-	-
4.527		4.327	$3\pi/2$

Table 3-II: Frequencies of a Vertically Hanging Chain

The explanation of this fact can be found in the way the tension is approaching zero at the bottom of a hanging chain. It varies parabolically near the bottom, compared to a linear variation for a vertically hanging chain. One of the lesser known characteristics of a first order WKB approximation is. as discussed before, that it is a valid approximation for singular perturbation problems with parabolic singularities. This may also explain why the derived solution remains valid in regions where the wave propagation speed is low.

The even modes have no counterpart in vertically hanging cables and

they must be always considered for top angles near, but never exactly equal to 90°. Clearly, at the midpoint region the equations break down when the angle is equal to 90°.

3.8 Extensible Cables

In this section the dynamic characteristics of elastic cables will be investigated using the same type of perturbation analysis as for the inextensible cables. The governing equations (3.1) can be written, using non-dimensional displacements and Lagrangian coordinate, as:

$$
\begin{aligned}
&- m\, L^2\, \omega^2\, \xi = \frac{dT_1}{d\sigma} - w_o\, L\, \cos\phi_o\, \phi_1 \\
&- M\, L^2\, \omega^2\, \eta = T_1 \frac{d\phi_o}{d\sigma} + T_o \frac{d\phi_1}{d\sigma} + w_o\, L\, \sin\phi_o\, \phi_1 \\
&\frac{d\xi}{d\sigma} - \eta \frac{d\phi_o}{d\sigma} = \frac{T_1}{E\cdot A} \\
&\frac{d\eta}{d\sigma} + \xi \frac{d\phi_o}{d\sigma} = \phi_1\, (1 + e_o)
\end{aligned}
\tag{3.74}
$$

The major change, compared to the inextensible governing equation is in the tangential compatibility relation, where allowance is made for stretching. The modification of this geometric condition will have a profound effect on the formation of modes.

The rather peculiar dynamic behavior of extensible cables has only recently received appropriate attention. Davenport [Davenport 65] is one of the first to discuss dynamic properties of extensible cables for the case of the

guys of a mast. Analytical solutions have been formulated for horizontal small sag cables by Simpson [Simpson 66] and independently, using a different method, by Irvine and Caughey [Irvine 74]. More recent work, of which this report is a part, was done in the Ocean Engineering Departement at M.I.T. See for instance [Triantafyllou 83] and [Triantafyllou 84].

As in the case of inextensible cables, the analysis of a horizontal, small sag cable, gives us valuable information about the more general cable behavior. The static relations derived in subsection 3.6.1 remain valid for the extensible cable.

3.8.1 Horizontal, Small Sag, Extensible Cable

The solution to this problem can be found in [Irvine 81], but will be rederived here, using the analogy with the inextensible case.

The derivation is completely similar with the one in subsection 3.6.2. To leading order the transverse dynamics can be written as:

$$- M \omega^2 L^2 \eta = \alpha T_{1o} + \frac{H}{1+e_o} \frac{d^2\eta}{d\sigma^2} + O(\epsilon^2) \tag{3.75}$$

To find the integration constant T_{1o}, we can use the tangential compatibility relation in (3.74), which can be integrated between the boundaries where $\xi(1/2) = \xi(-1/2) = 0$:

$$\frac{T_{1o}}{E \cdot A} = - \alpha \int_{-1/2}^{1/2} \eta(\sigma) \, d\sigma \tag{3.76}$$

In the above equation the implicit assumption was made that T_{1o} is constant, and this is a good assumption when no longitudinal dynamics of the cable are excited. An additional requirement for the validity of the theory is.

therefore, that the frequency is much lower than the first elastic eigenfrequency of the cable. This additional requirement was not needed in the case of an inextensible cable, because the relation between normal and tangential displacements are completely fixed due to the tangential compatibility relation.

The governing equation is obtained by substituting (3.76) in (3.75).

$$\frac{H}{1 + e_o} \cdot \frac{d^2\eta}{d\sigma^2} + M\,\omega^2\,L^2\,\eta = \alpha^2\,E{\cdot}A \int_{-1/2}^{1/2} \eta(\sigma)\,d\sigma \qquad (3.77)$$

where: $\alpha = \dfrac{w_o L}{H}$

The above equation is completely equivalent with the governing equation derived by Irvine and Caughey [Irvine 74]. The only minor difference can be found in the fact that Irvine introduced the concept of an effective length . It is the author's opinion that this introduction is not usefull and the actual unstretched length of the cable should be used consistently.

The eigenmodes of the above equations can easily be determined [Irvine 81]. The requirement for anti-symmetric eigenmodes is:

$$\sin \frac{kL}{2} = 0 \qquad (3.78)$$

$$\text{with:} \quad k = \omega \left[\frac{M(1+e_o)}{H} \right]^{1/2}$$

$$\simeq \omega \left[\frac{M}{H} \right]^{1/2}$$

The requirement for symmetric modes is:

$$\tan\left[\frac{kL}{2}\right] - \left[\frac{kL}{2}\right] + \frac{4}{\lambda^2}\left[\frac{kL}{2}\right]^3 = 0 \qquad (3.79)$$

where: $\lambda^2 = \left[\frac{w_o L}{H}\right]^2 \cdot \frac{E \cdot A_o}{H}$

$$A_o = (1 + e_o)A$$

λ^2 is proportional to the ratio of the elastic stiffness to the catenary stiffness. For an infinite elastic stiffness (an inextensible cable) the previous derived results are found. For a cable with infinite catenary stiffness (a perfectly extensible string) the symmetric string eigenmodes are found.

Equation (3.79) allows us to find the symmetric eigenfrequencies for the whole range of λ. For an extensive discussion of the properties of the eigenmodes and eigenfrequencies with varying λ see [Irvine 81] and [Veletsos 82]. We will limit ourselves to a plot of the eigenfrequencies versus λ. (See figure 3-8.) The points where the symmetric eigenfrequencies are equal to the anti-symmetric eigenfrequencies are called modal cross-over points.

It is also interesting to look at the modal shapes with varying λ. The transverse modal shapes for the first symmetric modes were plotted for various values of λ^2. (See figure 3-9)

The results obtained above are only valid for cables with end points on the same horizontal line and with a shallow sag. The case of an inclined cable, treated by Irvine approximately as an extension of the above theory [Irvine 78] is better treated with the more general analysis of the next section.

The theory presumes also that quasi-static stretch is present. If this is not the case, tangential dynamics will be excited and the above theory will not

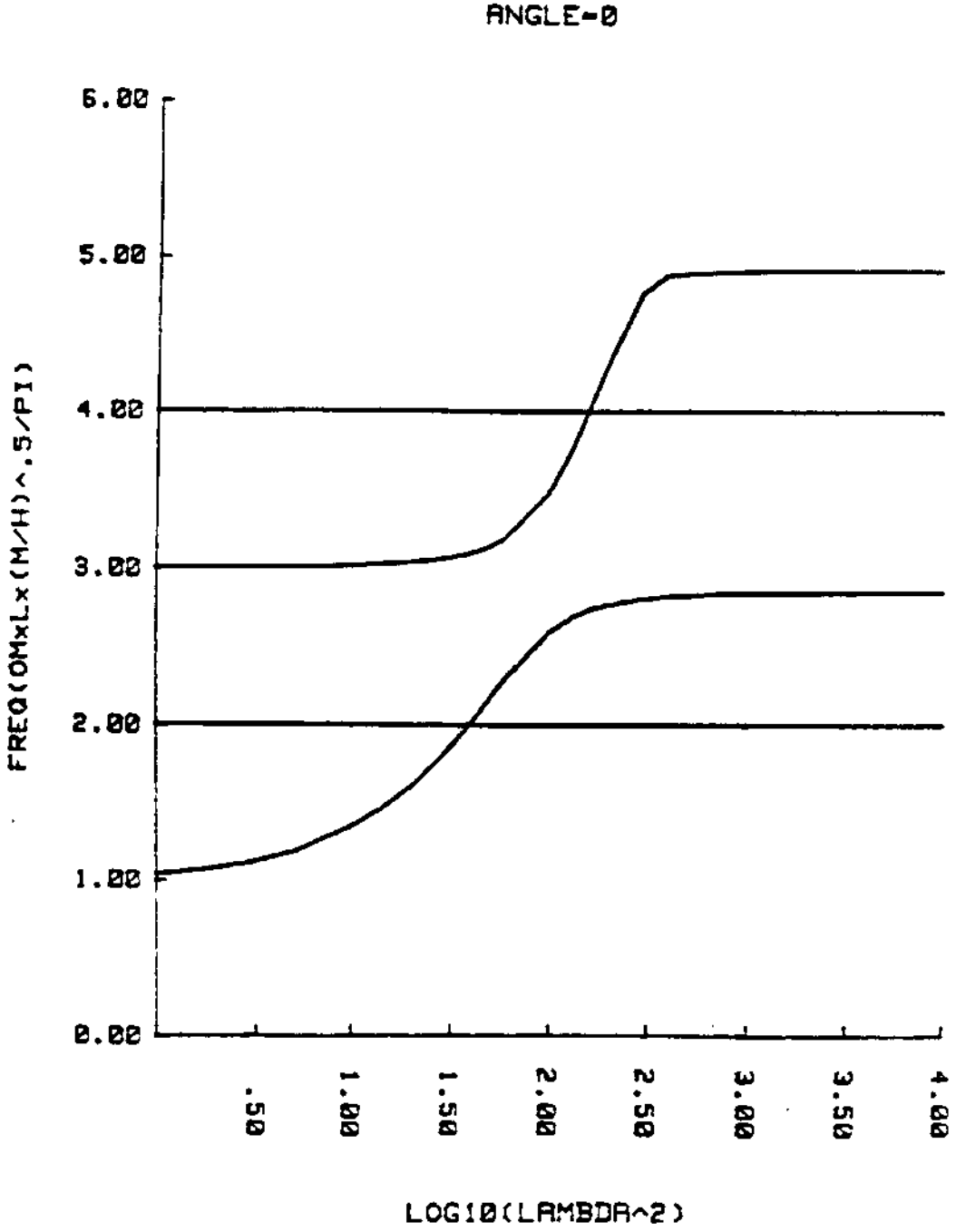


Figure 3-8: Cross-Over Phenomena for Small Sag Cables

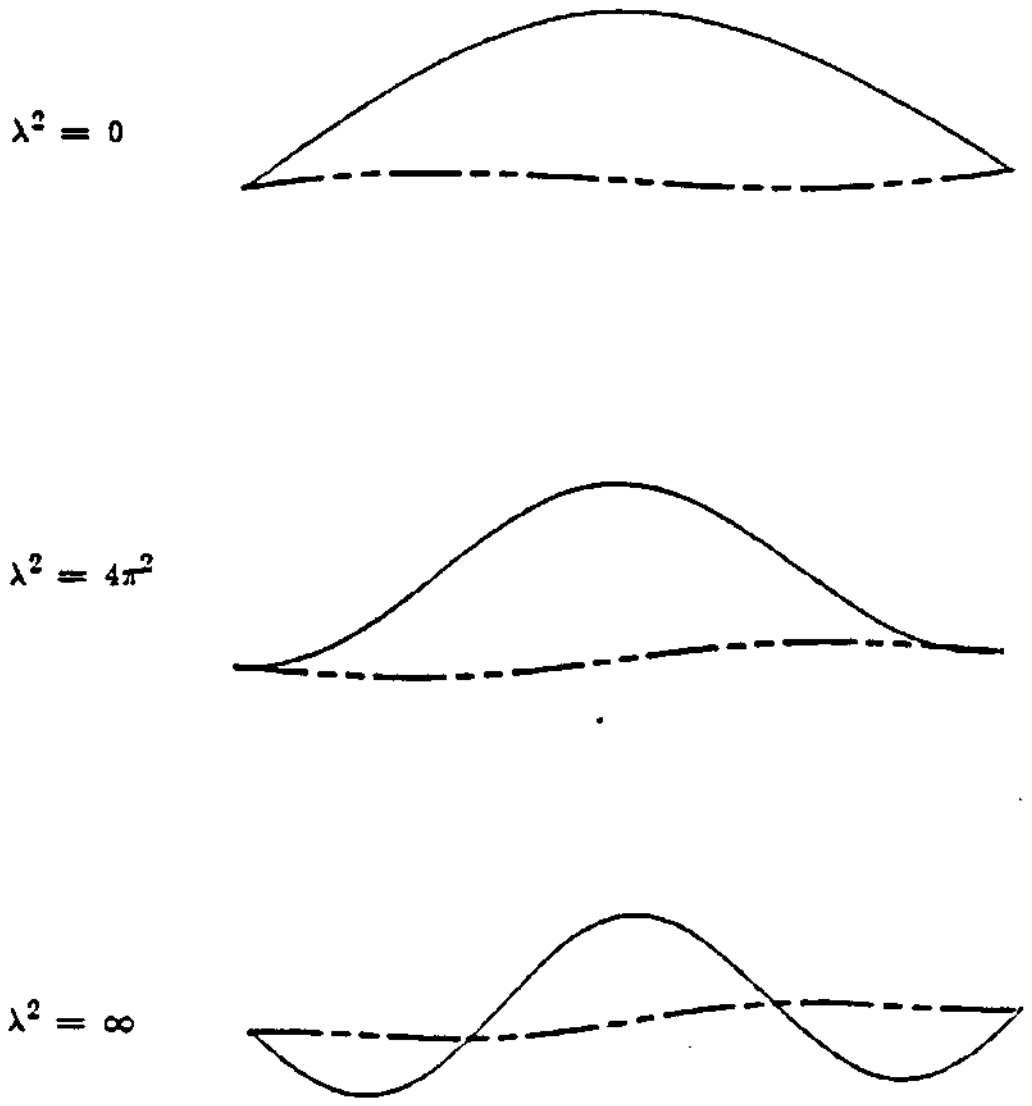


Figure 3-9: Symmetric Mode Shapes for a Sj

(Normal displacement: full line)
(10 × tangential displacement: dotted

be valid. The elastic wave speed is:

$$c_{el} = \left[\frac{E}{\rho} \right]^{1/2}$$

and the transverse wave speed is:

$$c_{tr} = \left[\frac{M(1+e_o)}{H} \right]^{1/2}$$

therefore:

$$\frac{c_{tr}}{c_{el}} = \left[\frac{h \cdot H}{E \cdot A} \right]^{1/2}$$

The ratio H/EA is in general very small, so that the effects are not important until the 15 - 20 th mode. It is important to note that if we vary λ to smaller values, $w_o L/H$ should be simultaneously reduced to smaller values in order not to violate the quasi-static stretching condition. In other words, for each geometric configuration, there is a minimal value of λ^2 below which the derivations above are not valid.

3.8.2 Orthogonality Condition

In the case of extensible cables, the four differential equations cannot be reduced to one equation as in the case of inextensible cables. This makes the analysis more difficult mathematically. The orthogonality condition for the modes, though, can be proven easily.

We define H(s,r) as the matrix of the influence functions corresponding to the static displacement components in the tangential and normal direction at a point s, due to unit forces in the normal and tangent direction at point r. (H(s,r) is the Green function for the problem.)

Rosenthal proved the following reciprocity relations for cables under static forces [Rosenthal 81], which can be seen as an application of Maxwell's reciprocity principle.

$$[H(s,r)] = [H(r,s)]^T$$

or explicity:

$$H_{11}(s,r) = H_{11}(r,s)$$

$$H_{22}(s,r) = H_{22}(r,s)$$

$$H_{12}(s,r) = H_{21}(r,s)$$

$$H_{21}(s,r) = H_{12}(r,s)$$

Using the above relations, it can be proved that the following orthogonality condition holds: (see Appendix D)

$$\int_0^1 (m\ \xi_i\ \xi_j + M\ \eta_i\ \eta_j)\ d\sigma = 0 \qquad i \neq j \tag{3.80}$$

where: m: mass of the cable, per unit length

M: mass plus added mass of the cable per unit length

ξ_i: tangential components of the i th mode

η_i: normal components of the i th mode

Although the above result is well known for the dynamics of rigid structures, it is believed that it has escaped the attention of previous researchers as applied to cables. In the case of non-extensible cables, the previously derived orthogonality condition is identical with the one derived above.

3.8.3 Derivation of an Approximate Solution

In the case of an inextensible cable, we were able to use a transformation of variables to reduce the governing equations to a single, fourth order differential equation. We could then solve the fourth order differential equation using a perturbation expansion.

For an extensible cable, the equations cannot be reduced to a single, higher order differential equation. We will therefore solve directly the set of differential equations using perturbation techniques. The procedure is more general in the sense that it can be applied to cases where the shape is not dominated by the weight forces [Triantafyllou 83], although the mathematical derivation is less elegant than in the case of an inextensible cable.

The assumptions to obtain the perturbation solution are in both cases completely equivalent and indeed the inextensible results are also obtained as a limiting case of the extensible perturbation theory.

To be able to use the perturbation expansions, two assumptions over the magnitude of the coefficients are necessary.

The first assumption is that the ratio of the wave speed of the elastic waves, versus the speed of the transverse waves is large. This assumption is valid for most cable applications where the material stretching is small. Indeed from chapter 1, we obtained as wave speeds:

$$c_{el} = \left[\frac{E}{\rho} \right]^{1/2} \tag{3.81}$$

$$c_{tr} = \left[\frac{T_o}{M(1+e_o)} \right]^{1/2}$$

therefore:

$$\frac{c_{el}}{c_{tr}} = \left[\frac{E \cdot A}{T_o} \frac{1}{h} \right]^{1/2}$$

where: $h = m/M$

h is a quantity of order 1, therefore the ratio of the wave speed is of the same order as the square root of the inverse of the static strain. Typically the elastic wave speed will be at least 20 times higher than the transverse one. The assumption can therefore be made that the solution consists of a part which is fast oscillating in space (small wave length, transverse waves) and a part which is slowly oscillating in space (large wave length, longitudinal waves).

The second assumption is that the static quantities in the equation are slowly varying in space compared to the transverse oscillation. In other words, the variation of the static tension and curvature is small over a transverse wave length. This assumption can be also stated as a requirement that the variation over the cable length of the static quantities is small and/or that the wave length of the transverse modes is small compared to the cable length. One of these two conditions must be met. Fortunately, it turns out that even for the first modes of oscillation the above conditions are generally satisfied. (See also the discussion on asymptotic strings with variable tension.)

Under the above conditions the four differential equations can be separated asymptotically in two second order differential equations. One will provide the transverse wave type of solution, which will be fast varying in space, while the other differential equation will provide slow solutions in space, which correspond to the elastic waves, or, in the limiting case of inextensible cables, to the instantaneous readjustement of the equilibrium position of the cable.

The equations can be written in non-dimensional form as:

$$
\begin{aligned}
&-\frac{m\,L^2\,\omega^2\,\xi}{T_c} = \frac{dT}{d\sigma} - \frac{w_o L}{T_c}\cos\phi_o\,\phi_1 \\
&-\frac{m\,L^2\,\omega^2\,\eta}{T_c} = T\,\frac{d\phi_o}{d\sigma} - \frac{d\phi_1}{d\sigma}\,\frac{T_o}{T_c} + \frac{w_o L}{T_c}\sin\phi_o\,\phi_1 \\
&\frac{d\xi}{d\sigma} - \eta\,\frac{d\phi_o}{d\sigma} = \frac{T_c}{E\cdot A}\,T \\
&\frac{d\eta}{d\sigma} + \xi\,\frac{d\phi_o}{d\sigma} = \phi_1(1+e_o)
\end{aligned}
\qquad (3.82)
$$

where: T_c: representive static tension along the cable = constant

$T_o(\sigma)$: static tension $dT_o/d\sigma = w_o L\sin\phi_o$

$T(\sigma)$: T_1/T_c non-dimensional dynamic tension

σ: non-dimensional Lagrangian coordinate

ξ: non-dimensional tangential displacement

η: non-dimensional normal displacement

3.8.4 Fast Varying Solution

We will assume that ξ, η are fast varying quantities, compared with the static quantities. We denote by ϵ a quantity which is an order of magnitude smaller than one. In the sequel we rewrite all variables in terms of new quantities wich are of order 1, multiplied by appropriate powers of ϵ to indicate their order of magnitude. (In order not to confuse the reader with different notations, $\bar{\xi}$, $\bar{\eta}$ will still denote tangential and normal displacements, but normalized to be of order 1.) For transverse waves, we know that the tangential displacements are an order of magnitude smaller than the normal displacements and, also that the dynamic tension is a second order effect. This can be expressed as:

$$\xi \rightarrow \epsilon^2 \bar{\xi}\,(\sigma)$$

$$\eta \rightarrow \epsilon \bar{\eta}(\sigma)$$

$$T \rightarrow \epsilon^2 \bar{T}(\sigma)$$

$$\phi \rightarrow \epsilon \bar{\phi}(\sigma)$$

These are the basic assumptions we make to obtain a solution of (3.82). Also, we denote as:

$$\bar{\omega}^2 = \frac{M\, L^2\, \omega^2}{T_c}$$

$$\frac{w_o L}{T_c} = \epsilon\, \bar{\alpha}$$

$$\frac{T_o}{E \cdot A} = \epsilon^2\, \bar{e}\,(\epsilon\sigma) \qquad (3.83)$$

$$\frac{d\phi_o}{d\sigma} = \epsilon\ \bar{\phi}_{oo}(\epsilon\sigma)$$

$$\frac{T_o(\sigma)}{T_c} = \bar{T}_o(\epsilon\sigma)$$

where now all the variables are expressed in terms of quantities of order 1 and powers of ϵ. It can be easily verified by substitution, that the governing equations, to second order, can be written as:

$$-\bar{\omega}^2\ \bar{\eta} = \frac{d}{d\sigma}\left[\ \bar{T}_o\ \frac{d\bar{\eta}}{d\sigma}\ \right] + O(\epsilon^2)$$

$$\frac{d\bar{\xi}}{d\sigma} = \bar{\phi}_{oo}\bar{\eta} + O(\epsilon^2) \tag{3.84}$$

The first equation is a simple string equation with variable tension. The unknown variable is fast varying compared to $\bar{T}_o$ and the non-dimensional quantity $\bar{\omega}^2$ is a large quantity when we study the eigenmodes. Therefore, the WKB solution previously discussed in this chapter can be used. However, reasonable care should be used because we want to determine both ξ and η with the same accuracy.

The reader will have noticed that in the case of a non-extensible cable the second order terms were kept in the normal displacement expression. Therefore, we proceed by deriving the equations in terms of $\bar{\xi}$ using the relation:

$$\bar{\eta} = \frac{1}{\bar{\phi}_{oo}}\frac{d\bar{\xi}}{d\sigma} \tag{3.85}$$

The equation for the fast solution in the tangential direction can then be written to second order as:

$$- \bar{\omega}^2 \, \bar{\xi} \; = \bar{T}_o \, \frac{d^2\bar{\xi}}{d\sigma^2} - 2\bar{T}_o \, \frac{\bar{\phi}_{o\sigma\sigma}}{\bar{\phi}_{o\sigma}} \, \frac{d\bar{\xi}}{d\sigma} + O(\epsilon^2) \tag{3.86}$$

Verification of (3.86) can easily be obtained by substituting (3.85) in (3.86). The solution to this problem can be obtained using the WKB method for large parameters.

To leading order:

$$\bar{\xi} \; = \bar{\phi}_{oo} \, (\bar{T}_o)^{1/4} \, \exp(\pm \, W) \tag{3.87}$$

where: $W = i \, \bar{\omega} \int_0^{\sigma} \frac{1}{(\bar{T}_o)^{1/2}} \, d\sigma$

This can be rewritten. in terms of dimensional quantities, as:

$$\xi = \frac{d\phi_o}{d\sigma} (T_o)^{1/4} \exp\Big[\pm i \, \omega \, L \int_0^{\sigma} \frac{1}{(T_o/M)^{1/2}} \, d\sigma \Big] \tag{3.88}$$

Using the relation (3.84), we obtain the solution in the normal direction as:

$$\eta = \frac{1}{(T_o)^{1/4}} \Big[\pm i \, \omega - \frac{7}{4} \frac{w_o \sin\phi_o}{(T_o \, M)^{1/2}} \Big] \cdot$$

$$\exp\Big[\pm i \, \omega \, L \int_0^{\sigma} \frac{1}{(T_o/M)^{1/2}} \, d\sigma \Big] \tag{3.89}$$

This result is identical with the result for inextensible cables, which was derived by applying the WKB method to the fourth order differential equation. i.e. the extensibility does not affect, to second order, the solution for transverse waves.

When the curvature is zero, the solution is the same as the WKB solution for a string with varying tension. The asymptotic WKB solutions

were found to be very accurate even if the assumption of slowly varying static quantities is violated. The dynamic angle can be obtained as:

$$\phi_1 = \frac{\partial \eta}{\partial \sigma} \tag{3.90}$$

The validity of the assumed order of magnitude of the dynamic variables can be checked using the obtained solution.

3.8.5 Slowly Varying Solution

We now assume that ξ, η are slowly varying quantities in space (compared to the fast transverse waves). The normal and tangential motions are now of the same order. The consistent perturbation assumption in this case are:

$$\xi \rightarrow \epsilon \bar{\xi}\,(\epsilon\sigma)$$

$$\eta \rightarrow \epsilon \bar{\eta}(\epsilon\sigma)$$

$$T \rightarrow \bar{T}(\epsilon\sigma)$$

$$\phi \rightarrow \epsilon^2\, \bar{\phi}(\epsilon\sigma)$$

The governing equations to second order can be written in terms of perturbation quantities:

$$\begin{aligned} &-\bar{\omega}^2\, \bar{\eta} = \bar{T}\, \bar{\phi}_{\sigma\sigma} + O(\epsilon^2) \\ &-h\, \bar{\omega}^2\, \bar{\xi} = \frac{d\bar{T}}{d\sigma} + O(\epsilon^2) \end{aligned} \tag{3.91}$$

where: $h = m/M$

The set of the above equations can be written as a single differential equation in terms of η. by using the tangential compatibility relation.

The compatibility can be rewritten in terms of perturbation quantities as:

$$\bar{T}\ \bar{e} = \frac{d\bar{\xi}}{d\sigma} - \bar{\phi}_{o\sigma}\ \bar{\eta} \tag{3.92}$$

Using (3.91) and (3.92) the approximate constitutive equation for the slow dynamics is obtained as:

$$\frac{d^2}{d\sigma^2}\left[\frac{\bar{\eta}}{\bar{\phi}_{o\sigma}}\right] - h\bar{\phi}^2_{o\sigma}\left[1 - \frac{\bar{\omega}^2\bar{e}}{\bar{\phi}^2_{o\sigma}}\right]\frac{\bar{\eta}}{\bar{\phi}_{o\sigma}} = 0 + O(\epsilon^2) \tag{3.93}$$

We can rewrite the governing equation as:

$$\frac{d^2}{d\sigma^2}\left[\frac{\eta}{\phi_{o\sigma}}\right] + Q(\sigma)\left[\frac{\eta}{\phi_{o\sigma}}\right] = 0 \tag{3.94}$$

where: $Q(\sigma) = -\frac{m}{M}\phi^2_{o\sigma}\left[1 - \frac{M\ \omega^2\ L^2}{E{\cdot}A\ \phi^2_{o\sigma}}\right]$

$$\phi_{o\sigma} = \frac{d\phi_o}{d\sigma}$$

The dynamic tension generated and the tangential displacement can be found as:

$$T_1 = -\ M\ \omega^2\ L^2\ \frac{\eta}{\phi_{o\sigma}}$$

$$\xi = h\ \frac{d}{d\sigma}\left[\frac{\eta}{\phi_{o\sigma}}\right] \tag{3.95}$$

If (3.94) can be solved, (3.95) can provide the other dynamic quantities, to obtain the complete slow solution.

The solution of equation (3.94) is unfortunately not possible analytically in the general case. The variation of the solution is of the same order as the variation of $Q(\sigma)$. Therefore a WKB type of solution cannot be used in this case.

The solution of equation (3.94) has been the subject of a major research effort at M.I.T, mainly by professor Triantafyllou. See [Triantafyllou 82b], [Triantafyllou 82c], [Triantafyllou 83] and [Triantafyllou 84].

The governing quantity is:

$$Q(\sigma) = - h \left[\phi_{o\sigma}^2 - \frac{M \omega^2 L^2}{E \cdot A} \right] \tag{3.96}$$

The only coefficient that is varying along the length is the curvature $\phi_{o\sigma}$. The solution of (3.94), therefore, strongly depends on the analytical functional form of the curvature.

The interesting features of the behavior of the slow equations are caused by the opposing effect of elasticity and curvature in (3.96). The sign of Q can be positive, negative or zero. The sign can even change at a point along the cable. This has a significant effect on the solution of the slow equation and on the overall cable behavior.

We will briefly discuss the different forms the solution can take, depending on the quantity $Q(\sigma)$. As mentioned in the introduction of chapter 3, only the weight dominated case will be considered.

For the weight dominated case, we can express the curvature in terms of the non-dimensional length using the following relation between static quantities:

$$\frac{d\phi_o}{d\sigma} = \alpha \cos^2\phi_o$$

$$\alpha = \frac{w_o L}{H} \qquad (3.97)$$

$$\tan\phi_o = \tan\phi_b + \frac{w_o L}{H}\sigma$$

then $\phi_{o\sigma}$ can be expressed as:

$$\frac{d\phi_o}{d\sigma} = \alpha \cos^2\left[\ \text{atan}\{\ \tan\phi_b + \frac{w_o L}{H}\sigma\ \}\ \right] \qquad (3.98)$$

where: ϕ_b is the static angle at the origin,
atan(x) denotes the inverse tangent function of x.

The curvature is a complicated function of σ, so that additional simplifications must be made to solve the governing slow equation analytically.

Large Sag Cable

In the case of a large sag cable, the elasticity effects are normally negligible, compared to the curvature effects. This can be expressed as:

$$\phi_{o\sigma}^2 >> \frac{M\ \omega^2\ L^2}{E{\cdot}A} \qquad (3.99)$$

which implies that the cable behaves essentially as an inextensible cable. (3.99) can, therefore, be used to determine whether the cable can be treated as inextensible or not. Note that the assumption of inextensibility depends on the frequency and for high frequencies the elasticity effects will be important. If (3.99) is valid, the governing equation (3.94) becomes:

$$\frac{d^2}{d\sigma^2}\left[\ \frac{\eta}{\phi_{o\sigma}}\ \right] - h\ \phi_{o\sigma}\ \eta = 0 \qquad (3.100)$$

By changing the independent variable from σ to ϕ_o, we can proceed to transform (3.100) into an equation expressed in terms of ϕ_o. We also use the

compatibility relation for inextensible cables, written in the form:

$$\eta = \frac{d\xi}{d\phi_o}$$

and the slow equation can now be written in terms of ξ as:

$$\frac{d^2\xi}{d\phi_o^2} - \frac{d\phi_{oo}/d\phi_o}{\phi_{oo}} \frac{d\xi}{d\phi_o} - h\ \xi = 0 \tag{3.101}$$

Using the fact that $\phi_{oo} = \alpha \cos^2\phi_o$ for a catenary, the final form for the slow solution for an inextensible cable is:

$$\frac{d^2\xi}{d\phi_o^2} + 2 \tan\phi_o \frac{d\xi}{d\phi_o} - h\ \xi = 0 \tag{3.102}$$

The solution of this equation has already been given in (3.60)-(3.63). The two independent solutions were obtained by series expansions around $h = 1$, where, fortunately, an approximate analytic solution to (3.102) can be found.

for $h = 1$

$$\xi_{s1} = \sin\phi_o$$

$$\xi_{s2} = (\phi_o - \pi/2)\sin\phi_o + \cos\phi_o$$

$$\eta_{s1} = \cos\phi_o$$

$$\eta_{s2} = (\phi_o - \pi/2)\cos\phi_o$$

The general solution, as already mentioned in paragraph 3.6.6.2 to $O(1 - h^3)$ is:

$$\xi_{s1} = (1-h^2) + h^2\sin\phi_o + \frac{h}{2}(h-1)\cos^2\phi_o + O(h^3)$$

$$\xi_{s2} = (1-h)\cos^3\phi_o + 3(3+h)\left[(\phi_o - \frac{\pi}{2})\sin\phi_o + \cos\phi_o\right] + O(h^3)$$

$$\eta_{s1} = \left[h^2 - h(h-1)\sin\phi_o\right]\cos\phi_o + O(h^3) \qquad (3.103)$$

$$\eta_{s2} = -3\cos\phi_o\left[(1-h)\cos\phi_o\sin\phi_o - (3+h)(\phi_o - \frac{\pi}{2})\right] + O(h^3)$$

The approach followed here provides, therefore, the same asymptotic solution, as the one for an inextensible cable, which was obtained by expanding a single fourth order governing equation. The agreement can be seen as a confirmation of the validity of the perturbation assumptions.

Small Sag Cable

In the case of a small sag cable, the curvature in the cable will not vary significantly and we can approximate it by a truncated Taylor series in terms of σ_o. To minimize the error in the approximation the expansion is made around a point near the middle of the cable. (See figure 3-10). We define a new origin of the Lagrangian coordinate at the point where the static angle is equal to the inclination angle of the line. The new non-dimensional Lagrangian coordinate is therefore defined as:

$$z = \sigma - \sigma_a$$

where σ_a is the length coordinate of a point where $\phi_o = \phi_a$, and which can be found from the static solution. The curvature can now be expanded around $z = 0$. The coefficients are only functions of the static quantities.

$$\phi_{o\sigma} = \phi_{o\sigma,a}\,(\,1 + a_1 z + a_2 z^2 + \ldots) \qquad (3.104)$$

By using the static solution, we obtain:

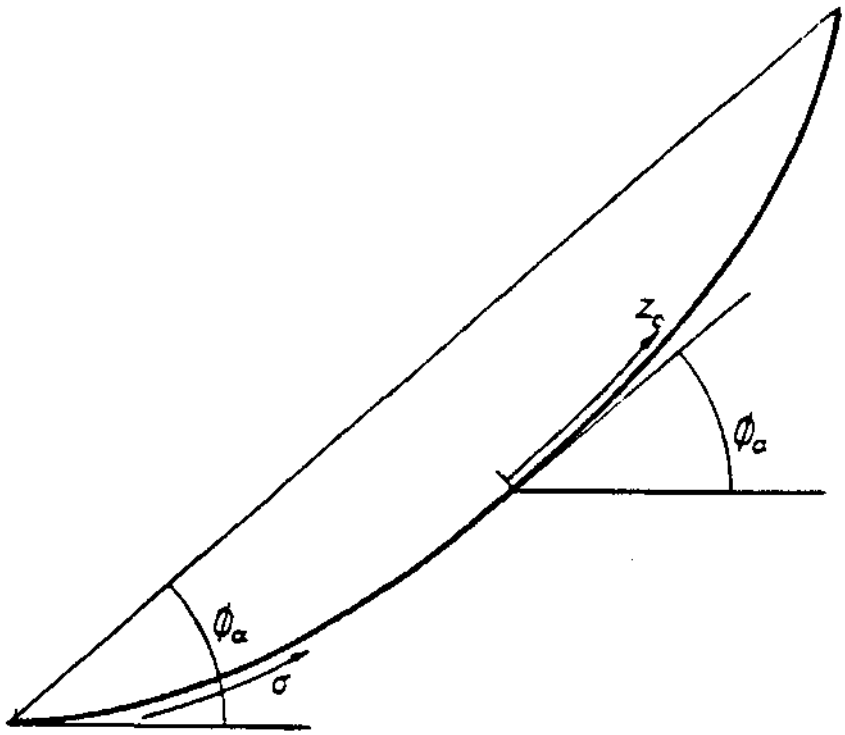


Figure 3-10: Definition of the New Lagrangian Coordinate

$$a_1 = -2\alpha \cdot \cos\phi_a \cdot \sin\phi_a$$

$$a_2 = \alpha^2 \cos^2\phi_a \cdot (3\sin^2\phi_a - \cos^2\phi_a) \qquad (3.105)$$

where: $\alpha = w_o L/H$

The quantity $Q(\sigma)$ can now be expressed in terms of z as:

$$Q(z) = -h\left[\phi^2_{o\sigma,a}\{1 + 2a_1 z + (a_1^2 + 2a_2)z^2\} - \frac{M\,\omega^2\,L^2}{E\cdot A}\right] \qquad (3.106)$$

or explicitly:

$$Q(z) = -h\left[\{\phi^2_{o\sigma,a} - \frac{M\,\omega^2\,L^2}{E\cdot A}\}\right.$$

$$- 4\alpha\,\phi^2_{o\sigma,a}\cos\phi_a \cdot \sin\phi_a\, z$$

$$+ \alpha^2\,\phi^2_{o\sigma,a}(10\cos^2\phi_a \cdot \sin^2\phi_a - 2\cos^4\phi_a)z^2$$

$$\left. + O(\alpha^3 z^3)\right] \qquad (3.107)$$

we write this as: $Q(z) = Q_o + Q_1 z + Q_2 z^2$ (3.108)

with: $Q_o = -h\left[\phi^2_{o\sigma,a} - \frac{M\,\omega^2\,L^2}{E\cdot A}\right]$

$$Q_1 = 4\alpha h\,\phi^2_{o\sigma,a}\cos\phi_a \cdot \sin\phi_a$$

$$Q_2 = -\alpha^2 h\,\phi^2_{o\sigma,a}(10\cos^2\phi_a \cdot \sin^2\phi_a - 2\cos^4\phi_a)$$

We are now in a position to obtain approximate solutions for a small sag cable. We will consider three different cases.

1. Q(z) is approximated as a constant.

$$Q(z) = - h \left[\phi_{oo,a}^2 - \frac{M \omega^2 L^2}{E \cdot A} \right] = Q_o$$

This is equivalent to assuming a constant curvature along the cable, i.e. the static shape is parabolic. The governing equation (3.100) can be written as:

$$\frac{d^2\eta}{dz^2} - h \left[\phi_{oo,a}^2 - \frac{M \omega^2 L^2}{E \cdot A} \right] \eta = 0 \qquad (3.109)$$

$$\frac{d^2\eta}{dz^2} + Q_o \eta = 0$$

where: $\phi_{oo} \cong \phi_{oo,a} \cong \alpha \cos^2 \phi_a$

The solution can be written as:

$$\eta(\sigma) = \exp \{ \pm (- Q_o)^{1/2} \sigma \}$$

$$\xi(\sigma) = \pm h (- Q_o)^{1/2} \exp \{ \pm (- Q_o)^{1/2} \sigma \} \qquad (3.110)$$

When the curvature is large compared to the elastic parameters, the solution is exponential, otherwise it provides elastic waves. The behavior of the slow solution can change from sinusoidal to exponential elastic waves, depending on the magnitude of the curvature and the elasticity.

The cross-over phenomena are predicted accurately with the above theory. Cross-over occurs when $Q_o = 0$, i.e.

$$\alpha^2 = \frac{M \omega^2 L^2}{E \cdot A}$$

For horizontal cables, the leading order approximation is correct to order $\alpha^2 z^2$, which explains the good accuracy of the horizontal small sag theory.

The leading order slow solution predicts a change of the type of solution (exponential to sinusoidal) uniformly over the cable.

2. Q(z) is approximated as a linear function.

The solution can now be written in terms of Airy and Bairy functions. (Airy functions of the first and second kind) for the normal displacement, and their derivates for the tangential displacement [Triantafyllou 84]. The parameter Q varies linearly with z only when the cable is inclined.

$$\begin{aligned} Q(z) &= Q_o + Q_1 z \\ &= Q_1 \left[z + \frac{Q_o}{Q_1} \right] \\ &= Q_1 (z - z_o) \end{aligned}$$

with: $z_o = - \dfrac{Q_o}{Q_1}$

The governing equation becomes:

$$\frac{d^2}{dz^2} \cdot \left[\frac{\eta}{\phi_{o\sigma}} \right] + Q_1 (z - z_o) \frac{\eta}{\phi_{o\sigma}} = 0 \qquad (3.111)$$

the solutions in this case are:

$$\eta(z) \begin{cases} = \phi_{o\sigma}(z) A_i \left[-Q_1^{1/3} (z - z_o) \right] \\ = \phi_{o\sigma}(z) B_i \left[-Q_1^{1/3} (z - z_o) \right] \end{cases}$$

$$\xi(z) \begin{cases} = - h\, Q_1^{1/3}\, \dot{A}_i \left[-Q_1^{1/3} (z - z_o) \right] \\ = - h\, Q_1^{1/3}\, \dot{B}_i \left[-Q_1^{1/3} (z - z_o) \right] \end{cases} \qquad (3.112)$$

This solution allows for a change from an exponential to a sinusoidal behavior along the cable length. If the transition point lies within the cable

span, the lower part behaves as an inextensible chain and the upper part as a taut wire.

3. Q(z) is approximated as a quadratic function.

The slow solution can be expressed in terms of parabolic cylinder functions for the normal displacement, and their derivatives for the tangential displacement.

The governing equation is:

$$\frac{d^2}{dz^2}\left[\frac{\eta}{\phi_{o\sigma}}\right] + \frac{\eta}{\phi_{o\sigma}}(Q_o + Q_1 z + Q_2 z^2) = 0 \tag{3.113}$$

This can be rewritten as:

$$\frac{d^2}{dv^2}\left[\frac{\eta}{\phi_{o\sigma}}\right] \mp (1/4\ v^2 \pm a)\frac{\eta}{\phi_{o\sigma}} = 0 \tag{3.114}$$

where: $v = \dfrac{z - b_o}{b_1}$

$$b_1 = \frac{1}{(4 \mid Q_2 \mid)^{1/4}}$$

$$b_o = -\frac{Q_1}{2Q_2}$$

$$a = -\frac{1}{(4 \mid Q_2 \mid)^{1/2}} \cdot \left[Q_o - \frac{Q_1^2}{4Q_2} \right]$$

The changes in sign are due to the use of an absolute value in the expression for b_1. The solution in this case is:

(3.115)

$$\eta(z) \begin{cases} = \phi_{oo} W_1(v,a) \\ = \phi_{oo} W_2(v,a) \end{cases}$$

$$\xi(z) \begin{cases} = h \dfrac{1}{b_1} \dfrac{dW_1}{dv} (v,a) \\ = h \dfrac{1}{b_1} \dfrac{dW_2}{dv} (v,a) \end{cases}$$

where W_1 , W_2 are the parabolic cylinder functions of the first kind for $Q_2 < 0$ and W_1 , W_2 are the parabolic cylinder functions of the second kind for $Q_2 > 0$.

3.8.6 Total solution

Two fast varying solutions were derived in subsection 3.8.4, and two slow solutions were derived in subsection 3.8.5. The total solution is obtained by a linear combination of the four solutions that satisfies the boundary conditions. The eigenfrequencies can be obtained by searching for the non-trivial solutions of the homogeneous problem.

3.8.7 Discussion and Validation

When we let the elastic stiffness go to infinity, the solutions obtained for extensible cables (both slow and fast) are identical to the results obtained from the perturbation expansion of the fourth order differential equation obtained for inelastic cables.

In the case of a shallow sag horizontal cable, by making the assumption used in subsection 3.8.1 of constant tension and quasi-static stretching, and by

using the leading order approximation of the slow solution, the shallow sag horizontal cable results are obtained [Irvine 74] [Triantafyllou 83]. The theory is more general, however, because it also predicts the eigenfrequencies of the elastic modes and the quasi-static stretching assumption need not be made. For deep sag, horizontal cables the derived perturbation theory predicts correctly the change of eigenfrequencies with increasing sag (see part II). The predictions are valid for extensible, as well as for inextensible cables. The theory is therefore more general than [Irvine 74] or [Saxon 53].

The shallow sag elastic cable theory has been applied by Irvine to inclined cables [Irvine 78]. The small sag approximation can be used approximately to predict the symmetric modes as follows:

$$\tan\left[\frac{k_* L}{2}\right] - \left[\frac{k_* L}{2}\right] + \frac{4}{\lambda_*^2}\left[\frac{k_* L}{2}\right]^3 = 0 \qquad (3.116)$$

with: $k_* = \omega \left[\frac{M}{H_*}\right]^{1/2}$

$$\lambda_*^2 = \left[\frac{w_o L}{H_*}\right]^2 \frac{E \cdot A_o}{H_*} \cdot \cos^2\phi_a$$

$$H_* = \frac{H}{\cos\phi_a}$$

ϕ_a = inclination angle between the cable chord and the horizontal

Again, cross-over of the modes is predicted, as shown in figure 3-8, where ω has been replaced with ω_* and λ^2 with λ_*^2. For inclined cables, however, the perturbation solution derived in previous sections should be used to reflect

the basic assymmetry introduced in the problem. The perturbation solutions predict hybrid mode formation and no cross-over phenomena. In figure 3-11 the eigenfrequencies are plotted versus λ_*^2. The inclination angle is 30^o and $w_0L/H_* = 0.5$. The eigenfrequencies are not crossing. The curve of the first mode is, at high values of λ_*^2, the continuation of the curve of the second mode, at low values of λ_*^2, and the second mode is, at high values of λ_*^2, the continuation of the curve of the first mode, at low values of λ_*^2. Figure 3-8 can be seen as the limit case of figure 3-11, for inclination angle 0^o and/or $w_0L/H_* = 0$, and can still be used to predict the eigenfrequencies accurately for moderately small values of w_0L/H_* and ϕ_a, except in the transition region. The size of the transition region increases with increasing values of w_0L/H_* and ϕ_a. It is interesting to note that for low values of λ_*^2 no curves were plotted. At such a low value of λ_*^2, the elastic strain in the cable becomes large, and cannot exist in real cables. Cut-off was selected at the point where the elastic wave speed is 10 times the transverse wave speed. This is approximately the minimal value for steel cables and chains used in mooring applications.

In figure 3-12 the shape of the first two modes for various values of λ_*^2 have been plotted. The symmetric[8] and anti-symmetric modes are changing over to hybrid modes in the transition region, and for high values of λ_*^2 to the anti-symmetric and symmetric mode of an inelastic chain. See also [Triantafyllou 84].

The application of numerical techniques to study the same phenomena

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[8]The reference to symmetric and anti-symmetric modes is of course approximate. The inclination of the cable destroys the fundamental symmetry of the problem.

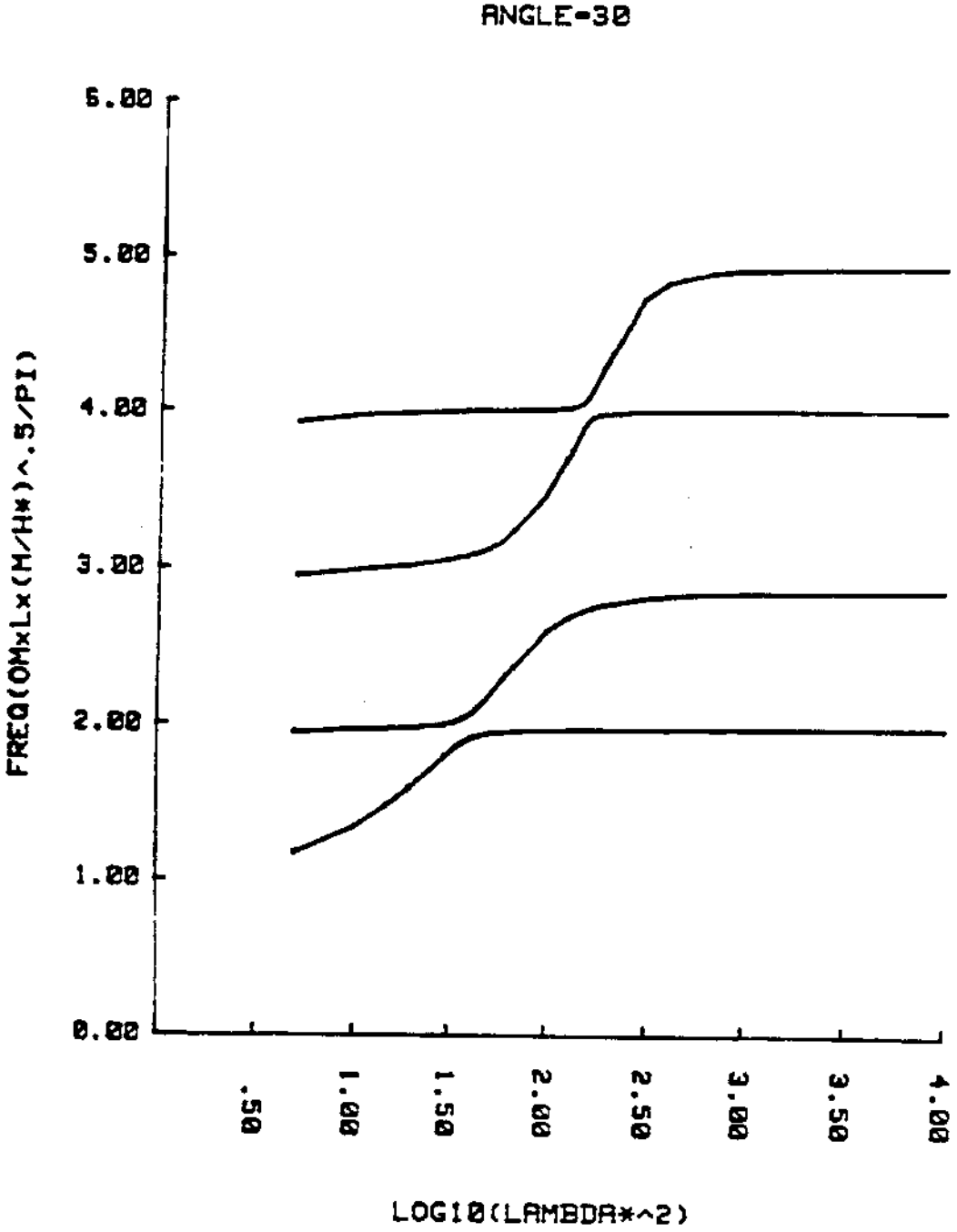

Figure 3-11: Eigenfrequencies of an Inclined Cable

($\phi_a = 30°$, $w_o L/H_* = 0.5$)

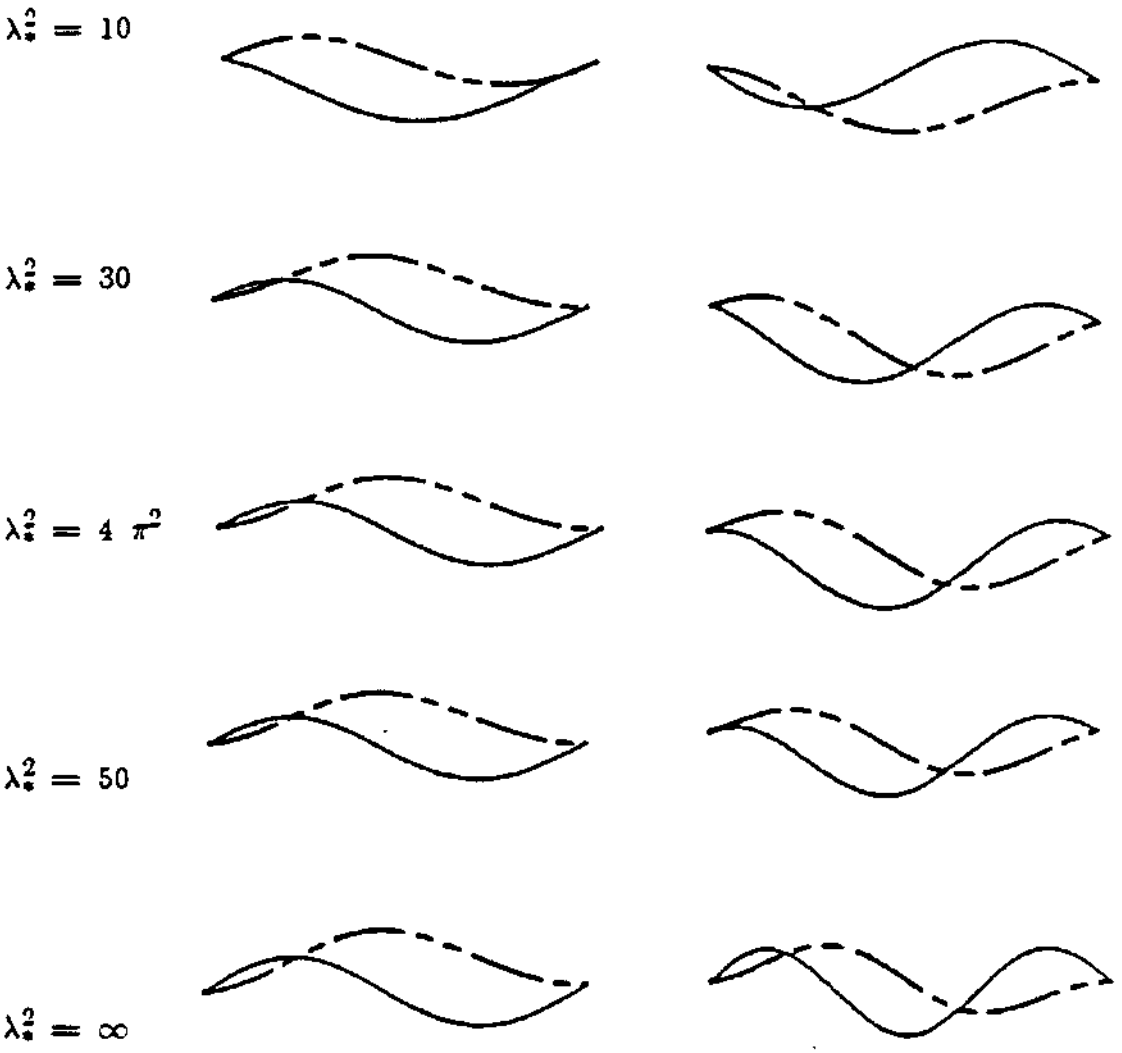

Figure 3-12: Formation of Hybrid Modes

($\phi_a = 30$, $w_o L/H_* = 0.5$)
(Normal displacement: solid line)
(10 × tangential displacement: dotted line)

confirms the existence of hybrid modes and the non-crossing of the modes (see part II). Yamaguchi [Yamaguchi 79] used Galerkin's method to solve for the eigenmodes. A cartesian description of the problem is used and the expansions are made using sinusoidal functions in space. The hybrid modes and the non-crossing of the modes are also obtained.

3.9 Numerical Solution of the Linearised Problem

The governing equations (3.1), which were solved using asymptotic methods, can also be solved using numerical methods. The governing equations written in the Fourier domain are (The solution method described here is also valid for cables which are subject to current forces):

$$
\begin{aligned}
-\, m\,\omega^2\, p &= \frac{dT_1}{ds} - T_o \frac{d\phi_o}{ds}\,\phi_1 \\
-\, M\,\omega^2\, q &= T_o \frac{d\phi_1}{ds} + T_1 \frac{d\phi_o}{ds} + \frac{dT_o}{ds}\,\phi_1 \qquad (3.117) \\
\frac{dp}{ds} - q \frac{d\phi_o}{ds} &= e_1 \\
\frac{dq}{ds} + p \frac{d\phi_o}{ds} &= (1 + e_o)\phi_1
\end{aligned}
$$

The problem is reduced to a set of four ordinary differential equations with two boundary conditions at each end.

Different methods can be used to solve the problem numerically. Using the linearity of the equations, shooting techniques can be used to reduce it to

an initial value problem, which can be solved by classical integration techniques.

Another possible method is the use of implicit differences, to reduce the problem to a set of linear algebraic equations. Unfortunately, when a realistic grid length is used, the number of equations becomes very large.

An explicit centered difference scheme was selected to solve this problem. The centered difference scheme allows a transfer matrix formulation of the problem. The boundary conditions can easily be handled in this formulation [Keller 69].

The linearized dynamic equations are rewritten in matrix form.

$$\begin{bmatrix} \frac{dT_1}{ds} \\ \frac{d\phi_1}{ds} \\ \frac{dp}{ds} \\ \frac{dq}{ds} \end{bmatrix} = \begin{bmatrix} 0 & T_o\, d\phi_o/ds & -m\omega^2 & 0 \\ -\frac{d\phi_o}{T_o ds} & -\frac{dT_o/ds}{T_o} & 0 & \frac{1}{T_o}(-M\omega^2) \\ \frac{1}{EA} & 0 & 0 & \frac{d\phi_o}{ds} \\ 0 & 1+e_o & \frac{d\phi_o}{ds} & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ \phi_1 \\ p \\ q \end{bmatrix}$$

We rewrite this as:

$$\frac{dy}{ds} = A(s)\, y(s) \tag{3.118}$$

$$\text{with: } y^T(s) = (T_1\ \phi_1\ p\ q\ )$$

Using centered differences, the following difference scheme is obtained:

$$y_{i+1} = \left[I - \frac{\Delta s_{i,\, i+1}}{2} A(s_{i+1}) \right]^{-1} \left[I + \frac{\Delta s_{i,\, i+1}}{2} A(s_i) \right] y_i$$

$$= B_i \, y_i \tag{3.119}$$

Due to the fact that the static variables are slowly changing with respect to the spatial coordinate, the difference scheme will give acceptable results, except when the wave length becomes of the same order as the grid length. The overall error is $O(\Delta s^2)$. To find the expression relating the two ends, we can write:

$$y_n = \left(\prod_{i=1}^{n-1} B_i \right) y_1 \tag{3.120}$$

We write:

$$\prod_{i=1}^{n-1} B_i = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \tag{3.121}$$

If motions are imposed at the upper end of the cable, the following relation is obtained:

$$\begin{bmatrix} T_{top} \\ \phi_{top} \\ p_{top} \\ q_{top} \end{bmatrix} = \begin{bmatrix} & \\ \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ & \end{bmatrix} \begin{bmatrix} T_{bot} \\ \phi_{bot} \\ 0 \\ 0 \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} p_{top} \\ q_{top} \end{bmatrix} = \alpha_{21} \, \alpha_{11}^{-1} \begin{bmatrix} T_{top} \\ \phi_{top} \end{bmatrix} \tag{3.122}$$

This provides a direct relation between the imposed motions at the top and the unknown dynamic angle and tension at the top. The matrices α_{11} and α_{21} are obtained from the matrix multiplication $\prod_{i=1}^{n-1} B_i$. These matrices can be calculated explicitly and they are functions of the static quantities, the frequency and the grid length.

The difference scheme can also be used to calculate the dynamic variables along the cable. When the displacements and the dynamic tension and angle at the top are known, the problem is an initial value problem which can be integrated directly.

The eigenfrequencies for the cable can be found by imposing zero motion boundary conditions at the top, and by searching for the frequencies which give the governing equations a non-trivial solution. This can be written according to (3.122) as:

$$\text{Det } [\ \alpha_{21}\ (\omega)] = 0 \tag{3.123}$$

A search of the roots of (3.123) is done by a root-finding method which locates the eigenfrequencies within a desired accuracy.

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Chapter 4

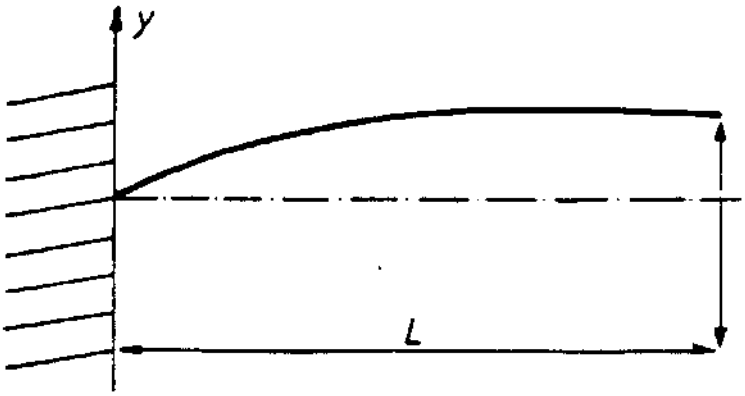
NON-LINEAR STRINGS

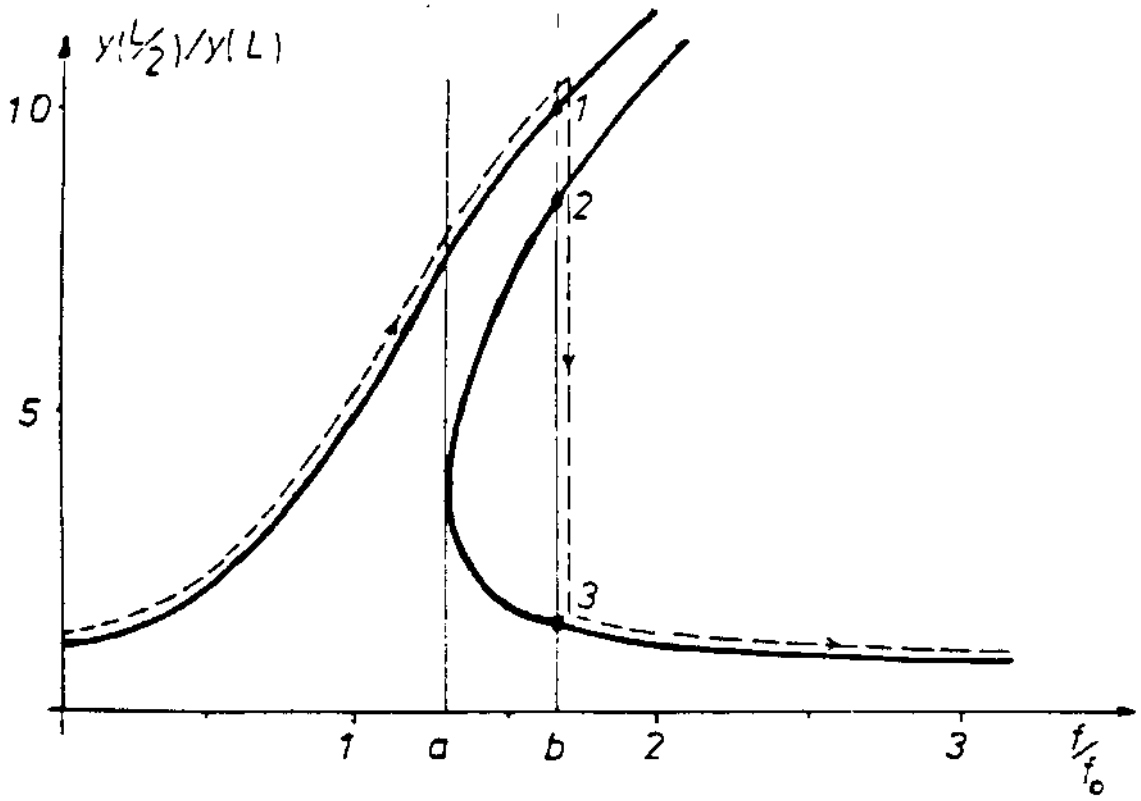
4.1 Introduction

The study of non-linear strings has attracted since the 1940's a great deal of attention. The linear string equations are valid only for small motions. while, when the motions become large, the dynamic tension generated by the motion must be included in the governing equations. The popularity of the subject is mainly due to the fact that the governing equations can be solved using perturbation analysis.

Carrier formulated the equations of motion for a non-linear string and solved the problem by means of a perturbation expansion in terms of the amplitude of motion. See [Carrier 45] and [Carrier 49]. Oplinger [Oplinger 60] studied the planar motion of a non-linear string using the method of separation of variables. He studied the motions of a string subject to an imposed motion of the boundary and he derived the response in the middle of the string. The response curves he obtained both analytically and experimentally, show a hardening spring type of effect.

In figure 4-1 the response of a non-linear string subject to a forced motion of one of its ends is shown. The response of the midpoint is given as a function of the ratio of the frequency to the first resonance frequency.





Taken from [Oplinger 60]

Figure 4-1: Response Curve for a Non-Linear String subject to Boundary Excitation

The parameter τ governs completely the non-linearity of the system. τ is given by:

$$\tau = \frac{E \cdot A}{T_o} \cdot \frac{(Ampl)^2}{2L^2} \tag{4.1}$$

The frequency response curve is bent to the right causing the resonance frequencies to shift to higher frequencies as the amplitude increases. True resonance is never obtained. Above a certain level, the response is multivalued and depends on how the exciting frequency was varied. Jump phenomena can occur, i.e. a sudden reduction in the amplitude of the response for a small increase in the frequency of excitation. Point 2 on the curve is an unstable point and cannot be achieved. When the excitation frequency is decreased an inverse jump (i.e. a sudden increase in amplitude) occurs.

Oplinger conducted also experiments to verify his results. The agreement of theory and experiments is remarkable. He also observed the spontaneous occurance of out-of-plane motion for high amplitudes.

The instabilities of the high-amplitude motion in a plane, and the resulting whirling motion, has been the focus of a major research effort. For a good review on this subject, see [Nayfeh 79].

Major contributions to the out-of-plane whirling motions can be found in [Murthy 65], [Miles 65], [Narasimha 68] and [Dickey 80]. Narasimha has also made in his paper an interesting derivation of the string equation, expressed in Eulerian coordinates.

Prediction of the accurate location of the jump phenomena can only be done when damping is included in the model. Anand [Anand 66] included viscous damping. A fluid drag type of damping was included by Hsu in his

study of parametric instabilities of a hanging string [Hsu 75]. This paper is particularly interesting as far as this study is concerned, because he uses the mode shapes directly in his solution method.

More recently, Richard and Anand studied the resonance phenomena caused by narrow band excitation [Richard 83]. Tagata [Tagata 83] studies the interesting subject of parametric excitation of a string of varying length.

In this work, we will only be concerned with the analysis of the in-plane motion of non-linear strings under excitation of one boundary. The main objective of the study is to illustrate the use of modal expansions to solve a non-linear problem. It should therefore not be considered as an in depth analysis of the subject.

4.2 Governing Equations

The string equation can be written in the transverse direction as:

$$M \frac{\partial^2 q}{\partial t^2} + b_d \frac{\partial q}{\partial t} \left| \frac{\partial q}{\partial t} \right| = \frac{\partial}{\partial x} \left[T(x,t) \frac{\partial q}{\partial x} \right] \tag{4.2}$$

q is the motion in the transverse direction, and x is the cartesian coordinate along the string in rest. In the above equation use was made of the fact that:

$$\frac{\partial x}{\partial s} \cong 1 \tag{4.3}$$

A quadratic damping of the Morison type was introduced, where b_d can be set in the form:

$$b_d = \frac{1}{2} \rho_w C_D D \tag{4.4}$$

It is convenient for further manipulation to write (4.2) as:

$$\frac{\partial^2\eta}{\partial t^2} + \frac{b_d}{M}\frac{\partial\eta}{\partial t}\left|\frac{\partial\eta}{\partial t}\right| = \frac{1}{ML^2}\frac{\partial}{\partial\sigma}\left[\, T(\sigma,t)\,\frac{\partial\eta}{\partial\sigma}\,\right] \tag{4.5}$$

where: η: the non-dimensional displacement q/L

σ: the non-dimensional length s/L

This equation is subject to boundary conditions:

$$\eta(0) = 0$$

$$\eta(1) = F(t) \tag{4.6}$$

A stress-strain relation for moderately large transverse displacements can be formulated as:

$$T = T_o(\sigma) + E{\cdot}A{\cdot}e$$

The compatibility relation, derived in chapter 1, can be written as:

$$e = \frac{\partial\xi}{\partial\sigma} + \frac{1}{2(1+e_o)}\left\{\left[\frac{\partial\xi}{\partial\sigma}\right]^2 + \left[\frac{\partial\eta}{\partial\sigma}\right]^2\right\} \tag{4.7}$$

Even for moderately large motions, the tangential motion is an order of magnitude smaller than the transverse motion. Therefore:

$$e \cong \frac{\partial\xi}{\partial\sigma} + \frac{1}{2}\left[\frac{\partial\eta}{\partial\sigma}\right]^2 \tag{4.8}$$

The strain can be assumed to be constant over the string length. This assumption is valid for excitation frequencies well below the first elastic eigenfrequency. Therefore:

$$e = \int_0^1 e\ d\sigma$$

$$= u(1) - u(0) + \frac{1}{2}\int_0^1 \eta_\sigma^2(\sigma)\ d\sigma \tag{4.9}$$

The governing equation can, therefore, be written as:

$$\frac{\partial^2\eta}{\partial t^2} + \frac{b_d}{M}\frac{\partial\eta}{\partial t}\left|\frac{\partial\eta}{\partial t}\right| = \frac{1}{ML^2}\frac{\partial}{\partial\sigma}\left[\, T_o(\sigma)\,\frac{\partial\eta}{\partial\sigma}\,\right] + \frac{E\cdot A}{ML^2}\,e\,\frac{\partial^2\eta}{\partial\sigma^2} \tag{4.10}$$

in which e is given by (4.9). This governing equation is subject to the boundary conditions (4.6). In equation (4.10) the static tension is allowed to vary along the length, which makes it more general than the case of a string under constant tension.

4.3 Expansions in Orthogonal Functions

We propose to expand the solution of the problem in the following form:

$$\eta(\sigma,t) = \eta_o(\sigma)\, f(t) + \sum_{i=1}^{n} \eta_i(\sigma)\, C_i(t) \tag{4.11}$$

where: $\eta_o(\sigma)$: the quasi-static solution, to take care of the inhomogeneous boundary conditions.

$\eta_i(\sigma)$: the modes, obtained from the linear eigenvalue problem with homogeneous boundary conditions.

The boundary condition is directly taken care of by introducing the quasi-static solution. The main advantage of using this technique is that for low frequency excitation only a very limited number of modes needs to be included.

4.4 The Quasi-Static Problem

The quasi-static problem consists of finding the solution to the following equation:

$$\frac{\partial}{\partial\sigma}\left[\, T_o\,(\sigma)\,\frac{\partial\eta}{\partial\sigma}\,\right] = 0 \qquad (4.12)$$

$$\eta(0) = 0 \qquad \eta(1) = 1$$

The solution to this problem can be written as:

$$\eta_o\,(\sigma) = C_1 \int_0^{\sigma} \frac{1}{T_o\,(\sigma)}\, d\sigma \qquad (4.13)$$

$$\text{where:}\quad C_1 = \left\{ \int_0^1 \frac{1}{T_o\,(\sigma)}\, d\sigma \right\}^{-1}$$

For a string under constant tension the solution is simply:

$$\eta_o\,(\sigma) = \sigma$$

4.5 The Eigenproblem

The linear eigenproblem associated with (4.10) can also be solved easily. The case of a string with varying static tension is a classical Sturm-Liouville problem:

$$\frac{1}{ML^2}\,\frac{\partial}{\partial\sigma}\left[\, T_o\,(\sigma)\,\frac{\partial\eta}{\partial\sigma}\,\right] = -\,\omega^2\,\eta \qquad (4.14)$$

$$\eta(0) = 0 \qquad \eta(1) = 0$$

The eigenfunctions can be obtained using analytical, perturbation or numerical methods. The WKB method, as a solution to this problem, has

been described in detail in chapter 3. The modes are orthogonal, with the following orthogonality condition: (η_i with $i \neq 0$ are the eigenmodes.)

$$\int_0^1 M\ \eta_i\ (\sigma)\ \eta_j\ (\sigma)\ d\sigma = 0 \qquad i \neq j \tag{4.15}$$

The eigenmodes are chosen to be orthonormal:

$$\int_0^1 M\ \eta_i^2(\sigma)\ d\sigma = 1 \tag{4.16}$$

4.6 Time Integration

The expansion (4.11) can be substituted in (4.10) where all of the space varying functions are known functions.

Substitution gives:

$$\eta_o\ \ddot{f} + \sum_{i=1}^{n} \eta_i\ \ddot{C}_i + \frac{b_d}{M} \left| \frac{\partial \eta}{\partial t} \right| \left[\eta_o\ \dot{f} + \sum_{i=1}^{n} \eta_i\ \dot{C}_i \right]$$

$$= - \sum_{i=1}^{n} \omega_i^2\ \eta_i\ C_i + \frac{E \cdot A}{ML^2}\ e \left[\eta_{o\sigma\sigma}\ f(t) + \sum_{i=1}^{n} \eta_{i\sigma\sigma}\ C_i\ (t) \right] \tag{4.17}$$

Multiplying with $M\eta_j$ and integrating over the domain gives the governing equation:

$$\ddot{C}_j + b_d \sum_{i=1}^{n} \int_0^1 \left| \frac{\partial \eta}{\partial t} \right| \eta_j \, \eta_i \, d\sigma \, \dot{C}_i$$

$$+ \sum_{i=1}^{n} \left[\delta_{ji} \, \omega_j^2 + \frac{E \cdot A}{L^2} \, e \int_0^1 \eta_{j\sigma} \, \eta_{i\sigma} \, d\sigma \right] C_i$$

$$= - \int_0^1 M \, \eta_o \, \eta_j \, d\sigma \, \ddot{f} - b \int_0^1 \left| \frac{\partial \eta}{\partial t} \right| \eta_o \, \eta_j \, d\sigma \, \dot{f}$$

$$- \frac{E \cdot A}{L^2} \, e \int_0^1 \eta_{o\sigma} \, \eta_{j\sigma} \, d\sigma \, f \qquad (4.18)$$

This sommation will be taken over a finite number n, so that a set of n non-linear differential equations with n unknown functions C_j is obtained. Several of the terms in the expansion are time varying. These are the coefficients involving $\partial\eta/\partial t$ and e, which are unknown.

Among the time integration schemes, which can be used to integrate (4.18) are: the explicit scheme, and the implicit scheme together with some iteration method. This will be discussed in great detail in chapter 5. Fortunately (4.18) has the characteristic that if the non-linearities are not dominant, the n equations are very weakly coupled. Even ignoring completely the coupling can lead to good practical results [Hsu 75].

In (4.18) integration by parts was used to reduce the order of spatial derivatives by one. This is a very important feature numerically and will be used in the more general cable dynamic problem extensively.

When a string with constant tension is simulated in time, the modes are simply sinuoidal functions and some of the cross-coupling terms between the equations disappear. An example of time simulation for a string with constant tension is discussed in part II.

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Chapter 5

NON-LINEAR DYNAMICS USING GALERKIN'S METHOD

5.1 Introduction

The advantages of using modal decomposition in linear dynamic analysis of structures are well documented [Clough 75] and [Bathe 82]. The dynamic equations of motion become uncoupled and, what is even more important, in many cases the number of degrees of freedom can be reduced drastically, without losing accuracy.

Modal expansions in the analysis of non-linear dynamics is much less developed. Some structural examples have been solved; buckling problems [Nickell 76], contact problems and earthquake excitation [Bathe 81]. The results were very encouraging. Although the problems considered were highly non-linear, a very small number of modes gave reasonable accuracy.

In riser dynamics Galerkin's method has been used by several researchers. Kirk [Kirk 79] assumes the deflection as a series of sinusoids. Dareing [Dareing 79] solved the riser problem using a modal decomposition and he found that under regular wave excitation few modes are needed. To uncouple the equations he used a uniform equivalent damping over the riser. The full, non-linear drag term was used in the modal expansion of Y. C. Kim [Kim 83], who investigated, also, the effect of second order geometric terms. Fast convergence of the modal expansion is reported. Applications of modal expansions in non-linear strings have been described in

chapter 4.

The use of modal analysis in linearised, small sag cable dynamics was developed by Irvine [Irvine 76]. He shows the influence of the ratio of elastic and geometric stiffness on the modal dynamic tension. See also [Veletsos 82]. Hagedorn [Hagedorn 80] includes non-linearities up to third order in the modal expansions of a small sag cable. The resulting equations were solved using a perturbation solution. In a very interesting paper, written in Japanese and therefore not well known in this continent, Yamaguchi [Yamaguchi 79] uses Galerkin's method with sinusoidal functions to solve the general linear cable problem. He uses this method to calculate eigenfrequencies and good agreement with previous results is reported. The use of modal expansion for a cable without using the small sag approximation has, to the author's knowledge, not been reported.

5.2 Galerkin's Method

The Galerkin method will be explained using a simple one-dimensional example, although the concept can be applied to problems in several dimensions. For more details, see [Gotlieb 77]. The problem considered is a mixed initial boundary value problem.

$$\frac{\partial u}{\partial t} = L(x,t)\ u(x,t) + f(x,t)$$

$$u(x_1,t) = 0 \qquad u(x_2,t) = 0 \qquad t > 0$$

$$u(x,o) = g(x) \qquad x_1 < x < x_2$$

The operator L is a linear differential operator in x. Only the homogeneous problem is considered, because the inhomogeneous problem can be

rewritten as a homogeneous one by adding an appropriate, but otherwise arbitrary function, satisfying the inhomogeneous boundary conditions.

The solution is expanded in the form of a truncated series:

$$u_n\ (x,t) = \sum_{i=0}^{n} a_i\ (t)\ \phi_i\ (x)$$

where ϕ_i are assumed to be linearly independent space functions, satisfying the homogeneous boundary conditions.

The residual error, by substituting the truncated series in the governing equation, is:

$$R_n = \frac{\partial u_n}{\partial t} - L_n - f$$

Several methods can be used to minimize this error over the solution domain. In Galerkin's method the error is weighted with the trial functions and the averaged value is required to be 0. This gives:

$$\sum_{i=0}^{n} \int \phi_j\ \phi_i\ dx\ \frac{\partial a_i}{\partial t} = \sum_{i=0}^{n} \int \phi_j\ L\ \phi_i\ dx\ a_i + \int \phi_j\ f\ dx \qquad j=1...n$$

Several types of functions can be used in the expansions. If the eigenfunctions of the problem are used, the above equations become uncoupled, resulting in a significant simplification of the time integration scheme.

5.3 Linearised Dynamics using Modal Expansions

The modal expansions can be applied directly to the linearised dynamic equations. The external forces can be of arbitrary form and they are a function of time. Because the static forces are in equilibrium (see chapter 2) only dynamic forces must be considered. The governing dynamic equations are:

$$m \frac{\partial^2 p}{\partial t^2} = \frac{\partial T_1}{\partial s} - T_o \frac{\partial \phi_o}{\partial s} \phi_1 + F_t(t)(1 + e)$$

$$M \frac{\partial^2 q}{\partial t^2} = \frac{\partial \phi_o}{\partial s} T_1 + \frac{\partial T_o}{\partial s} \phi_1 + T_o \frac{\partial \phi_1}{\partial s} + F_n(t)(1 + e)$$

$$\frac{\partial p}{\partial s} - q \frac{\partial \phi_o}{\partial s} = e_1 \qquad (5.1)$$

$$\frac{\partial q}{\partial s} + p \frac{\partial \phi_o}{\partial s} = \phi_1 (1+e_o)$$

These equations are supplemented with the linear tension-strain relation:

$$T_1 = E \cdot A \cdot e_1 \qquad (5.2)$$

To be able to use modal expansions, it is more convenient to eliminate the dynamic tension out of the relations (5.1) and (5.2):

$$m \frac{\partial^2 p}{\partial t^2} - \frac{\partial}{\partial s}\left[E \cdot A \left(\frac{\partial p}{\partial s} - q \frac{\partial \phi_o}{\partial s}\right) \right]$$

$$+ T_o \frac{\partial \phi_o}{\partial s} \cdot \frac{1}{1+e_o} \left(\frac{\partial q}{\partial s} + p \frac{\partial \phi_o}{\partial s}\right) = F_t(t)(1 + e)$$

$$M \frac{\partial^2 q}{\partial t^2} - \frac{\partial \phi_o}{\partial s} E \cdot A \left(\frac{\partial p}{\partial s} - q \frac{\partial \phi_o}{\partial s}\right) - \frac{\partial T_o}{\partial s} \cdot \frac{1}{1+e_o} \left(\frac{\partial q}{\partial s} + p \frac{\partial \phi_o}{\partial s}\right) \tag{5.3}$$

$$- T_o \frac{\partial}{\partial s}\left[\frac{1}{1+e_o} \left(\frac{\partial q}{\partial s} + p \frac{\partial \phi_o}{\partial s}\right)\right] = F_n(t)(1 + e)$$

To have a complete formulation of the dynamic problem, four boundary conditions are needed. In this work we will assume imposed top motions and a fixed end at the bottom. The boundary conditions are then:

$$\begin{aligned} p(0) &= 0 & p(L) &= f(t) \\ q(0) &= 0 & q(L) &= g(t) \end{aligned} \tag{5.4}$$

we expand the solutions in the following form:

$$\begin{aligned} p(s,t) &= \sum_{i=0}^{n} c_i(t)\, p_i(s) \\ q(s,t) &= \sum_{i=0}^{n} c_i(t)\, q_i(s) \end{aligned} \tag{5.5}$$

where the functions in (5.5) must be determined to satisfy the governing equations (5.3) and the boundary conditions (5.4).

5.3.1 Solution of the Linearised Quasi-Static Problem

The boundary conditions (5.4) will be satisfied by introducing the quasi-static solutions in the proposed expansions. The quasi-static solutions are

obtained by neglecting the inertia and external forces in (5.1) or (5.3) and solving the resulting equations:

$$0 = \frac{\partial T_1}{\partial s} - T_o \frac{\partial \phi_o}{\partial s} \phi_1$$

$$0 = \frac{\partial \phi_o}{\partial s} T_1 + \frac{\partial T_o}{\partial s} \phi_1 + T_o \frac{\partial \phi_1}{\partial t}$$

$$\phi_1 = \frac{1}{1+e_o} \cdot \left[\frac{\partial q}{\partial s} + p \frac{\partial \phi_o}{\partial s} \right] \tag{5.6}$$

$$e_1 = \frac{\partial p}{\partial s} - q \frac{\partial \phi_o}{\partial s}$$

$$T_1 = E \cdot A \cdot e_1$$

with boundary conditions:

$$p(0) = 0 \qquad p(L) = f(t)$$

$$q(0) = 0 \qquad q(L) = g(t)$$

We can separate the time and space dependence by writing the solutions, making use of the linearity, in the following form:

$$p_{qs} = f(t)\, p_{qs1}(s) + g(t)\, p_{qs2}(s)$$

$$q_{qs} = f(t)\, q_{qs1}(s) + g(t)\, q_{qs2}(s) \tag{5.7}$$

where p_{qs1} , q_{qs1} must satisfy:

$$p_{qs1}(0) = 0 \qquad p_{qs1}(L) = 1$$

$$q_{qs1}(0) = 0 \qquad q_{qs1}(L) = 0$$

and p_{qs2} , q_{qs2} must satisfy:

$$p_{qs2}(0) = 0 \qquad p_{qs2}(L) = 0$$

$$q_{qs2}(0) = 0 \qquad q_{qs2}(L) = 1$$

The quantities p_{qs},q_{qs} above are the solution to the very slow motion of a cable forced to move with unit amplitude in the tangential and normal direction, respectively. They are the zero frequency limit of the linear dynamic problem. The solution can be obtained by using methods similar to the ones described in chapter 3, because the equations are in the same form as the equations providing the mode shapes. In this work, numerical central differences were used to obtain quasi-static solutions.

We can now try expansions of the solutions in the following form:

$$p(s,t) = f(t)\, p_{qs1}(s) + g(t)\, p_{qs2}(s) + \sum_{i=1}^{n} C_i(t)\, p_i(s)$$

$$q(s,t) = f(t)\, q_{qs1}(s) + g(t)\, q_{qs2}(s) + \sum_{i=1}^{n} C_i(t)\, q_i(s) \qquad (5.8)$$

where $p_i(s)$, $q_i(s)$ are the components of the i th mode with fixed end conditions.

5.3.2 Solution of the Eigenvalue Problem

The eigenmodes of the cable are obtained by solving the following eigenvalue problem.

$$- m \omega^2 p - \frac{\partial}{\partial s}\left[E \cdot A \left(\frac{\partial p}{\partial s} - q \frac{\partial \phi_o}{\partial s}\right) \right] + T_o \frac{\partial \phi_o}{\partial s} \cdot \frac{1}{1+e_o} \left(\frac{\partial q}{\partial s} + p \frac{\partial \phi_o}{\partial s}\right) = 0 \qquad (5.9)$$

$$- M \omega^2 q - \frac{\partial \phi_o}{\partial s} E \cdot A \left(\frac{\partial p}{\partial s} - q \frac{\partial \phi_o}{\partial s}\right) - \frac{\partial T_o}{\partial s} \cdot \frac{1}{1+e_o} \cdot \left[\frac{\partial q}{\partial s} + p \frac{\partial \phi_o}{\partial s} \right] - T_o \frac{\partial}{\partial s}\left[\frac{1}{1+e_o} \left(\frac{\partial q}{\partial s} + p \frac{\partial \phi_o}{\partial s}\right) \right] = 0$$

with the following boundary conditions:

$$p(0) = q(0) = 0 \qquad p(L) = q(L) = 0$$

Again, the methods described in chapter 3 can be used to solve this problem. The results of the analysis are the eigenfrequencies and the eigenmodes p_i , q_i, as well as the modal dynamic tension and the modal dynamic angle, T_i , ϕ_i respectively.

The eigenmodes are orthogonal to each other, in the following form:

$$\int_0^L (m\ p_i p_j + M\ q_i q_j\ )\ ds = 0 \qquad i \neq j \qquad (5.10)$$

In the sequel, the modes are also assumed to be normalised so that:

$$\int_0^L (m\ p_i^2 + M\ q_i^2\ )\ ds = 1 \qquad (5.11)$$

5.3.3 Substitution in the Solution

The proposed solution (5.8)

$$p(s,t) = f(t)\ p_{qs1}(s) + g(t)\ p_{qs2}(s) + \sum_{i=1}^{n} C_i(t)\ p_i(s)$$

$$q(s,t) = f(t)\ q_{qs1}(s) + g(t)\ q_{qs2}(s) + \sum_{i=1}^{n} C_i(t)\ q_i(s)$$

can now be substituted in (5.3), taking into account the quasi-static equations and the eigenvalue equations:

$$m\left[\ddot{f}p_{qs1} + \ddot{g}p_{qs2} + \sum_{i=1}^{n} p_i\ddot{C}_i\right] + \sum_{i=1}^{n} m\,\omega_i^2\,p_i\,C_i = F_t(1+e)$$

$$M\left[\ddot{f}q_{qs1} + \ddot{g}q_{qs2} + \sum_{i=1}^{n} q_i\ddot{C}_i\right] + \sum_{i=1}^{n} M\,\omega_i^2\,q_i\,C_i = F_n(1+e) \qquad (5.12)$$

Multiplying the first equation in (5.12) with p_i and the second equation in (5.12) with q_i and integrating over the length, we obtain:

$$\ddot{C}_i + \omega_i^2\,C_i = \mathcal{F}_i - \mathcal{A}_i\,\ddot{f} - \mathcal{B}_i\,\ddot{g} \qquad (5.13)$$

where: $\mathcal{F}_i = \int (F_t\,p_i + F_n\,q_i)\,(1+e)\,ds$

$$\mathcal{A}_i = \int (m\,p_{qs1}\,p_i + M\,q_{qs1}\,q_i)\,ds$$

$$\mathcal{B}_i = \int (m\,p_{qs2}\,p_i + M\,q_{qs2}\,q_i)\,ds$$

To make the description more complete, we introduce a linear modal damping. The equations (5.13) can then be written in its final form as:

$$\ddot{C}_i + 2\,\xi_i\,\omega_i\,\dot{C}_i + \omega_i^2\,C_i = \mathcal{F}_i - \mathcal{A}_i\,\ddot{f} - \mathcal{B}_i\,\ddot{g} \qquad (5.14)$$

5.4 Time Integration

In this analysis Newmark's method was selected. It is an implicit time integration scheme which is unconditionally stable. In the case of a modal description, no additional computational effort is needed compared with an explicit scheme, so that it is unquestionably preferable to use an implicit scheme. Newmark's method was developed for direct time integration of very large systems of linear equations, where it gives excellent results. (see [Bathe 82])

The Newmark integration scheme evaluates the equilibrium equations at a time $t + \Delta t$ as follows:

$$
\begin{aligned}
{}^{t+\Delta t}\dot{X} &= {}^{t}\dot{X} + \left[(1 - \delta)\, {}^{t}\ddot{X} + \delta\, {}^{t+\Delta t}\ddot{X} \right] \Delta t \\
{}^{t+\Delta t}X &= {}^{t}X + {}^{t}\dot{X}\, \Delta t + \left[(1/2 - \alpha)\, {}^{t}\ddot{X} + \alpha\, {}^{t+\Delta t}\ddot{X} \right] \Delta t^2
\end{aligned}
\tag{5.15}
$$

where α and δ are parameters which can be selected to give an optimal combination of accuracy and stability. Writing the velocity and acceleration in terms of displacement, we obtain, using the conventions in [Bathe 82]:

$$
\begin{aligned}
a_0 &= \frac{1}{\alpha \Delta t^2} \\
a_1 &= \frac{\delta}{\alpha \Delta t} \\
a_2 &= \frac{1}{\alpha \Delta t} \\
a_3 &= \frac{1}{2\alpha} - 1
\end{aligned}
$$

$$a_4 = \frac{\delta}{\alpha} - 1$$

$$a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right)$$

$$a_6 = \Delta t\ (1-\delta)$$

$$a_7 = \delta \Delta t$$

$$\begin{aligned} {}^{t+\Delta t}\ddot{x} &= a_0\ (\ {}^{t+\Delta t}x - {}^{t}x\ ) - a_2\ {}^{t}\dot{x} - a_3\ {}^{t}\ddot{x} \\ {}^{t+\Delta t}\dot{x} &= -\ a_1^t\ x - a_4\ {}^{t}\dot{x} - a_5\ {}^{t}\ddot{x} \end{aligned} \qquad (5.16)$$

Evaluation of the equations of motion at time $t+\Delta t$ gives:

$$\begin{aligned} &{}^{t+\Delta t}\ddot{C}_i + 2\xi_i\ \omega_i\ {}^{t+\Delta t}\dot{C}_i + \omega_i^2\ {}^{t+\Delta t}C_i \\ &\qquad = {}^{t+\Delta t}\mathcal{F}_i - {}^{t+\Delta t}\ddot{f}\ \mathcal{A}_i - {}^{t+\Delta t}\ddot{g}\ \mathcal{B}_i \end{aligned} \qquad (5.17)$$

Substitution of (5.16) in (5.17) gives:

$$\begin{aligned} &(\ a_0 + a_1\ 2\xi_i\ \omega_i + \omega_i^2\ )\ {}^{t+\Delta t}C_i \\ &= (\ a_0\ {}^{t}C_i + a_2\ {}^{t}\dot{C}_i + a_3\ {}^{t}\ddot{C}_i\ ) \\ &+ 2\xi_i\ \omega_i\ (a_1\ {}^{t}C_i + a_4\ {}^{t}\dot{C}_i + a_5\ {}^{t}\dot{C}_i\ ) \\ &+ {}^{t+\Delta t}\mathcal{F}_i - {}^{t+\Delta t}\ddot{f}\ \mathcal{A}_i - {}^{t+\Delta t}\ddot{g}\ \mathcal{B}_i \end{aligned} \qquad (5.18)$$

When the above equations (5.18) are solved for each time step, a time history of the modal displacements is obtained. A complete solution of the linearised cable problem can be constructed as follows:

$$p(s,t) = f(t)\ p_{qs1}\ (s) + g(t)\ p_{qs2}\ (s) + \sum_{i=1}^{n} p_i\ (s)\ C_i\ (t)$$

$$q(s,t) = f(t)\ q_{qs1}\ (s) + g(t)\ q_{qs2}\ (s) + \sum_{i=1}^{n} q_i\ (s)\ C_i\ (t)$$

$$\phi_1(s,t) = f(t)\ \phi_{qs1}\ (s) + g(t)\ p_{qs2}\ (s) + \sum_{i=1}^{n} \phi_i\ (s)\ C_i\ (t) \qquad (5.19)$$

$$T_1(s,t) = f(t)\ T_{qs1}\ (s) + g(t)\ T_{qs2}\ (s) + \sum_{i=1}^{n} T_i\ (s)\ C_i\ (t)$$

where p_i , q_i , ϕ_i , T_i are the modal components of the tangential displacement, normal displacement, dynamic angle and dynamic tension respectively.

The Newmark integration scheme is unconditionally stable when $\delta \geq 0.5$ and $\alpha \geq 0.25(\delta+0.5)^2$, which means that the solution does not grow without bound for any initial condition and for any Δt. Newmark introduced his method originally with $\delta = 0.5$ and $\alpha = 0.25$. This method is called the constant average acceleration method (also trapezoidal rule). (See figure 5-1)

It can be shown [Bathe 82] that using this selection, an undamped system shows no amplitude decay and only a phase shift due to time integration errors appears, while no numerical damping is introduced by the scheme. If numerical damping is desired, the value of δ must be taken greater than 0.5 and the value of α should be varied accordingly. Due to its favorable characteristics, the constant average acceleration scheme will be used.

The following relations are obtained for (5.16) using $\delta = 0.5$ and $\alpha = 0.25$:

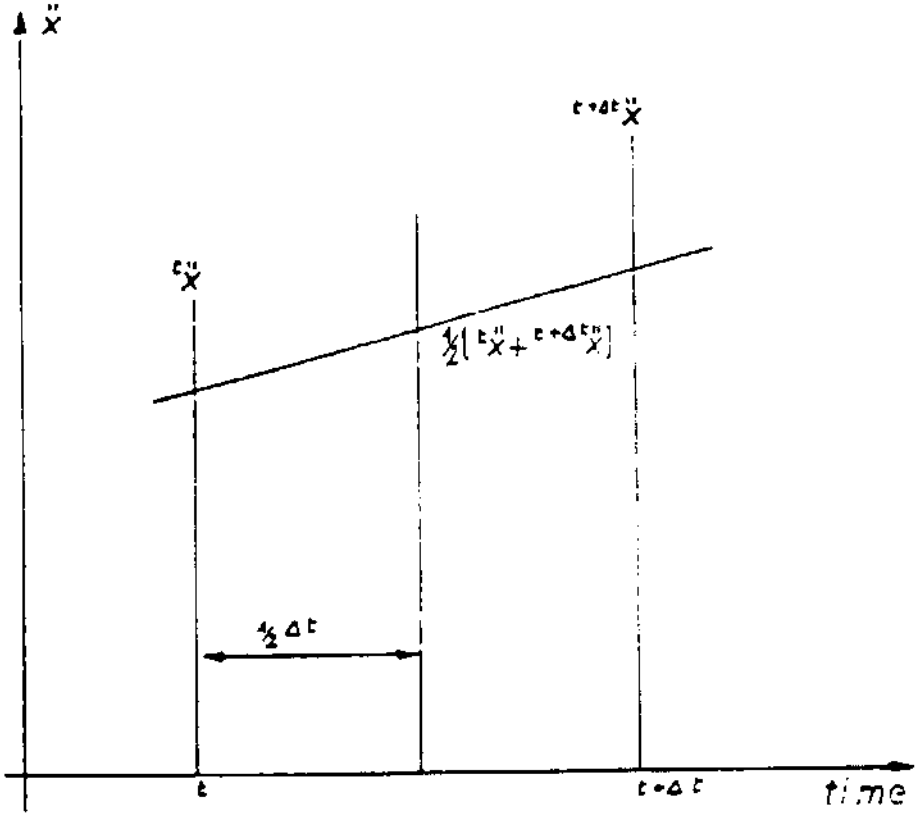

Figure 5-1: Newmark Constant Acceleration Scheme [Bathe 82]

$$
\begin{aligned}
{}^{t+\Delta t}\ddot{x} &= (0.25\Delta^2 t)^{-1}\,({}^{t+\Delta t}x - {}^{t}x) - (0.25\Delta t)^{-1}\,{}^{t}\dot{x} - {}^{t}\ddot{x} \\
{}^{t+\Delta t}\dot{x} &= {}^{t}\dot{x} + 0.5\Delta t\,({}^{t}\ddot{x} + {}^{t+\Delta t}\ddot{x})
\end{aligned}
\tag{5.20}
$$

5.5 Non-Linear External Forces

5.5.1 Description

A full description of all the external forces was given in chapter 1. In the linear problem only the dynamic component of these forces should be included.

The only important force for most applications is the hydrodynamic loading. In this work, we use, apart from the added mass force, a quadratic drag force for the hydrodynamic loading, as described in chapter 1. Other forms of loading can easily be implemented. The dynamic component of the external force terms can be written as:

$$
\begin{aligned}
F_{dt}(1+e) &= 0.5\,\rho_w\,C_{Dt}(Re)\,D_o\,(U\cos\phi - v_t)\,|U\cos\phi - v_t|\,(1+e/2) \\
&\quad - 0.5\,\rho_w\,C_{dt}(Re)\,D_o\,U\cdot\cos\phi_o\,|\,U\cdot\cos\phi_o\,|(1+e/2)
\end{aligned}
\tag{5.21}
$$

$$
\begin{aligned}
F_{dn}(1+e) &= -0.5\,\rho_w\,C_{Dn}(Re)\,D_o\,(U\sin\phi + v_n)|U\sin\phi + v_n|(1+e/2) \\
&\quad + 0.5\,\rho_w\,C_{dn}(Re)\,D_o\,U\cdot\sin\phi_o\,|\,U\cdot\sin\phi_o\,|(1+e/2)
\end{aligned}
$$

5.5.2 Time Integration in the Presence of a Non-Linear Force

The equation of motion, using Newmark's method has been written as:

$$(a_0 + a_1 2\xi_i \omega_i + \omega_i^2)\ {}^{t+\Delta t}C_i$$

$$= (a_0\ {}^{t}C_i + a_2\ {}^{t}\dot{C}_i + a_3\ {}^{t}\ddot{C}_i)$$

$$+ 2\xi_i \omega_i (a_1\ {}^{t}C_i + a_4\ {}^{t}\dot{C}_i + a_5\ {}^{t}\dot{C}_i)$$

$$+ {}^{t+\Delta t}\mathcal{F}_i - {}^{t+\Delta t}\ddot{f}\ A_i - {}^{t+\Delta t}\ddot{g}\ B_i$$

In the case of the hydrodynamic drag force, the modal force is given by:

$${}^{t+\Delta t}\mathcal{F}_i = \int_0^L ({}^{t+\Delta t}F_{dt}\ p_i + {}^{t+\Delta t}F_{dn}\ q_i)\ (1 + {}^{t+\Delta t}e)\ ds \qquad (5.22)$$

The forces are not known at time $t+\Delta t$, because they depend in a highly non-linear way on the relative velocities, therefore some approximation of the hydrodynamic drag force is required.

$${}^{t+\Delta t}\dot{\mathcal{F}}_i = \int_0^L ({}^{t+\Delta t}\dot{F}_{dt}\ p_i + {}^{t+\Delta t}\dot{F}_{dn}\ q_i)\ (1 + \dot{e})\ ds \qquad (5.23)$$

Several methods can be used to make estimations. The easiest approximation is simply to use the forces calculated at the previous time step. Therefore:

$${}^{t}\dot{\mathcal{F}}_i = \int_0^L ({}^{t}F_{dt}\ p_i + {}^{t}F_{dn}\ q_i)\ (1 + {}^{t}e)\ ds \qquad (5.24)$$

Improvements on this first estimation can be achieved by using an iteration procedure to approximate the modal forces.

The complete governing equations for the modal components with equilibrium iterations can be written as:

$$(a_0 + a_1 2\xi_i \omega_i + \omega_i^2) \; {}^{t+\Delta t}C_i^k$$

$$= (a_0 \; {}^{t}C_i + a_2 \; {}^{t}\dot{C}_i + a_3 \; {}^{t}\ddot{C}_i)$$

$$+ 2\xi_i \omega_i (a_1 \; {}^{t}C_i + a_4 \; {}^{t}\dot{C}_i + a_5 \; {}^{t}\dot{C}_i)$$

$$+ {}^{t+\Delta t}\mathcal{F}_i^{k-1} - {}^{t+\Delta t}\ddot{f} \; A_i - {}^{t+\Delta t}\ddot{g} \; B_i \qquad (5.25)$$

$$\text{with:} \quad \mathcal{F}_i^{k-1} = \int_0^L ({}^{t+\Delta t}F_{dt} \; p_i + {}^{t+\Delta t}F_{dn} \; q_i) (1 + {}^{t+\Delta t}e) \; ds$$

$$\text{with:} \quad \mathcal{F}_i^{o} = \int_0^L ({}^{t}F_{dt} \; p_i + {}^{t}F_{dn} \; q_i) (1 + {}^{t}e) \; ds$$

5.5.3 Discussion

Equation (5.25) describes completely the dynamics of a two-dimensional cable with a non-linear time varying force. The only restriction is the assumption of small motions, which means that the dynamic tension must be small compared to the static tension. The dynamic angle must also be small.

The complete solution procedure is described in table 5-I. The major advantage of the method is that a minimal number of modal components can be selected to represent the cable motions accurately.

It is the author's opinion that this procedure can also be used to solve the inverse problem. Knowing the cable motions, study the hydrodynamic forces in their modal components. This could enable us to make some progress in the understanding of the hydrodynamic loading of flexible structures.

Solution Method

1. Evaluation of the Static Equilibrium

2. Calculation of quasi-static solution due to top motion

3. Calculation of eigenfrequencies and modes

4. Time simulation

   For each step ;

   Evaluation of the force

Iterative Evaluation of the modal force

Procedure Evaluation of the modal components

   Evaluation of the displacements.

Table 5-I: Time simulation using Modal Expansion

5.6 Non-Linear Governing equations

The full non-linear equations of motion were derived in detail in chapter 2. The two dimensional equations of motion are written in a coordinate system fixed with respect to the equilibrium state. In our analysis we selected the static equilibrium as the reference coordinate of the system. (See figure 5-2)

p: tangential coordinate on static configuration

q: normal coordinate on static configuration

ϕ_1: angle between tangent on dynamic and static configuration

5.6.1 Dynamic Equations

The dynamic equilibrium, as outlined in chapter 2, can be used along the p and q directions to find the equations:

$$m\frac{\partial^2 p}{\partial t^2} = \frac{\partial T}{\partial s}\cdot\cos\phi_1 - T\frac{\partial \phi}{\partial s}\cdot\sin\phi_1 + \sum_{i=0}^{n} F_{ext,p,i}(1+e)$$

$$m\frac{\partial^2 q}{\partial t^2} = \frac{\partial T}{\partial s}\cdot\sin\phi_1 + T\frac{\partial \phi}{\partial s}\cdot\cos\phi_1 + \sum_{i=0}^{n} F_{ext,q,i}(1+e) \qquad (5.26)$$

These can be rewritten as:

$$m\frac{\partial^2 p}{\partial t^2} = \frac{\partial}{\partial s}(T\cos\phi_1) - T\frac{\partial \phi_o}{\partial s}\sin\phi_1 + \sum_{i=0}^{n} F_{ext,p,i}(1+e)$$

$$m\frac{\partial^2 q}{\partial t^2} = \frac{\partial}{\partial s}(T\sin\phi_1) + T\frac{\partial \phi_o}{\partial s}\cos\phi_1 + \sum_{i=0}^{n} F_{ext,q,i}(1+e) \qquad (5.27)$$

The external forces on the cable consist of the following components

- Gravity and buoyancy forces

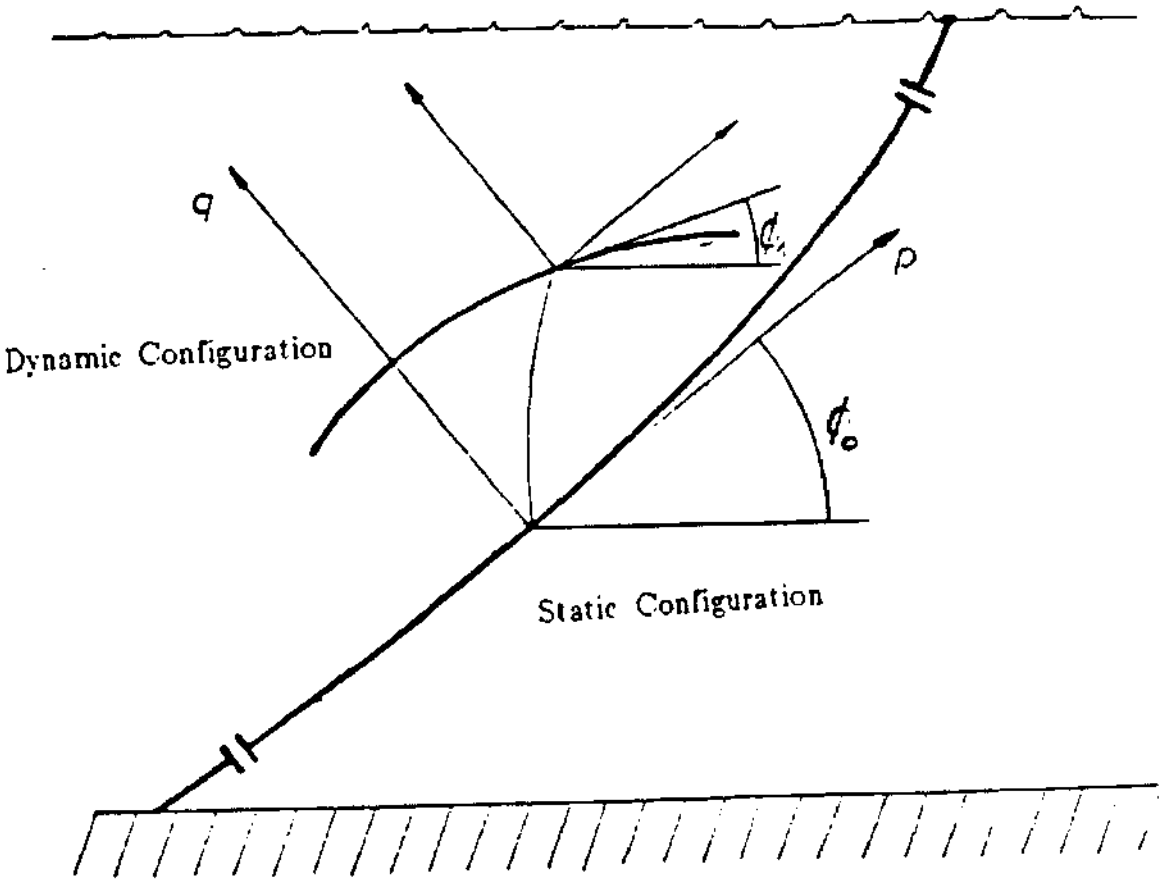

Figure 5-2: Coordinate System (p,q)

- The added mass force

- The non-linear drag force

Gravity and buoyancy forces

The gravity and buoyancy forces act in the vertical direction when the effective tension concept is used. (See chapter 2) The net force acting on the cable is given by:

$$\begin{aligned} F_{ext,p}\ (1+e) &= -\ w_o \cdot \sin\phi_o \\ F_{ext,q}\ (1+e) &= -\ w_o \cdot \cos\phi_o \end{aligned} \tag{5.28}$$

where w_o is the net weight

The added mass force

The added mass force as derived in chapter 2, can be written in the p and q directions as:

$$\begin{aligned} F_{ext,p}\ (1+e) = -\ m_a \Big[& \frac{\partial^2 p}{\partial t^2} \cdot \sin^2\phi_1 - \frac{\partial^2 q}{\partial t^2} \cdot \cos\phi_1 \cdot \sin\phi_1 \\ & + \frac{\partial p}{\partial t} \cdot \cos\phi_1 \cdot \sin\phi_1 \cdot \frac{\partial \phi_1}{\partial t} + \frac{\partial q}{\partial t} \cdot \sin^2\phi_1 \cdot \frac{\partial \phi_1}{\partial t} \Big] \end{aligned}$$

$$\begin{aligned} F_{ext,q}\ (1+e) = -\ m_a \Big[& -\frac{\partial^2 p}{\partial t^2} \cdot \sin\phi_1 \cdot \cos\phi_1 + \frac{\partial^2 q}{\partial t^2} \cdot \cos^2\phi_1 \\ & - \frac{\partial p}{\partial t} \cdot \cos^2\phi_1 \cdot \frac{\partial \phi_1}{\partial t} - \frac{\partial q}{\partial t} \cdot \sin\phi_1 \cdot \cos\phi_1 \frac{\partial \phi_1}{\partial t} \Big] \end{aligned} \tag{5.29}$$

The added mass force, written in the reference coordinate system, is a complicated non-linear function of velocities, accelerations and angular orientation. This is mainly due to the appearance of Coriolis and centripetal

terms. For small dynamic angles this force will be reduced to a normal component.

$$F_{2ext,q}\ (1+e) = -\ m_a \frac{\partial^2 q}{\partial t^2} \tag{5.30}$$

For a large dynamic angle, we rewrite the normal added mass force term as follows:

$$F_{2ext,q}\ (1+e) = -m_a \frac{\partial^2 q}{\partial t^2} - m_a\Big[-\frac{\partial^2 p}{\partial t^2}\sin\phi_1\cdot\cos\phi_1 + \frac{\partial^2 q}{\partial t^2}(\cos^2\phi_1 - 1) - \frac{\partial p}{\partial t}\cdot\cos^2\phi_1\frac{\partial\phi_1}{\partial t} - \frac{\partial q}{\partial t}\sin\phi_1\cdot\cos\phi_1\frac{\partial\phi_1}{\partial t}\Big] \tag{5.31}$$

The added mass can be considered to be the sum of a linear term in the q direction plus correction terms. The linear term can be included in the inertia term.

The added mass term is small compared with the mass term for chains and wires, therefore the influence of the additional added mass terms on the motion is believed to be minimal (see [Barr 74]).

The Non-linear Drag Force

The drag force, as described in chapter 1, can be written in the (p,q) coordinate system as:

$$\begin{aligned} F_{ext,p}\ (1+e) = {} & 0.5\ \rho_w\ C_{Dn}\ (Re)\ D_o\ (U\sin\phi + v_n)\ |\ U\sin\phi + v_n\ | \\ & (1+e/2)\cdot\sin\phi_1 \\ & + 0.5\ \rho_w\ C_{Dt}\ (Re)\ D_o\ (U\cos\phi - v_t)\ |U\cos\phi - v_t| \\ & (1+e/2)\cdot\cos\phi_1 \end{aligned}$$

$$F_{ext,q}(1+e) = -0.5\rho_w C_{Dn}(Re) D_o (U\sin\phi + v_n)|U\sin\phi + v_n| \cdot (1+e/2)\cdot\cos\phi_1 + 0.5\rho_w C_{Dt}(Re) D_o (U\cos\phi - v_t)|U\cos\phi - v_t| \cdot (1+e/2)\cdot\sin\phi_1 \quad (5.32)$$

where: $\phi = \phi_o + \phi_1$

$$v_t = \frac{\partial p}{\partial t}\cdot\cos\phi_1 + \frac{\partial q}{\partial t}\cdot\sin\phi_1$$

$$v_n = -\frac{\partial p}{\partial t}\cdot\sin\phi_1 + \frac{\partial q}{\partial t}\cdot\cos\phi_1$$

5.6.2 The Compatibility Relations

The non-linear compatibility relations were derived in chapter 2 and are given as:

$$(1+e)\cos\phi_1 = (1+e_o) + \left[\frac{\partial p}{\partial s} - q\phi_{os}\right]$$

$$(1+e)\sin\phi_1 = \left[\frac{\partial q}{\partial s} + p\phi_{os}\right] \quad (5.33)$$

5.6.3 Summary

The equations in 5.6.1 and 5.6.2 give a complete description of the full non-linear two dimensional dynamic problem, with the external forces described in 5.6.1 A constitutive law, relating the strain and the tension and the boundary conditions must be added to make the formulation complete.

5.7 Newton-Raphson Method

The equations of motion of a cable are described by a set of non-linear partial differential equations of the hyperbolic type. In this work an incremental linearised description of the non-linear equations is used to solve the problem iteratively.

The partial differential equations are linearised in an incremental way. The result of the linearised problem is used to obtain an estimation of the solution of the non-linear problem, which is refined in succesive iterations until the error is below an acceptable bound. The linearisation of the partial differential equations is obtained by using a Taylor expansion in several variables.

The resulting linearised partial differential equations are solved by using the modal superposition technique described in sections 5.3 - 5.5.

The method of incremental linearisation of the partial differential equations is a suitable technique to solve a non-linear set of equations in several variables, as explained in the sequel.

In the Newton-Raphson method the non-linear set of equations is expanded in Taylor series [Dahlquist 74]. If we represent the set of non-linear equations as $\mathbf{f}(\mathbf{x})$, Taylor's formula gives:

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^k) + \dot{\mathbf{f}}(\mathbf{x}^k)(\mathbf{x} - \mathbf{x}^k) + O(|\mathbf{x} - \mathbf{x}^k|)^2$$

where $\dot{\mathbf{f}}(\mathbf{x})$ is a $n \times n$ matrix, the Jacobian with element

$$\dot{f}_{ij}(x) = \frac{\partial f_i}{\partial x_j}(x) \qquad 1 \leq i, j \leq n$$

We try to find a solution to the problem $\mathbf{f(x)} = 0$. If $\mathbf{x}_k$ is close to the solution of $\mathbf{f(x)} = 0$, we obtain a better approximation as:

$$\dot{\mathbf{f}}(\mathbf{x}^k)(\mathbf{x}^{k+1} - \mathbf{x}^k) + \mathbf{f}(\mathbf{x}^k) = 0$$

which we can solve for $\mathbf{x}^{k+1}$, because in the case of non-linear functions, this is a set of n linear equations. In the case of non-linear partial differential equations, we obtain a set of linear partial differential equations as will be shown later.

The evaluation of the Jacobian of a $n \times n$ system can be quite complicated and time consuming. Therefore it can be advantageous to use the same Jacobian during several iterations. This method is known as the modified Newton-Raphson method and can be written as:

$$\dot{\mathbf{f}}(\mathbf{x}^p)(\mathbf{x}^{k+1} - \mathbf{x}^k) + \mathbf{f}(\mathbf{x}^k) = 0 \qquad k = p, p+m$$

In some cases it can be appropriate to evaluate the Jacobian only at the first iteration step. This can be done when the solution is close to the initial estimation and the non-linearities are not too severe. This can be written as:

$$\dot{\mathbf{f}}(\mathbf{x}^o)(\mathbf{x}^{k+1} - \mathbf{x}^k) + \mathbf{f}(\mathbf{x}^k) = 0$$

The selection of the most advantageous method depends strongly on the nature of the problem. An illustration of the modified Newton-Raphson scheme for a single function is shown in figure 5-3.

There can be a problem with the convergence of the Newton-Raphson scheme. This is especially true for concave curves, as for example in the case of stiffening systems. Also, if the initial estimate is not close to the root, the convergence can be very slow, as compared to the quadratic convergence of the

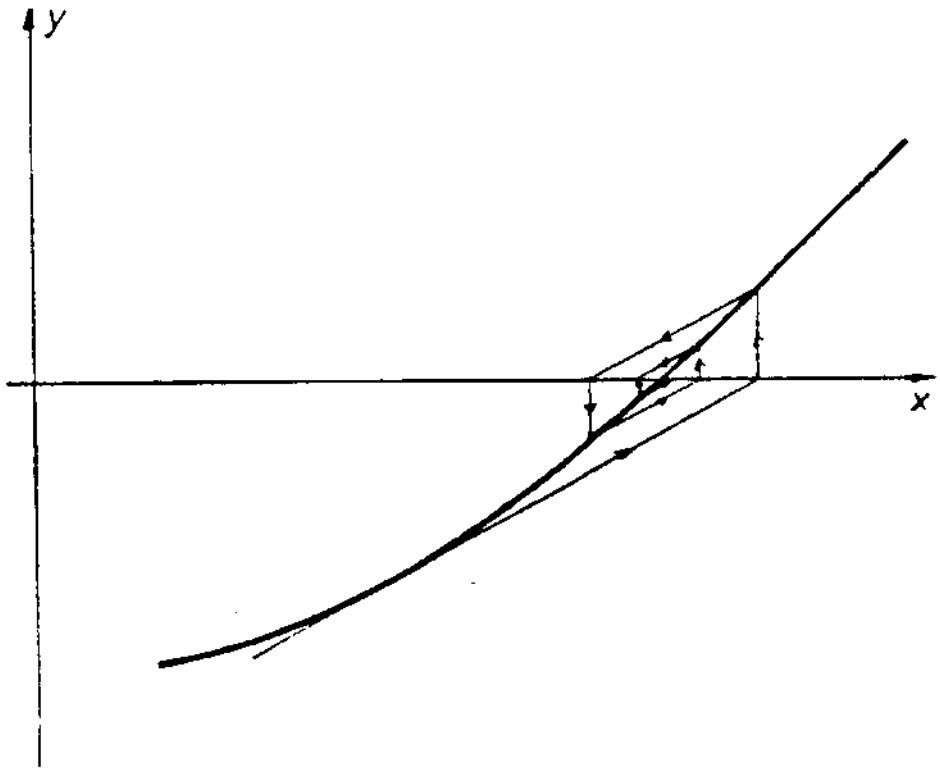

Figure 5-3: Convergence of Newton-Raphson Method

full Newton-Raphson method.

The modified Newton-Raphson method has been applied extensively to partial differential equations using non-linear finite element methods. When no updating of the Jacobian is performed, the method is known as the initial stiffness method [Bathe 80] and [Bathe 81].

In the case of time domain analysis, the initial stiffness method will always convergence for sufficiently small time steps [Bathe 82], although in some cases of severe non-linearities, time steps will be extremely small.

The initial stiffness method has been applied to cables with relative success [Larsen 82]. The method is espacially attractive for use with modal expansions, because no update of the modes is needed in the time simulation.

5.8 Incremental Formulation

We will use an incremental formulation of the governing equations, using a modified Newton-Raphson scheme. The Jacobian will be evaluated at a reference state, which was selected to be the static configuration of the cable. In this work no updating of the reference state will be performed. The formulation takes fully into account the non-linearities of the problem by using an iterative procedure to obtain the force balance.

5.8.1 Incremental Formulation of the Equations of Motion

The non-linear equations of motion can be written, including the added mass in the normal inertia term, as:

$$m \frac{\partial^2 p}{\partial t^2} = \frac{\partial}{\partial s} (T \cdot \cos\phi_1) + F_{ext,p}(1+e) - T \frac{\partial \phi_o}{\partial s} \cdot \sin\phi_1$$

$$M \frac{\partial^2 q}{\partial t^2} = \frac{\partial}{\partial s} (T \cdot \sin\phi_1) + F_{ext,q}(1+e) + T \frac{\partial \phi_o}{\partial s} \cdot \cos\phi_1 \qquad (5.34)$$

The only non-linear terms are the restoring force terms, which are:

$$F_{int,t} = \frac{\partial}{\partial s} (T \cdot \cos\phi_1) - T \cdot \frac{\partial \phi_o}{\partial s} \cdot \sin\phi_1$$

$$F_{int,n} = \frac{\partial}{\partial s} (T \cdot \sin\phi_1) + T \cdot \frac{\partial \phi_o}{\partial s} \cdot \cos\phi_1 \qquad (5.35)$$

We will linearise those terms, using the estimation of terms from a previous iteration plus a linearised part, while the Jacobian is evaluated in the reference state. This gives:

$$F_{int,t}^{k+1} = \frac{\partial \Delta T}{\partial s} - T_o \frac{\partial \phi_o}{\partial s} \Delta\phi_1 + \frac{\partial}{\partial s} (T^k \cdot \cos\phi_1^k) - T^k \frac{\partial \phi_o}{\partial s} \cdot \sin\phi_1^k$$

$$F_{int,n}^{k+1} = \frac{\partial \phi_o}{\partial s} \Delta T + \frac{\partial T_o}{\partial s} \Delta\phi_1 + T_o \frac{\partial \Delta\phi_1}{\partial s} + \frac{\partial}{\partial s} (T^k \cdot \sin\phi_1^k) + T^k \cdot \frac{\partial \phi_o}{\partial s} \cos\phi_1^k \qquad (5.36)$$

Therefore the incremental formulation of the equations of motion becomes:

$$m \left[\frac{\partial^2 p}{\partial t^2} \right]^{k+1} = \frac{\partial \Delta T}{\partial s} - T_o \frac{\partial \phi_o}{\partial s} \Delta\phi_1 + \frac{\partial}{\partial s} (T^k \cdot \cos\phi_1^k) + F_{ext,t}(1+e^k) - T^k \frac{\partial \phi_o}{\partial s} \cdot \sin\phi_1^k$$

$$M\left[\frac{\partial^2 q}{\partial t^2}\right]^{k+1} = \frac{\partial \phi_o}{\partial s}\Delta T + \frac{\partial T_o}{\partial s}\Delta\phi_1 + T_o\frac{\partial \Delta\phi_I}{\partial s} + \frac{\partial}{\partial s}(T^k \sin\phi_1^k) + F_{ext,n}(1+e^k) + T^k\frac{\partial \phi_o}{\partial s}\cos\phi_1^k \tag{5.37}$$

5.8.2 Incremental Formulation of the Compatibility Relations

The non-linear compatibility relations are as given in 5.6.2

$$(1+e)\cos\phi_1 = (1+e_o) + \left[\frac{\partial p}{\partial s} - q\phi_{os}\right]$$

$$(1+e)\sin\phi_1 = \left[\frac{\partial q}{\partial s} + p\phi_{os}\right] \tag{5.33}$$

The incremental form of the compatibility relations is:

$$\Delta e = \frac{\partial \Delta p}{\partial s} - \Delta q\phi_{os} + \left[\frac{\partial p^k}{\partial s} - q^k\phi_{os}\right] + (1+e_o) - (1+e^k)\cos\phi_1^k$$

$$(1+e_o)\Delta\phi_1 = \frac{\partial \Delta q}{\partial s} + \Delta p\phi_{os} + \left[\frac{\partial q^k}{\partial s} + p^k\phi_{os}\right] - (1+e^k)\sin\phi_1^k \tag{5.38}$$

5.8.3 Incremental Formulation of the Force-Displacement Relation

The most frequently used force-displacement relations can be written as:

$$T = f(e)$$

This can be written incrementally as:

$$\Delta T = \left[\frac{\partial f}{\partial e}\right]_{e_o}\Delta e + \left[f(e^k) - T^k\right] \tag{5.39}$$

To keep things relatively simple, we will use, without destroying the generality of the solution, a linear force-displacement relation.

$$\Delta T = A \cdot E \cdot \Delta e \tag{5.40}$$

5.8.4 Incremental Formulation of the Governing Equations

We can substitute (5.38) and (5.40) in (5.37) to obtain the incremental formulation of the governing equations. We obtain:

$$\begin{aligned}
m \frac{\partial^2 p}{\partial t^2} - \frac{\partial}{\partial s}\left[EA\left(\frac{\partial \Delta p}{\partial s} - \Delta q \phi_{os}\right)\right] + T_o \frac{\partial \phi_o}{\partial s}\left[\frac{1}{1+e_o}\left(\frac{\partial \Delta q}{\partial s} + \Delta p \phi_{os}\right)\right] \\
= \frac{\partial}{\partial s}(T^k \cos\phi_1^k) + F^k_{ext,t}(1+e^k) - T^k \frac{\partial \phi_o}{\partial s} \sin\phi_1^k \\
+ \frac{\partial}{\partial s}\left[EA\left[\frac{\partial^k p}{\partial s} - q^k \phi_{os} + (1+e_o) - (1+e^k)\cos\phi_1^k\right]\right] \\
- T_o \frac{\partial \phi_o}{\partial s}\left[\frac{1}{1+e_o}\left[\frac{\partial^k q}{\partial s} + p^k \phi_{os} - (1+e^k) \cdot \sin\phi_1^k\right]\right]
\end{aligned} \tag{5.41}$$

$$\begin{aligned}
M \frac{\partial^2 q}{\partial t^2} - \frac{\partial \phi_o}{\partial s}\left[EA\left(\frac{\partial \Delta p}{\partial s} - \Delta q \phi_{os}\right)\right] - \frac{\partial T_o}{\partial s}\left[\frac{1}{1+e_o}\left(\frac{\partial \Delta q}{\partial s} + \Delta p \phi_{os}\right)\right] \\
- T_o \frac{\partial}{\partial s}\left[\frac{1}{1+e_o}\left(\frac{\partial \Delta q}{\partial s} + \Delta p \phi_{os}\right)\right] \\
= \frac{\partial}{\partial s}(T^k \sin\phi_1^k) + F^k_{ext,n}(1+e^k) + T^k \frac{\partial \phi_o}{\partial s} \cos\phi_1^k \\
+ \frac{\partial \phi_o}{\partial s}\left[EA\left[\frac{\partial^k p}{\partial s} - q^k \phi_{os} + (1+e_o) - (1+e^k)\cos\phi_1^k\right]\right] \\
+ \frac{\partial}{\partial s}\left[T_o \frac{1}{1+e_o}\left[\frac{\partial^k q}{\partial s} + p^k \phi_{os} - (1+e^k) \cdot \sin\phi_1^k\right]\right]
\end{aligned}$$

These are the complete incremental equations using the modified Newton-Raphson method. Note that in the left hand side are the linearised dynamic equations. To obtain the linearisation around the reference state we did not make any assumption about the reference state, so this can be any dynamic

equilibrium position.

5.9 Solution using the Modes

As in the case of the linear dynamic problem, we look for a solution.

$$p(s,t) = f(t)\ p_{qs1}(s) + g(t)\ p_{qs2}(s) + \sum_{i=1}^{n} C_i(t)\ p_i(s)$$

$$q(s,t) = f(t)\ q_{qs1}(s) + g(t)\ q_{qs2}(s) + \sum_{i=1}^{n} C_i(t)\ q_i(s) \tag{5.42}$$

where p_{qs1} , q_{qs2} and p_{qs1} , q_{qs2} are the two quasi-static solutions for the reference state and p_i , q_i are mode shapes for the reference state. As discussed previously the reference state was chosen to be the static equilibrium configuration.

Because the quasi-static solution and the mode shapes satisfy the compatibility relations, the following relations are valid:

$$e^{k}_{lin} - e_o = \frac{\partial p^k}{\partial s} - q^k \frac{\partial \phi_o}{\partial s}$$

$$\phi^{k}_{1\ lin} = \frac{\partial q^k}{\partial s} + p^k \frac{\partial \phi_o}{\partial s} \tag{5.43}$$

where e^{k}_{lin} , $\phi^{k}_{1\ lin}$ are the strain and the dynamic tension obtained by linear superposition.

We can now obtain the modal equations of motion by multiplying equations (5.41) with the modal shapes, adding them up and integrating. The following relations are used to simplify the equations:

$$\int_0^L p_i \frac{\partial A}{\partial s}\, ds = - \int_0^L A \frac{\partial p_i}{\partial s}\, ds$$

$$\int_0^L q_i \frac{\partial A}{\partial s}\, ds = - \int_0^L A \frac{\partial q_i}{\partial s}\, ds$$

$$e_i = \frac{\partial p_i}{\partial s} - q_i \frac{\partial \phi_o}{\partial s}$$

$$\phi_{1i} = \frac{\partial q_i}{\partial s} + p_i \frac{\partial \phi_o}{\partial s}$$

After some manipulation, the following modal equations are obtained:

$$\ddot{C}_i^{k+1} + 2\xi\, \omega_i\, \dot{C}_i^{k+1} + \omega_i^2\, \Delta C_i = - A_i\, \ddot{f} - B_i\, \ddot{g}$$

$$+ \int_0^L \left\{ \left[F^k_{ext,t} (1+e^k) + \frac{\partial T_o}{\partial s} \right] p_i + \left[F^k_{ext,n} (1+e^k) + T_o \frac{\partial \phi_o}{\partial s} \right] q_i \right\} ds$$

$$- \int_0^L e_i \, (T^k \cos\phi_1^k - T_o)\, ds - \int_0^L \phi_{1i} \left[T^k \sin\phi_1^k (1+e_o) \right] ds$$

$$- \int_0^L e_i \left[EA[(e^k_{lin} + 1) - (1+e^k)\cos\phi_1^k] \right] ds$$

$$- \int_0^L \phi_{1i} \left[T_o \, [\phi_{1\,lin}^k (1+e_o) - (1+e^k)\sin\phi_1^k] \right] ds \qquad (5.45)$$

This equation is the full incremental representation of the non-linear, two-dimensional cable problem using modal expansion. The right hand side of (5.45) contains known quantities at the iteration step k+1.

Explanation of terms:

Terms to take account of the boundary conditions:

$$- A_i \ddot{f} - B_i \ddot{g}$$

Dynamic modal force:

$$+ \int_0^L \left\{ \left[F^k_{ext,t} (1+e^k) + \frac{\partial T_o}{\partial s} \right] p_i \right.$$

$$\left. + \left[F^k_{ext,n} (1+e^k) + T_o \frac{\partial \phi_o}{\partial s} \right] q_i \right.$$

Force terms to take care of the increments of the internal forces and the non-linearities in the tension terms:

$$- \int_0^L e_i \, (T^k \cos\phi^k_1 - T_o) ds - \int_0^L \phi_{1i} \left[T^k \sin\phi^k_1 (1+e_o) \right] ds$$

Force terms to take care of non-linearities in the compatibility equations:

$$- \int_0^L e_i \left[EA\{(e^k_{lin} + 1) - (1+e^k)\cos\phi^k_1\} \right] ds$$

$$- \int_0^L \phi_{1i} \left[T_o \{\phi^k_{1\,lin}(1+e_o) - (1+e^k)\sin\phi^k_1\} \right] ds$$

5.10 Time Integration of the Equations in Incremental Form

The time integration can be done as described in 5.4. Written in an incremental way, (5.16) become:

$$
\begin{aligned}
{}^{t+\Delta t}\ddot{C}_i &= a_o \left({}^{t+\Delta t}C_i^{k-1} + \Delta C_i - {}^{t}C_i \right) - a_2 \, {}^{t}\dot{C}_i - a_3 \, {}^{t}\ddot{C}_i \\
{}^{t+\Delta t}\dot{C}_i &= a_1 \left({}^{t+\Delta t}C_i^{k-1} + \Delta C_i - {}^{t}C_i \right) - a_4 \, {}^{t}\dot{C}_i - a_5 \, {}^{t}\ddot{C}_i
\end{aligned}
\qquad (5.46)
$$

When we substitute in (5.45), the equations can be solved directly for the modal increments ΔC_i.

5.11 Non-Linear Boundary Conditions

The time integration scheme discussed in the preceding section is valid for the following boundary conditions.

- Prescribed motions at the top of the cable in function of time. The motions plus the first and second time derivative are required.

- The bottom end of the cable is a fixed point.

In offshore applications, other boundary conditions, than the one described above, might be applicable. We will limit ourselve to the case of a cable which can lay on the ocean floor. In most mooring line application this situation occurs. The non-linear modal expansion technique can still be used, when the bottom interaction forces are modeled as non-linear external forces on the cable.

As reference state around which the modes are expanded, the static configuration with the cable partially on the sea bottom is used. To be able to use as few modes as possible, the static configuration should only include the part of the cable which is moving from the bottom, and not the part which remains on the bottom even during large motions. This requires an

estimation of maximum lift-off length before the time simulation. The adequacy of the initial guess can be checked by looking at the dynamic angle at the bottom point which should be very small during the whole time simulation.

The expansions of the solutions, as derived previously can still be used:

$$p(s,t) = f(t)\, p_{qs1}(s) + g(t)\, p_{qs2}(s) + \sum_{i=1}^{n} C_i(t)\, p_i(s)$$

$$q(s,t) = f(t)\, q_{qs1}(s) + g(t)\, q_{qs2}(s) + \sum_{i=1}^{n} C_i(t)\, q_i(s) \qquad (5.47)$$

where $p_i(s)$, $q_i(s)$ are the components of the i th mode with fixed end conditions, for the appropriate reference system selected.

The bottom interaction forces can be modeled appropriately as vertical external forces. This includes a linear of non-linear spring plus a damping force. The damping force is necessary to reduce the numerical instabilities of a large spring force.

$$F_{y,\text{bound.cond.}} = 0 \qquad \text{for } y>0$$

$$F_{y,\text{bound.cond.}} = -k(y)\, y - c\, \dot{y} \quad \text{for } y<0 \qquad (5.48)$$

(y distance of cable segment to sea bottom)

The non-linear boundary forces are obtained by calculating the vertical displacements of the cable at each iteration step. The vertical forces are projected on the reference coordinate system and the modal contributions of these non-linear boundary forces can be obtained.

It is important to realize that the accuracy and efficiency of the modal

expansion scheme depends on the appropriate selection of the reference coordinate system used. The inclusion of long portions of cable which remains on the bottom during the time simulation will require a substantial increase in the number of modes considered to obtain accurate motion and tension response.

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Chapter 6

LINEARIZATION OF THE DRAG FORCES AND THE NONLINEAR BOTTOM BOUNDARY CONDITION

6.1 Introduction

In this chapter we examine the possibility of using an equivalent linearization technique to model the drag forces. We will limit the analysis to cables with a sinusoidal excitation at the top. When an equivalent linearization technique is used, the motion equations can be solved in the frequency domain, reducing considerably the calculation time.

The cable experiences a significant nonlinear drag force when subjected to a top excitation. The drag force is a function of the relative velocity between the cable and the current. In this chapter we decompose this nonlinear force into a mean force plus an equivalent linear damping force by minimizing the quadratic error between the equivalent representation and the nonlinear drag force averaged over one period. The mean force can be included in the static analysis. Equivalent linearized cable dynamic equations can be formulated using the governing equations in chapter 1 and the damping coefficients derived in this chapter. These equivalent linearized dynamic equations are easily solvable in the frequency domain.

The equivalent linearization technique has been applied to the OTEC cold water pipe dynamics [Paulling 79] and riser dynamics [Krolikowski 80]. See also [Triantafyllou 82].

6.2 Nonlinear Drag Forces in Two-dimensions

The drag force, as described in chapter 1, can be written in the (p,q) coordinate system as:

$$
\begin{aligned}
F_{ext,p}\,(1+e) = {} & 0.5\,\rho_w\,C_{Dn}\,(Re)\,D_o\,(U\sin\phi + v_n)\,|\,U\sin\phi + v_n\,| \\
& (1+e/2)\cdot\sin\phi_1 \\
& + 0.5\,\rho_w\,C_{Dt}\,(Re)\,D_o\,(U\cos\phi - v_t)\,|U\cos\phi - v_t| \\
& (1+e/2)\cdot\cos\phi_1
\end{aligned}
\qquad (6.1)
$$

$$
\begin{aligned}
F_{ext,q}(1+e) = {} & -0.5\rho_w\,C_{Dn}\,(Re)\,D_o\,(U\sin\phi + v_n)\,|U\sin\phi + v_n| \\
& (1+e/2)\cdot\cos\phi_1 \\
& + 0.5\,\rho_w\,C_{Dt}\,(Re)\,D_o\,(U\cos\phi - v_t)\,|U\cos\phi - v_t| \\
& (1+e/2)\cdot\sin\phi_1
\end{aligned}
$$

where: $\phi = \phi_o + \phi_1$

$$v_t = \frac{\partial p}{\partial t}\cdot\cos\phi_1 + \frac{\partial q}{\partial t}\cdot\sin\phi_1$$

$$v_n = -\frac{\partial p}{\partial t}\cdot\sin\phi_1 + \frac{\partial q}{\partial t}\cdot\cos\phi_1$$

We can rewrite these equations in nondimensional form using the relative velocities between the cable and the current as;

$$
\begin{aligned}
F_p &= \frac{F_p}{0.5\,\rho_w\,D_o\,U|U|} \\
&= C_{Dn}\,A\,|A|\,\sin\phi_1 + C_{Dt}\,B\,|B|\,\cos\phi_1
\end{aligned}
$$

$$F_q = \frac{F_q}{0.5\ \rho_w\ D_o\ U|U|} \qquad (6.2)$$

$$= - C_{Dn}\ A\ |A|\ \cos\phi_1 + C_{Dt}\ B\ |B|\ \sin\phi_1$$

where:

$$A = (\sin\phi_o + \frac{1}{U}\frac{\partial q}{\partial t})\ \cos\phi_1 + (\cos\phi_o - \frac{1}{U}\frac{\partial p}{\partial t})\ \sin\phi_1$$

$$B = (\cos\phi_o - \frac{1}{U}\frac{\partial p}{\partial t})\ \cos\phi_1 - (\sin\phi_o + \frac{1}{U}\frac{\partial q}{\partial t})\ \sin\phi_1$$

In the above equations the assumption is made that the influence of the strain on the drag forces is negligible. The following new variables are introduced and substituted for the absolute signs.

$$\mathrm{sgn}(A)=\frac{|A|}{A} \qquad \mathrm{sgn}(B)=\frac{|B|}{B} \qquad (6.3)$$

The nonlinear nondimensionalized drag forces can now be written as:

$$F_p = C_{Dn}\ A^2\ \mathrm{sgn}(A)\ \sin\phi_1 + C_{Dt}\ B^2\ \mathrm{sgn}(B)\ \cos\phi_1 \qquad (6.4)$$

$$F_q = - C_{Dn}\ A^2\ \mathrm{sgn}(A)\ \cos\phi_1 + C_{Dt}\ B^2\ \mathrm{sgn}(B)\ \sin\phi_1$$

6.3 Assumptions

Several assumptions are made in order to approximate the nonlinear drag force with an equivalent linearized force.

1. The linearized cable equations, except for the drag forces, are valid. The motions are small and the dynamic tension generated is small compared to the static tension.

2. The excitation is sinusoidal

3. The tangential drag forces are small compared to the normal drag forces ($C_{Dt} = O(\epsilon)$).

Under the above conditions, the two dimensional sinusoidal motions can be written as:

$$\begin{aligned} p &= p_a \cos(\omega t+\epsilon_1) \\ q &= q_a \cos(\omega t+\epsilon_2) \\ \phi_1 &= \phi_a \cos(\omega t+\epsilon_3) \end{aligned} \tag{6.5}$$

where: p_a, q_a and ϕ_a are small quantities of order ϵ

and ϵ_1, ϵ_2 and ϵ_3 represent the phase angles.

6.4 Approximation of the Nonlinear Drag Force for Small Motions

The expressions for the nonlinear drag forces can be simplified significantly if small motions and a small tangential drag coefficient are assumed. Generally we can distinguish three cases as a function of the ratio of the cable velocity to the current velocity.

$$\begin{aligned} &\frac{\omega p_a}{U} \text{ and } \frac{\omega q_a}{U} \ll 1 \\ &\frac{\omega p_a}{U} \text{ and } \frac{\omega q_a}{U} \simeq 1 \\ &\frac{\omega p_a}{U} \text{ and } \frac{\omega q_a}{U} \gg 1 \end{aligned} \tag{6.6}$$

The two limiting cases, which represent the large current and the small

current velocity (compared to the cable velocity) are studied here.

6.4.1 Large Current Velocity

For large current:

$$\frac{\omega p_a}{U} \text{ and } \frac{\omega q_a}{U} \simeq \epsilon$$

The drag forces can be approximated to $O(\epsilon^2)$ as:

$$\begin{aligned} F_p(t) \simeq\ & C_{Dn} \sin\phi_o |\sin\phi_o| \phi_a \cos(\omega t+\epsilon_3) \\ & + C_{Dt} \cos\phi_o |\cos\phi_o| \end{aligned} \tag{6.7}$$

$$\begin{aligned} F_q(t) \simeq\ & - C_{Dn}\{ \sin\phi_o |\sin\phi_o| - 2 \frac{\omega q_a}{U} | \sin\phi_o| \sin(\omega t+\epsilon_1) \\ & + 2 |\sin\phi_o| \cos\phi_o \phi_a \cos(\omega t+\epsilon_3)\} \end{aligned}$$

6.4.2 Small Current Velocity

For small current:

$$\frac{U}{\omega p_a} \text{ and } \frac{U}{\omega q_a} \simeq \epsilon$$

The drag forces can be approximated to O(1):

$$F_p(t) \simeq - C_{Dn} \left| \frac{\omega q_a}{U} \sin(\omega t+\epsilon_2) \right| \frac{\omega q_a}{U} \sin(\omega t+\epsilon_2)\ \phi_a \cos(\omega t+\epsilon_3)$$

$$+ C_{Dt} \left| \frac{\omega p_a}{U} \sin(\omega t+\epsilon_1) \right| \frac{\omega p_a}{U} \sin(\omega t+\epsilon_1)$$

(6.8)

$$F_q(t) \simeq C_{Dn} \left| \frac{\omega q_a}{U} \sin(\omega t+\epsilon_2) \right| \frac{\omega q_a}{U} \sin(\omega t+\epsilon_2)$$

6.5 Method of Least Squares

We propose to approximate the above equations by forces proportional to the cable velocities as follows;

$$F_{ep} = F_{Fop} + C_p \frac{1}{U} \frac{\partial p}{\partial t}$$

$$F_{eq} = F_{Foq} + C_q \frac{1}{U} \frac{\partial q}{\partial t} \qquad (6.9)$$

In order to minimize the integral of the square of the residual over the period interval the Method of Least squares is used. The residuals are defined as:

$$E_p = F_p - F_{ep}$$

$$E_q = F_q - F_{eq} \qquad (6.10)$$

The following conditions must be satisfied:

$$\frac{\partial \bar{E}_p^2}{\partial F_{Fop}} = 0 \qquad \frac{\partial \bar{E}_p^2}{\partial C_p} = 0$$

$$\frac{\partial \bar{E}_p^2}{\partial F_{Foq}} = 0 \qquad \frac{\partial \bar{E}_p^2}{\partial C_q} = 0 \tag{6.11}$$

6.6 Integral of the Square of the Residual over One Period

6.6.1 Large Current

$$\bar{E}_p^2 = 1/(2\pi) \int_0^{2\pi} \{ C_{Dn} \sin\phi_o |\sin\phi_o| \phi_a \cos(\omega\tau+\epsilon_3) + C_{Dt} \cos\phi_o |\cos\phi_o| - F_{Fop} + C_q \frac{\omega p_a}{U} \sin(\omega\tau+\epsilon_1)\}^2 \, d\tau \tag{6.12}$$

If the method of least squares is applied, we can find the equivalent mean static force and the linear force as follows:

$$F_{Fop} = C_{Dt} \cos\phi_o |\cos\phi_o|$$

$$C_p = - C_{Dn} \sin\phi_o |\sin\phi_o| \phi_a \sin(\epsilon_1-\epsilon_3) \frac{U}{\omega p_a} \tag{6.13}$$

Similarly,

$$F_{Foq} = - C_{Dn} \sin\phi_o |\sin\phi_o|$$

$$C_q = - 2 C_{Dn} |\sin\phi_o| + 2 C_{Dn} |\sin\phi_o| \cos\phi_o \phi_a \sin(\epsilon_2-\epsilon_3) \frac{U}{\omega q_a} \tag{6.14}$$

6.6.2 Small Current or No External Current

In this case we obtain:

$$F_{Fop} = 0 \qquad F_{Foq} = 0$$

$$C_p = - \; C_{Dt} \frac{8}{3\pi} \frac{\omega p_a}{U} \qquad C_q = - \; C_{Dn} \frac{8}{3\pi} \frac{\omega q_a}{U} \tag{6.15}$$

6.7 Linearized Equivalent Damping Coefficients

From the above relations we can obtain the equivalent damping coefficients written in dimensional form:

$$F_p = - \; b_p \frac{\partial p}{\partial t}$$

$$F_q = - \; b_q \frac{\partial q}{\partial t} \tag{6.16}$$

where b_p and b_q are the equivalent damping coefficients obtained from the equations (6.13), (6.14) and (6.15). The final forms for the equivalent damping coefficients can be obtained for each case as:

For large current:

$$b_p = 0.5 \; \rho_w \; C_{Dn} \; D \; \left(\frac{|U \sin\phi_o| \; U \sin\phi_o}{\omega p_a} \right) \phi_a \sin(\epsilon_1 - \epsilon_3)$$

$$b_q = \rho_w \; C_{Dn} \; D \; \{ |U \sin\phi_o| - \left(\frac{|U \sin\phi_o| \; U \sin\phi_o}{\omega q_a} \right) \phi_a \sin(\epsilon_2 - \epsilon_3) \} \tag{6.17}$$

For small current:

$$b_p = \rho_w \, C_{Dt} \, D \, \frac{4}{3\pi} \, \omega p_a$$

$$b_q = \rho_w \, C_{Dn} \, D \, \frac{4}{3\pi} \, \omega q_a \qquad (6.18)$$

6.8 Numerical Solution of the Cable Dynamic Equations using the Linearized Equivalent Damping Force

The three-dimensional linear dynamics of a cable with a two-dimensional static configuration have been studied in chapter 2 and the uncoupled in-plane and out-of-plane governing equations have been obtained.

The equivalent linearization of the nonlinear hydrodynamic drag forces can be used directly in the equations obtained in chapter 2. The centered difference scheme formulated in chapter 3 has been modified to include the equivalent dampingterms. The matrix form is the same as that in chapter 3 except for the linearized damping forces. The matrix formulation is now in terms of complex quantities due to the damping forces and has been implemented numerically. To find the correct estimation of the linearized damping coefficient an iteration procedure starting from an assumed initial amplitude is used. This numerical procedure has been implemented in a computer code.

6.9 Linearization of the Nonlinear Boundary Conditions

6.9.1 Introduction

When a cable is laying partially on the bottom, dynamic motion causes constant changes in length of the cable on the bottom (See figure 6-1). The

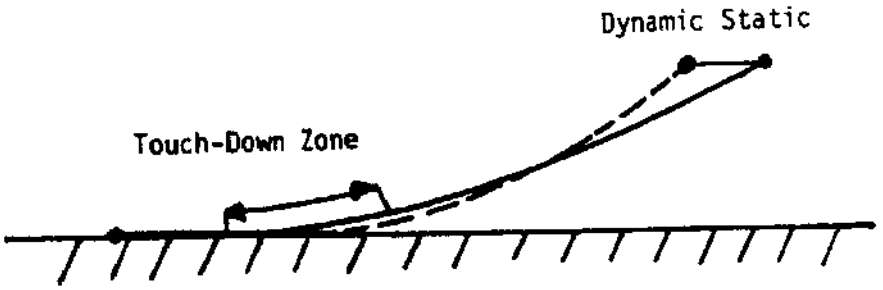

Figure 6-1: Bottom-cable Interaction

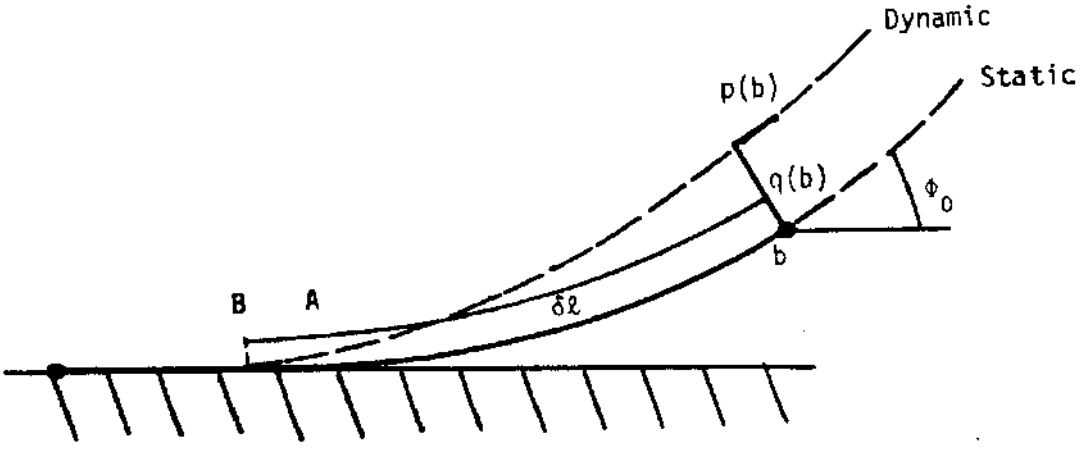

Figure 6-2: Motion Kinematics

touch-down point moves constantly along the length of the cable. Mathematically this can be expressed as a moving boundary condition. It is not practical to use a moving boundary condition in a linearized theory. We propose therefore to replace the moving boundary condition by a fixed boundary condition on the cable at a point which is just outside the touch-down zone. The boundary condition has to be specified in such a way that it models correctly the bottom part of the cable.

6.9.2 Kinematics

To be able to approximate the dynamics of the bottom part of the cable, some a priori assumptions about the motion behavior have to be made. The assumption is made that the shape of the cable between the new boundary point and the bottom is parabolic, with the curvature equal to the static curvature at the boundary point. The variation of curvature along the bottom part of the cable is assumed to be small (See figure 6-2). If we assume subsonic motion of the bottom part of the cable the slope of the mooring line will be zero at the bottom, therefore:

$$\alpha = \frac{\partial \phi}{\partial s} = \frac{\partial q_s}{\partial s} = \frac{q_s(b)}{\delta l} \tag{6.19}$$

Since the curvature is constant we obtain:

$$q(s) = q(b) \left(\frac{s}{\delta l} \right)^2 \tag{6.20}$$

Within this linearized theory, we also make the assumptions:

$$q_t(s) = q_t(b) \left(\frac{s}{\delta l} \right)^2 \tag{6.21}$$

$$q_{tt}(s) = q_{tt}(b) \left(\frac{s}{\delta l} \right)^2 \tag{6.22}$$

6.9.3 Dynamic Equilibrium

The inertia forces in the tangential direction are assumed to neglibly small. In the normal direction, we obtain:

$$F_{\text{bound.seg. inertia,q}} = \frac{d}{dt} \int_0^{\delta l} M \, q_t \, ds \tag{6.23}$$

To linear order this can be written as:

$$F_{\text{bound.seg. inertia,q}} = 1/3 \; M \; q_{tt}(b) \; b \tag{6.24}$$

Equilibrium in the normal direction gives:

$$T_o(b) \; \phi_1(b) = k_{q,eq} \; q(b) + 1/3 \; M \; q_{tt}(b) \tag{6.25}$$

Equilibrium in the tangential direction gives:

$$T_1(b) = k_{p,eq} \; p(b) \tag{6.26}$$

In this new equivalent boundary conditions, we can select the stiffness parameters $k_{q,eq}$, $k_{p,eq}$ to model correctly the bottom-cable interaction.

6.10 references

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Chapter 7

TERMINAL IMPEDANCES

7.1 Introduction

The concept of termination impedances has been widely used in the design of guyed masts (See for example [Davenport 65]). In mooring line design, the termination impedances can also be useful. In this chapter we will discuss some introductory concepts about terminal impedances. This will allow us to calculate the eigenfrequencies and the linear response for multi-leg systems. The concept discussed here is only valid for linear systems. In the case where equivalent drag damping is used, the superposition principle is not valid anymore, so that the combined response can only be obtained by adding the equivalent impedance coefficients with the correct motion for each leg of a multi-leg system. Further research in the domain of equivalent impedance coefficients for multi-leg systems is certainly required.

7.2 Terminal Impedances in Two-Dimensions

The upper end of the cable is excited by an externally imposed harmonic motion. The mooring line termination impedances are defined in the following way:

$$\begin{bmatrix} S_{xx}(\omega) & S_{xy}(\omega) \\ S_{yx}(\omega) & S_{yy}(\omega) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \end{bmatrix} \tag{7.1}$$

where: x: complex amplitude motion, horizontal direction

y: complex amplitude motion, vertical direction

F_x: complex amplitude force, horizontal direction

F_y: complex amplitude force, vertical direction

The resonance frequencies are the poles of the impedance transfer functions. The mooring line admittance matrix can be calculated as the inverse of the impedance matrix. In practical applications, the above transfer matrices can be used to find body motions and dynamic tensions.

At the top of the cable, sinusoidal motions in the x direction and the y direction are imposed. The dynamic forces at the top are obtained as:

$$S_{xx} = [\, T_{11} \cdot \cos\phi_o - T_o \cdot \sin\phi_o \cdot \phi_{11}]/A_x$$

$$S_{yx} = [\, T_{11} \cdot \sin\phi_o + T_o \cdot \cos\phi_o \cdot \phi_{11}]/A_x \qquad (7.2)$$

where: T_{11}, ϕ_{11} are the dynamic tension and the dynamic angle, respectively, caused by external motion in the x direction

$$S_{xy} = [\, T_{12} \cdot \cos\phi_o - T_o \cdot \sin\phi_o \cdot \phi_{12}/A_y$$

$$S_{yy} = [\, T_{12} \cdot \sin\phi_o + T_o \cdot \cos\phi_o \cdot \phi_{12}]/A_y \qquad (7.3)$$

where: T_{12}, ϕ_{12} are the dynamic tension and the dynamic angle, respectively, caused by external motion in the y direction

The dynamic tension at the top due to unit motion in the x and the y directions is also a very important transfer function. It indicates how much

dynamic tension is generated in the cable by a unit amplitude motion, and can be used directly in fatigue analysis. The dynamic tension transfer functions are defined as;

$$S_{Tx} = T_{11}/A_x$$

$$S_{Ty} = T_{12}/A_y \tag{7.4}$$

The solutions of the above equations can be obtained by solving the linearized dynamic equations in the frequency domain. Solutions were obtained by using the perturbation method described in chapter 3 and also by using the finite difference scheme. An example of terminal impedances can be found in part II. When equivalent linearization is used, formulas (7.2) and (7.3) remains valid for a particular amplitude of motion, but the superposition principle can not be used anymore.

7.3 Terminal Impedances in Three-Dimensions

The concept of termination impedances derived for two-dimensional cable dynamics can easily be extended to three dimensional cable dynamics. By using the Eulerian angles and the transformations of coordinates (as derived in chapter 1), the tension force at the top of the cable can be decomposed as:

$$F_x = F_{xo} + F_{x1} = (T_o + T_1) \cdot \cos(\phi_o+\phi_1) \cdot \cos(\theta_o+\theta_1)$$

$$F_y = F_{yo} + F_{y1} = (T_o + T_1) \cdot \sin(\phi_o+\phi_1) \tag{7.5}$$

$$F_z = F_{zo} + F_{z1} = - (T_o + T_1) \cdot \cos(\phi_o+\phi_1) \cdot \sin(\theta_o+\theta_1)$$

where the subscript o denotes static quantities and the subscript 1 denotes dynamic quantities. Assuming small motions, the equations (7.5) can be linearized and the dynamic forces can be obtained as:

$$F_{x1} = - T_o\theta_1\cos\phi_o\sin\theta_o - T_o\phi_1\sin\phi_o\cos\theta_o + T_1\cos\phi_o\cos\theta_o$$

$$F_{y1} = T_1\sin\phi_o + T_o\phi_1\cos\phi_o \tag{7.6}$$

$$F_{z1} = T_o\phi_1\sin\phi_o\sin\theta_o - T_o\theta_1\cos\phi_o\cos\theta_o - T_1\cos\phi_o\sin\theta_o$$

where T_1, ϕ_1, θ_1 are the dynamic tension force and the dynamic angle respectively. The mooring line termination impedances are defined in the same way as in the two dimensional case:

$$\begin{bmatrix} S_{xx}(\omega) & S_{xy}(\omega) & S_{xz}(\omega) \\ S_{yx}(\omega) & S_{yy}(\omega) & S_{yz}(\omega) \\ S_{zx}(\omega) & S_{zy}(\omega) & S_{zz}(\omega) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{z1} \end{bmatrix} \tag{7.7}$$

where:x: amplitude of motion, x direction in absolute coordinate system

y: amplitude of motion,vertical direction in absolute coordinate system

z: amplitude of motion, z direction in absolute coordinate system

F_x: force amplitude, x direction in absolute coordinate system

F_y: force amplitude, vertical direction in absolute coordinate system

F_z: force amplitude, z direction in absolute coordinate system

(all amplitudes are complex)

7.3.1 Terminal Impedances in Absolute Coordinate System

For harmonic motions, the above matrix equations can be expanded for a three-dimensional cable as (see figure 7-1):

$$S_{xx} = [- T_o\theta_{11}\cos\phi_o\sin\theta_o - T_o\phi_{11}\sin\phi_o\cos\theta_o + T_{11}\cos\phi_o\cos\theta_o]/A_x$$

$$S_{yx} = [T_{11}\sin\phi_o + T_o\phi_{11}\cos\phi_o]/A_x$$

$$S_{zx} = [T_o\phi_{11}\sin\phi_o\sin\theta_o - T_o\theta_{11}\cos\phi_o\cos\theta_o - T_{11}\cos\phi_o\sin\theta_o]/A_x$$

$$S_{xy} = [- T_o\theta_{12}\cos\phi_o\sin\theta_o - T_o\phi_{12}\sin\phi_o\cos\theta_o + T_{12}\cos\phi_o\cos\theta_o]/A_y$$

$$S_{yy} = [T_{12}\sin\phi_o + T_o\phi_{12}\cos\phi_o]/A_y \qquad (7.8)$$

$$S_{zy} = [T_o\phi_{12}\sin\phi_o\sin\theta_o - T_o\theta_{12}\cos\phi_o\cos\theta_o - T_{12}\cos\phi_o\sin\theta_o]/A_y$$

$$S_{xz} = [- T_o\theta_{13}\cos\phi_o\sin\theta_o - T_o\phi_{13}\sin\phi_o\cos\theta_o + T_{13}\cos\phi_o\cos\theta_o]/A_z$$

$$S_{yz} = [T_{13}\sin\phi_o + T_o\phi_{13}\cos\phi_o]/A_z$$

$$S_{zz} = [T_o\phi_{13}\sin\phi_o\sin\theta_o - T_o\theta_{13}\cos\phi_o\cos\theta_o - T_{13}\cos\phi_o\sin\theta_o]/A_z$$

where T_{1j}, θ_{1j} and ϕ_{1j} are the complex amplitudes of dynamic tension force and dynamic angles due to external motion in the j-direction. A_x, A_y and A_z are the complex amplitudes of externally imposed harmonic motions.

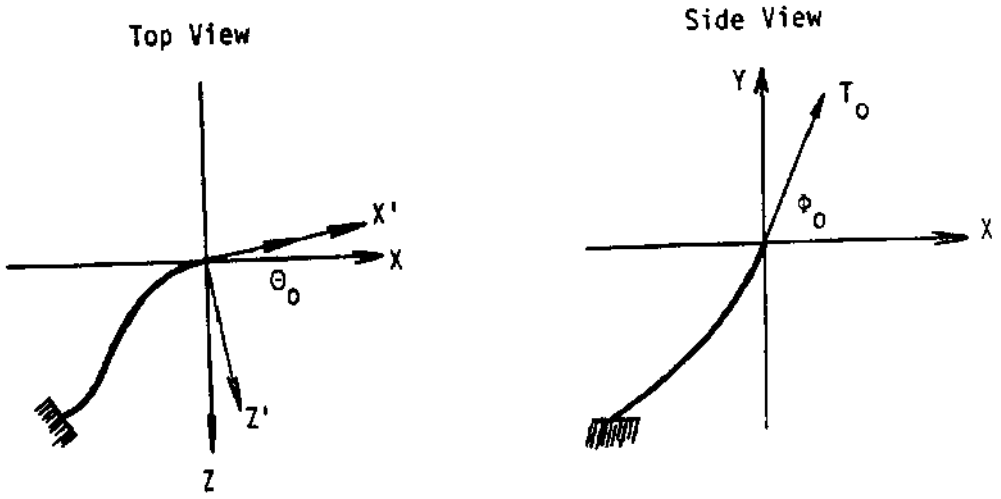

Figure 7-1: Impedance function in Global Coordinates for a Cable with Three-dimensional Shape

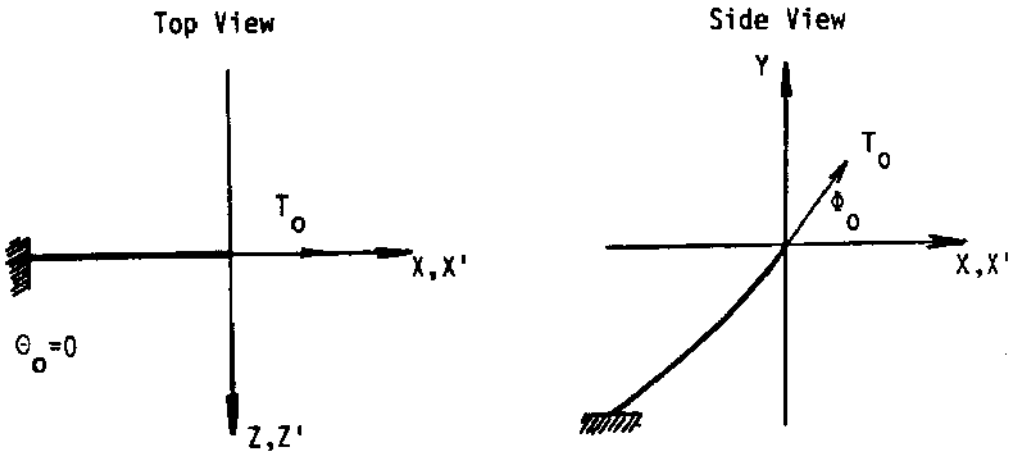

Figure 7-2: Impedance function in Local Coordinates for a Cable with Two-dimensional Shape

7.3.2 Terminal Impedances in Local Coordinate System for a Cable with a Two-dimensional Static Shape

From the linear dynamic analysis of the cable with an in-plane static shape the in-plane motions are known to be uncoupled from the out-of-plane motions. In other words there is no driving mechanism in the out-of-plane direction due to in-plane motions. (See figure 7-2) This simplifies significantly the terminal impedance functions. Following simplification can be made when the local coordinate system is used (see 7-2):

$$\theta_{11} = \theta_{12} = 0$$
$$T_{13} = 0 \qquad (7.9)$$
$$\phi_{13} = 0$$

The compact forms of the termination impedances are obtained as:

$$S_{x'x'} = [\, - T_o\, \phi_{11} \sin\phi_o + T_{11} \cos\phi_o \,]/A_{x'}$$

$$S_{y'x'} = [\, T_{11} \sin\phi_o + T_o\, \phi_{11} \cos\phi_o]/A_{x'}$$

$$S_{z'x'} = 0$$

$$S_{x'y'} = [\, - T_o\, \phi_{12} \sin\phi_o + T_{12} \cos\phi_o \,]/A_{y'}$$

$$S_{y'y'} = [\, T_{12} \sin\phi_o + T_o\, \phi_{12} \cos\phi_o \,]/A_{y'} \qquad (7.10)$$

$$S_{z'y'} = 0$$

$$S_{x'z'} = 0$$

$$S_{y'z'} = 0$$

$$S_{z'z'} = - T_o \theta_{13} \cos\phi_o / A_{z'}$$

7.4 Terminal Impedances of multi-leg systems

7.4.1 Multi-leg System Impedances in Absolute Coordinate System

From the definition of termination impedances for three-dimensional cable dynamics, the final forms for multi-leg systems are expressed in summation of each terminal impedance as follows (See figure 7-3):

$$S_{xx} = \sum_{i=1}^{n} S^i_{xx} \quad S_{xy} = \sum_{i=1}^{n} S^i_{xy} \quad S_{xz} = \sum_{i=1}^{n} S^i_{xz}$$

$$S_{yx} = \sum_{i=1}^{n} S^i_{yx} \quad S_{yy} = \sum_{i=1}^{n} S^i_{yy} \quad S_{yz} = \sum_{i=1}^{n} S^i_{yz} \qquad (7.11)$$

$$S_{xz} = \sum_{i=1}^{n} S^i_{xz} \quad S_{yz} = \sum_{i=1}^{n} S^i_{yz} \quad S_{zz} = \sum_{i=1}^{n} S^i_{zz}$$

where: superscript (i) represents the ith cable of the multi-leg systems

7.4.2 Multi-leg System Impedances starting from Impedances for a Cable with a Two-dimensional Static Shape

By using the specialized forms of equations (7.10) the total termination impedances of multi-leg systems where each cable has an in-plane static configuration can be obtained as (See 7-4):

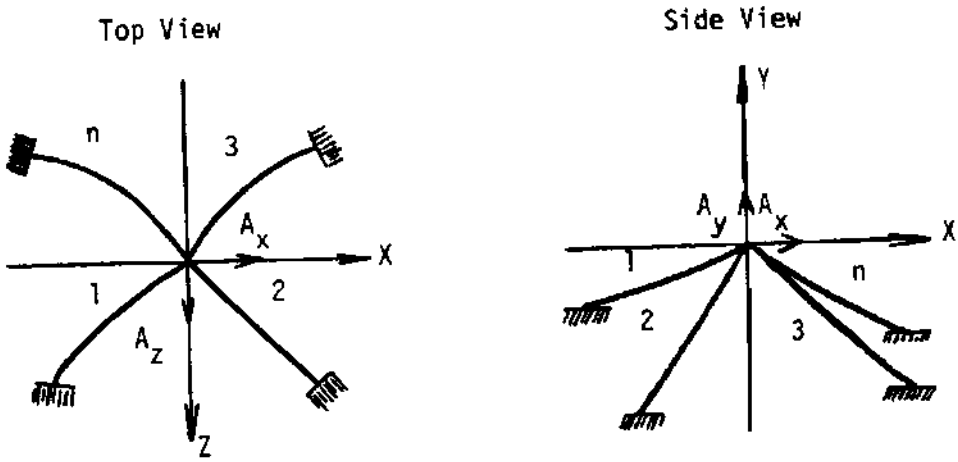

Figure 7-3: Impedance Function for a Multi-leg System (Three-Dimensional Static Shape)

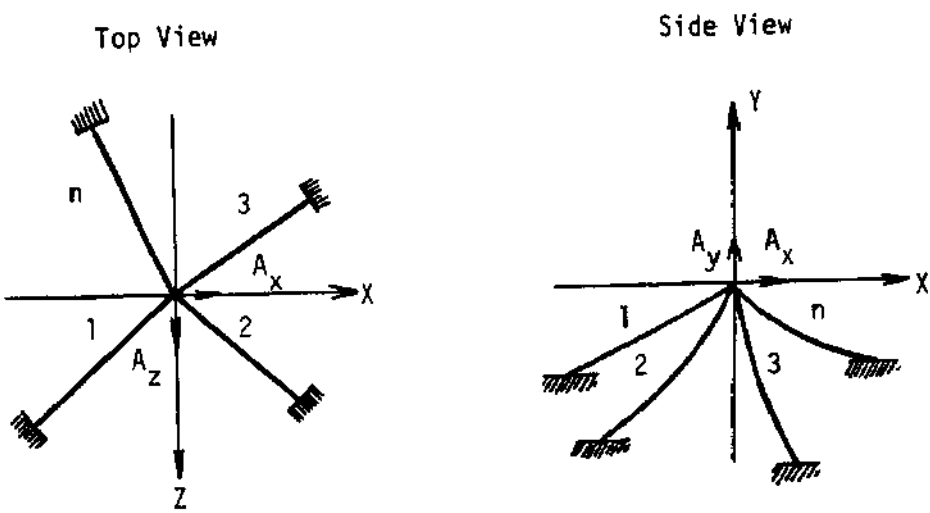

Figure 7-4: Impedance Function for a Multi-leg System (Two-Dimensional Static Shape)

$$S_{xx} = \sum_{i=1}^{n} S^i_{x'x'} \cos^2\theta^i_o + \sum_{i=1}^{n} S^i_{z'z'} \sin^2\theta^i_o$$

$$S_{yx} = \sum_{i=1}^{n} S^i_{y'x'} \cos\theta^i_o$$

$$S_{zx} = \sum_{i=1}^{n} (- S^i_{x'x'} + S^i_{z'z'}) \cos\theta^i_o \sin\theta^i_o$$

$$S_{xy} = \sum_{i=1}^{n} S^i_{x'y'} \cos\theta^i_o$$

$$S_{yy} = \sum_{i=1}^{n} S^i_{y'y'} \qquad (7.12)$$

$$S_{zy} = \sum_{i=1}^{n} (- S^i_{x'y'}) \sin\theta^i_o$$

$$S_{xz} = \sum_{i=1}^{n} (- S^i_{x'x'} + S^i_{z'z'}) \cos\theta^i_o \sin\theta^i_o$$

$$S_{yz} = \sum_{i=1}^{n} (- S^i_{y'x'}) \sin\theta^i_o$$

$$S_{zz} = \sum_{i=1}^{n} S^i_{x'x'} \sin^2\theta^i_o + \sum_{i=1}^{n} S^i_{z'z'} \cos^2\theta^i_o$$

This formulation is strictly valid only for the linear dynamics of a mooring system. If the equivalent drag damping is used, the evaluation of the impedance function of each leg must be done at the correct total imposed motion to obtain the correct total equivalent impedance function.

The general formulation for a multi-leg system is identical to (7.11) when the static shape of each cable is two-dimensional and no equivalent damping is used, using the assumptions (7.9).

7.5 Application to a Two-leg System with an In-plane Static Configuration

The impedance functions for the in-plane motions are completely uncoupled from the out-of-plane motions (assuming a linear system). The total impedance function can be obtained by simple addition of the two single line impedance functions (see figure 7-5):

$$\begin{bmatrix} S^1_{xx} + S^2_{xx} & S^1_{xy} - S^2_{xy} \\ S^1_{yx} - S^2_{yx} & S^1_{yy} + S^2_{yy} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \end{bmatrix} \tag{7.13}$$

Especially, if the two legs are symmetric, the above relation is reduced to the simple form:

$$\begin{bmatrix} 2\,S^1_{xx} & 0 \\ 0 & 2\,S^1_{yy} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \end{bmatrix} \tag{7.14}$$

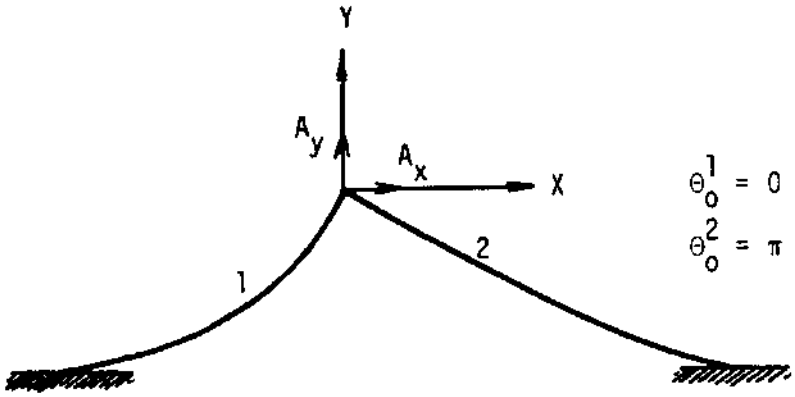

Figure 7-5: Impedance function for a Two-leg System

7.6 References

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Chapter 8

IMPULSIVE MOTIONS OF A HORIZONTAL, SHALLOW SAG CABLE, SUBJECT TO A HORIZONTAL MOTION AT ONE END

8.1 Governing Equations

A cable is suspended between two supports at the same elevation, one end is held fixed and the other is forced to move with some prescribed motion. In what follows the motions of this cable are simulated using the superposition of the eigenmode shapes as derived in chapter 5. The simulations of this section are different, however, than in the rest of the report, because of the importance of the elastic cable modes, which allows the simulation of travelling elastic waves.

Non-dimensionalizing displacements with respect to the cable length yields the equations of motion for a shallow sag horizontal cable as;

$$\frac{m\,L^2}{H}\frac{\partial^2 p}{\partial t^2} = \frac{EA}{H}\frac{\partial^2 p}{\partial s^2} - \alpha\left(\frac{EA}{H} + \frac{1}{1+e_o}\right)\frac{\partial q}{\partial s} \tag{8.1}$$

$$\frac{M\,L^2}{H}\frac{\partial^2 q}{\partial t^2} = \frac{EA}{H\,(1+e_o)}\frac{\partial^2 q}{\partial s^2} + \alpha\left(\frac{EA}{H} + \frac{1}{1+e_o}\right)\frac{\partial p}{\partial s} + \frac{L}{H}F_d \tag{8.2}$$

For the cable considered in this analysis there are also the following boundary and initial conditions:

$$p(0,t) = 0 \tag{8.3}$$

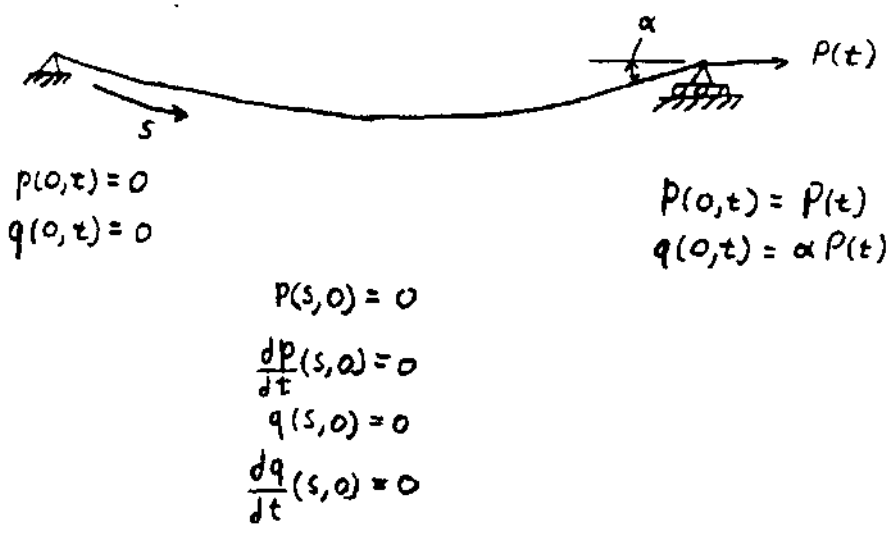

Figure 8-1: Horizontal Shallow Sag Cable with Horizontal End Excitation

$$q(0,t) = 0 \tag{8.4}$$

$$p(1,t) = P(t) \tag{8.5}$$

$$q(1,t) = \alpha\ P(t) \tag{8.6}$$

$$p(s,0) = q(s,0) = \dot{p}(s,0) = \dot{q}(s,0) = 0 \tag{8.7}$$

In order to use the technique of mode superposition on this set of cable equations it is first necessary to find the eigenfrequencies and eigenmodes of the cable configuration. This is accomplished by assuming that the motions are harmonic, the boundary conditions are homogeneous and by neglecting the normal drag. With these assumptions the equations of motion become,

$$-\frac{m\ L^2}{H}\ \omega^2\ \xi(s) = \frac{EA}{H}\ \frac{\partial^2 \xi}{\partial s^2} - \alpha\ \left(\frac{EA}{H} + \frac{1}{1+e_o}\right)\ \frac{\partial \eta}{\partial s} \tag{8.8}$$

$$-\frac{M\ L^2}{H}\ \omega^2\ \eta(s) = \frac{EA}{H\ (1+e_o)}\ \frac{\partial^2 \eta}{\partial s^2} + \alpha\ \left(\frac{EA}{H} + \frac{1}{1+e_o}\right)\ \frac{\partial \xi}{\partial s} + \frac{L}{H}\ F_d \tag{8.9}$$

where the following substitutions have been made,

$$p = \xi(s)\ e^{i\omega t}$$

$$q = \eta(s)\ e^{i\omega t} \tag{8.10}$$

The eigenmodes and eigenfrequencies are the combinations of ω_n, ξ_n, η_n which satisfy equations (8.8) and (8.9) and which have been derived in this report. Applying these results to a shallow sag cable leads to the following two transcendental equations for the natural frequencies of the horizontal cable,

$$\tan(\bar{\omega}/2) = \frac{Q\ \bar{\omega}}{\alpha^2}\ \tanh\ (\ Q/2\ ) \tag{8.11}$$

and,

$$-\frac{Q\,\bar{\omega}}{\alpha^2}\tan(\bar{\omega}/2) = \tanh\,(\,Q/2\,) \tag{8.12}$$

where;

$$Q = (\,\alpha^2 - \frac{H}{EA}\,\bar{\omega}^2\,)^{1/2}$$

$\bar{\omega}^2 = \omega^2\,L^2\,(M/H)$, non-dimensional frequency

Equation (8.11) is the equation for the even symmetric frequencies of the cable and equation (8.12) is the expression for the odd antisymmetric frequencies of the cable. The corresponding mode shapes are found as,

$$\xi_n = A\,[\,\sin(\bar{\omega}_n s) - \frac{\sin(\bar{\omega}_n/2)}{\sinh(Q_n/2)}\sinh\,(Q_n s)]$$

$$\eta_n = A\,[\,\frac{\bar{\omega}_n}{\alpha}\cos(\bar{\omega}_n s) - \frac{\alpha\,\sin(\bar{\omega}_n/2)}{Q_n\sinh(Q_n/2)}\cosh\,(Q_n s)] \tag{8.13}$$

for the even symmetric frequencies and,

$$\xi_n = A\,[\,\cos(\bar{\omega}_n s) - \frac{\cos(\bar{\omega}_n/2)}{\cosh(Q_n/2)}\cosh\,(Q_n s)]$$

$$\eta_n = A\,[\,-\frac{\bar{\omega}_n}{\alpha}\sin(\bar{\omega}_n s) + \frac{\alpha\,\cos(\bar{\omega}_n/2)}{Q_n\cosh(Q_n/2)}\sinh\,(Q_n s)] \tag{8.14}$$

for the odd antisymmetric frequencies, where the constant A is chosen to satisfy the orthonormality condition, as derived in chapter 3.

$$\int_0^1 (m\xi_n\xi_m + M\eta_n\eta_m)ds = 1 \quad n=m$$
$$= 0 \quad \text{otherwise} \tag{8.15}$$

The next step in the analysis is to convert the inhomogeneous boundary value problem described by equations (8.1) through (8.7) to a homogeneous boundary value problem. This is accomplished by finding a quasi-static solution which satisfies the boundary conditions identically and then assuming a solution of the form:

$$p(s,t) = p_{qs}(s)P(t) + \sum_{i=1}^{n} C_n(t)\ \xi_n(s)$$
$$q(s,t) = q_{qs}(s)P(t) + \sum_{i=1}^{n} C_n(t)\ \eta_n(s) \tag{8.16}$$

The quasi-static solutions for the horizontal shallow sag cable may be found as;

$$p_{qs}(s)=(c_2s/2+c_3)\sin(\alpha s)+(-\ c_1s/2+c_4)\cos(\alpha s)$$
$$q_{qs}(s)=(c_3+c_2s/2+\ c_1/\alpha\ [\beta-1/2])\cos(\alpha s)\ -\ (c_4+c_2/\alpha[\beta-1/2]-c_1s/2)\sin(\alpha s) \tag{8.17}$$
where $\beta = H/EA$

and the constants c_1, c_2, c_3 and c_4 depend on the boundary conditions.

Having the quasi-static solution, the expressions in (8.16) are plugged in the original set of equations (8.1) through (8.7) yielding a new set of differential equations describing the cable motion:

$$\frac{m\,L^2}{H}\sum_{i=1}^{n}\frac{\partial^2 c_n}{\partial t^2}\,\xi_n = \frac{EA}{H}\sum_{i=1}^{n}\frac{\partial^2 \xi_n}{\partial s^2} - \alpha\left(\frac{EA}{H}+\frac{1}{1+e_o}\right)\sum_{i=1}^{n} c_n\frac{\partial \eta_n}{\partial s}$$

$$+\frac{M\,L^2}{H}\,p_{qs}(s)\,\frac{\partial^2 P}{\partial t^2} \tag{8.18}$$

$$\frac{M\,L^2}{H}\sum_{i=1}^{n}\frac{\partial^2 c_n}{\partial t^2}\,\eta_n = \frac{EA}{H\,(1+e_o)}\sum_{i=1}^{n} c_n\frac{\partial^2 \eta_n}{\partial s^2} + \alpha\left(\frac{EA}{H}+\frac{1}{1+e_o}\right)\sum_{i=1}^{n} c_n\frac{\partial \eta_n}{\partial s}$$

$$+\frac{L}{H}F_d + \frac{m\,L^2}{H}\,q_{qs}(s)\,\frac{\partial^2 P}{\partial t^2} \tag{8.19}$$

Multiplying equation (8.18) by $\xi_m(s)$ and equation (8.19) by $\eta_m(s)$, integrating over the length of the cable and making suitable use of equations (8.8), (8.9) and (8.15), the original partial differential equations are reduced to a set of equations for the time dependent coefficients, $c_n(t)$:

$$\frac{\partial^2 c_m}{\partial t^2} = -\,c_m\,\omega_m^2 + \int_0^1 \frac{\partial^2 P}{\partial t^2}\,\frac{m\,L^2}{H}\,p_{qs}(s)\,\xi_m(s)\,ds$$

$$+\int_0^1 \frac{\partial^2 P}{\partial t^2}\,\frac{M\,L^2}{H}\,q_{qs}(s)\,\eta_m(s)\,ds + \int_0^1 \frac{L}{H}\,F_d\,\eta_m(s)ds \tag{8.20}$$

With initial conditions which must be derived in a similar fashion and are found to be:

$$c_m(0)=0$$

$$\frac{\partial c_m(0)}{\partial t} = -\,\frac{\partial P(0)}{\partial t}\int_0^1 (p_{qs}(s)\,\xi_m + q_{qs}(s)\,\eta_m)\,ds \tag{8.21}$$

The remaining problem which must be adressed is the effect of the drag term,

F_d. If the drag term where a linear function of the displacement, the drag term could be lumped into a linear modal damping coefficient, which would account for the structural damping as well as the linear drag. However, the drag of the cable will, in general be a non-linear function of the normal displacement which requires an iterative procedure to find the correct motion. Neglecting the drag term in the first approach and rewriting the expression for $c_m(t)$ in state variable form, we obtain

$$\frac{\partial c_m^1}{\partial t} = c_m^2$$

$$\frac{\partial c_m^2}{\partial t} = - \omega^2 c_m^1 + \frac{M L^2}{H} \int_0^1 (q_{qs}(s) \eta_m(s) + p_{qs}(s) \xi) ds \frac{\partial^2 P}{\partial t^2} \tag{8.22}$$

With the initial conditions:

$$c_m^1(0) = 0$$

$$c_m^2(0) = - \frac{\partial P(0)}{\partial t} \int_0^1 (q_{qs}(s) \eta_m(s) + p_{qs}(s) \xi) ds \tag{8.23}$$

in which form a numerical integration scheme may easily be applied. Following the advice of Hornbeck [Hornbeck 77] numerical integration is carried out using a fourth order Runge-Kutta method for the first four start-up steps and the remaining integration is carried out with a 4th order Adams Predictor-Corrector method. An application of the above procedure can be found in part II.

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Appendix A

PROOF OF THE COMPATIBILITY RELATIONS IN TERMS OF VELOCITIES

Let $\vec{v}(t,p)$ denote the velocity of a material point of the cable. Then compatibility provides the following relation:

$$\frac{D\vec{v}}{Ds} = \frac{\partial e}{\partial t}\vec{t} + (1 + e)(\vec{\omega}\times\vec{t}) \tag{A.1}$$

We can prove this as follows: let $\vec{r}(p,t)$ denote the vector from the origin of a Cartesian system to a material point. Then, according to the definition of the tangential vector (see also [Hildebrand 48]):

$$\vec{t}\,(t,p) = \frac{D\vec{r}\,(t,p)}{Dp} \tag{A.2}$$

At time $t + \delta t$ the segment has stretched by a certain amount:

$$\delta p(s,\ t+\delta t) \simeq \delta s \left[\ 1 + e + \frac{\partial e}{\partial t}\,\delta t\ \right]$$

$$\simeq \frac{\delta p(s,t)}{1+e}\left[\ 1 + e + \frac{\partial e}{\partial t}\,\delta t\ \right]$$

$$\simeq \delta p(s,t)\left[\ 1 + \frac{\delta t}{1+e}\,\frac{\partial e}{\partial t}\ \right] \tag{A.3}$$

where relation (A.3) is exact in the limit as $\delta t \to 0$. Note also that it is convenient to revert to the Langrangian coordinate s, which is time invariant. At time $t+\delta t$ therefore, the tangential vector is:

$$\vec{t}\ (t+\delta t,s) = \frac{D\vec{r}(t+\delta t,s)}{Dp(t+\delta t,s)} \tag{A.4}$$

while:

$$\vec{r}(t+\delta t,s) \simeq \vec{r}(t,s) + \frac{\partial \vec{r}}{\partial t}(t,s) \cdot \delta t = \vec{r}(t,s) + \vec{v}(t,s) \cdot \delta t \tag{A.5}$$

a relation which is, again, exact as $\delta t \to 0$. So finally

$$\vec{t}\ (t+\delta t,\ s) \simeq \frac{1}{1 + \frac{\delta t}{1+e}\frac{\partial e}{\partial t}} \cdot \frac{D}{Dp}\left[\ \vec{r} + \vec{v} \cdot \delta t\ \right] \tag{A.6}$$

or:

$$\frac{D\vec{t}}{Dt} = \lim_{\delta t \to 0} \frac{1}{\delta t}\left[\ \vec{t}\,(t+\delta t,s) - \vec{t}\,(t,s)\ \right] = \frac{D\vec{v}}{Dp} - \frac{\partial e}{\partial t}\frac{\vec{t}}{1+e} \tag{A.7}$$

but from (1.8) we obtain that:

$$\frac{D\vec{t}}{Dt} = \vec{\omega} \times \vec{t} \tag{A.8}$$

therefore:

$$\frac{D\vec{v}}{Dp} = \frac{\vec{t}}{1+e} \cdot \frac{\partial e}{\partial t} + \vec{\omega} \times \vec{t} \tag{A.9}$$

When this is rewritten in the unstretched Lagrangian coordinate, (A.1) is obtained.

Appendix B

HIGHER ORDER WKB APPROXIMATION OF THE STRING EQUATION

The string equation with varying coeficients can be written as:

$$\frac{d}{ds}\left[T(\sigma) \frac{dq}{d\sigma} \right] + \omega^2 L^2 M(\sigma)\, q = 0 \qquad (B.1)$$

with: $M(\sigma)$ and $T(\sigma)$ slowly varying.

and/or: $\dfrac{\omega^2 M L^2}{T}$ large.

(σ is the non-dimensional length)

The WKB approximation is of the following form:

$$q = \frac{1}{(M \cdot T)^{1/4}} e^{\pm i\omega L(g_o + g_2/\omega^2)} \qquad (B.2)$$

where: $g_o = \int_0^\sigma (M/T)^{1/2}\, d\sigma$

$$g_2 = \int_0^\sigma \left[-\frac{1}{8} \frac{\ddot{T}}{(T \cdot M)^{1/2}} + \frac{1}{32} \frac{\dot{T}^2}{(M)^{1/2} \cdot T^{3/2}} - \frac{1}{16} \frac{\dot{M} \cdot \dot{T}}{M^{3/2} \cdot (T)^{1/2}} + \frac{5}{32} \frac{\dot{M}^2 \cdot (T)^{1/2}}{M^{5/2}} - \frac{1}{8} \frac{\ddot{M} \cdot (T)^{1/2}}{M^{3/2}} \right] d\sigma$$

Appendix C

ORTHOGONALITY OF THE MODES OF INEXTENSIBLE CABLES

The self-adjoint differential equation was obtained in subsection 3.6.4 as:

$$\left[\cos\phi_o \, \xi^{II} \right]^{II} + \left[2\cos\phi_o + \frac{\lambda^2}{\cos^2\phi_o} \, \xi^{I} \right]^{I} - \frac{h\lambda^2}{\cos^2\phi_o} \, \xi = 0 \qquad \text{(C.1)}$$

Assume ξ_i , ξ_j eigenmodes with eigenvalues λ_i , λ_j. We can write for mode i:

$$\left[\cos\phi_o \, \xi_i^{II} \right]^{II} + \left[2\cos\phi_o \, \xi_i^{I} \right]^{I}$$

$$= \lambda_i^2 \left\{ - \left[\frac{\xi_i^{I}}{\cos^2\phi_o} \right]^{I} + \frac{h}{\cos^2\phi_o} \, \xi_i \right\} \qquad \text{(C.2)}$$

Multiplication with ξ_j and integration over the domain gives:

$$\int_{\phi_{bot}}^{\phi_{top}} \xi_j \left\{ \left[\cos\phi_o \, \xi_i^{II} \right]^{II} + \left[2\cos\phi_o \, \xi_i^{I} \right]^{I} \right\} d\phi_o$$

$$= \lambda_i^2 \int_{\phi_{bot}}^{\phi_{top}} \left\{ - \xi_j \left[\frac{\xi_i^{I}}{\cos^2\phi_o} \right]^{I} + \frac{h\xi_i\xi_j}{\cos^2\phi_o} \right\} d\phi_o \qquad \text{(C.3)}$$

Using partial integration, this can be written as:

$$\int_{\phi_{bot}}^{\phi_{top}} \left[\cos\phi_o \, \xi_j^{II} \, \xi_i^{II} - 2\cos\phi_o \, \xi_j^{I} \, \xi_i^{I} \right] d\phi_o$$

$$= \lambda_i^2 \int_{\phi_{bot}}^{\phi_{top}} \left[\frac{\xi_j^{I} \, \xi_i^{I}}{\cos^2\phi_o} + h \, \frac{\xi_j \, \xi_i}{\cos^2\phi_o} \right] d\phi_o \qquad \text{(C.4)}$$

The same operation can be performed by multiplying ξ_i with the equation

for the mode ξ_j and integrating:

$$\int_{\phi_{bot}}^{\phi_{top}} \left[\cos\phi_o \, \xi_i^{II} \, \xi_j^{II} - 2\cos\phi_o \, \xi_i^{I} \, \xi_j^{I} \right] d\phi_o$$

$$= \lambda_j^2 \int_{\phi_{bot}}^{\phi_{top}} \left[\frac{\xi_i^{I} \, \xi_j^{I}}{\cos^2\phi_o} + h \, \frac{\xi_i \, \xi_j}{\cos^2\phi_o} \right] d\phi_o$$

Substracting (C.4) from (C.5) gives:

$$\int_{\phi_{bot}}^{\phi_{top}} \left[\frac{\xi_i^{I} \, \xi_j^{I}}{\cos^2\phi_o} + h \, \frac{\xi_j \, \xi_i}{\cos^2\phi_o} \right] d\phi_o = 0 \qquad \text{for } i \neq j$$

which was to be proved.

Using the relation that:

$$\xi_i^{I} = \eta_i \qquad\qquad \xi_j^{I} = \eta_j$$

and

$$\frac{d\phi_o}{d\sigma} = \alpha \cos^2\phi_o \qquad\qquad h = \frac{m}{M}$$

we obtain:

$$\int_0^1 (m \, \xi_i \, \xi_j + M \, \eta_i \, \eta_j) \, d\sigma = 0 \qquad \text{for } i \neq j$$

Appendix D

ORTHOGONALITY OF THE MODES OF EXTENSIBLE CABLES

Using influence functions, the requirement for the modes of an extensible cable can be written as [Rosenthal 81]:

$$[\alpha_i\ (s)] = \int_0^L [H(s,r)]\ \omega_i^2\ [m(r)]\ [\alpha_i\ (r)]\ dr \tag{D.1}$$

where in the two dimensional case, with the components written in the tangential and normal direction, we have:

$$[\alpha_i\ (s)]^T = [p_i\ (s) \quad q_i\ (s)] \tag{D.2}$$

$$[H(s,r)] = \begin{bmatrix} H_{11}\ (s,r) & H_{12}\ (s,r) \\ H_{21}\ (s,r) & H_{22}\ (s,r) \end{bmatrix} \tag{D.3}$$

$$[m(r)] = \begin{bmatrix} m(r) & 0 \\ 0 & M(r) \end{bmatrix} \tag{D.4}$$

therefore: $[m(r)]^T = [m(r)]$

$$[H(s,r)]^T = [H(r,s)] \tag{D.5}$$

Multiplying (D.1) with the transpose of a the modal matrix of a different mode and the mass matrix, and integrating, we obtain:

$$\int_0^L [\alpha_j(s)]^T\ [m(s)]\ [\alpha_i(s)]\ ds$$

$$= \omega_i^2 \int_0^L [\alpha_j(s)]^T\ [m(s)]\ ds \int_0^L [H(s,r)]\ [m(r)]\ [\alpha_i(r)]\ dr$$

Due to (D.5) this is equivalent to:

$$\int_0^L [\alpha_j(s)]^T\ [m(s)]\ [\alpha_i(s)]\ ds$$

$$= \omega_i^2 \int_0^L [\alpha_i(r)]^T\ [m(r)]\ dr \int_0^L [H(r,s)]\ [m(s)]\ [\alpha_j(s)]\ ds$$

We can apply (D.6) also to the jth mode. Subtracting the result from (D.7) when ω_i is different from ω_j, then:

$$\int_0^L [\alpha_j(s)]^T\ [m(s)]\ [\alpha_i(s)]\ ds = 0 \qquad i \neq j$$

or, explicity in non-dimensional form:

$$\int_0^1 (m\ \xi_i\ \xi_j + M\ \eta_i\ \eta_j\ )\ d\sigma = 0 \qquad i \neq j$$

Appendix E

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