UNH SEA GRANT PROGRAMS

UNH-RAYTHEON SEA GRANT PROJECT

TECHNICAL REPORT

Sound Transmission In Liquid-Viscoelastic Multilayer Media

Gary K. Stewart
Mechanics Research Laboratory

A Report of a
Cooperative University-Industry Research Project
between

University of New Hampshire
Durham, New Hampshire 03824

Raytheon Company
Portsmouth, R. I.
02871

UNIVERSITY of NEW HAMPSHIRE
DURHAM, NEW HAMPSHIRE. 03824

Report No. UNH-SG116

July 1973

A TECHNICAL REPORT TO

THE NATIONAL SEA GRANT PROGRAM

OF

THE NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION

U. S. DEPARTMENT OF COMMERCE

SOUND TRANSMISSION IN LIQUID-VISCOELASTIC MULTILAYER MEDIA

by

Gary K. Stewart

Mechanics Research Laboratory

July 1973

This work is a result of research sponsored by NOAA Office of Sea Grant, Department of Commerce, under Grant No. NG 30-72. The U. S. Government is authorized to produce and distribute reprints for governmental purposes notwithstanding any copyright notation that may appear hereon.

Approved:

Musa Yildiz - Technical Director

Cooperating Institutions

University of New Hampshire
Durham, New Hampshire 03824

Submarine Signal Division
Raytheon Company
Portsmouth, Rhode Island 02871

ACKNOWLEDGEMENTS

The author wishes to acknowledge with gratitude the inspiration and leadership provided by Dr. Asim Yildiz, Professor of Mechanics, who directed this thesis. His keen insight into this physical problem and suggestions as to the method of approach to the solution proved to be invaluable.

It is a pleasure to express appreciation to Dr. Allen Magnuson, who contributed to the theoretical development of the subject.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Figure

NOMENCLATURE

ϕ_o - potential function of the liquid field

ϕ_{Ln} - potential function of the compressional field for the n th layer of the viscoelastic medium

ϕ_{Tn} - potential function of the shear field for the n th layer of the viscoelastic medium

A_o, B_o - amplitude functions of the liquid field

A_{Ln}, B_{Ln} - amplitude functions of the compressional field for the n th layer of the viscoelastic medium

A_{Tn}, B_{Tn} - amplitude functions of the shear field for the n th layer of the viscoelastic medium

ζ - transformation parameter for the Hankel transform

k_o - wave number of the liquid field

k_{Ln} - complex wave number of the compressional field for the n th layer of the viscoelastic medium

k_{Tn} - complex wave number of the shear field for the n th layer of the viscoelastic medium

ω - transformation parameter for the Fourier transform

c_o - sound velocity in the liquid medium

c_{Ln} - complex sound velocity in the compressional field for the n th layer of the viscoelastic medium

c_{Tn} - complex sound velocity in the shear field for the n th layer of the viscoelastic medium

$\bar{\lambda}_n, \bar{\mu}_n$ - complex Lamé parameters of the n th layer

ρ_n - density of the n th layer

σ_{ij} - stress tensor

$\bar{u}$ - displacement vector

r, θ, z - subscripts denoting components in the radial, circumferential and longitudinal directions for a cylindrical coordinate system

ABSTRACT

An expression for the acoustic response of a liquid layer overlying a multilayer viscoelastic medium is determined. The excitation is provided by a point source in the liquid layer. The output relationship of the system is expressed as a multiple integral using Fourier transforms for the time domain and Hankel transforms for the spatial domain. In this boundary-value problem the theories of fluid dynamics and elasticity provide the basis for describing the hydrodynamic and viscoelastic fields. The mathematical model utilizes assumptions, most of which have proven to accurately describe actual physical observations, particularly with respect to seismic work in geophysics. New techniques in this approach include the introduction of complex wave numbers to describe the damping of the system using viscoelastic theory. Also a scheme is developed, using recursion relations between adjacent layers, whereby the potential of the liquid layer can be found easily. The liquid layer and each viscoelastic layer is considered to be homogeneous and the interfaces are assumed to be plane and parallel.

CHAPTER I

INTRODUCTION

I. INTRODUCTION

Investigations of reflections from an ocean subbottom covered with a liquid layer have been pursued for a long time. The determination of the subbottom structure and its mechanical properties are problems of extreme importance and require fine theoretical as well as fine experimental results. The present method of approach to this problem area is largely inspired by elasticity theoreticians. Indeed, Lamb's 1904 work [10]on the investigation of the earth's elastic properties is generalized here by covering the earth's surface with a finite depth of liquid otherwise extending to infinity. Since Lamb's investigation a great deal of work has been done in this area by Haskell [3], Thompson [21], Tolstoy [20] and others who have used plane wave approximation methods or the ray theoretical approach, as opposed to field theoretical formalisms, without gaining a keen insight into the phenomenological aspects of the situation. Others have shown that the ray theory results can be obtained from the field theoretical approach by applying proper approximation techniques. These arguments suggest that the field theoretical approach, which is based on the conservation laws of nature, is more general and more sensitive to the physical situation, and thus gives a more complete picture of nature. For this reason, field theory serves as the foundation of this thesis.

New concepts, that contribute to the improvement of modeling and solving this class of subbottom probing problems, are introduced in this thesis. The basic improvement is in the subbottom model which in previous studies has been considered to be a perfectly elastic medium or layers, whereas in this investigation, the subbottom model is chosen to be viscoelastic medium. This is a much better representation of reality. This generalization is well justified, but from the mathematical and numerical calculation point of view it becomes rather costly. Thus the results are new and facilitate efforts in the fields of oceanography, geophysics, and marine technology on the understanding and identification of soil mechanical properties of the subbottom. Also, the results may open new possibilities for further extrapolations of the subject matter from the geophysical point of view.

In addition to the improvement of subbottom modeling, a breakthrough in multilayer analysis makes it possible to avoid the medium characteristic matrix inversion problem consequently saving considerable efforts in the analysis of n-layer problems. This is accomplished by a simple and new boundary-iterative formalism which contains matrix theory as the underlying algebraic structure.

The theoretical development of this thesis begins with the introduction of the field equations necessary to describe the hydrodynamic and viscoelastic models, followed by chapters detailing the steps that lead to the solution of the one-layer problem, and finally the n-layer problem. Since we shall investigate an axially symmetric isotropic medium, the mathematical formalism is expressed in terms of cylindrical coordinates.

CHAPTER II

THEORETICAL DEVELOPMENT

II. THEORETICAL DEVELOPMENT

1. Field Equations

We have at our disposal two fields, namely the liquid and the viscoelastic fields. The wave propagation properties of these fields can be best represented by the following wave equations (See Appendix A).

$$(\nabla^2+k_o^2)\phi_o=0 \tag{1}$$

$$(\nabla^2+k_L^2)\phi_L=0 \tag{2a}$$

$$(\nabla^2+k_T^2)\phi_T=0 \tag{2b}$$

Here the o, L, and T indices refer to the liquid, viscoelastic longitudinal and viscoelastic transverse, respectively. Equations (1) and (2) were obtained from the field equations by taking the Fourier transforms in time. Wave numbers for the viscoelastic field equations are complex because of the complex moduli of the Lamé parameters whereas the velocity of sound in the liquid layer is assumed to be real and constant:

$$k_o=\omega/c_o \qquad k_L=\omega/c_L \qquad k_T=\omega/c_T$$

$$\bar{\lambda}=\lambda'+i\omega\lambda'' \qquad \bar{\mu}=\mu'+i\omega\mu'' \tag{3}$$

$$c_o^2=\lambda_o/\rho_o \qquad c_L^2=(\bar{\lambda}+2\bar{\mu})/\rho \qquad c_T^2=\bar{\mu}/\rho$$

Here we have defined the hydrodynamic field from the elasticity point of view by denoting c_o^2 equated to λ_o/ρ_o where λ_o is the newly defined Lame constant for the liquid. Since neither viscosity (no shear waves are supported by the liquid) nor damping effects are considered in the liquid, we can write the acoustic field wave equations from the viscoelastic wave equations by simply setting $\bar{\mu}(\omega)=0$ and retaining the real part of $\bar{\lambda}(\omega)$. This is very convenient for the stress field descriptions of these two distinct fields as will be apparent in the forthcoming discussions.

Longitudinal (compressional) and transverse (shear) waves are not coupled in an infinitely extended medium when the regime is linear (Hooke's regime). However, coupling between these two waves will occur due to the existence of the liquid-solid interface. In such a boundary coupling case longitudinal and vertical shear waves will be coupled. Because their polarization planes are common, energy transfer from one polarization to the other becomes possible.

2. Approach to the Solution

Due to the isotropy of the liquid layer and the viscoelastic halfspace, there exists an axial symmetry in the problem. This suggests that the proper coordinate system is the cylindrical coordinate system. While in general the problem can be solved in any coordinate system (physical laws being independent of the choice of the coordinate system), this specific choice of coordinates makes the calculations considerably simpler. Thus equations (1) and (2) need to be written in cylindrical coordinates. Starting with equation (1),

$$(\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_\theta^2+\partial_z^2+k_o^2)\phi_o=0$$

the ∂_θ will vanish due to axial symmetry, therefore

$$(\partial_r^2+\frac{1}{r}\partial_r+\partial_z^2+k_o^2)\phi_o=0 \tag{4}$$

Similarly equations (2a) and (2b) will read:

$$(\partial_r^2+\frac{1}{r}\partial_r+\partial_z^2+k_L^2)\phi_L=0 \tag{5}$$

$$(\partial_r^2+\frac{1}{r}\partial_r+\partial_z^2+k_T^2)\phi_T=0 \tag{6}$$

We would like to use boundary conditions which are on the boundary planes perpendicular to the z-axis. It is obvious that there are no boundaries (discontinuities) in the radial direction, thus we can convert r-dependent operators into constants by using the following Hankel transform pair (see, e. g., Sneddon [17]):

$$\begin{aligned}\bar{X}(\zeta)&=\int_0^\infty X(r)J_o(\zeta r)r\,dr\\ X(r)&=\int_0^\infty \bar{X}(\zeta)J_o(\zeta r)\zeta\,d\zeta\end{aligned} \tag{7}$$

In the Hankel transform pair ζ represents the transformation parameter. The kernel $J_o(\zeta r)$ is chosen as the zeroth order Bessel function because of the angular symmetry of the problem. Note that the Hankel transform provides the following identity:

$$\int_0^\infty (\partial_r^2+\frac{1}{r}\partial_r)X(r)J_o(\zeta r)rdr=-\zeta^2\bar{X}(\zeta) \tag{8}$$

Therefore, equations (4), (5) and (6) will be reduced to the following forms:

$$[\partial_z^2-(\zeta^2-k_o^2)]\bar{\phi}_o(\zeta,z,\omega)=0 \tag{9}$$

$$[\partial_z^2-(\zeta^2-k_L^2)]\bar{\phi}_L(\zeta,z,\omega)=0 \tag{10}$$

$$[\partial_z^2-(\zeta^2-k_T^2)]\bar{\phi}_T(\zeta,z,\omega)=0 \tag{11}$$

Now, the solutions of these differential equations are obvious:

$$\bar{\phi}_o(\zeta,z,\omega)=A_o(\zeta,\omega)e^{-a_o z}+B_o(\zeta,\omega)e^{a_o z} \tag{12}$$

$$\bar{\phi}_L(\zeta,z,\omega)=A_L(\zeta,\omega)e^{-a_L z}+B_L(\zeta,\omega)e^{a_L z} \tag{13}$$

$$\bar{\phi}_T(\zeta,z,\omega)=A_T(\zeta,\omega)e^{-a_T z}+B_T(\zeta,\omega)e^{a_T z} \tag{14}$$

$$a_o=\sqrt{\zeta^2-k_o^2} \qquad a_L=\sqrt{\zeta^2-k_L^2} \qquad a_T=\sqrt{\zeta^2-k_T^2} \tag{15}$$

The exponential terms containing the two integrations parameters $A(\zeta,\omega)$ and $B(\zeta,\omega)$ represent downward and upward traveling waves, respectively. In forthcoming discussions of the one-layer problem and the multilayer problem we will see that the two integration parameters $A_o(\zeta,\omega)$ and $B_o(\zeta,\omega)$ for the liquid layer stay in the picture. However, for viscoelastic layer solutions, the $B_L(\zeta,\omega)$ and $B_T(\zeta,\omega)$ exponential solutions drop out for the bottom layer due to the non-reflective property of the semiinfinite medium.

3. Boundary Conditions

Boundary conditions are required to evaluate the integration parameters that arise in the solut[illegible] of the field equations. Later when we consider the pr[illegible]em of a source suspended in the liquid layer, the boundary conditions will lead to a dispersion relation which represents the forced oscillations of the viscoelastic half-space covered with a finite height liquid layer. Two types of boundary conditions arise. One is a result of the continuity of mass density. This boundary condition implies that the displacement is continuous across the interface between two different media, or

$$\begin{aligned} u_{ni} &= u_{n(i+1)} \\ u_{ti} &= u_{t(i+1)} \end{aligned} \tag{16}$$

at the boundary between the i th layer and the (i+1) th layer, where u_n is the component of the displacement normal to the boundary, and u_t is the tangential component of the displacement. The second type of boundary condition arises from the conservation of linear momentum law. The statement of this boundary condition is that the stress tensor is continuous across the boundary, or

$$\begin{aligned} \sigma_{nni} &= \sigma_{nn(i+1)} \\ \sigma_{tni} &= \sigma_{tn(i+1)} \end{aligned} \tag{17}$$

where σ_{nn} is the stress normal to the boundary, and σ_{tn} is the shear stress at the boundary.

4. Stress and Displacement Field Equations

In order to use the boundary conditions suggested in the previous section it is necessary to determine the stress and displacement fields completely in both the liquid and viscoelastic layers. The stress tensor σ_{ij} is related to the strain tensor ε_{ij} by the well-known relation [11],

$$\sigma_{ij}=\bar{\lambda}e\delta_{ij}+2\bar{\mu}\varepsilon_{ij} \tag{18}$$

where
$$e=\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{zz}=\nabla.\bar{u} \tag{19}$$

and
$$\bar{u}=u_r\bar{e}_r+u_\theta\bar{e}_\theta+u_z\bar{e}_z$$

The strain tensor in cylindrical coordinates (r,θ,z) is:

$$\begin{aligned}
&\varepsilon_{rr}=\partial_r u_r && 2\varepsilon_{r\theta}=\partial_r u_\theta-\frac{u_\theta}{r}+\frac{1}{r}\partial_\theta u_r \\
&\varepsilon_{\theta\theta}=\frac{1}{r}\partial_\theta u_\theta+\frac{u_r}{r} && 2\varepsilon_{rz}=\partial_z u_r+\partial_r u_z \\
&\varepsilon_{zz}=\partial_z u_z && 2\varepsilon_{\theta z}=\frac{1}{r}\partial_\theta u_z+\partial_z u_\theta
\end{aligned} \tag{20}$$

Substituting equations (19) and (20) into the stress-strain relation given by (18) yields:

$$\begin{aligned}
&\sigma_{rr}=\bar{\lambda}(\nabla.\bar{u})+2\bar{\mu}\partial_r u_r \\
&\sigma_{\theta\theta}=\bar{\lambda}(\nabla.\bar{u})+2\bar{\mu}(\frac{1}{r}\partial_\theta u_\theta+\frac{u_r}{r}) \\
&\sigma_{zz}=\bar{\lambda}(\nabla.\bar{u})+2\bar{\mu}\partial_z u_z \\
&\sigma_{r\theta}=\bar{\mu}(\partial_r u_\theta-\frac{u_\theta}{r}+\frac{1}{r}\partial_\theta u_r) \\
&\sigma_{rz}=\bar{\mu}(\partial_z u_r+\partial_r u_z) \\
&\sigma_{\theta z}=\bar{\mu}(\frac{1}{r}\partial_\theta u_z+\partial_z u_\theta)
\end{aligned} \tag{21a-f}$$

For convenience of the calculations, the stress field components in the liquid and the viscoelastic media will be written in terms of the scalar potentials defined by equations (1) and (2).

From the hydrodynamic field equation derivations of Appendix A we use

$$\bar{u}_o=\nabla\bar{\Phi}_o \tag{22}$$

for the liquid layer which results in the following stress field,

$$\begin{aligned}
\sigma_{rr}&=\lambda_o\nabla^2\bar{\Phi}_o\\
\sigma_{\theta\theta}&=\lambda_o\nabla^2\bar{\Phi}_o\\
\sigma_{zz}&=\lambda_o\nabla^2\bar{\Phi}_o\\
\sigma_{r\theta}&=0\\
\sigma_{rz}&=0\\
\sigma_{\theta z}&=0
\end{aligned} \tag{23a-f}$$

where $\mu_o=0$ because neither viscosity nor damping effects are considered in the liquid. The components of the displacement field are determined from equation (22),

$$\begin{aligned}
u_r&=\partial_r\bar{\Phi}_o\\
u_\theta&=\frac{1}{r}\partial_\theta\bar{\Phi}_o\\
u_z&=\partial_z\bar{\Phi}_o
\end{aligned} \tag{24a-c}$$

where $u_{\theta}=0$ due to the axial symmetry of the problem.

Similarly, from the viscoelastic field equation derivations of Appendix A, we know that it is preferable to solve the vector Helmholtz equations by using scalar potentials. The longitudinal and (vertical shear) transverse parts can be represented by the following expressions,

$$\bar{u}=\bar{u}_L+\bar{u}_T$$

$$\bar{u}_L=\nabla\bar{\phi}_L \qquad (25a\text{-}c)$$

$$\bar{u}_T^{VS}=\nabla x\nabla x\bar{e}_z\bar{\phi}_T$$

where $\bar{e}_z$ is the unit vector in the z- direction for cylindrical coordinates. Since the stress tensor is a function of strain, and strain is a function of displacement, it is necessary to describe the displacement field for the viscoelastic field first (See Appendix B).

$$u_r=\partial_r(\bar{\phi}_L+\partial_z\bar{\phi}_T)$$

$$u_\theta=\frac{1}{r}\partial_\theta(\bar{\phi}_L+\partial_z\bar{\phi}_T) \qquad (26a\text{-}c)$$

$$u_z=\partial_z\bar{\phi}_L+k_T^2\bar{\phi}_T+\partial_z^2\bar{\phi}_T$$

Again, due to the axial symmetry in the problem, $u_\theta=0$. Substituting equations (25), (26) and (2) into the stress-displacement relation given by (21) yields:

$$\sigma_{rr}=-\bar{\lambda}k_L^2\bar{\Phi}_L+2\bar{\mu}\partial_r^2(\bar{\Phi}_L+\partial_z\bar{\Phi}_T)$$

$$\sigma_{\theta\theta}=-\bar{\lambda}k_L^2\bar{\Phi}_L+2\bar{\mu}\frac{1}{r}\partial_r(\bar{\Phi}_L+\partial_z\bar{\Phi}_T)$$

$$\sigma_{zz}=-\bar{\lambda}k_L^2\bar{\Phi}_L+2\bar{\mu}[\partial_z^2\bar{\Phi}_L+\partial_z(k_T^2\bar{\Phi}_T+\partial_z^2\bar{\Phi}_T)]$$

$$\sigma_{r\theta}=0 \qquad (27a\text{-}f)$$

$$\sigma_{rz}=\bar{\mu}\partial_r(2\partial_z\bar{\Phi}_L+k_T^2\bar{\Phi}_T+2\partial_z^2\bar{\Phi}_T)$$

$$\sigma_{\theta z}=0$$

Now the stress and displacement fields for the liquid and viscoelastic layers are completely defined. It is worthwhile to observe that when the second Lamé parameter becomes zero in equations (27), we obtain equations (23). It was mentioned previously that formulating the hydrodynamic field in terms of the elasticity field makes the calculations easy.

5. Source Representation

In the problems under consideration in the next two sections a point source will be suspended in the liquid layer. The effect that the presence of the source has on the differential equation solutions, equations (12), (13) and (14), is the addition of a source term, in potential form, to the solution for the liquid layer. The source term for an unbounded homogeneous medium may be obtained, following Sommerfeld [18], as

$$\phi_s(r,z,\omega)=\frac{e^{-ik_oR}}{R}=\int_0^\infty J_o(\zeta r)\frac{e^{-a_o|z-z_s|}}{a_o}\zeta d\zeta \tag{28}$$

where

$$R=\sqrt{r^2+(z-z_s)^2}$$

In order to add this source potential to the solution for the liquid layer it must undergo a Hankel transformation as did the previous differential equation for the liquid layer. We find that this can be done simply because Sommerfeld's source representation, equation (28), is conveniently in the form of a Hankel transformation. The Hankel transform of equation (28) is

$$\bar{\Phi}_s(\zeta,z,\omega)=\frac{e^{-a_o|z-z_s|}}{a_o} \tag{29}$$

and now equation (12) may be written as:

$$\bar{\Phi}_o(\zeta,z,\omega)=A_o(\zeta,\omega)e^{-a_oz}+B_o(\zeta,\omega)e^{a_oz}+\frac{e^{-a_o|z-z_s|}}{a_o} \tag{30}$$

6. Solution of the One-layer Problem

This section is concerned with obtaining the formal solution for the problem of a source suspended in a liquid layer over a semiinfinite viscoelastic subbottom. The geometry of the problem is shown in figure 1. Our concern will be with adapting the information of the previous sections to describe the problem at hand. Recalling equations (30), (14) and (13), we may write the transformed potential functions for this problem as special cases. For the liquid only a longitudinal field potential exists. We write equation (30) as

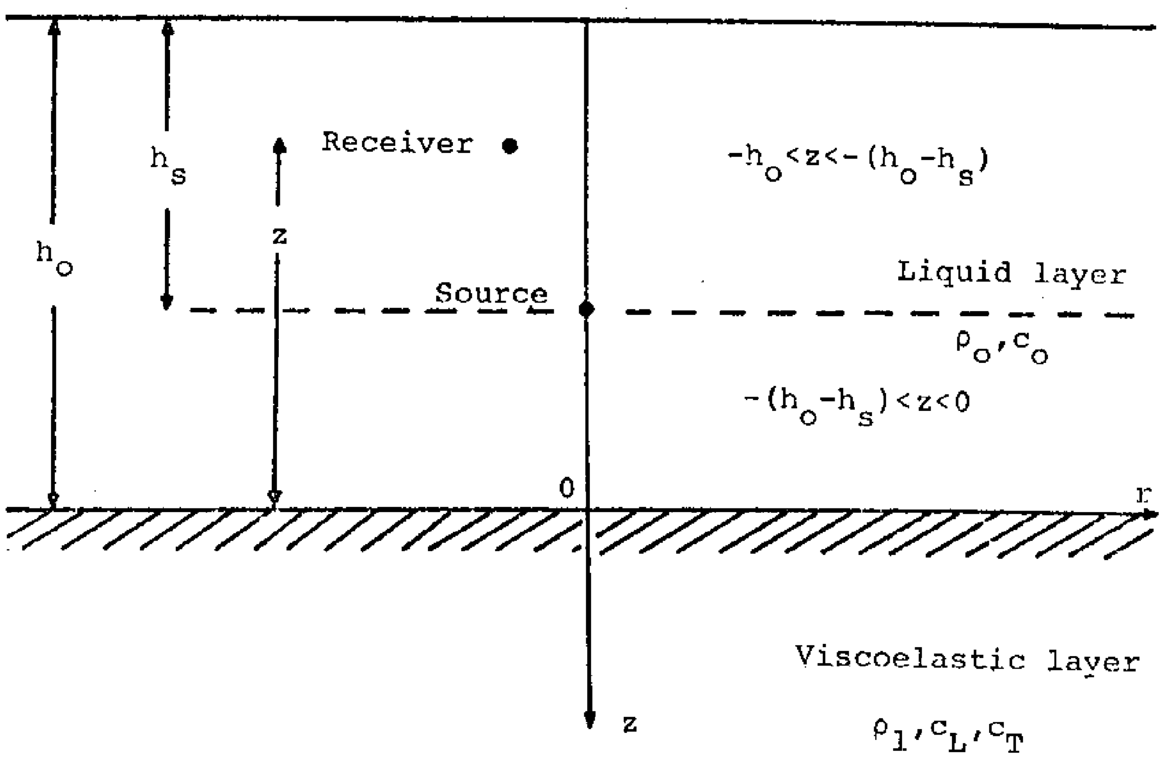
$$\bar{\phi}_o(\zeta,z,\omega)=A_o(\zeta,\omega)e^{-a_oz}+B_o(\zeta,\omega)e^{a_oz}+\frac{e^{-a_o|z+(h_o-h_s)|}}{a_o} \tag{31a}$$

where in equation (29) $z_s=(h_o-h_s)$ from figure 1. Furthermore, before the application of the boundary conditions, it will be necessary for us to classify the liquid potential $\bar{\phi}_o$ as a potential above the source $\bar{\phi}_o^I$ and a potential below the source $\bar{\phi}_o^{II}$, where

$$\bar{\phi}_o^I=A_oe^{-a_oz}+B_oe^{a_oz}+\frac{e^{a_o[z+(h_o-h_s)]}}{a_o} \qquad z<-(h_o-h_s) \tag{31b}$$

$$\bar{\phi}_o^{II}=A_oe^{-a_oz}+B_oe^{a_oz}+\frac{e^{-a_o[z+(h_o-h_s)]}}{a_o} \qquad z>-(h_o-h_s) \tag{31c}$$

Two potential functions exist in the viscoelastic subbottom, namely one for the longitudinal field and one for the transverse field. Since the subbottom is unbounded in the z- direction, the parts of equations (13) and (14) representing upward traveling waves vanish due to the non-reflective



LIQUID LAYER OVERLYING ONE VISCOELASTIC LAYER

FIG. 1

property mentioned previously, and equations (13) and (14) become:

$$\bar{\phi}_L(\zeta,z,\omega)=A_L(\zeta,\omega)e^{-a_L z} \tag{32}$$

$$\bar{\phi}_T(\zeta,z,\omega)=A_T(\zeta,\omega)e^{-a_T z} \tag{33}$$

We must solve for the four integration constants A_o, B_o, B_L and B_T using the applicable boundary conditions. The boundary conditions that will prove to be most useful to us (see, e. g., Ewing, Jardetsky, and Press [2]) are at $z=-h_o$,

$$(\sigma_{zz})_o=0 \tag{34}$$

for the water surface, and at z=0,

$$(\sigma_{zz})_o=(\sigma_{zz})_1 \tag{35}$$

$$(u_z)_o=(u_z)_1 \tag{36}$$

$$(\sigma_{rz})_o=(\sigma_{rz})_1=0 \tag{37}$$

for the liquid-viscoelastic interface, where the subscript o refers to the liquid and 1 refers to the subbottom. We recall that the liquid cannot sustain a shear stress. Applying the boundary condition at $z=-h_o$ to the expression for the potential above the source in the liquid gives:

$$0=A_o e^{a_o h_o}+B_o e^{-a_o h_o}+\frac{e^{-a_o h_s}}{a_o} \tag{38}$$

If one eliminates B_o from this, the expression for the liquid potentials above and below the source may be written as;

$$\bar{\phi}_o^{I}=-2A_o e^{a_o h_o}\sinh[a_o(z+h_o)] \tag{39}$$

$$\bar{\phi}_o^{II}=-2A_o e^{a_o h_o}\sinh[a_o(z+h_o)]-\frac{2}{a_o}\sinh[a_o(z+h_o-h_s)] \tag{40}$$

By applying the three boundary conditions at z=0, and using the expressions for the stress and displacement fields in equations (23), (24) and (27), we may write the following matrix expression for the unknowns A_o, A_L and A_T (See Appendix C).

$$\begin{bmatrix} -2a_o e^{a_o h_o}\cosh(a_o h_o) & a_L & -\zeta^2 \\ 0 & -2a_L & (2\zeta^2-k_T^2) \\ 2\rho_o\omega^2 e^{a_o h_o}\sinh(a_o h_o) & -\bar{\mu}_1(2\zeta^2-k_T^2) & 2\bar{\mu}_1 a_T\zeta^2 \end{bmatrix}\begin{bmatrix} A_o \\ A_L \\ A_T \end{bmatrix} = \begin{bmatrix} 2\cosh a_o(h_o-h_s) \\ 0 \\ -\frac{2\rho_o\omega^2}{a_o}\sinh a_o(h_o-h_s) \end{bmatrix} \tag{41}$$

We are interested in the solution for the acoustic field in the liquid, so we solve for A_o in equation (41) using Cramer's rule.

$$A_o = \frac{\Delta_1}{\Delta_o} \tag{42}$$

where
$$\Delta_1 = \begin{vmatrix} 2\cosh[a_o(h_o-h_s)] & a_L & -\zeta^2 \\ 0 & -2a_L & (2\zeta^2-k_T^2) \\ -\frac{2\rho_o\omega^2}{a_o}\sinh[a_o(h_o-h_s)] & -\bar{\mu}_1(2\zeta^2-k_T^2) & 2\bar{\mu}_1 a_T\zeta^2 \end{vmatrix} \tag{43}$$

and
$$\Delta_o = \begin{vmatrix} -2a_o e^{a_o h_o}\cosh(a_o h_o) & a_L & -\zeta^2 \\ 0 & -2a_L & (2\zeta^2-k_T^2) \\ 2\rho_o\omega^2 e^{a_o h_o}\sinh(a_o h_o) & -\bar{\mu}_1(2\zeta^2-k_T^2) & 2\bar{\mu}_1 a_T\zeta^2 \end{vmatrix} \tag{44}$$

Expanding equations (43) and (44) gives:

$$\Delta_1 = \frac{2}{a_o}\left\{\frac{\rho_o\omega^4 a_L}{c_T^2}\sinh a_o(h_o-h_s) + a_o\rho_1 c_T^2[(2\zeta^2-k_T^2)^2-4a_L a_T\zeta^2]\cosh a_o(h_o-h_s)\right\} \tag{45}$$

$$\Delta_o = -2e^{a_o h_o}\left\{\frac{\rho_o\omega^4 a_L}{c_T^2}\sinh(a_o h_o) + a_o\rho_1 c_T^2[(2\zeta^2-k_T^2)^2-4a_L a_T\zeta^2]\cosh(a_o h_o)\right\} \tag{46}$$

Substituting equations (45) and (46) into (42) yields:

$$A_o=-\frac{e^{-a_o h_o}}{a_o}\left\{\frac{k_T^4 a_L \sinh a_o(h_o-h_s)+a_o\frac{\rho_1}{\rho_o}[(2\zeta^2-k_T^2)^2-4a_L a_T\zeta^2]\cosh a_o(h_o-h_s)}{k_T^4 a_L \sinh(a_o h_o)+a_o\frac{\rho_1}{\rho_o}[(2\zeta^2-k_T^2)^2-4a_L a_T\zeta^2]\cosh(a_o h_o)}\right\} \tag{47}$$

In the discussion of the multilayer problem that lies ahead we will find it helpful if A_o is expressed in a more compact form. We may write a condensed version of equation (47) as

$$A_o=-\frac{e^{-a_o h_o}}{a_o}\frac{\Delta_1'}{\Delta_o'} \tag{48}$$

where Δ_1' and Δ_o' are clearly defined in equation (47). Now that A_o has been determined, the expressions for the liquid potentials above and below the source immediately follow:

$$\bar{\phi}_o^{I}=\frac{2\sinh[a_o(z+h_o)]}{a_o}\times$$
$$\left\{\frac{k_T^4 a_L \sinh[a_o(h_o-h_s)]+a_o\frac{\rho_1}{\rho_o}[(2\zeta^2-k_T^2)^2-4a_L a_T\zeta^2]\cosh[a_o(h_o-h_s)]}{k_T^4 a_L \sinh(a_o h_o)+a_o\frac{\rho_1}{\rho_o}[(2\zeta^2-k_T^2)^2-4a_L a_T\zeta^2]\cosh(a_o h_o)}\right\} \tag{49}$$

$$\bar{\phi}_o^{II}=\frac{2\sinh(a_o h_s)}{a_o}\times$$
$$\left\{\frac{-k_T^4 a_L \sinh(a_o z)+a_o\frac{\rho_1}{\rho_o}[(2\zeta^2-k_T^2)^2-4a_L a_T\zeta^2]\cosh(a_o z)}{k_T^4 a_L \sinh(a_o h_o)+a_o\frac{\rho_1}{\rho_o}[(2\zeta^2-k_T^2)^2-4a_L a_T\zeta^2]\cosh(a_o h_o)}\right\} \tag{50}$$

These results agree with those predicted by Ewing, Jardetsky, and Press [2], and Officer [15].

The potentials in the liquid layer having been determined, we are primarily interested in the pressure recorded by the receiver in the liquid. The state of stress described by equations (23) is clearly hydrostatic, and the pressure is taken as the negative of the stress. Making use of identities from equation (1) and (3),

$$\nabla^2\phi_o=-k_o^2\phi_o$$
$$\lambda_o k_o^2=\rho_o\omega^2$$

the pressure in the liquid in terms of potential is just

$$\bar{P}_o(\zeta,z,\omega)=\rho_o\omega^2\bar{\phi}_o \tag{51}$$

In the experimental program it seems logical that the receiver will be suspended above the source for optimum reception, so equation (50) combined with equation (51) is the formal solution to the problem. To obtain the expression for the output pressure as $P_o(r,z,t)$, where the r and t dependence has been recovered, we must perform the inverse Hankel transform in space and the inverse Fourier transform in time.

$$\bar{P}_o(r,z,\omega)=\rho_o\omega^2\int_0^\infty \bar{\phi}_o(\zeta,z,\omega)J_o(\zeta r)\zeta d\zeta \tag{52}$$

$$P_o(r,z,t)=\frac{1}{2\pi}\rho_o\int_0^\infty d\omega e^{i\omega t}\omega^2\int_0^\infty \bar{\phi}_o(\zeta,z,\omega)J_o(\zeta r)\zeta d\zeta \tag{53}$$

Evaluation of the double integral in equation (53) may be performed numerically, or the integration can be done in the complex plane using Cauchy's theorem.

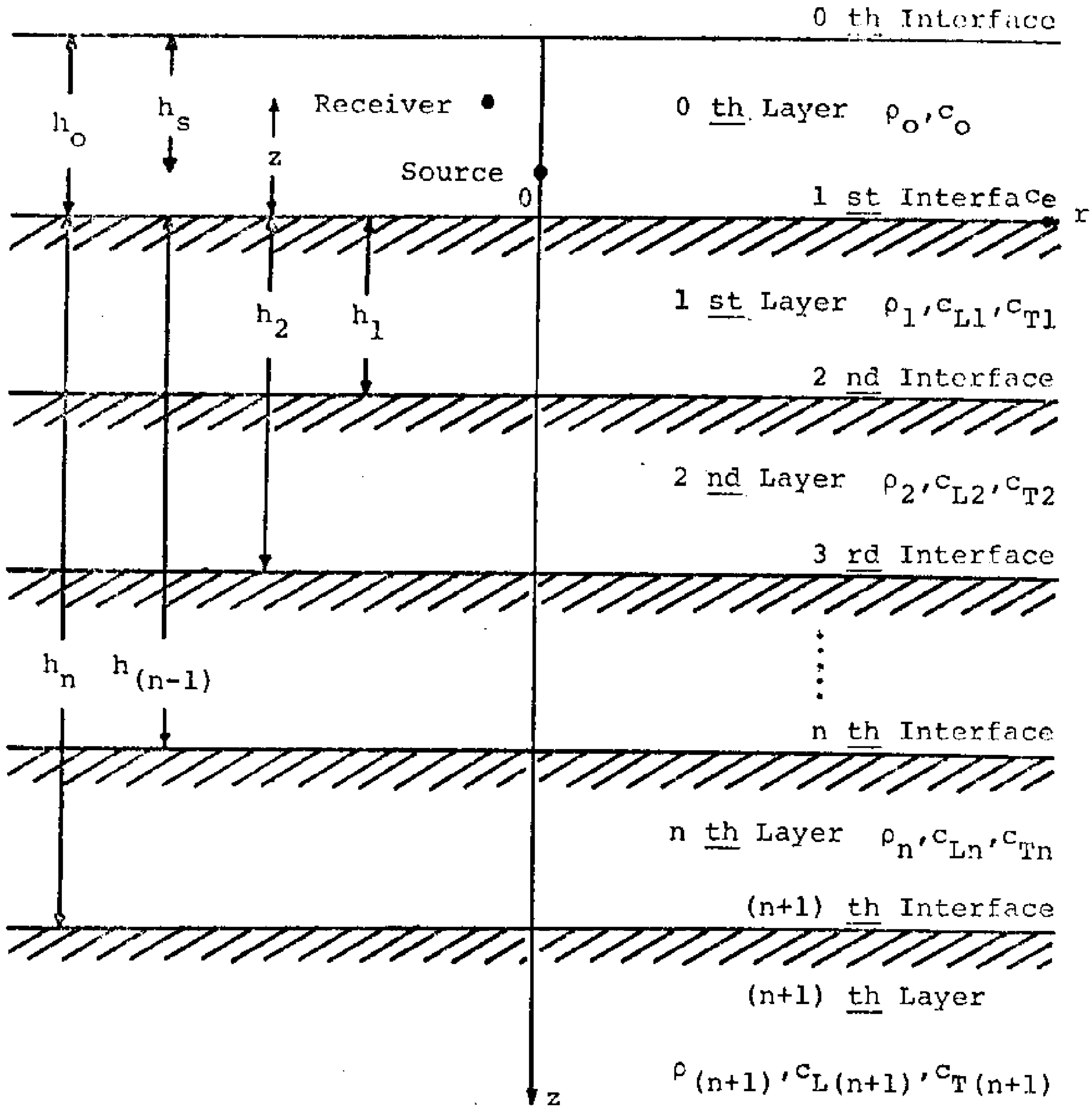
7. General Solution of the Multilayer Problem

The present problem is a generalization of the problem treated in the previous section. The viscoelastic subbottom is assumed to consist of n parallel layers as shown in figure 2. If we attempt to solve the n-layer problem by continuing along the lines of section six, for each additional layer we consider, the dispersion matrix of equation (41) will increase dimensionally from a 3x3 matrix to a [3+(n-1)4]x[3+(n-1)4] matrix. For example, the two viscoelastic layer case would result in a 7x7 dispersion matrix, etc. It becomes apparent that for multilayer problems the complexity of the calculations involved increases dramatically. In fact it will be helpful to employ computer techniques for these problems, however, when the matrices involved become very large, computer time increases and memory space becomes exhausted. Therefore, the primary purpose of this section will be to obtain a formal solution of the multilayer problem, through the development of recursion relations between adjacent layers, whereby matrix size does not increase beyond 4x4.

Similar to the discussion in the previous section, we write the liquid and viscoelastic layer potentials as follows:

0 th Layer (liquid layer)

$$\bar{\phi}_o = A_o e^{-a_o z} + B_o e^{a_o z} + \frac{e^{-a_o |z+(h_o-h_s)|}}{a_o}$$



LIQUID LAYER OVERLYING A MULTILAYERED SUBBOTTOM

FIG. 2

1 st Layer

$$\bar{\phi}_{L1}=A_{L1}e^{-a_{L1}z}+B_{L1}e^{a_{L1}z}$$

$$\bar{\phi}_{T1}=A_{T1}e^{-a_{T1}z}+B_{T1}e^{a_{T1}z}$$

n th Layer

$$\bar{\phi}_{Ln}=A_{Ln}e^{-a_{Ln}z}+B_{Ln}e^{a_{Ln}z}$$

$$\bar{\phi}_{Tn}=A_{Tn}e^{-a_{Tn}z}+B_{Tn}e^{a_{Tn}z}$$

(n+1) th Layer

$$\bar{\phi}_{L(n+1)}=A_{L(n+1)}e^{-a_{L(n+1)}z}$$

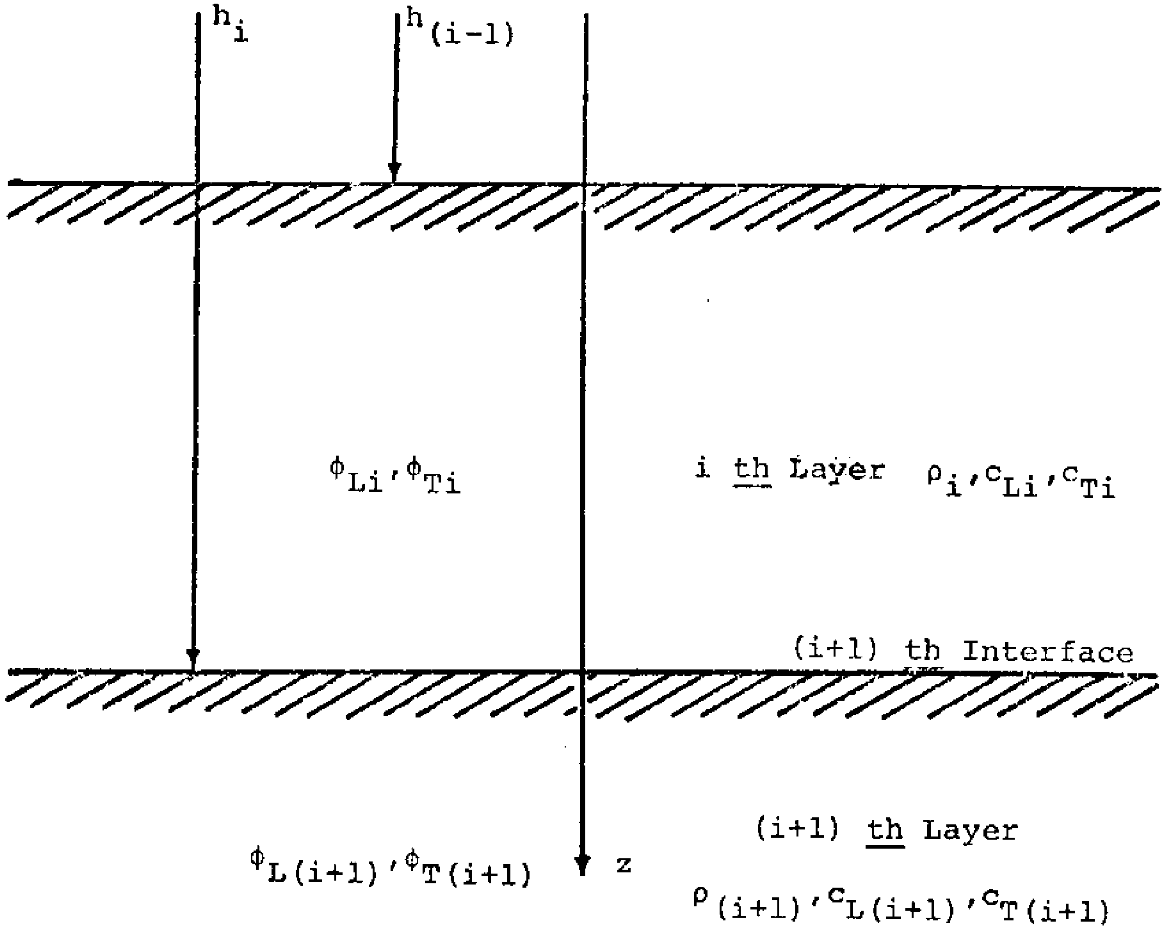
$$\bar{\phi}_{T(n+1)}=A_{T(n+1)}e^{-a_{T(n+1)}z}$$

The (n+1) th layer is a halfspace, so no B_L and B_T terms exist due to the convergence requirement mentioned previously.

For each interface the boundary conditions discussed in section three are applied. The boundary conditions are that the radial u_r and vertical u_z components of the displacement and the normal σ_{zz} and shear σ_{rz} stresses are continuous at the interface separating two different media. If the most general case is taken, say for the (i+1) th interface (see figure 3), then the boundary conditions can be written from equations (26) and (27):

i) u_r continuous at $z=h_i$:

$$\bar{\phi}_{Li}+\partial_z\bar{\phi}_{Ti}=\bar{\phi}_{L(i+1)}+\partial_z\bar{\phi}_{T(i+1)}$$



INTERFACE BETWEEN THE i th AND (i+1) th LAYERS

FIG. 3

or, for compactness, where $(i+1)=i'$:

$$A_{Li}e^{-a_{Li}h_i}+B_{Li}e^{a_{Li}h_i}-a_{Ti}A_{Ti}e^{-a_{Ti}h_i}+a_{Ti}B_{Ti}e^{a_{Ti}h_i}=$$

$$A_{Li'}e^{-a_{Li'}h_i}+B_{Li'}e^{a_{Li'}h_i}-a_{Ti'}A_{Ti'}e^{-a_{Ti'}h_i}+a_{Ti'}B_{Ti'}e^{a_{Ti'}h_i} \quad (54)$$

ii) u_z continuous at $z=h_i$:

$$\partial_z\bar{\phi}_{Li}+(\partial_z^2\bar{\phi}_{Ti}+k_{Ti}^2\bar{\phi}_{Ti})=\partial_z\bar{\phi}_{Li'}+(\partial_z^2\bar{\phi}_{Ti'}+k_{Ti'}^2\bar{\phi}_{Ti'})$$

note that

$$(\partial_z^2+k_{Ti}^2)\bar{\phi}_{Ti}=\zeta^2\bar{\phi}_{Ti}$$

or

$$-a_{Li}A_{Li}e^{-a_{Li}h_i}+a_{Li}B_{Li}e^{a_{Li}h_i}+\zeta^2(A_{Ti}e^{-a_{Ti}h_i}+B_{Ti}e^{a_{Ti}h_i})=$$

$$-a_{Li'}A_{Li'}e^{-a_{Li'}h_i}+a_{Li'}B_{Li'}e^{a_{Li'}h_i}+\zeta^2(A_{Ti'}e^{-a_{Ti'}h_i}+B_{Ti'}e^{a_{Ti'}h_i}) \quad (55)$$

iii) σ_{zz} continuous at $z=h_i$:

$$-\bar{\lambda}_i k_{Li}^2\bar{\phi}_{Li}+2\bar{\mu}_i[\partial_z^2\bar{\phi}_{Li}+\partial_z(\partial_z^2\bar{\phi}_{Ti}+k_{Ti}^2\bar{\phi}_{Ti})]=$$

$$-\bar{\lambda}_{i'}k_{Li'}^2\bar{\phi}_{Li'}+2\bar{\mu}_{i'}[\partial_z^2\bar{\phi}_{Li'}+\partial_z(\partial_z^2\bar{\phi}_{Ti'}+k_{Ti'}^2\bar{\phi}_{Ti'})]$$

or

$$\bar{\mu}_i[(2\zeta^2-k_{Ti}^2)(A_{Li}e^{-a_{Li}h_i}+B_{Li}e^{a_{Li}h_i})+2a_{Ti}\zeta^2(-A_{Ti}e^{-a_{Ti}h_i}+B_{Ti}e^{a_{Ti}h_i})]=$$

$$\bar{\mu}_{i'}[(2\zeta^2-k_{Ti'}^2)(A_{Li'}e^{-a_{Li'}h_i}+B_{Li'}e^{a_{Li'}h_i})+$$

$$2\zeta^2a_{Ti'}(-A_{Ti'}e^{-a_{Ti'}h_i}+B_{Ti'}e^{a_{Ti'}h_i})] \quad (56)$$

iv) σ_{rz} continuous at $z=h_i$:

$$\bar{\mu}_i[2\partial_z\bar{\phi}_{Li}+(2\zeta^2-k_{Ti}^2)\bar{\phi}_{Ti}]=\bar{\mu}_{i'}[2\partial_z\bar{\phi}_{Li'}+(2\zeta^2-k_{Ti'}^2)\bar{\phi}_{Ti'}]$$

or

$$\bar{\mu}_i[2a_{Li}(-A_{Li}e^{-a_{Li}h_i}+B_{Li}e^{a_{Li}h_i})+(2\zeta^2-k_{Ti}^2)(A_{Ti}e^{-a_{Ti}h_i}+B_{Ti}e^{a_{Ti}h_i})]=$$

$$\bar{\mu}_{i'}[2a_{Li'}(-A_{Li'}e^{-a_{Li'}h_i}+B_{Li'}e^{a_{Li'}h_i})+$$

$$(2\zeta^2-k_{Ti'}^2)(A_{Ti'}e^{-a_{Ti'}h_i}+B_{Ti'}e^{a_{Ti'}h_i})] \quad (57)$$

The four equations (54-57) in the eight unknowns A_{Li}, B_{Li}, A_{Ti}, B_{Ti}, $A_{L(i+1)}$, $B_{L(i+1)}$, $A_{T(i+1)}$ and $B_{T(i+1)}$ may be written in matrix form as follows

$$a_{(i+1)}\bar{A}_i=B_{(i+1)}\bar{A}_{(i+1)} \quad (58)$$

where $a_{(i+1)}$ and $B_{(i+1)}$ are 4x4 matrices and $\bar{A}_i$ is a column vector for the coefficients of the potentials for the i th layer,

$$\bar{A}_i=\begin{bmatrix}A_{Li}\\B_{Li}\\A_{Ti}\\B_{Ti}\end{bmatrix} \quad (59)$$

and similarly, for $\bar{A}_{(i+1)}$,

$$\bar{A}_{(i+1)}=\begin{bmatrix} A_{L(i+1)} \\ B_{L(i+1)} \\ A_{T(i+1)} \\ B_{T(i+1)} \end{bmatrix} \tag{60}$$

The matrix $a_{(i+1)}$ is constructed from the coefficients from equations (54-57) of the terms associated with the elements of the $\bar{A}_i$ vector. The exponential functions may be factored out of the $a_{(i+1)}$ matrix, resulting in the following expression

$$a_{(i+1)}=a'_{(i+1)}e_{(i+1)} \tag{61}$$

where

$$a'_{(i+1)}=\begin{bmatrix} 1 & 1 & -a_{Ti} & a_{Ti} \\ -a_{Li} & a_{Li} & \zeta^2 & \zeta^2 \\ \bar{\mu}_i(2\zeta^2-k_{Ti}^2) & \bar{\mu}_i(2\zeta^2-k_{Ti}^2) & -2\bar{\mu}_i a_{Ti}\zeta^2 & 2\bar{\mu}_i a_{Ti}\zeta^2 \\ -2\bar{\mu}_i a_{Li} & 2\bar{\mu}_i a_{Li} & \bar{\mu}_i(2\zeta^2-k_{Ti}^2) & \bar{\mu}_i(2\zeta^2-k_{Ti}^2) \end{bmatrix} \tag{62}$$

and

$$e_{(i+1)}=\begin{bmatrix} e^{-a_{Li}h_i} & 0 & 0 & 0 \\ 0 & e^{a_{Li}h_i} & 0 & 0 \\ 0 & 0 & e^{-a_{Ti}h_i} & 0 \\ 0 & 0 & 0 & e^{a_{Ti}h_i} \end{bmatrix} \tag{63}$$

An expression for the $B_{(i+1)}$ matrix may be written similar to equation (61)

$$B_{(i+1)}=B'_{(i+1)}e'_{(i+1)} \tag{64}$$

where

$$B'_{(i+1)}=a'_{(i+2)} \tag{65}$$

and

$$e'_{(i+1)} = \begin{bmatrix} e^{-a_{L(i+1)}h_i} & 0 & 0 & 0 \\ 0 & e^{a_{L(i+1)}h_i} & 0 & 0 \\ 0 & 0 & e^{-a_{T(i+1)}h_i} & 0 \\ 0 & 0 & 0 & e^{a_{T(i+1)}h_i} \end{bmatrix} \quad (66)$$

From equation (65), it is seen that the $B'_{(i+1)}$ matrix is formed by replacing the subscripts (i) in equation (62) with (i+1). Using equations (61) and (64) in equation (58) gives

$$[a'_{(i+1)}e_{(i+1)}]\bar{A}_i=[B'_{(i+1)}e'_{(i+1)}]\bar{A}_{(i+1)} \quad (67)$$

Equation (58) or (67) is a recurrence relation relating the coefficients of the i th layer's potentials to the (i+1) th layer's potentials. This recurrence relation can be successively applied for the n-layer case, i. e., referring to figure 2:

i) for the (n+1) th interface

$$a_{(n+1)}\bar{A}_n=B_{(n+1)}\bar{A}_{(n+1)} \quad (68)$$

where

$$\bar{A}_{(n+1)}=\begin{bmatrix} A_{L(n+1)} \\ 0 \\ A_{T(n+1)} \\ 0 \end{bmatrix} \qquad \bar{A}_n=\begin{bmatrix} A_{Ln} \\ B_{Ln} \\ A_{Tn} \\ B_{Tn} \end{bmatrix}$$

ii) for the n th interface

$$a_n\bar{A}_{(n-1)}=B_n\bar{A}_n \quad (69)$$

iii) for the (n-1) th interface

$$a_{(n-1)}\bar{A}_{(n-2)}=B_{(n-1)}\bar{A}_{(n-1)} \quad (70)$$

iv) for the third interface

$$a_3\bar{A}_2=B_3\bar{A}_3 \quad (71)$$

v) for the second interface

$$a_2\bar{A}_1=B_2\bar{A}_2 \quad (72)$$

vi) the first interface is a special case, since

$$\bar{A}_o=\begin{bmatrix} A_{Lo} \\ B_{Lo} \\ 0 \\ 0 \end{bmatrix}$$

and $\mu_o=0$. In addition, a source term must be included in the equations for the boundary conditions of the first interface. Now solving for $\bar{A}_1$ in equation (72),

$$\bar{A}_1=a_2^{-1}B_2\bar{A}_2 \quad (73)$$

where a_2^{-1} is the inverse of the a_2 matrix. Form equation (71)

$$\bar{A}_2=a_3^{-1}B_3\bar{A}_3 \quad (74)$$

substituting equation (74) into equation (73) gives:

$$\bar{A}_1=[a_2^{-1}B_2][a_3^{-1}B_3]\bar{A}_3 \quad (75)$$

This process can be repeated for all layers, resulting in

$$\bar{A}_1=[a_2^{-1}B_2][a_3^{-1}B_3][a_4^{-1}B_4]\ldots.[a_n^{-1}B_n][a_{(n+1)}^{-1}B_{(n+1)}]\bar{A}_{(n+1)} \quad (76)$$

This result implies that the potential coefficients of the first solid layer are related to the potential coefficients of the last, (n+1) th, layer by a matrix expression of the form

$$\bar{A}_1=M\bar{A}_{(n+1)} \quad (77)$$

where

$$M=[a_2^{-1}B_2][a_3^{-1}B_3]\ldots.[a_{(n+1)}^{-1}B_{(n+1)}]$$

is a 4x4 matrix. Denoting the element of M in the i th row and j th column by m_{ij}, we have

$$M=\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \quad (78)$$

From equation (68), we see that it is possible to write the elements of $\bar{A}_1$ in equation (77) as

$$\begin{aligned} A_{L1}&=m_{11}A_{L(n+1)}+m_{13}A_{T(n+1)} \\ B_{L1}&=m_{21}A_{L(n+1)}+m_{23}A_{T(n+1)} \\ A_{T1}&=m_{31}A_{L(n+1)}+m_{33}A_{T(n+1)} \\ B_{T1}&=m_{41}A_{L(n+1)}+m_{43}A_{T(n+1)} \end{aligned} \quad (79)$$

where $B_{L(n+1)} = B_{T(n+1)} = 0$.

At this time we will consider the first interface and introduce the source term into the equations for the boundary conditions. Recall that the potentials for the liquid layer are

$$\bar{\phi}_o^I = -2A_o e^{a_o h_o} \sinh[a_o(z+h_o)] \qquad (39)$$

for $z<-(h_o-h_s)$, i. e., above the source, and

$$\bar{\phi}_o^{II} = -2A_o e^{a_o h_o} \sinh[a_o(z+h_o)] - \frac{2}{a_o} \sinh[a_o(z+h_o-h_s)] \qquad (40)$$

for $z>-(h_o-h_s)$, i. e., below the source. In both equations (39) and (40), the boundary condition at the 0 th interface (the water surface) has been used to eliminate B_o in equation (30). The applicable boundary condition on the water surface is that σ_{zz} is zero, or equivalently, that $\bar{\phi}_o=0$. We may write the expressions for the potentials in the first solid layer as:

$$\bar{\phi}_{L1} = A_{L1} e^{-a_{L1} z} + B_{L1} e^{a_{L1} z} \qquad (13)$$

$$\bar{\phi}_{T1} = A_{T1} e^{-a_{T1} z} + B_{T1} e^{a_{T1} z} \qquad (14)$$

The three boundary conditions that apply to the first interface are at z=0:

$$(\sigma_{zz})_o = (\sigma_{zz})_1 \qquad (35)$$

$$(u_z)_o = (u_z)_1 \qquad (36)$$

$$(\sigma_{rz})_o = (\sigma_{rz})_1 = 0 \qquad (37)$$

Applying equations (13), (14) and (40) to the boundary conditions results in three equations in the five unknowns A_o, A_{L1}, A_{T1}, B_{L1} and B_{T1}, so the system of three equations is indeterminate. The equations for the boundary conditions at z=0 can be arranged in matrix form as follows (See Appendix D):

$$\begin{bmatrix} -2a_o e^{a_o h_o}\cosh(a_o h_o) & a_{L1} & -a_{L1} & -\zeta^2 & -\zeta^2 \\ -2\rho_o \omega^2 e^{a_o h_o}\sinh(a_o h_o) & \bar{\mu}_1(2\zeta^2-k_{T1}^2) & \bar{\mu}_1(2\zeta^2-k_{T1}^2) & -2\bar{\mu}_1 a_{T1}\zeta^2 & 2\bar{\mu}_1 a_{T1}\zeta^2 \\ 0 & -2a_{L1} & 2a_{L1} & (2\zeta^2-k_{T1}^2) & (2\zeta^2-k_{T1}^2) \end{bmatrix} \begin{bmatrix} A_o \\ A_{L1} \\ B_{L1} \\ A_{T1} \\ B_{T1} \end{bmatrix} = \begin{bmatrix} 2\cosh[a_o(h_o-h_s)] \\ \dfrac{2\rho_o\omega^2}{a_o}\sinh[a_o(h_o-h_s)] \\ 0 \end{bmatrix} \qquad (80)$$

The indeterminacy in equation (80) can be eliminated by applying equations (79). Equation (79) and (80), when combined, yield a determinate system of seven independent equations in seven unknowns. We eliminate the variables A_{L1}, B_{L1}, A_{T1} and B_{T1} in equation (80), using equations (79), giving the following matrix expression.

$$\begin{bmatrix} -2a_o e^{a_o h_o}\cosh(a_o h_o) & a_{L1}(m_{11}-m_{21})-\zeta^2(m_{31}+m_{41}) & a_{L1}(m_{13}-m_{23})-\zeta^2(m_{33}+m_{43}) \\ -2\rho_o\omega^2 e^{a_o h_o}\sinh(a_o h_o) & \bar{\mu}_1[(m_{11}+m_{21})(2\zeta^2-k_{T1}^2)-(m_{31}-m_{41})2a_{T1}\zeta^2] & \bar{\mu}_1[(m_{13}+m_{23})(2\zeta^2-k_{T1}^2)-(m_{33}-m_{43})2a_{T1}\zeta^2] \\ 0 & -2a_{L1}(m_{11}-m_{21})+(2\zeta^2-k_{T1}^2)(m_{31}+m_{41}) & -2a_{L1}(m_{13}-m_{23})+(2\zeta^2-k_{T1}^2)(m_{33}+m_{43}) \end{bmatrix} \begin{bmatrix} A_o \\ A_{L(n+1)} \\ A_{T(n+1)} \end{bmatrix} = \begin{bmatrix} 2\cosh[a_o(h_o-h_s)] \\ \frac{2\rho_o\omega^2}{a_o}\sinh[a_o(h_o-h_s)] \\ 0 \end{bmatrix} \tag{81a}$$

Equation (81a) may be rewritten as

$$\begin{bmatrix} -2a_o e^{a_o h_o}\cosh(a_o h_o) & b_{12} & b_{13} \\ -2\rho_o\omega^2 e^{a_o h_o}\sinh(a_o h_o) & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} A_o \\ A_{L(n+1)} \\ A_{T(n+1)} \end{bmatrix} = \begin{bmatrix} 2\cosh[a_o(h_o-h_s)] \\ \frac{2\rho_o\omega^2}{a_o}\sinh[a_o(h_o-h_s)] \\ 0 \end{bmatrix} \tag{81b}$$

where

$$b_{12}=a_{L1}(m_{11}-m_{21})-\zeta^2(m_{31}+m_{41})$$

$$b_{13}=a_{L1}(m_{13}-m_{23})-\zeta^2(m_{33}+m_{43})$$

$$b_{22}=\bar{\mu}_1[(m_{11}+m_{21})(2\zeta^2-k_{T1}^2)-(m_{31}-m_{41})2a_{T1}\zeta^2]$$

$$b_{23}=\bar{\mu}_1[(m_{13}+m_{23})(2\zeta^2-k_{T1}^2)-(m_{33}-m_{43})2a_{T1}\zeta^2]$$

$$b_{32}=-2a_{L1}(m_{11}-m_{21})+(2\zeta^2-k_{T1}^2)(m_{31}+m_{41})$$

$$b_{33}=-2a_{L1}(m_{13}-m_{23})+(2\zeta^2-k_{T1}^2)(m_{33}+m_{43})$$

We are interested in the solution for the acoustic field in the liquid, so we solve for A_o in equation (81b) using Cramer's rule

$$A_o = -\frac{e^{-a_o h_o}}{a_o}\,\frac{\Delta_1'}{\Delta_o'} \tag{48}$$

where

$$\Delta_1' = \begin{vmatrix} a_o\cosh[a_o(h_o-h_s)] & b_{12} & b_{13} \\ \rho_o\omega^2\sinh[a_o(h_o-h_s)] & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{vmatrix} \tag{82a}$$

and

$$\Delta_o' = \begin{vmatrix} a_o\cosh(a_o h_o) & b_{12} & b_{13} \\ \rho_o\omega^2\sinh(a_o h_o) & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{vmatrix} \tag{83a}$$

Expanding equations (82a) and (83a) gives

$$\Delta_1' = a_o\cosh[a_o(h_o-h_s)]\{4\bar{\mu}_1 a_{L1}(2\zeta^2-k_{T1}^2)C_1+\bar{\mu}_1(2\zeta^2-k_{T1}^2)^2C_2+4\bar{\mu}_1 a_{L1}a_{T1}\zeta^2C_3$$
$$+4\bar{\mu}_1 a_{T1}\zeta^2(2\zeta^2-k_{T1}^2)C_4\}-\rho_o\omega^2\sinh[a_o(h_o-h_s)]\{-a_{L1}k_{T1}^2C_5\} \tag{82b}$$

and

$$\Delta_o' = a_o\cosh(a_o h_o)\{4\bar{\mu}_1 a_{L1}(2\zeta^2-k_{T1}^2)C_1+\bar{\mu}_1(2\zeta^2-k_{T1}^2)^2C_2+4\bar{\mu}_1 a_{L1}a_{T1}\zeta^2C_3$$
$$+4\bar{\mu}_1 a_{T1}\zeta^2(2\zeta^2-k_{T1}^2)C_4\}-\rho_o\omega^2\sinh(a_o h_o)\{-a_{L1}k_{T1}^2C_5\} \tag{83b}$$

where

$$C_1=m_{11}m_{23}-m_{13}m_{21}$$

$$C_2=(m_{11}+m_{21})(m_{33}+m_{43})-(m_{31}+m_{41})(m_{13}+m_{23})$$

$$C_3=(m_{13}-m_{23})(m_{31}-m_{41})-(m_{11}-m_{21})(m_{33}-m_{43}) \qquad (84a\text{-}e)$$

$$C_4=m_{41}m_{33}-m_{31}m_{43}$$

$$C_5=(m_{11}-m_{21})(m_{33}+m_{43})-(m_{31}+m_{41})(m_{13}-m_{23})$$

It should be noted that equation (48) reduces to the result for the n=0 case (one viscoelastic layer) developed in the previous section. For n=0, equation (79) reduces to a trivial identity,

$$A_{L1}=A_{L1}$$

$$A_{T1}=A_{T1}$$

or $m_{11}=1$ and $m_{33}=1$, and $m_{ij}=0$ for $i\neq j$. Setting $m_{11}=1$, $m_{33}=1$ and $m_{ij}=0$ for $i\neq j$ in equations (84) gives the values of C_{1-5}

$$C_1=0$$

$$C_2=1$$

$$C_3=-1$$

$$C_4=0$$

$$C_5=1$$

in equations (82b) and (83b) which result in the expression for A_o for one viscoelastic layer. Of course, since this new method yields an expression for A_o for the one-layer case equivalent to equation (47), the subsequent double integral for P_o will be identical to equation (53). Thus, we have developed a scheme whereby, determining the components of the M matrix of equation (78) and using equation (81a), A_o can be found. Let's look at a multilayer problem using this scheme.

We will consider two viscoelastic layers to illustrate the use of the more general method described in this section. The geometry of the problem is shown in figure 4, which is a special case of the n-layer problem, figure 2, with n=1. The expression for the coefficient of the acoustic potential, A_o, is taken from equation (48).

$$A_o=-\frac{e^{-a_o h_o}}{a_o}\frac{\Delta_1'}{\Delta_o'} \tag{48}$$

The expansions for Δ_1' and Δ_o' involve the terms C_{1-5}, which in turn involve m_{ij} factors. The m_{ij} factors must be calculated for n=1, with m_{ij} defined by equations (77-79). For n=1, these equations become

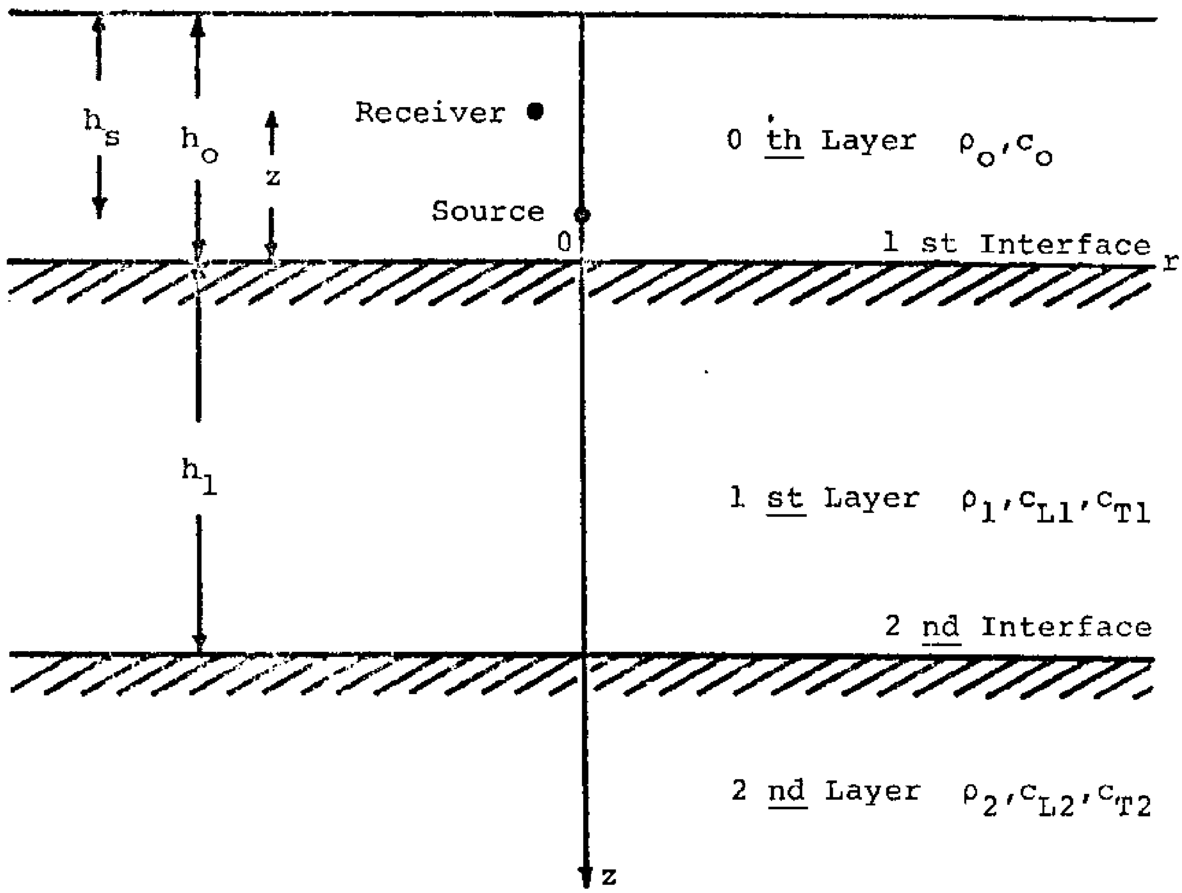
$$\bar{A}_1=M\bar{A}_2 \tag{85}$$

where

$$M=[a_2^{-1}B_2] \tag{86}$$

and

$$\begin{aligned} A_{L1}&=m_{11}A_{L2}+m_{13}A_{T2}\\ B_{L1}&=m_{21}A_{L2}+m_{23}A_{T2}\\ A_{T1}&=m_{31}A_{L2}+m_{33}A_{T2}\\ B_{T1}&=m_{41}A_{L2}+m_{43}A_{T2} \end{aligned} \tag{87}$$



LIQUID LAYER OVERLYING TWO VISCOELASTIC LAYERS

FIG. 4

To find A_o for the two-layer case only the m_{ij} coefficients appearing in equation (87) need to be calculated. The B_2 term in equation (86) can be written from equations (64) and (65) as

$$B_2 = B'_2 e'_2 \tag{88}$$

where

$$B'_2 = a'_3 \tag{89}$$

From equation (62)

$$B'_2 = \begin{bmatrix} 1 & 1 & -a_{T2} & a_{T2} \\ -a_{L2} & a_{L2} & \zeta^2 & \zeta^2 \\ \bar{\mu}_2(2\zeta^2-k_{T2}^2) & \bar{\mu}_2(2\zeta^2-k_{T2}^2) & -2\bar{\mu}_2 a_{T2}\zeta^2 & 2\bar{\mu}_2 a_{T2}\zeta^2 \\ -2\bar{\mu}_2 a_{L2} & 2\bar{\mu}_2 a_{L2} & \bar{\mu}_2(2\zeta^2-k_{T2}^2) & \bar{\mu}_2(2\zeta^2-k_{T2}^2) \end{bmatrix} \tag{90}$$

and from equation (66)

$$e'_2 = \begin{bmatrix} e^{-a_{L2}h_1} & 0 & 0 & 0 \\ 0 & e^{a_{L2}h_1} & 0 & 0 \\ 0 & 0 & e^{-a_{T2}h_1} & 0 \\ 0 & 0 & 0 & e^{a_{T2}h_1} \end{bmatrix} \tag{91}$$

The term a_2^{-1} in equation (86) must be computed. From equations (61), (62) and (63)

$$a_2 = a'_2 e_2 \tag{92}$$

where

$$a'_2 = \begin{bmatrix} 1 & 1 & -a_{T1} & a_{T1} \\ -a_{L1} & a_{L1} & \zeta^2 & \zeta^2 \\ \bar{\mu}_1(2\zeta^2-k_{T1}^2) & \bar{\mu}_1(2\zeta^2-k_{T1}^2) & -2\bar{\mu}_1 a_{T1}\zeta^2 & 2\bar{\mu}_1 a_{T1}\zeta^2 \\ -2\bar{\mu}_1 a_{L1} & 2\bar{\mu}_1 a_{L1} & \bar{\mu}_1(2\zeta^2-k_{T1}^2) & \bar{\mu}_1(2\zeta^2-k_{T1}^2) \end{bmatrix} \tag{93}$$

and

$$e_2=\begin{bmatrix} e^{-a_{L1}h_1} & 0 & 0 & 0 \\ 0 & e^{a_{L1}h_1} & 0 & 0 \\ 0 & 0 & e^{-a_{T1}h_1} & 0 \\ 0 & 0 & 0 & e^{a_{T1}h_1} \end{bmatrix} \tag{94}$$

To obtain a_2^{-1}, from equation (92), we make use of the matrix relation for the inverse of the product of two matrices.

$$a_2^{-1}=(e_2)^{-1}(a_2')^{-1} \tag{95}$$

From equations (85), (88) and (95)

$$M=[(e_2)^{-1}(a_2')^{-1}][B_2'e_2'] \tag{96}$$

After considerable algebra the m_{ij} coefficients of equation (87) and, finally, C_{1-5} from equations (84) can be calculated (See Appendix E). Again our scheme has given us an expression for A_o and subsequently for the double integral P_o. The results agree with those obtained by expanding the 7x7 dispersion matrix as prescribed by the earlier method used.

CHAPTER III

RESULTS AND DISCUSSION

III. RESULTS AND DISCUSSION

1. Summary

The expression for the acoustic response due to a point source excitation in a liquid layer overlying a multilayer viscoelastic subbottom has been determined. The expression implicitly includes the contribution of the viscoelastic subbottom lying below the liquid, a result of coupling phenomenon between adjacent layers. The input-output pressure relationship appears in general integral equation form, a double integral in fact, due to Fourier and Hankel transformations, in the temporal and spatial domains respectively. The stress and displacement fields in both the liquid and viscoelastic media were determined as a necessity of applying the boundary conditions at existing interfaces.

The primary innovations in the multilayer techniques used include the development of recursion relations between adjacent layers to find the liquid layer potential more easily and the introduction of complex wave numbers to describe the damping of the viscoelastic medium. The problem of expressing the liquid layer potential for multilayer problems has been reduced to determining eight components of a 4x4 matrix and using these in a simple matrix equation. A convenience of the method developed is that no matrix used exceeds 4x4 dimensions, allowing the employment of a computer to aid in the calculation of potentials with a minimum of time and cost. The existence of complex wave numbers the expression for the liquid layer potential indicates that evaluation of the double integral can be done performing an integration in the complex plane using Cauchy's theorem. Complex variable techniques include the algebraic search for roots and branch cuts. The integration will yield a functional relationship between the unknown Lame constants and the density of the viscoelastic subbottom.

2. Recommendations

The scope of the present investigation is limited due to the simplifications in the model used in this treatment, the assumption of plane boundaries, etc. The model could be made more realistic by including the effects of medium inhomogeneities and bottom roughness. Medium inhomogeneities can be incorporated into the model by introducing perturbation techniques. In this case it seems likely that perturbation would be performed about the density parameters. Bottom roughness can be accounted for in a more sophisticated model by employing statistical methods.

The advances in multilayer analysis, introduced by this thesis, suggest the use of computer studies for the solution of n-layer problems. The computer may also play an important role evaluating the double integral, obtained in the formal solution, either by numerical methods or complex integration. An investigation of the limiting case, where the depth of water covering the viscoelastic layers becomes infinite, would be helpful for modeling tests where a single short pulse is reflected off the subbottom and the first bottom return is analyzed. This type of test is the most frequently performed and simplest to analyze.

BIBLIOGRAPHY

1. Bateman, H., Partial Differential Equations of Mathematical Physics, Dover Publications, Inc., New York, 1944.

2. Ewing, W. M., Jardetzky, W. S. and Press, F., Elastic Waves in Layered Media, McGraw-Hill Book Co., Inc., New York, 1957.

3. Haskell, N. A., "The Dispersion of Surface Waves in Multi-layered Media", Bull. Seism. Soc. Amer., Vol. 43, 1953, Pp. 17-34.

4. Honda, H. and Nakamura, K., "Notes on the Reflection and Refraction of the SH Pulse Emitted from a Point Source", Science Repts. Tohoku Univ., Fifth Ser., Geophys., Vol. 5, 1953, Pp. 163-166.

5. ______________ "On the Reflection and Refraction of the Explosive Sounds at the Ocean Bottom", Science Repts. Tohoku Univ., Fifth Ser., Geophys., Vol. 4, 1953, Pp. 125-133.

6. Jardetzky, W. S., "Period Equation for an n-Layered Halfspace and Some Related Questions, Columbia Univ. Lamont Geol. Obs. Tech. Rept. Seismology 29, 1953.

7. Jeffreys, H., Cartesian Tensors, Cambridge University Press, London, 1931, Pp. 66-70.

8. Kolsky, H., Stress Waves in Solids, Dover Publications, Inc., New York, 1963.

9. Lamb H., Hydrodynamics, 6th ed., Dover Publications, Inc., New York, 1945.

10. ________, "On the Propagation of Tremors Over the Surface of an Elastic Solid", Phil. Trans. Roy. Soc., Series A, Vol. 203, 1904, Pp. 1-42.

11. Landau, L. D. and Lifshitz, E. M., Theory of Elasticity, Permagon Press, Addison-Wesley Publishing Co., Inc., New York, 1959.

12. Morse, P. M. and Feshback, H., Mathematical Methods of Theoretical Physics, Vol. 2, McGraw-Hill Book Co., Inc., New York, 1953.

13. Love, A. E. H., A Treatise on the Mathematical Theory of Elasticity, 4th ed., Dover Publications, Inc., New York, 1944.

14. Munroe, M. E., Modern Multidimensional Calculus, Addison-Wesley Publishing Co., Inc., Reading Mass., 1963.

15. Officer, C. B., Introduction to the Theory of Sound Transmission, with Application to the Ocean, McGraw-Hill Book Co., Inc., New York, 1958.

16. Redwood, M., Mechanical Waveguides, Permagon Press, New York, 1960.

17. Sneddon, I. N., Fourier Transforms, McGraw-Hill Book Co., Inc., New York, 1951.

18. Sommerfeld, A., Partial Differential Equations in Physics, Academic Press, Inc., New York, 1949.

19. Stratton, J. A., Electromagentic Theory, McGraw-Hill Book Co., Inc., New York, 1951.

20. Thompson, W. T., "Transmission of Elastic Waves Through a Stratified Solid Medium", Journal of Applied Physics, Vol. 21, 1950, Pp. 89-93.

21. Tolstoy, I., "Dispersive Properties of a Fluid Layer Overlying a Semi-infinite Elastic Solid, Bull. Seism. Soc. Amer., Vol. 44, 1954, Pp. 493-512.

22. Yildiz, A., "Scattering of Plane Plasma Waves from a Plasma Sphere", Il Nuovo Cimento, Series X, Vol. 30, 1963, Pp. 1182-1207.

23. Glanz, F., Magnuson, A. and Nichols, E., "Acoustic Responses of a Viscoelastic Semiinfinite Medium Covered with a Liquid Layer", Technical Memorandum - I, University of New Hampshire, August, 1970.

24. Azzi, V. and Magnuson, A., "Acoustic Response of a Viscoelastic Semiinfinite Medium to a Source in a Covering Liquid Layer", Technical Memorandum - II, University of New Hampshire, August, 1970.

APPENDICES

APPENDIX A

Derivation of the Hydrodynamic and the Viscoelastic Field Equations

The equation of motion (balance of the rate of change of linear momentum) describing a fluid, known as the Navier-Stokes equation reads

$$\rho(\partial_t v^k + v^l \partial_l v^k) - \partial_l \sigma_{kl} = \rho F^k \qquad \text{(A-1a)}$$

where F^k is the body force resulting from an external field, σ_{kl} is the stress tensor, and v^k is the velocity vector. Since we are considering a hydrodynamic field, which cannot sustain shear forces, the stress tensor does not contain a deviatoric part, thus

$$\sigma_{kl} = -p\delta^{kl}$$

and the Navier-Stokes equation becomes

$$\rho(\partial_t v^k + v^l \partial_l v^k) + p\delta^{kl} = \rho F^k \qquad \text{(A-1b)}$$

Two other relationships that prove to be helpful in describing the hydrodynamic field are the continuity equation (conservation of mass density),

$$\partial_t \rho + \partial_k (\rho v^k) = 0 \qquad \text{(A-2)}$$

and the equation of state (constitutive relation),

$$\partial^k p=(\partial p/\partial \rho)_T \partial^k \rho \tag{A-3a}$$

or

$$\partial^k p=c_o^2 \partial^k \rho \tag{A-3b}$$

Now, we linearize equations (A-1), (A-2) and (A-3) by defining a set of perturbation parameters,

$$v^k(\bar{r},t)=0+\tilde{v}^k(\bar{r},t) \tag{A-4a}$$

$$p(\bar{r},t)=p_o+\tilde{p}(\bar{r},t) \tag{A-4b}$$

$$\rho(\bar{r},t)=\rho_o+\tilde{\rho}(\bar{r},t) \tag{A-4c}$$

where for the no-flow regime, $v_o^k=0$ and the superscripts $\sim$ refer to the fluctuating part of the variable functions. We obtain

$$\rho_o \partial_t \tilde{v}^k+\partial^k \tilde{p}=\bar{S}_1 \tag{A-5a}$$

$$\partial_t \tilde{\rho}+\rho_o \partial_k \tilde{v}^k=\bar{S}_2 \tag{A-5b}$$

$$\partial^k \tilde{p}-c_o^2 \partial^k \tilde{\rho}=0 \tag{A-5c}$$

where $\bar{S}_1$ and $\bar{S}_2$ are force terms or equivalent source terms composed of higher order non-linear terms responsible for turbulence.

By omitting these higher order terms and returning from tensor notation to vector notation we have the following homogeneous equations:

$$\rho_o \partial_t \bar{v} + \nabla p = 0 \qquad \text{(A-6a)}$$

$$\rho_o \nabla . \bar{v} + \partial_t \rho = 0 \qquad \text{(A-6b)}$$

$$\nabla p - c_o^2 \nabla \rho = 0 \qquad \text{(A-6c)}$$

Subtracting equation (A-6c) from equation (A-6a) and taking the time derivative we obtain

$$\rho_o \partial_t^2 \bar{v} + c_o^2 \nabla \partial_t \rho = 0 \qquad \text{(A-7a)}$$

Substituting for $\partial_t \rho$ from equation (A-6b) we obtain

$$\rho_o \partial_t^2 \bar{v} + c_o^2 \nabla (-\rho_o \nabla . \bar{v}) = 0 \qquad \text{(A-7b)}$$

or

$$\nabla (\nabla . \bar{v}) - \frac{1}{c_o^2} \partial_t^2 \bar{v} = 0 \qquad \text{(A-7c)}$$

Thus far we have developed the hydrodynamic field from the fluid dynamics point of view. Due to forthcoming boundary condition considerations, however, it is convenient to describe the fluid from the elasticity point of view in order to establish a basis of comparison between the viscoelastic and fluid media. We accomplish this simply by describing the fluid in terms of displacement. Since $\bar{v} = \partial_t \bar{u}$, equation (A-7c) may be written as

$$\nabla (\nabla . (\partial_t \bar{u}_o)) - \frac{1}{c_o^2} \partial_t^2 (\partial_t \bar{u}_o) = 0 \qquad \text{(A-7d)}$$

At this time if we define the Fourier transform pair, linking the time and frequency domains, as

$$\bar{X}(\omega)=\int_{-\infty}^{\infty} X(t)e^{-i\omega t}dt$$
$$X(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{X}(\omega)e^{i\omega t}d\omega \qquad \text{(A-8)}$$

we obtain from equation (A-7d) the following vector equation:

$$\nabla(\nabla.\bar{u}_o)+k_o^2\bar{u}_o=0 \qquad \text{(A-7e)}$$

If we define the displacement potential for the fluid as

$$\bar{u}_o=\nabla\phi_o \qquad \text{(A-9)}$$

it follows that

$$\nabla(\nabla^2\phi_o)+k_o^2\nabla\phi_o=0 \qquad \text{(A-10a)}$$

or

$$\nabla[(\nabla^2+k_o^2)\phi_o]=0 \qquad \text{(A-10b)}$$

Without loss of generality, we may write the scalar wave equation for the fluid as

$$(\nabla^2+k_o^2)\phi_o=0 \qquad \text{(A-11)}$$

where the integration constant arbitrarily has been set equal to zero.

We are also interested in deriving an expression for the dynamic behavior of a homogeneous, isotropic visco-elastic solid of density ρ. The equation of motion describing the viscoelastic field reads

$$\rho\partial_t^2 u^k - \partial_l \sigma_{kl} = \rho F^k \tag{A-12a}$$

where F^k is the body force resulting from an external field, σ_{kl} is the stress tensor, and u^k is the displacement vector. We are primarily interested in the solution of the homogeneous form of the viscoelastic field equation, since solutions of the inhomogeneous equation may be obtained by superposition. We write the homogeneous viscoelastic field equation as:

$$\rho\partial_t^2 u^k - \partial_l \sigma_{kl} = 0 \tag{A-12b}$$

The constitutive relation in the Hookean regime is

$$\sigma_{kl} = E_{klmn}\varepsilon_{mn} \tag{A-13}$$

where for a homogeneous and isotropic medium one writes

$$E_{klmn} = \lambda(g_{kl}g_{mn}) + \mu(g_{km}g_{ln} + g_{kn}g_{lm}) \tag{A-14}$$

and for a linear regime

$$\varepsilon_{mn} = \frac{1}{2}(\partial_n u_m + \partial_m u_n) \tag{A-15}$$

is the strain tensor. In equation (A-14), λ and μ are known as Lamé constants, and for the Voigt viscoelastic model they become time dependent operators

$$\lambda=\lambda' + \lambda'' \partial_t$$
$$\mu=\mu' + \mu'' \partial_t \tag{A-16}$$

Substituting equations (A-13), (A-14) and (A-15) into equation (A-12b) and noting that the metric g_{kl} in Euclidean space is nothing but Kronecker delta, δ_{kl}, we obtain

$$\rho\partial_t^2\bar{u}-(\lambda+\mu)\nabla(\nabla.\bar{u})-\mu\nabla^2\bar{u}=0 \tag{A-17}$$

Taking the Fourier transform in time, according to the trans- from pair defined by equations (A-8), we obtain

$$(\nabla^2+k_T^2)\bar{u}-(1-\frac{k_T^2}{k_L^2})\nabla(\nabla.\bar{u})=0 \tag{A-18}$$

where

$$k_L=\omega/c_L \qquad k_T=\omega/c_T$$
$$\bar{\lambda}=\lambda' + i\omega\lambda'' \qquad \bar{\mu}=\mu' + i\omega\mu'' \tag{A-19}$$
$$c_L^2=(\bar{\lambda}+2\bar{\mu})/\rho \qquad c_T^2=\bar{\mu}/\rho$$

Separating $\bar{u}$ into longitudinal and transverse parts,

$$\bar{u}=\bar{u}_L+\bar{u}_T \tag{A-20}$$

and performing vector manipulations on equation (A-18), we obtain two vector Helmholtz equations as follows:

$$(\nabla^2+k_L^2)\bar{u}_L=0 \qquad \text{(A-21a)}$$

$$(\nabla^2+k_T^2)\bar{u}_T=0 \qquad \text{(A-21b)}$$

It is preferable to solve these vector wave equations by using scalar potentials. The longitudinal and the (vertical shear) transverse parts can be represented by the following expressions

$$\bar{u}_L=\nabla\phi_L \qquad \text{(A-22a)}$$

$$\bar{u}_T^{VS}=\nabla x\nabla x\bar{e}_z\phi_T \qquad \text{(A-22b)}$$

where $\bar{e}_z$ is the unit vector in the z-direction for cylindrical coordinates, and ϕ_L and ϕ_T are known to satisfy the following relations

$$(\nabla^2+k_L^2)\phi_L=0 \qquad \text{(A-23a)}$$

$$(\nabla^2+k_T^2)\phi_T=0 \qquad \text{(A-23b)}$$

which will be used in our calculations. In general there are two types of transverse shear waves, the horizontal shear and the vertical shear, but, due to the type of excitation introduced in the problem at hand, a dilitational point source, the theory of elasticity predicts that we should consider only the vertical shear component.

APPENDIX B

Derivation of the Displacement Field for the Viscoelastic Medium

The superposition of equations (A-22) describes the displacement field for the viscoelastic medium. Since these equations define the longitudinal and transverse displacements implicitly, we need to develop explicit expressions and substitute these into equation (A-20). Considering the longitudinal displacements first we obtain

$$\bar{u}_L = \nabla\phi_L = \partial_r\phi_L\bar{e}_r + \frac{1}{r}\partial_\theta\phi_L\bar{e}_\theta + \partial_z\phi_L\bar{e}_z \tag{B-1}$$

The transverse displacement given by equation (A-22b) will be developed in accordance with the following formula,

$$\begin{aligned}\nabla\times\nabla\times\bar{A} = {} & \frac{1}{h_2h_3}[\partial_{u_2}\{\frac{h_3}{h_1h_2}[\partial_{u_1}(h_2A_2)-\partial_{u_2}(h_1A_1)]\} \\ & -\partial_{u_3}\{\frac{h_2}{h_1h_3}[\partial_{u_3}(h_1A_1)-\partial_{u_1}(h_3A_3)]\}]\bar{e}_1 \\ & -\frac{1}{h_1h_3}[\partial_{u_1}\{\frac{h_3}{h_1h_2}[\partial_{u_1}(h_2A_2)-\partial_{u_2}(h_1A_1)]\} \\ & -\partial_{u_3}\{\frac{h_1}{h_2h_3}[\partial_{u_2}(h_3A_3)-\partial_{u_3}(h_2A_2)]\}]\bar{e}_2 \\ & +\frac{1}{h_1h_2}[\partial_{u_1}\{\frac{h_2}{h_1h_3}[\partial_{u_3}(h_1A_1)-\partial_{u_1}(h_3A_3)]\} \\ & -\partial_{u_2}\{\frac{h_1}{h_2h_3}[\partial_{u_2}(h_3A_3)-\partial_{u_3}(h_2A_2)]\}]\bar{e}_3\end{aligned} \tag{B-2}$$

which is true for orthogonal coordinates. For cylindrical coordinates, in particular, we have

$$h_1=1 \qquad h_2=r \qquad h_3=1$$
$$u_1=r \qquad u_2=\theta \qquad u_3=z \qquad \text{(B-3)}$$

By letting $\bar{A}=\bar{e}_z\phi_T$ we can easily obtain the components of the transverse vertical shear,

$$\bar{u}_T^{VS}=\nabla x\nabla x\bar{e}_z\phi_T=\partial^2_{rz}\phi_T\bar{e}_r+\frac{1}{r}\partial^2_{\theta z}\phi_T\bar{e}_\theta-(\partial^2_r\phi_T+\frac{1}{r}\partial_r\phi_T+\frac{1}{r^2}\partial^2_\theta\phi_T)\bar{e}_z \qquad \text{(B-4a)}$$

or by substituting an identity from the scalar Helmholtz equation

$$\bar{u}_T^{VS}=\partial^2_{rz}\phi_T\bar{e}_r+\frac{1}{r}\partial^2_{\theta z}\phi_T\bar{e}_\theta+(\partial^2_z\phi_T+k_T^2\phi_T)\bar{e}_z \qquad \text{(B-4b)}$$

Thus, the total displacement described by equation (A-20) becomes equation (26) of the text.

APPENDIX C

Application of the Boundary Conditions for the One-layer Problem

We obtain the 3x3 dispersion matrix in the text by applying three boundary conditions, equations (35), (36) and (37), at $z=0$, and using the expressions that describe the stress and displacement fields in terms of potential, given by equations (23), (24) and (27). Of course, we will use the expression for the liquid potential below the source, while applying these boundary conditions at the liquid-solid interface. Applying equations (35), (36) and (37), respectively, we obtain:

$$\{2\rho_o\omega^2 e^{a_o h_o}\sinh(a_o h_o)\}A_o+\frac{2\rho_o\omega^2}{a_o}\sinh[a_o(h_o-h_s)]=$$

$$\{2\bar{\mu}_1 a_L^2-\bar{\lambda}_1 k_L^2\}A_L-\{2\bar{\mu}_1 a_T(k_T^2+a_T^2)\}A_T \qquad \text{(C-1)}$$

$$-\{a_L\}A_L+\{k_T^2+a_T^2\}A_T=-\{2a_o e^{a_o h_o}\cosh(a_o h_o)\}A_o-2\cosh[a_o(h_o-h_s)] \qquad \text{(C-2)}$$

$$-\{2a_L\}A_L+\{k_T^2+2a_T^2\}A_T=0 \qquad \text{(C-3)}$$

Simplifying with the aid of the following identities,

$$2\bar{\mu}_1 a_L^2-\bar{\lambda}_1 k_L^2=\bar{\mu}_1(2\zeta^2-k_T^2) \qquad \text{(C-4a)}$$

$$k_T^2+a_T^2=\zeta^2 \qquad \text{(C-4b)}$$

$$k_T^2+2a_T^2=(2\zeta^2-k_T^2) \qquad \text{(C-4c)}$$

and arranging our equations in a fashion such that only source terms appear on the right hand side of the equation, we obtain matrix equation (41) in the text.

APPENDIX D

Application of the Boundary Conditions for the Multilayer Problem

In order to obtain matrix equation (80) of the text we need only refer to Appendix C and recognize that the solution of the more general multilayer problem requires that we retain the B_{L1} and B_{T1} terms which disappeared in the one-layer problem due to convergence requirements. For this reason we have a 3x5 medium characteristic matrix for the multilayer problem as opposed to the 3x3 matrix for the one-layer problem.

APPENDIX E

Solution for the Acoustic Potential for the Two-layer Problem

Recall that in Section 7 of the text we developed a scheme whereby, determining the components of the first and third columns of the M matrix of equation (78) and using equation (81a), the acoustic potential can be found for any number of layers. Note that the text ended after formulating the solution for the M matrix in the two-layer problem under consideration. At this point considerable algebra is required to obtain the components of the M matrix, and finally C_{1-5}.

From equation (96) of the text, the expression for M for the two-layer problem is

$$M=[(e_2)^{-1}(a_2')^{-1}][B_2'e_2'] \qquad \text{(E-1)}$$

In the text we found each of the matrices e_2, e_2', a_2' and B_2', so we need only perform the matrix inversion process on e_2 and a_2' and multiply properly to obtain M.

$$(e_2)^{-1}=\begin{bmatrix} e^{a_{L1}h_1} & 0 & 0 & 0 \\ 0 & e^{-a_{L1}h_1} & 0 & 0 \\ 0 & 0 & e^{a_{T1}h_1} & 0 \\ 0 & 0 & 0 & e^{-a_{T1}h_1} \end{bmatrix} \qquad \text{(E-2)}$$

$$(a_2')^{-1}=\frac{1}{2\bar{\mu}_1 a_{L1} a_{T1} k_{T1}^2}\begin{bmatrix} 2\bar{\mu}_1 a_{L1} a_{T1}\zeta^2 & \bar{\mu}_1 a_{T1}(2\zeta^2-k_{T1}^2) & -a_{L1}a_{T1} & -a_{T1}\zeta^2 \\ 2\bar{\mu}_1 a_{L1} a_{T1}\zeta^2 & -\bar{\mu}_1 a_{T1}(2\zeta^2-k_{T1}^2) & -a_{L1}a_{T1} & a_{T1}\zeta^2 \\ \bar{\mu}_1 a_{L1}(2\zeta^2-k_{T1}^2) & 2\bar{\mu}_1 a_{L1} a_{T1} & -a_{L1} & -a_{L1}a_{T1} \\ -\bar{\mu}_1 a_{L1}(2\zeta^2-k_{T1}^2) & 2\bar{\mu}_1 a_{L1} a_{T1} & a_{L1} & -a_{L1}a_{T1} \end{bmatrix} \qquad \text{(E-3)}$$

$$[(e_2)^{-1}(a_2')^{-1}]=\frac{1}{2\bar{\mu}_1 a_{L1} a_{T1} k_{T1}^2}\times$$

$$\begin{bmatrix} 2\bar{\mu}_1 a_{L1} a_{T1}\zeta^2 e^{a_{L1}h_1} & \bar{\mu}_1 a_{T1}(2\zeta^2-k_{T1}^2)e^{a_{L1}h_1} & -a_{L1}a_{T1}e^{a_{L1}h_1} & -a_{T1}\zeta^2 e^{a_{L1}h_1} \\ 2\bar{\mu}_1 a_{L1} a_{T1}\zeta^2 e^{-a_{L1}h_1} & -\bar{\mu}_1 a_{T1}(2\zeta^2-k_{T1}^2)e^{-a_{L1}h_1} & -a_{L1}a_{T1}e^{-a_{L1}h_1} & a_{T1}\zeta^2 e^{-a_{L1}h_1} \\ \bar{\mu}_1 a_{L1}(2\zeta^2-k_{T1}^2)e^{a_{T1}h_1} & 2\bar{\mu}_1 a_{L1} a_{T1} e^{a_{T1}h_1} & -a_{L1}e^{a_{T1}h_1} & -a_{L1}a_{T1}e^{a_{T1}h_1} \\ -\bar{\mu}_1 a_{L1}(2\zeta^2-k_{T1}^2)e^{-a_{T1}h_1} & 2\bar{\mu}_1 a_{L1} a_{T1} e^{-a_{T1}h_1} & a_{L1}e^{-a_{T1}h_1} & -a_{L1}a_{T1}e^{-a_{T1}h_1} \end{bmatrix}$$

(E-4)

$$[B_2' e_2'] = \begin{bmatrix} e^{-a_{L2}h_1} & e^{a_{L2}h_1} & -a_{T2}e^{-a_{T2}h_1} & a_{T2}e^{a_{T2}h_1} \\ -a_{L2}e^{-a_{L2}h_1} & a_{L2}e^{a_{L2}h_1} & \zeta^2 e^{-a_{T2}h_1} & \zeta^2 e^{a_{T2}h_1} \\ \bar{\mu}_2(2\zeta^2-k_{T2}^2)e^{-a_{L2}h_1} & \bar{\mu}_2(2\zeta^2-k_{T2}^2)e^{a_{L2}h_1} & -2\bar{\mu}_2 a_{T2}\zeta^2 e^{-a_{T2}h_1} & 2\bar{\mu}_2 a_{T2}\zeta^2 e^{a_{T2}h_1} \\ -2\bar{\mu}_2 a_{L2}e^{-a_{L2}h_1} & 2\bar{\mu}_2 a_{L2}e^{a_{L2}h_1} & \bar{\mu}_2(2\zeta^2-k_{T2}^2)e^{-a_{T2}h_1} & \bar{\mu}_2(2\zeta^2-k_{T2}^2)e^{a_{T2}h_1} \end{bmatrix} \tag{E-5}$$

We are interested in the eight elements comprising the first and third columns of the M martix:

$$m_{11} = \frac{e^{(a_{L1}-a_{L2})h_1}}{2\bar{\mu}_1 a_{L1} a_{T1} k_{T1}^2}\{2\bar{\mu}_1 a_{L1} a_{T1}\zeta^2 - \bar{\mu}_1 a_{T1} a_{L2}(2\zeta^2-k_{T1}^2) - \bar{\mu}_2 a_{L1} a_{T1}(2\zeta^2-k_{T2}^2) + 2\bar{\mu}_2 a_{T1} a_{L2}\zeta^2\} \tag{E-6a}$$

$$m_{21} = \frac{e^{-(a_{L1}+a_{L2})h_1}}{2\bar{\mu}_1 a_{L1} a_{T1} k_{T1}^2}\{2\bar{\mu}_1 a_{L1} a_{T1}\zeta^2 + \bar{\mu}_1 a_{T1} a_{L2}(2\zeta^2-k_{T1}^2) - \bar{\mu}_2 a_{L1} a_{T1}(2\zeta^2-k_{T2}^2) - 2\bar{\mu}_2 a_{T1} a_{L2}\zeta^2\} \tag{E-6b}$$

$$m_{31} = \frac{e^{(a_{T1}-a_{L2})h_1}}{2\bar{\mu}_1 a_{L1} a_{T1} k_{T1}^2}\{\bar{\mu}_1 a_{L1}(2\zeta^2-k_{T1}^2) - 2\bar{\mu}_1 a_{L1} a_{T1} a_{L2} - \bar{\mu}_2 a_{L1}(2\zeta^2-k_{T2}^2) + 2\bar{\mu}_2 a_{L1} a_{T1} a_{L2}\} \tag{E-6c}$$

$$m_{41}=\frac{e^{-(a_{T1}+a_{L2})h_1}}{2\bar{\mu}_1 a_{L1} a_{T1} k_{T1}^2}\{-\bar{\mu}_1 a_{L1}(2\zeta^2-k_{T1}^2)-2\bar{\mu}_1 a_{L1} a_{T1} a_{L2}+\bar{\mu}_2 a_{L1}(2\zeta^2-k_{T2}^2)+2\bar{\mu}_2 a_{L1} a_{T1} a_{L2}\} \quad \text{(E-6d)}$$

$$m_{13}=\frac{e^{(a_{L1}-a_{T2})h_1}}{2\bar{\mu}_1 a_{L1} a_{T1} k_{T1}^2}\{-2\bar{\mu}_1 a_{L1} a_{T1} a_{T2}\zeta^2+\bar{\mu}_1 a_{T1}\zeta^2(2\zeta^2-k_{T1}^2)+2\bar{\mu}_2 a_{L1} a_{T1} a_{T2}\zeta^2-\bar{\mu}_2 a_{T1}\zeta^2(2\zeta^2-k_{T2}^2)\} \quad \text{(E-6e)}$$

$$m_{23}=\frac{e^{-(a_{L1}+a_{T2})h_1}}{2\bar{\mu}_1 a_{L1} a_{T1} k_{T1}^2}\{-2\bar{\mu}_1 a_{L1} a_{T1} a_{T2}\zeta^2-\bar{\mu}_1 a_{T1}\zeta^2(2\zeta^2-k_{T1}^2)+2\bar{\mu}_2 a_{L1} a_{T1} a_{T2}\zeta^2+\bar{\mu}_2 a_{T1}\zeta^2(2\zeta^2-k_{T2}^2)\} \quad \text{(E-6f)}$$

$$m_{33}=\frac{e^{(a_{T1}-a_{T2})h_1}}{2\bar{\mu}_1 a_{L1} a_{T1} k_{T1}^2}\{-\bar{\mu}_1 a_{L1} a_{T2}(2\zeta^2-k_{T1}^2)+2\bar{\mu}_1 a_{L1} a_{T1}\zeta^2+2\bar{\mu}_2 a_{L1} a_{T2}\zeta^2-\bar{\mu}_2 a_{L1} a_{T1}(2\zeta^2-k_{T2}^2)\} \quad \text{(E-6g)}$$

$$m_{43}=\frac{e^{-(a_{T1}+a_{T2})h_1}}{2\bar{\mu}_1 a_{L1} a_{T1} k_{T1}^2}\{\bar{\mu}_1 a_{L1} a_{T2}(2\zeta^2-k_{T1}^2)+2\bar{\mu}_1 a_{L1} a_{T1}\zeta^2-2\bar{\mu}_2 a_{L1} a_{T2}\zeta^2-\bar{\mu}_2 a_{L1} a_{T1}(2\zeta^2-k_{T2}^2)\} \quad \text{(E-6h)}$$

Recalling from the text that equations (84a-e) express C_{1-5} as a function of m_{ij}, we obtain the following expressions for C_{1-5} for the two-layer problem:

$$C_1=\frac{e^{-(a_{L2}+a_{T2})h_1}}{2\bar{\mu}_1^2 a_{L1}k_{T1}^4}\{2\bar{\mu}_1^2 a_{L2}a_{T2}\zeta^2(2\zeta^2-k_{T1}^2)-4\bar{\mu}_1\bar{\mu}_2 a_{L2}a_{T2}\zeta^4-2\bar{\mu}_1^2\zeta^4(2\zeta^2-k_{T1}^2)$$

$$+\bar{\mu}_1\bar{\mu}_2\zeta^2(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2)-2\bar{\mu}_1\bar{\mu}_2 a_{L2}a_{T2}\zeta^2(2\zeta^2-k_{T1}^2)+4\bar{\mu}_2^2 a_{L2}a_{T2}\zeta^4$$

$$+2\bar{\mu}_1\bar{\mu}_2\zeta^4(2\zeta^2-k_{T2}^2)-\bar{\mu}_2^2\zeta^2(2\zeta^2-k_{T2}^2)^2\} \qquad \text{(E-7a)}$$

$$C_2=\frac{e^{-(a_{L2}+a_{T2})h_1}}{\bar{\mu}_1^2 a_{L1}a_{T1}k_{T1}^4}\{\sinh(a_{L1}h_1)\sinh(a_{T1}h_1)[\bar{\mu}_1^2 a_{L2}a_{T2}(2\zeta^2-k_{T1}^2)^2-4\bar{\mu}_1\bar{\mu}_2 a_{L2}a_{T2}\zeta^2(2\zeta^2-k_{T\cdot}^2)$$

$$+4\bar{\mu}_2^2 a_{L2}a_{T2}\zeta^4-\bar{\mu}_1^2\zeta^2(2\zeta^2-k_{T1}^2)^2+2\bar{\mu}_1\bar{\mu}_2\zeta^2(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2)$$

$$-\bar{\mu}_2^2\zeta^2(2\zeta^2-k_{T2}^2)^2]$$

$$+\cosh(a_{L1}h_1)\sinh(a_{T1}h_1)[\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2)+4\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}\zeta^4$$

$$-2\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}\zeta^2(2\zeta^2-k_{T2}^2)-2\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}\zeta^2(2\zeta^2-k_{T1}^2)]$$

$$+\cosh(a_{L1}h_1)\cosh(a_{T1}h_1)[4\bar{\mu}_1^2 a_{L1}a_{T1}\zeta^4-4\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T1}\zeta^2(2\zeta^2-k_{T2}^2)$$

$$+\bar{\mu}_2^2 a_{L1}a_{T1}(2\zeta^2-k_{T2}^2)^2-4\bar{\mu}_1^2 a_{L1}a_{T1}a_{L2}a_{T2}\zeta^2+8\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T1}a_{L2}a_{T2}\zeta^2$$

$$-4\bar{\mu}_2^2 a_{L1}a_{T1}a_{L2}a_{T2}\zeta^2]$$

$$+\sinh(a_{L1}h_1)\cosh(a_{T1}h_1)[4\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}\zeta^4+\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2)$$

$$-2\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}\zeta^2(2\zeta^2-k_{T1}^2)-2\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}\zeta^2(2\zeta^2-k_{T2}^2)]\} \qquad \text{(E-7b)}$$

$$C_1=\frac{e^{-(a_{L2}+a_{T2})h_1}}{2\bar{\mu}_1^2 a_{L1}k_{T1}^4}\{2\bar{\mu}_1^2 a_{L2}a_{T2}\zeta^2(2\zeta^2-k_{T1}^2)-4\bar{\mu}_1\bar{\mu}_2 a_{L2}a_{T2}\zeta^4-2\bar{\mu}_1^2\zeta^4(2\zeta^2-k_{T1}^2)$$

$$+\bar{\mu}_1\bar{\mu}_2\zeta^2(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2)-2\bar{\mu}_1\bar{\mu}_2 a_{L2}a_{T2}\zeta^2(2\zeta^2-k_{T1}^2)+4\bar{\mu}_2^2 a_{L2}a_{T2}\zeta^4$$

$$+2\bar{\mu}_1\bar{\mu}_2\zeta^4(2\zeta^2-k_{T2}^2)-\bar{\mu}_2^2\zeta^2(2\zeta^2-k_{T2}^2)^2\} \qquad \text{(E-7a)}$$

$$C_2=\frac{e^{-(a_{L2}+a_{T2})h_1}}{\bar{\mu}_1^2 a_{L1}a_{T1}k_{T1}^4}\{\sinh(a_{L1}h_1)\sinh(a_{T1}h_1)[\bar{\mu}_1^2 a_{L2}a_{T2}(2\zeta^2-k_{T1}^2)^2-4\bar{\mu}_1\bar{\mu}_2 a_{L2}a_{T2}\zeta^2(2\zeta^2-k_{T1}^2)$$

$$+4\bar{\mu}_2^2 a_{L2}a_{T2}\zeta^4-\bar{\mu}_1^2\zeta^2(2\zeta^2-k_{T1}^2)^2+2\bar{\mu}_1\bar{\mu}_2\zeta^2(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2)$$

$$-\bar{\mu}_2^2\zeta^2(2\zeta^2-k_{T2}^2)^2]$$

$$+\cosh(a_{L1}h_1)\sinh(a_{T1}h_1)[\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2)+4\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}\zeta^4$$

$$-2\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}\zeta^2(2\zeta^2-k_{T2}^2)-2\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}\zeta^2(2\zeta^2-k_{T1}^2)]$$

$$+\cosh(a_{L1}h_1)\cosh(a_{T1}h_1)[4\bar{\mu}_1^2 a_{L1}a_{T1}\zeta^4-4\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T1}\zeta^2(2\zeta^2-k_{T2}^2)$$

$$+\bar{\mu}_2^2 a_{L1}a_{T1}(2\zeta^2-k_{T2}^2)^2-4\bar{\mu}_1^2 a_{L1}a_{T1}a_{L2}a_{T2}\zeta^2+8\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T1}a_{L2}a_{T2}\zeta^2$$

$$-4\bar{\mu}_2^2 a_{L1}a_{T1}a_{L2}a_{T2}\zeta^2]$$

$$+\sinh(a_{L1}h_1)\cosh(a_{T1}h_1)[4\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}\zeta^4+\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2)$$

$$-2\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}\zeta^2(2\zeta^2-k_{T1}^2)-2\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}\zeta^2(2\zeta^2-k_{T2}^2)]\} \qquad \text{(E-7b)}$$

$$C_3=\frac{e^{-(a_{L2}+a_{T2})h_1}}{\bar{\mu}_1^2 a_{L1} a_{T1} k_{T1}^4}\{\sinh(a_{L1}h_1)\sinh(a_{T1}h_1)[4\bar{\mu}_1^2 a_{L1}a_{T1}a_{L2}a_{T2}\zeta^2-8\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T1}a_{L2}a_{T2}\zeta^2$$

$$+4\bar{\mu}_2^2 a_{L1}a_{T1}a_{L2}a_{T2}\zeta^2-4\bar{\mu}_1^2 a_{L1}a_{T1}\zeta^4+4\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T1}\zeta^2(2\zeta^2-k_{T2}^2)$$

$$-\bar{\mu}_2^2 a_{L1}a_{T1}(2\zeta^2-k_{T2}^2)^2]$$

$$+\cosh(a_{L1}h_1)\sinh(a_{T1}h_1)[2\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}\zeta^2(2\zeta^2-k_{T2}^2)+2\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}\zeta^2(2\zeta^2-k_{T1}^2)$$

$$-4\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}\zeta^4-\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2)]$$

$$+\cosh(a_{L1}h_1)\cosh(a_{T1}h_1)[\bar{\mu}_1^2\zeta^2(2\zeta^2-k_{T1}^2)^2-2\bar{\mu}_1\bar{\mu}_2\zeta^2(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2)$$

$$+\bar{\mu}_2^2\zeta^2(2\zeta^2-k_{T2}^2)^2-\bar{\mu}_1^2 a_{L2}a_{T2}(2\zeta^2-k_{T1}^2)^2+4\bar{\mu}_1\bar{\mu}_2 a_{L2}a_{T2}\zeta^2(2\zeta^2-k_{T1}^2)$$

$$-4\bar{\mu}_2^2 a_{L2}a_{T2}\zeta^4]$$

$$+\sinh(a_{L1}h_1)\cosh(a_{T1}h_1)[2\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}\zeta^2(2\zeta^2-k_{T1}^2)+2\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}\zeta^2(2\zeta^2-k_{T2}^2)$$

$$-\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2)-4\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}\zeta^4]\} \qquad \text{(E-7c)}$$

$$C_4=\frac{e^{-(a_{L2}+a_{T2})h_1}}{2\bar{\mu}_1^2 a_{T1}k_{T1}^4}\{2\bar{\mu}_1^2 a_{L2}a_{T2}(2\zeta^2-k_{T1}^2)-2\bar{\mu}_1\bar{\mu}_2 a_{L2}a_{T2}(2\zeta^2-k_{T1}^2)-2\bar{\mu}_1^2\zeta^2(2\zeta^2-k_{T1}^2)$$

$$+2\bar{\mu}_1\bar{\mu}_2\zeta^2(2\zeta^2-k_{T2}^2)-4\bar{\mu}_1\bar{\mu}_2 a_{L2}a_{T2}\zeta^2+4\bar{\mu}_2^2 a_{L2}a_{T2}\zeta^2+\bar{\mu}_1\bar{\mu}_2(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2)$$

$$-\bar{\mu}_2^2(2\zeta^2-k_{T2}^2)^2\} \qquad \text{(E-7d)}$$

$$
\begin{aligned}
C_5 = \frac{e^{-(a_{L2}+a_{T2})h_1}}{\bar{\mu}_1^2 a_{L1} a_{T1} k_{T1}^4} \{ & \sinh(a_{L1}h_1)\sinh(a_{T1}h_1)[\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2)+4\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}\zeta^4 \\
& -2\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}\zeta^2(2\zeta^2-k_{T2}^2)-2\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T2}\zeta^2(2\zeta^2-k_{T1}^2)] \\
& +\cosh(a_{L1}h_1)\sinh(a_{T1}h_1)[\bar{\mu}_1^2 a_{L2}a_{T2}(2\zeta^2-k_{T1}^2)^2-4\bar{\mu}_1\bar{\mu}_2 a_{L2}a_{T2}\zeta^2(2\zeta^2-k_{T1}^2) \\
& +4\bar{\mu}_2^2 a_{L2}a_{T2}\zeta^4-\bar{\mu}_1^2\zeta^2(2\zeta^2-k_{T1}^2)^2+2\bar{\mu}_1\bar{\mu}_2\zeta^2(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2) \\
& -\bar{\mu}_2^2\zeta^2(2\zeta^2-k_{T2}^2)^2] \\
& +\cosh(a_{L1}h_1)\cosh(a_{T1}h_1)[4\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}\zeta^4+\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}(2\zeta^2-k_{T1}^2)(2\zeta^2-k_{T2}^2) \\
& -2\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}\zeta^2(2\zeta^2-k_{T1}^2)-2\bar{\mu}_1\bar{\mu}_2 a_{T1}a_{L2}\zeta^2(2\zeta^2-k_{T2}^2)] \\
& +\sinh(a_{L1}h_1)\cosh(a_{T1}h_1)[4\bar{\mu}_1^2 a_{L1}a_{T1}\zeta^4-4\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T1}\zeta^2(2\zeta^2-k_{T2}^2) \\
& +\bar{\mu}_2^2 a_{L1}a_{T1}(2\zeta^2-k_{T2}^2)^2-4\bar{\mu}_1^2 a_{L1}a_{T1}a_{L2}a_{T2}\zeta^2+8\bar{\mu}_1\bar{\mu}_2 a_{L1}a_{T1}a_{L2}a_{T2}\zeta^2 \\
& -4\bar{\mu}_2^2 a_{L1}a_{T1}a_{L2}a_{T2}\zeta^2]\}
\end{aligned}
\tag{E-7e}
$$

Now that C_{1-5} have been determined, the problem is essentially solved. We can evaluate equations (82b) and (83b) in the text and substitute into equation (48) in the text to obtain the coefficient of the acoustic potential, A_o. Then the liquid layer potentials may be written from equations (39) and (40) in the text.

Report Distribution

	Copies
External Distribution	
Sea Grant Depository	1
Sea Grant 70's	1
Office of Sea Grant	5
Robert Wildman - National Sea Grant Program Office	1
NOAA - Technical Information Division	50
Rogers Rutter - Office of State Planning	2
Raytheon Distribution	
Arthur S. Westneat	10
Internal Distribution	
Vice Provost Robert Faiman	1
Dean Richard S. Davis	1
Dean William H. Drew	1
Dean Maynard Heckel	1
Prof. E. E. Allmendinger	1
Prof. William Mosberg	1
Dimond Library	1
News Bureau	1
Project Team	11