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TECHNICAL REPORT

The Normal Incidence Acoustic Response for a Liquid Overlying a Viscoelastic Halfspace

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A Report of a
Cooperative University-Industry Research Project
between

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THE NORMAL INCIDENCE ACOUSTIC RESPONSE FOR A LIQUID OVERLYING A VISCOELASTIC HALFSPACE

by

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July 1973

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Nomenclature

ζ	--	transformation parameter for Fourier-Bessel transform
$\underline{G}$	--	Fourier-Bessel transformed Green's Function
G	--	inverse Fourier-Bessel transformed Green's Function
K_o	--	wave number of the liquid field
K_L	--	complex wave number of the compressional field in the viscoelastic medium
K_T	--	complex wave number of the shear field in the viscoelastic medium
c_o	--	sound velocity in the liquid field
c_L	--	complex sound velocity of the compressional field in the viscoelastic medium
c_T	--	complex sound velocity of the shear field in the viscoelastic medium
ω	--	frequency
ρ_o	--	density of the liquid field
ρ_1	--	density of the viscoelastic medium
h_o	--	depth of the liquid
$a_{o,L,T}$	--	functions of the transformation parameter ζ
λ_o	--	Lamé constant of the liquid field
λ_1, μ_1	--	complex Lamé parameters of the viscoelastic medium
m	--	ratio of viscoelastic medium density to liquid density
r, θ, z	--	radial, circumferential, and longitudinal direction components for a cylindrical coordinate system
ε	--	perturbation parameters

ABSTRACT

Using complex variable techniques, the inverse Bessel transformation is performed to obtain the actual Green's Function expression characterizing a semi-infinite liquid overlying a viscoelastic halfspace. The two media are assumed to be homogeneous and the discontinuity between them is considered to be plane. The integral representing the inverse transform is evaluated for normal incidence, where excitation is provided by a simple harmonic point source in the liquid. The resulting response is the sum of a direct wave, i.e., a wave passing directly from the source to the receiver; and a reflected wave term. The actual Green's Function is then separated into real and imaginary components, so that the effect of introducing viscoelasticity into the model may subsequently be analyzed by computer methods. Damping in the viscoelastic layer is assumed to be small for our frequence range.

CHAPTER I

INTRODUCTION

The necessity to develop economically feasible means to classify and extract subbottom sediments has increased steadily in recent years. Coupled with mineral, sand, and gravel extraction is the desire to determine the engineering properties of the sediments for offshore construction purposes. Some data on the elastic properties of ocean sediments has recently been obtained by Hamilton [4]. The sediments analyzed were from North Pacific areas, however, the measured and computed properties should be valid for similar sediments elsewhere. Table 1 indicates Hamilton's results which are of interest in our theoretical formulation.

Using a simple harmonic point source for excitation, we will develop an acoustic response system for a semi-infinite liquid overlying a viscoelastic halfspace. The theoretical model employed in this thesis is governed closely by the experimental viewpoint. Surface reflections occur well after first returns for near bottom sensing, thus enabling us to consider the hydrodynamic field as being infinite in depth. To account for attenuation phenomena, it is desirable to consider Voigt damping in the viscoelastic field. This is introduced by taking the Lamé parameters to be of the form $\lambda = \lambda' + \lambda'' \frac{\partial}{\partial t}$, or $\lambda = \lambda' + i\omega\lambda''$ in the frequency domain.

Theoretical development begins by computing the proper Fourier-Bessel transformed Green's Function for one viscoelastic layer and infinite liquid depth from the general expression obtained by Magnuson and Stewart [8]. Subsequently, the proper contour is chosen and the inverse transform is performed. Higher order branch line contributions are expressed as a series

Table 1. Average Measured and Computed Elastic Constants for North Pacific Sediments on the Continental Terrace (from Hamilton [5])

Sediment Type	Measured Values Porosity (%)	Density (g/cc)	C_L (m/sec)	Computed Values Poisson's Ratio	C_T (m/sec)
Sand:					
Coarse	38.6	2.03	1836	.491	250
Fine	43.9	1.98	1742	.469	382
Very Fine	47.4	1.91	1711	.453	503
Silty Sand	52.8	1.83	1677	.457	457
Sandy Silt	68.3	1.56	1552	.461	379
Sand-Silt-Clay	67.5	1.58	1578	.463	409
Clayey Silt	75.0	1.43	1535	.478	364
Silty Clay	76.0	1.42	1519	.480	287

of Gamma Functions, and perturbation theory is used to compute the undamped Stoneley wave velocity.

Previous investigations of this type have been undertaken by Pekeris [10], who investigated the response due to a point source in a liquid overlying another liquid. Similarly Press and Ewing [2] investigated the model discussed in this thesis, but neglected branch line contributions. The branch line integrals were later evaluated by Honda and Nakamura [5]. Most recently, Magnuson and Stewart [8] have developed a general multilayer recurrence relation suitable for computer analysis.

CHAPTER II

THEORETICAL DEVELOPMENT

1. Green's Function Formalism

We will determine the actual Green's Function for a semi-infinite liquid overlying a viscoelastic halfspace for the special case of normal incidence. The general Fourier-Bessel transformed Green's Function as taken from equation (18) of Magnuson and Stewart [8] reads as follows:

$$\underline{G}(\zeta,z_>,z_<,\omega) = \frac{2}{4\pi a_o} \sinh[a_o(h_o-z_>)]\{\frac{K_1 a_o \cosh[a_o z_<]-K_2\rho_o\omega^2 \sinh[a_o z_<]}{K_1 a_o \cosh[a_o h_o]-K_2\rho_o\omega^2 \sinh[a_o h_o]}\} \quad (1.1)$$

where for one viscoelastic layer

$$K_1 = \rho_1 c_T^2[(2\zeta^2-K_T^2)^2 - 4a_L a_T\zeta^2] \quad (1.1\text{-a})$$

and $$K_2 = -a_L K_T^2 \quad (1.1\text{-b}).$$

The functions $a_{o,L,T}$ are given by the following expressions:

$$a_o = \sqrt{\zeta^2-K_o^2} \quad (1.1\text{-c})$$

$$a_L = \sqrt{\zeta^2-K_L^2} \quad (1.1\text{-d})$$

$$a_T = \sqrt{\zeta^2-K_T^2} \quad (1.1\text{-e}) .$$

For the case of an unbounded fluid, h_o is taken to infinity and equation (1.1) (upon expansion of sinh and cosh terms) reduces to:

$$\underline{G}(\zeta,z_>,z_<,\omega) = \frac{1}{4\pi a_o}[e^{-a_o(z_>-z_<)}+e^{-a_o(z_>+z_<)}(\frac{K_1 a_o+K_2\rho_o\omega^2}{K_1 a_o-K_2\rho_o\omega^2})] \quad (1.2)$$

The first term in equation (1.2) represents the wave travelling directly from the source to the receiver, while the second term represents the contribution due to the viscoelastic halfspace. Substituting equations (1.1-a) and (1.1-b) into equation (1.1) and noting from Figure 1 that $z_{>} = z_{max} = h$ and $z_{<} = z_{min} = z$, the Green's Function becomes:

$$\underline{G}(\zeta,z,h,\omega) = \frac{1}{4\pi}\left[\frac{e^{-a_o(h-z)}}{a_o} + \frac{e^{-a_o(h+z)}}{a_o}\,\frac{N(\zeta^2)}{D(\zeta^2)}\right] \tag{1.3}$$

where

$$N(\zeta^2) = a_o m[(2\zeta^2-K_T^2)^2-4a_L a_T\zeta^2] - a_L K_T^4 \tag{1.3-a}$$

and

$$D(\zeta^2) = a_o m[(2\zeta^2-K_T^2)^2-4a_L a_T\zeta^2] + a_L K_T^4 \tag{1.3-b}$$

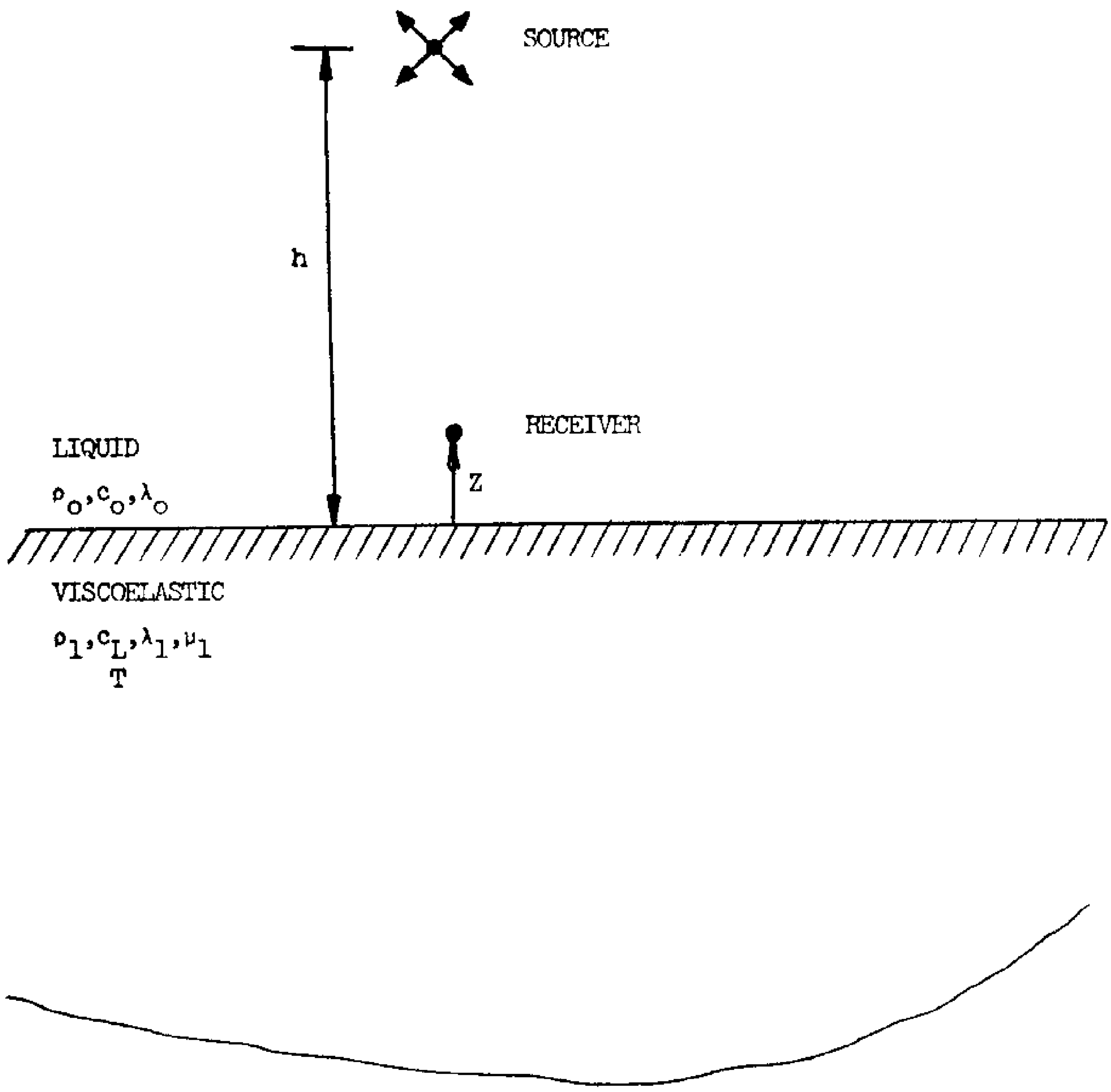
Performing the inverse transform on the primary stimulation or direct wave term in equation (1.3) will yield according to Sommerfeld [11]:

$$\int_0^\infty \frac{1}{4\pi a_o}\, e^{-a_o(h-z)} J_o(\zeta r)\zeta d\zeta = \frac{1}{4\pi(h-z)}\, e^{-iK_o(h-z)} \tag{1.4}$$

The main objective of this investigation is to determine the inverse transform for the second term in equation (1.3). Noting this residual term as $\underline{G}'$ we may write:

$$G'(r,z,h,\omega) = \int_0^\infty \underline{G}'(\zeta,z,h,\omega)\, J_o(\zeta r)\zeta d\zeta \tag{1.5}$$

For the case of normal incidence $(r=o)$, $J_o(\zeta r) \to 1$, and equation (1.5)



Geometry of Normal Incidence Acoustic Response System:
Semi-Infinite Liquid Over a Viscoelastic Halfspace

FIGURE 1

simplifies to:

$$G' = \int_0^{\infty} \underline{G}' \; \zeta d\zeta \qquad (1.6)$$

2. Application of Contour Integration

We choose to integrate the integral (1.6) in the complex ($\xi = \zeta + i\eta$) plane. We write a contour integral from equation (1.6) as follows:

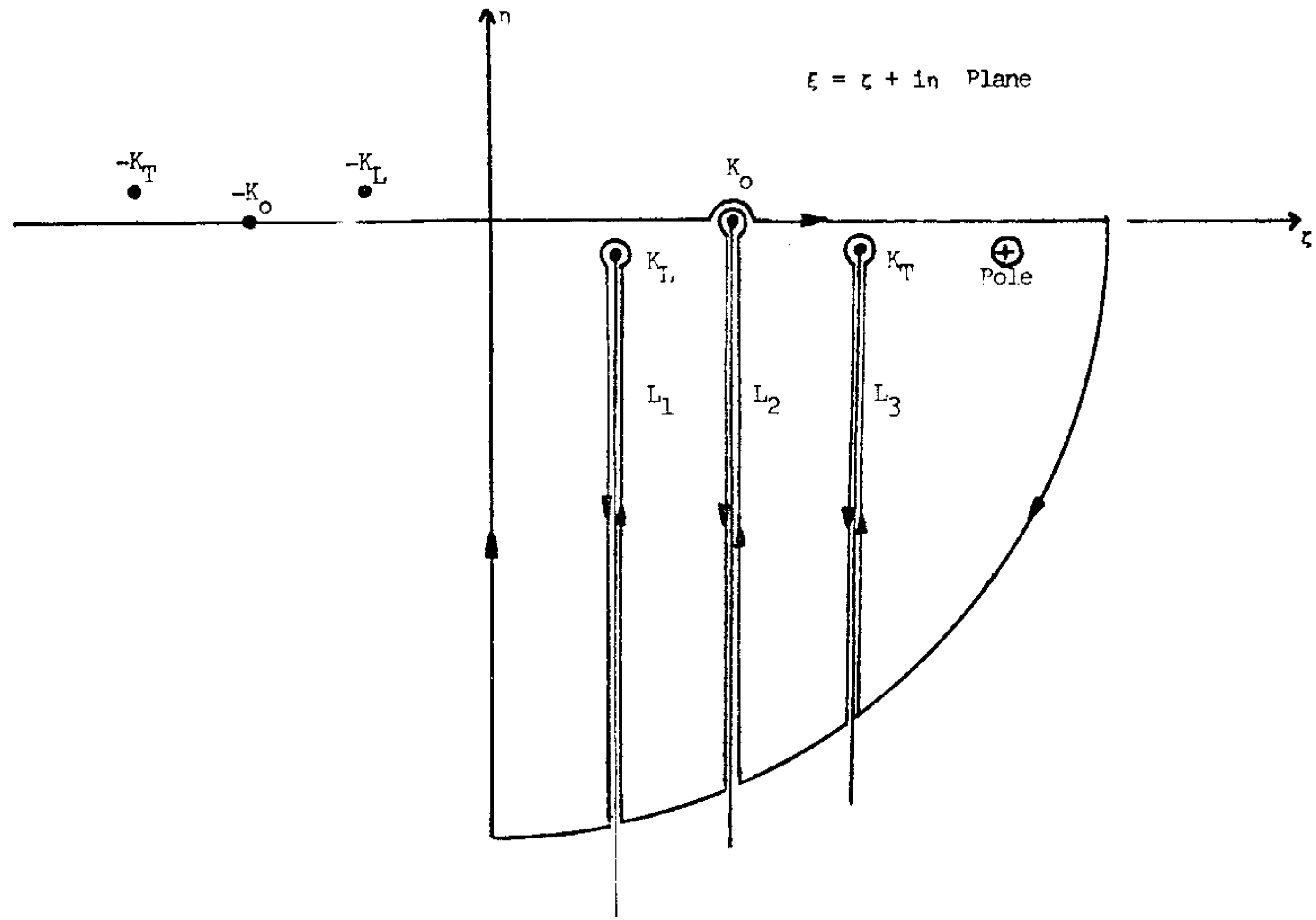
$$I = \oint_0^{\infty} \frac{e^{-a_o(h+z)}}{a_o} \frac{N(\xi^2)}{D(\xi^2)} \xi d\xi \qquad (2.1)$$

Branch point singularities of the integrand occur at $\xi = \pm K_{o,L,T}$. The poles of equation (2.1) are given by

$$D(\xi^2) = 0 \qquad (2.2)$$

Strick and Ginsbarg [12] numerically obtained one real root, representing a Stoneley wave contribution, for equation (2.2). The effect of the Voigt type damping employed in this treatment is that the pole and $K_{L,T}$ were pulled slightly off the real axis into the fourth quadrant.

Having determined the singularities in equation (2.1), we must now select an appropriate contour. Careful examination of the exponential term in equation (2.1) shows that we must keep $R_e\{a_o\} > 0$ for convergence. Sommerfeld's [11] radiation condition is satisfied by keeping $I_m\{a_o\} > 0$. We draw the contour as shown in Figure 2. Applying Cauchy's theorem to equation (2.1) for the path shown in Figure 2 and noting that the contribution



Contour of Integration for Equation (2.1)

FIGURE 2

along the quadrant vanishes (see Appendix A) we obtain as follows:

$$\oint_0^\infty \frac{e^{-a_o(z+h)}}{a_o} \frac{N(\xi^2)}{D(\xi^2)} \xi d\xi = \int_0^\infty \frac{e^{-a_o(z+h)}}{a_o} \frac{N(\zeta^2)}{D(\zeta^2)} \zeta d\zeta + \int_{-i\infty}^0 \frac{e^{-a_o(z+h)}}{a_o} \frac{N[(-i\eta)^2]}{D[(-i\eta)^2]} i\eta d i\eta +$$

$$I_{L_1} + I_{L_2} + I_{L_3} = -2\pi i \times \text{Residue} \tag{2.3}$$

Solving for the real axis contribution, which represents the Green's Function, will yield

$$\int_0^\infty \frac{e^{-a_o(z+h)}}{a_o} \frac{N(\zeta^2)}{D(\zeta^2)} \zeta d\zeta = - \int_{-i\infty}^0 \frac{e^{-a_o(z+h)}}{a_o} \frac{N[(-i\eta)^2]}{D[(-i\eta)^2]} i\eta d i\eta - I_{L_1} - I_{L_2} - I_{L_3} - 2\pi i R \tag{2.4}$$

3. Evaluation of Residue Contribution

The residue term in equation (2.4) is given by the formula:

$$R = \lim_{\xi \to \xi_o} \left[\frac{e^{-a_o(z+h)}}{a_o} (\xi - \xi_o) \frac{N(\xi^2)}{D(\xi^2)} \xi\right] \tag{3.1}$$

where ξ_o is the value at which $D(\xi^2) \to 0$. It should be clear that the residue in its present form is indeterminate. Applying L'Hospital's Rule we obtain:

$$R = \lim_{\xi \to \xi_o} \left[\frac{e^{-a_o(z+h)}}{a_o} \frac{\xi N(\xi^2)}{\frac{d}{d\xi} D(\xi^2)}\right] \tag{3.1-a}$$

In general any point in the ξ plane represents a wave number for a certain mode of vibration; i.e., $\xi = \frac{\omega}{c}$. At the pole, the phase velocity represents the propagation speed of Stoneley waves at the interface. To evaluate the residue contribution, we first determine the phase velocity of these surface waves. From equation (1.3-b) we write:

$$D(\xi^2) = 0 = m(\xi^2-K_o^2)^{1/2}[(2\xi^2-K_T^2)^2-4\xi^2(\xi^2-K_L^2)^{1/2}(\xi^2-K_T^2)^{1/2}] + K_T^4(\xi^2-K_L^2)^{1/2} \tag{3.2}$$

Recalling the expressions for the wave numbers:

$$K_o = \frac{\omega}{c_o} \tag{3.3-a}$$

$$K_L = \frac{\omega}{c_L} \tag{3.3-b}$$

$$K_T = \frac{\omega}{c_T} \tag{3.3-c}$$

Using equations (3.3a-c), equation (3.2) simplifies to read

$$0 = m(1-(\frac{c}{c_o})^2)^{1/2}[(2-(\frac{c}{c_T})^2)^2 - 4(1-(\frac{c}{c_L})^2)^{1/2}(1-(\frac{c}{c_T})^2)^{1/2}] + (\frac{c}{c_T})^4(1-(\frac{c}{c_L})^2)^{1/2} \tag{3.4}$$

Equation (3.4) represents the frequency independent modal equation at the pole. The undamped phase velocity is determined by using perturbation techniques (see Appendix B). The result is that the phase velocity equals the transverse wave velocity to first order in ϵ. The fact that

$c = c_T$ enables us to conclude that:

$$a_T = (\xi^2 - \frac{\omega^2}{c_T^2})^{1/2} = (\frac{\omega^2}{c^2} - \frac{\omega^2}{c_T^2})^{1/2} = 0 \qquad (3.5)$$

The residue is evaluated by first computing $\frac{d}{d\xi} D(\xi^2)$. Using the chain rule we may write

$$\frac{d}{d\xi} D(\xi^2) = \frac{d\psi}{d\xi} \cdot \frac{dD(\psi)}{d\psi} \qquad (3.6)$$

where

$$\psi = \xi^2 \qquad (3.6\text{-a})$$

and

$$D(\psi) = (\psi - K_o^2)^{1/2} m[(2\psi - K_T^2)^2 - 4\psi(\psi - K_L^2)^{1/2}(\psi - K_T^2)^{1/2}] + K_T^4(\psi - K_L^2)^{1/2} \qquad (3.6\text{-b})$$

Using equations (3.6-a) and (3.6-b) we may expand equation (3.6) as follows:

$$\frac{d}{d\xi} D(\xi^2) = 2\xi\{\frac{m}{2a_o}[(2\psi - K_T^2)^2 - 4\psi(\psi - K_L^2)^{1/2}(\psi - K_T^2)^{1/2}] +$$
$$ma_o[4(2\psi - K_T^2) - 4(a_L a_T + \frac{a_T\psi}{2a_L} + \frac{a_L\psi}{2a_T})] + \frac{K_T^4}{2a_L}\} \qquad (3.7)$$

Applying the result in equation (3.5) to equation (3.7) we note that the term $\frac{a_L \psi}{2a_T} \to \infty$. It follows from equation (3.1-a) that the residue contribution vanishes. Computing the exact value of the phase velocity from equation (3.4) would yield a small residue contribution.

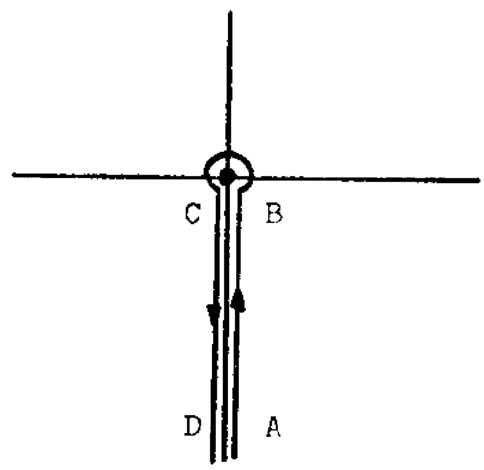
4. Branch Line Integrations

We wish to evaluate the line integrals L_1, L_2, and L_3 in equation (2.4) on the paths shown in Figure 2. We begin by discussing the integral for the branch point at $\xi = K_o$.

1) Line Integral for Path L_2: From equation (2.4) we may write the integral as

$$I_{L_2} = \int_{L_2} \frac{e^{-a_o(z+h)}}{a_o} \frac{N(\xi^2)}{D(\xi^2)} \xi d\xi \tag{4.1}$$

The path of integration for equation (4.1) is indicated below.



We recall that a_o and ξ are related as follows:

$$a_o^2 = \xi^2 - K_o^2 \tag{4.2}$$

It follows that

$$a_o da_o = \xi d\xi \tag{4.3}$$

Applying equations (4.2) and (4.3) to the integral (4.1) we may change variables of integration so that the integral reads

$$I_{L_2} = \int_{L_2} e^{-a_o(z+h)} \frac{N(a_o^2)}{D(a_o^2)} da_o \tag{4.4}$$

Along the path AB we write

$$a_o = -i\eta_o$$

where η_o is the distance from the branch point. The argument of a_o increases by 2π when passing from AB to CD . Hence, we may say that along CD

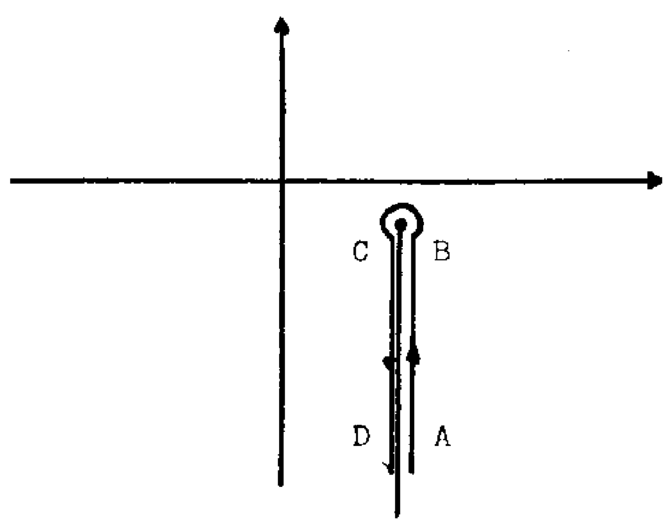
$$a_o = -i\eta_o e^{i2\pi}$$

It should be clear that $a_o^2 = -\eta_o^2$ on both sides of the branch cut. Integrating along η_o gives us symbolically

$$I_{L_2} = \int_{AB} (\ )d\eta_o + \int_{CD} (\ )d\eta_o = \int_{\infty}^{0} (\ )d\eta_o + \int_{0}^{\infty} (\ )d\eta_o = 0 \qquad (4.5)$$

Contributions on the two sides of the cut cancel, resulting in $I_{L_2} = 0$.

2) Line Integral for Path L_1: The path of integration L_1 for the branch point $\xi = K_L$ is indicated below.



Again we choose to integrate with respect to the variable a_o . From equations (2.4) and (4.3), the integral reads

$$I_{L_2} = \int_{L_2} e^{-a_o(z+h)} \frac{N(a_o^2)}{D(a_o^2)} da_o \tag{4.6}$$

The value of the integration variable a_o at the branch point is given by

$$\alpha_L = (\xi^2-K_o^2)^{1/2} = (K_L^2-K_o^2)^{1/2} \tag{4.7}$$

Using equation (4.7), on the portion AB we write

$$a_o = \alpha_L - i\eta_L \tag{4.8}$$

and on CD

$$a_o = \alpha_L - i\eta_L e^{i2\pi} \tag{4.9}$$

Applying the change in variables for the quantity $(\xi^2-K_L^2)^{1/2}$ will yield the following on AB:

$$(\xi^2-K_L^2)^{1/2} = (-2i\eta_L\alpha_L-\eta_L^2)^{1/2} = a_L \tag{4.10}$$

and on CD:

$$(\xi^2-K_L^2)^{1/2} = e^{i\pi}(-2i\eta_L\alpha_L-\eta_L^2)^{1/2} = -a_L \tag{4.11}$$

Since the quantity a_L changes sign from one side of the cut to the other, there is a discontinuity in the integrand. Using equations (4.10) and (4.11) one writes the integral (4.6) as follows:

$$I_{L_2} = \int_{\alpha_L-i\infty}^{\alpha_L} e^{-a_o(z+h)}\frac{N(a_o^2,a_L)}{D(a_o^2,a_L)} da_o + \int_{\alpha}^{\alpha_L-i\infty} e^{-a_o(z+h)}\frac{N(a_o^2,-a_L)}{D(a_o^2,-a_L)} da_o =$$

$$\int_{\alpha_L}^{\alpha_L - i\infty} e^{-a_o(z+h)}\left[\frac{N(a_o^2,-a_L)}{D(a_o^2,-a_L)} - \frac{N(a_o^2,a_L)}{D(a_o^2,a_L)}\right] = \int_{\alpha_L}^{\alpha_L - i\infty} e^{-a_o(z+h)} a_L F(a_o^2) da_o \qquad (4.12)$$

We now expand

$$F(a_o^2) = F((\alpha_L - i\eta_L)^2) = A + B\eta_L + C\eta_L^2 + \cdots \qquad (4.13\text{-a})$$

and

$$a_L = (-2i\eta_L\alpha_L - \eta_L^2)^{1/2} = i^{3/2}(2\alpha_L)^{1/2}(\eta_L)^{1/2}\left(1 - \frac{i\eta_L}{2\alpha_L}\right) =$$

$$i^{3/2}(2\alpha_L)^{1/2}(\eta_L)^{1/2}\left[1 - \frac{i\eta_L}{4\alpha_L} + \cdots\right] \qquad (4.13\text{-b})$$

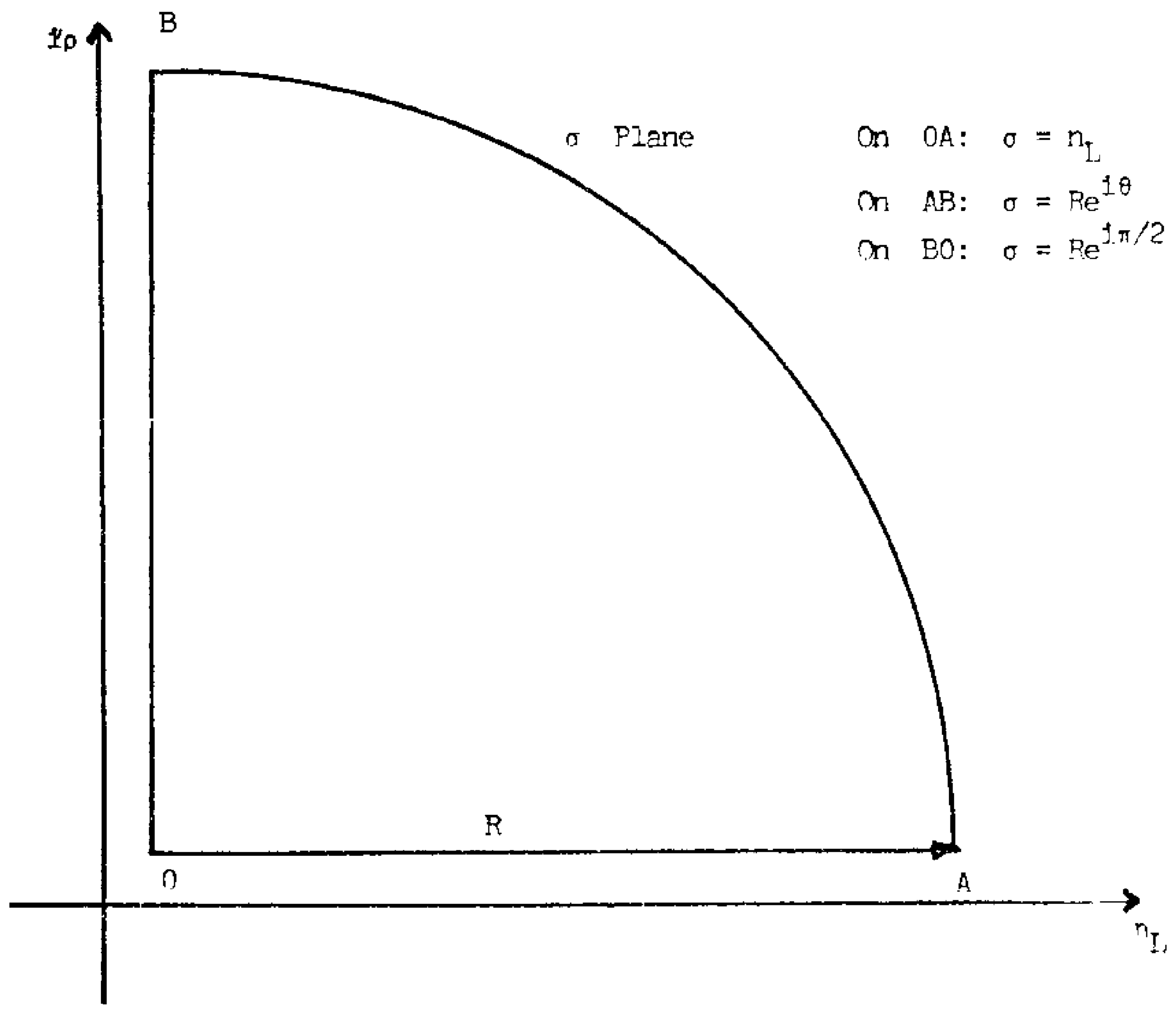
Recalling that $da_o = -id\eta_L$, we integrate equation (4.12) along η_L and obtain

$$I_{L_2} = -i\int_0^\infty e^{-(\alpha_L - i\eta_L)(z+h)}\left[i^{3/2}(2\alpha_L)^{1/2}(\eta_L)^{1/2}\left(1 - \frac{i\eta_L}{4\alpha_L} + \cdots\right)(A + B\eta_L + \cdots)d\eta_L\right.$$

$$= i^{1/2}(2\alpha_L)^{1/2} e^{-\alpha_L(z+h)}\int_0^\infty e^{i\eta_L(z+h)}[A_1' + A_2'\eta_L + A_3'\eta_L^2 + \cdots](\eta_L)^{1/2} d\eta_L$$

$$= i^{1/2}(2\alpha_L)^{1/2} e^{-\alpha_L(z+h)}\int_0^\infty e^{i\eta_L(z+h)} A_n' \eta_L^{\frac{2n-1}{2}} d\eta_L \qquad (4.14)$$

To obtain the expression for the integrals in equation (4.14), we choose to integrate in the complex $(\sigma = \eta_L + i\rho)$ plane on the contour shown in Figure 3. Applying Cauchy's Theorem around the path, and noting that



Contour of Integration for Equation (4.14)

FIGURE 3

the contribution along the quadrant again vanishes, we obtain the following:

$$\int e^{i\sigma(z+h)}\sigma^{\frac{2n-1}{2}}\,d\sigma = \int_0^\infty e^{-\eta_L(z+h)}\eta_L^{\frac{2n-1}{2}}\,d\eta_L + \int_\infty^0 e^{iRe^{i\pi/2}(z+h)}(Re^{i\pi/2})^{\frac{2n-1}{2}}e^{i\pi/2}dR=0$$

Solving for the real axis contribution in equation (4.15) will yield

$$\int_0^\infty e^{i\eta_L(z+h)}\eta_L^{\frac{2n-1}{2}}\,d\eta_L = i\int_0^\infty e^{i\pi/4(2n-1)}e^{-R(z+h)}R^{\frac{2n-1}{2}}\,dR \tag{4.16}$$

Recalling the expression for the Gamma Function

$$\Gamma(z) = \int_0^\infty e^{-u}u^{z-1}du \tag{4.17}$$

Applying equation (4.17) to equation (4.16) will yield

$$\int_0^\infty e^{i\eta_L(z+h)}\eta_L^{\frac{2n-1}{2}}\,d\eta_L = \sum_{n=1}^\infty \frac{ie^{i\frac{\pi}{4}(2n-1)}}{z+h}\Gamma(n+\tfrac{1}{2}) \tag{4.18}$$

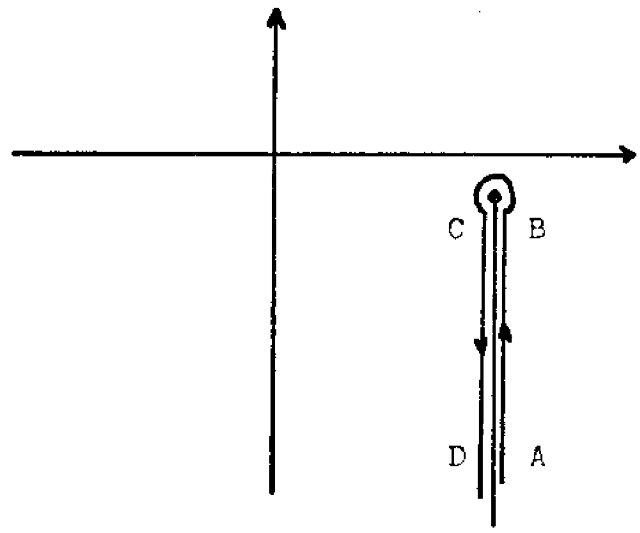
The constants A_n' in equation (4.14) must be determined to complete the solution of the branch line integral L_2. It should be clear that the lowest order constant A_1' may be obtained by evaluating equation (4.13-a) at the branch point $(\eta_L=0)$. The result (see Appendix C) is given by

$$A_1' = \frac{4K_T^4}{m\alpha_L(2K_L^2-K_T^2)} \tag{4.19}$$

Using the results of equations (4.18) and (4.19) the branch line integral in lowest order form will read as follows:

$$I_{L_2} = i^{3/2}(2\alpha_L)^{1/2} e^{-\alpha_L(z+h)} \left[\frac{4K_T^4}{m\alpha_L(2K_L^2-K_T^2)}\right] \frac{e^{i\pi/4}}{z+h}(\pi)^{1/2} \qquad (4.20)$$

3) Line Integral for Path L_3: The path of integration L_3 for the branch point $\xi = K_T$ is indicated below.



The integral upon changing variables will read

$$I_{L_3} = \int_{L_3} e^{-a_o(z+h)} \frac{N(a_o^2)}{D(a_o^2)} da_o \qquad (4.21)$$

The value of the variable a_o at the branch point is given by

$$\alpha_T = (K_T^2-K_o^2)^{1/2} \qquad (4.22)$$

Using equation (4.22), on the portion AB we may write

$$a_o = \alpha_T - i\eta_T \qquad (4.23)$$

and on CD

$$a_o = \alpha_T - i\eta_T e^{i2\pi} \qquad (4.24)$$

Applying the change in variables for the quantity $(\xi^2-K_T^2)^{1/2}$ will yield the following on AB:

$$(\xi^2-K_T^2)^{1/2} = (-2i\eta_T\alpha_T-\eta_T^2)^{1/2} = a_T \tag{4.25}$$

and on CD:

$$(\xi^2-K_T^2)^{1/2} = e^{i\pi}(-2i\eta_T\alpha_T-\eta_T^2) = -a_T \tag{4.26}$$

Applying these results to the integral exactly as done in the preceding section will yield

$$I_{L_3} = i^{1/2}(2\alpha_T)^{1/2}e^{-\alpha_T(z+h)}\int_0^\infty e^{i\eta_T(z+h)}B_n'\eta_T^{\frac{2n-1}{2}}d\eta_T \tag{4.27}$$

It should be clear that the integral terms are exactly of the same form as equation (4.18). The lowest order constant B_1' is now determined (see Appendix D), and the branch line integral I_{L_3} is written in lowest order form as:

$$I_{L_3} = -i^{3/2}(2\alpha_T)^{1/2}e^{-\alpha_T(z+h)}\left[\frac{16\alpha_T m a_L^2}{K_T^2(\alpha_T m+a_L)^2}\right]\frac{e^{i\pi/4}(\pi)^{1/2}}{z+h} \tag{4.28}$$

5. <u>Evaluation of the Integral Along the Imaginary Axis</u>

From equation (2.3), the integral along the imaginary axis in Figure 2 is given by the following expression:

$$I_{i\eta} = \int_{-i\infty}^{0}\frac{e^{-a_o(z+h)}}{a_o}\frac{N(-\eta^2)}{D(-\eta^2)}\,i\eta d(i\eta) \tag{5.1}$$

If we integrate along the variable η , equation (5.1) becomes

$$I_{1_\eta} = \int_{-\infty}^{0} \frac{e^{-i(\eta^2+K_o^2)^{1/2}(z+h)}}{i(\eta^2+K_o^2)^{1/2}} \frac{N(-\eta^2)}{D(-\eta^2)} -\eta d\eta = \int_{0}^{-\infty} \frac{e^{-i(\eta^2+K_o^2)^{1/2}(z+h)}}{i(\eta^2+K_o^2)^{1/2}} \frac{N(-\eta^2)}{D(-\eta^2)} d\eta \quad (5.2)$$

Since the integrand in equation (5.2) is odd in η , the upper limit may be changed from $-\infty \to +\infty$ without any loss of generality. Equation (5.2) now reads

$$I_{1_\eta} = \int_{0}^{\infty} \frac{e^{-i(\eta^2+K_o^2)^{1/2}(z+h)}}{i(\eta^2+K_o^2)^{1/2}} \frac{N(-\eta^2)}{D(-\eta^2)} \eta d\eta \quad (5.2\text{-a})$$

where from equations (1.3-a) and (1.3-b)

$$N(-\eta^2) = i[(\eta^2+K_o^2)^{1/2} m((2\eta^2+K_T^2)^2 - 4\eta^2(\eta^2+K_L^2)^{1/2}(\eta^2+K_T^2)^{1/2}) - K_T^4(\eta^2+K_L^2)^{1/2}] \quad (5.3\text{-a})$$

and

$$D(-\eta^2) = i[(\eta^2+K_o^2)^{1/2} m((2\eta^2+K_T^2)^2 - 4\eta^2(\eta^2+K_L^2)^{1/2}(\eta^2+K_T^2)^{1/2}) + K_T^4(\eta^2+K_L^2)^{1/2}] \quad (5.3\text{-b})$$

Since the integrand of equation (5.2-a) is of the form $\int_a^b e^{ixh(\eta)} g(\eta) d\eta$, where x is large, it is desirable to integrate by the method of stationary phase. The major contribution to the integral results from the point of stationarity, i.e., $h(\eta) = 0$. From equation (5.2-a) we write

$$h(\eta) = (\eta^2+K_o^2)^{1/2} = K_o(1+\eta^2 K_o^{-2})^{1/2} \quad (5.4\text{-a})$$

It follows that

$$h'(\eta) = \frac{\eta}{K_o(1+\eta^2 K_o^{-2})^{1/2}} \quad (5.4\text{-b})$$

and

$$h''(\eta) = \frac{-\eta^2}{K_o^2}(1+\ ^2K_o^{-2})^{-3/2} + \frac{1}{K_o}(1+\eta^2K_o^{-2})^{1/2} \qquad (5.4\text{-c})$$

From equation (5.4-b) we note that the point of stationarity is given by $\eta_o = 0$. Expanding equation (5.4-a) about this point will yield

$$h(\eta) = h(\eta_o) + \frac{h''(\eta_o)}{2}(\eta-\eta_o)^2 + \cdots = K_o + \frac{\eta^2}{2K_o} + \cdots \qquad (5.5)$$

From equation (5.2-a), the function $g(\eta)$ is given by

$$g(\eta) = \frac{N(-\eta^2)}{D(-\eta^2)} \frac{1}{(\eta^2+K_o^2)^{1/2}} \qquad (5.6)$$

Expanding equation (5.6) about the point of stationarity will yield

$$g(\eta_o) = g(0) + \frac{d}{d\eta_1} g(\eta^2)\ \frac{\eta}{2} + \cdots \qquad (5.7)$$

where from equation (5.6)

$$g(0) = \frac{1}{K_o}\frac{N(0)}{D(0)} = \frac{1}{K_o}\left[\frac{\rho_1 c_L - \rho_o c_o}{\rho_1 c_L + \rho_o c_o}\right] \qquad (5.8)$$

Substituting the expanded functions in equations (5.5) and (5.7) into equation (5.2-a) will yield

$$I_{i_n} = \frac{-ie^{-iK_o(z+h)}}{K_o}\int_0^\infty e^{-i/2K_o\eta^2(z+h)}\left[\frac{\rho_1 c_L - \rho_o c_o}{\rho_1 c_L + \rho_o c_o}\right]\eta d\eta \qquad (5.9)$$

The integrand of equation (5.9) is of the form $e^u du$. Applying this result, the integral is evaluated as follows:

$$I_{1n} = \frac{e^{-iK_o(z+h)}}{z+h}\left[\frac{\rho_1 c_L - \rho_o c_o}{\rho_1 c_L + \rho_o c_o}\right] \int_0^\infty e^{\frac{-i\eta^2(z+h)}{2K_o}} \frac{-i\eta(z+h)}{K_o}\, d\eta =$$

$$\frac{e^{-iK_o(z+h)}}{z+h}\left[\frac{\rho_1 c_L - \rho_o c_o}{\rho_1 c_L + \rho_o c_o}\right] e^{\frac{-i(z+h)}{2K_o}\eta^2} \Bigg|_0^\infty =$$

$$\frac{e^{-iK_o(z+h)}}{z+h}\left[\frac{\rho_1 c_L - \rho_o c_o}{\rho_1 c_L + \rho_o c_o}\right] \qquad (5.10)$$

6. The Complete Green's Function

Substituting the results of equations (4.20), (4.28), and (5.10) into equation (2.4) enables us to express the residual Green's Function term by the following relation:

$$G'(r,z,h,\omega) = -i^{3/2}(2\alpha_L)^{1/2} e^{-\alpha_L(z+h)}\left[\frac{4K_T^4}{m\alpha_L(2K_L^2-K_T^2)}\right]\frac{e^{i\pi/4}}{z+h}\pi^{1/2} +$$

$$i^{3/2}(2\alpha_T)^{1/2} e^{-\alpha_T(z+h)}\left[\frac{16\alpha_T m a_L^2}{K_T^2(\alpha_T m+a_L)^2}\right]\frac{e^{i\pi/4}\pi^{1/2}}{z+h} \frac{-e^{-iK_o(z+h)}}{(z+h)}\left[\frac{\rho_1 c_L - \rho_o c_o}{\rho_1 c_L + \rho_o c_o}\right] \qquad (6.1)$$

The first term of equation (6.1) correspoinding to the branch cut for the singularity at $\xi = K_L$ in Figure 2 warrants further investigation. We recall the expressions for the branch point singularities in terms of

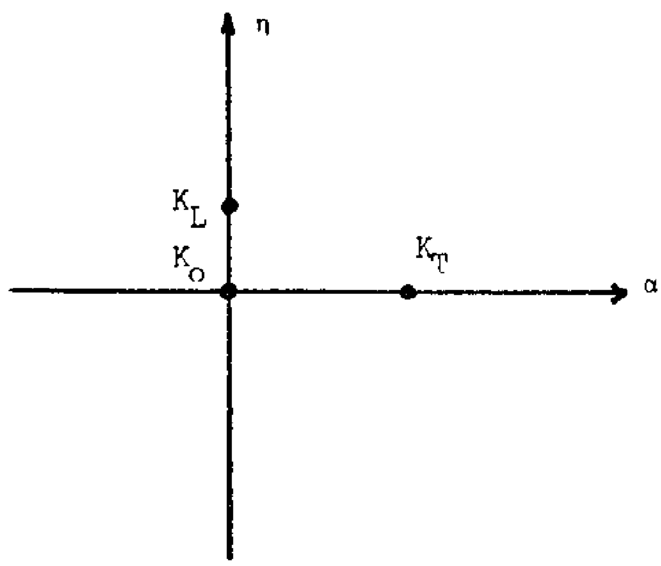
a_o

$$a_o = (\xi^2 - K_o^2)^{1/2} = 0 \qquad (6.2\text{-a})$$

$$\alpha_L = (K_L^2 - K_o^2)^{1/2} \qquad (6.2\text{-b})$$

$$\alpha_T = (K_T^2 - K_o^2)^{1/2} \qquad (6.2\text{-c})$$

The singularities of equation (6.2) are mapped into the a_o plane as shown in the following diagram.



Coordinates in the a_o plane have special significance in the contour integral of equation (2.4). The condition $Re\{a_o\} > 0$ accounts for wave attenuation. Since the branch point K_L appears on the imaginary axis in the a_o plane, we are compelled to define this branch point as an improper singularity. Applying this result, equation (6.1) becomes

$$G'(r,z,h,\omega) = \frac{i^{3/2} e^{i\pi/4} e^{-\alpha_T(h+z)} (2\alpha_T)^{1/2} 16m\pi^{1/2} \alpha_T a_L^2}{(h+z) K_T^2 (\alpha_T m + a_L)^2} - \frac{e^{-iK_o(h+z)}}{(h+z)} \left[\frac{\rho_1 c_L - \rho_o c_o}{\rho_1 c_L + \rho_o c_o}\right] \qquad (6.3)$$

The complete Green's Function, obtained by adding Sommerfeld's result for the direct wave contribution (given in equation (1.4)) reads as follows:

$$G(r,z,h,\omega) = \frac{1}{4\pi}\left\{\frac{e^{-iK_o(h-z)}}{(h-z)} - \frac{e^{-iK_o(h-z)}}{h+z}\left[\frac{\rho_1 c_L - \rho_o c_o}{\rho_1 c_L + \rho_o c_o}\right] - \frac{e^{-\alpha_T(h+z)}\, 16\sqrt{2}\, m^{1/2}\, \alpha_T^{3/2}\, a_L^2}{(h+z)K_T^2(\alpha_T m + a_L)^2}\right\} \tag{6.4}$$

Equation (6.4) represents the lowest order form of the Green's Function. In order to determine elastic versus viscoelastic effects in subsequent computer analyses, equation (6.4) must be separated into real and imaginary components (see Appendix E). The result for the elastic contribution is given by

$$G_E = \frac{1}{4\pi}\left\{\frac{e^{-iK_o(h-z)}}{(h-z)} - \frac{e^{-iK_o(h+z)}}{(h-z)}\left[\frac{(\rho_1\omega)^2 - (\rho_o c_o K_{Lo})^2}{(\rho_1\omega + \rho_o c_o K_{Lo})^2}\right] - \right.$$

$$\frac{e^{-\alpha_T(h+z)}}{} \frac{16\sqrt{2}\, m^{1/2}\, \beta_2^2 \beta_1^{3/2}\, K_{To}(m\beta_1+\beta_2)}{[K_{To}(m\beta_1+\beta_2)^2]}$$

and for the viscoelastic

$$G_V = \frac{i\varepsilon}{4\pi}\left\{- \frac{e^{-iK_o(h+z)}}{(h+z)}\left[\frac{2\omega\rho_1\rho_o c_o K_L'}{(\rho_1\omega+\rho_o c_o K_{Lo})^2}\right]\right.$$

$$+ \frac{e^{-\alpha_T(h+z)}\, 16\sqrt{2}\, m^{1/2}\pi\, \beta_2[K_{To}(m\beta_1+\beta_2)(2\beta_1^{3/2}\beta_3 - \frac{3}{2}\beta_2 K_{To}K_T']}{(h+z)[K_{To}(m\beta_1+\beta_2)]^3}$$

$$\left. - \frac{e^{-\alpha_T(h+z)}\, 32\sqrt{2}\, m^{1/2}\pi\, \beta_1^{3/2}\, \beta_2^2[K_T'(m\beta_1+\beta_2)+K_{To}(\beta_3+mK_{To}K_T')]}{(h+z)[K_{To}(m\beta_1+\beta_2)]^3}\right\}$$

CHAPTER III

RESULTS AND DISCUSSION

The expression for the actual Green's Function characterizing a semi-infinite liquid overlying a viscoelastic halfspace (for the special case of normal incidence) has been determined. The resultant Fourier integral is expressed as the sum of a direct wave contribution, a branch line integration, and an imaginary axis integral. The branch line integral was evaluated and the result is expressed as a series of Gamma Functions (equation 4.18). Results for the branch line integration concur with those of Honda and Nakamura [5], except that in our case there is no radial dependence and the wave numbers of the viscoelastic field are complex. The integral along the imaginary axis was shown to be proportional to the plane wave reflection coefficient $[\frac{\rho_1 c_L - \rho_o c_o}{\rho_1 c_L + \rho_o c_o}]$. The phase velocity of the Stoneley waves at the interface was determined using perturbation techniques. The result was that the Stoneley waves propagated at a speed equal to that of the transverse shear waves. The analytic determination of the Stoneley waves may suggest a basis for computer analysis of the Stoneley wave equation.

The Green's Function obtained in this thesis clearly indicates the feasibility of classifying subbottom sediments in terms of their physical parameters. We have determined the properties of the system and must now analyze the various outputs. The theoretical model employed in this thesis has been closely governed by the experimental viewpoint, since normal incidence testing may be accomplished from a moving research vessel. Recent computer analyses at the University of New Hampshire indicate that the normal incidence case may be valid for incidence angles as large as 18°.

Subsequent analyses should account for a corrugated interface and inhomogeneities in the viscoelastic medium. These generalizations may be introduced using statistical methods and perturbation theory. Using computer analysis and the work of Magnuson and Stewart [8], the model should be modified to account for the effects of an unlimited number of layers.

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APPENDIX A

VANISHING INTEGRAND ALONG THE QUADRANT

Along the quadrant in Figure 2 the exponential of the integrand in equation (2.3) of the text is simplified as follows:

$$e^{-a_o(z+h)} = e^{-\xi(z+h)} = e^{-R(\cos\theta+i\sin\theta)(z+h)} = e^{-R\cos\theta(z+h)}\, e^{-iR\sin\theta(z+h)} \qquad \text{(A-1)}$$

As R is taken to infinity, $e^{-iR\sin\theta(z+h)}$ would represent a rapidly oscillating function with self-cancelling contributions. Simultaneously, the $e^{-R\cos\theta(z+h)}$ term converges to zero. Thus we may conclude there is no contribution to the integral along the quadrant.

APPENDIX B

ANALYTICAL DETERMINATION OF STONELEY WAVE VELOCITY USING PERTURBATION TECHNIQUES

We note that for our particular case of interest, as seen from the values in Table 1, the following inequalities hold:

$$c_L > c_o > c_T \qquad \text{(B-1a)}$$

$$\mu_1 << \lambda_1 \qquad \text{(B-1b)}$$

From equation (B-1b) we define the small parameter

$$\varepsilon_1 = \frac{\mu_1}{\lambda_1} \qquad \text{(B-2)}$$

Recalling the expressions for the propagation velocities

$$c_o^2 = \frac{\lambda_o}{\rho_o} \qquad \text{(B-3a)}$$

$$c_L^2 = \frac{\lambda_1 + 2\mu_1}{\rho_1} \qquad \text{(B-3b)}$$

$$c_T^2 = \frac{\mu_1}{\rho_1} \qquad \text{(B-3c)}$$

The ratio of the squares of the propagation velocities in the viscoelastic medium may be approximated as follows:

$$\frac{c_T^2}{c_L^2} = \frac{\mu_1}{\lambda_1 + \mu_1} = \frac{\lambda_1\varepsilon_1}{\lambda_1 + 2\lambda_1\varepsilon_1} \simeq \varepsilon_1 \qquad \text{(B-4)}$$

or

$$c_T^2 = \varepsilon_1 c_L^2 \qquad \text{(B-4a)}$$

Using the results of Hamilton [4] in Table 1 the compressional wave velocities in the two media may be related by a second perturbation parameter as follows:

$$c_L^2 = c_o^2(1+\varepsilon_2) \qquad \text{(B-5)}$$

Table 2 contains explicit values of the perturbation parameters ε_1 and ε_2 for the sediments considered in Table 1. Also included are the corresponding Stoneley wave velocities as obtained from the experimental data of Strick and Ginsbarg [12]. We recall that the modal equation at the pole is given by

$$0 = m(1-(\frac{c}{c_o})^2)^{1/2}[(2-(\frac{c}{c_T})^2)^2-4(1-(\frac{c}{c_L})^2)^{1/2}(1-(\frac{c}{c_T})^2)^{1/2}]+(\frac{c}{c_T})^4(1-(\frac{c}{c_L})^2)^{1/2} \qquad \text{(B-6)}$$

We may formally apply perturbation theory to the problem by approximating the square of the phase velocity as follows:

$$c^2 = c'^2 + \varepsilon_1 c''^2 \qquad \text{(B-7)}$$

From equation (B-4a) we note that the zeroth order phase velocity is obtained from equation (B-6) by setting $c_T^2 = 0$. The modal equation reduces to

$$m(1-(\frac{c'}{c_o})^2)^{1/2} + (1-(\frac{c'}{c_L})^2)^{1/2} = 0 \qquad \text{(B-8a)}$$

or

$$m = \frac{-(1-(\frac{c'}{c_L})^2)^{1/2}}{(1-(\frac{c'}{c_o})^2)^{1/2}} \qquad \text{(B-8b)}$$

Recalling that $c_L > c_o$, we consider the various possible values for

TABLE 2. Perturbation Parameters and Stoneley Wave Velocities for North Pacific Sediments

(From Hamilton [4] and Strick and Ginsbarg [12].)

Sediment Type	ϵ_1	ϵ_2	$c_{STONELEY}$(m/sec)
Sand:			
Coarse	.0184	.5100	224
Fine	.0485	.3560	341
Very Fine	.0865	.2880	445
Silty Sand	.0750	.2450	405
Sandy Silt	.0595	.0755	330
Sand-Silt-Clay	.0649	.1100	358
Clayey Silt	.0430	.0489	314
Silty Clay	.0358	.0267	248

c' as follows:

(A) If $c' > c_L$, the right-hand side of equation (B-8b) is negative.

(B) If $c' < c_o$, the right-hand side of equation (B-8b) is negative.

(C) If $c_o < c' < c_L$, the right-hand side of equation (B-8b) is imaginary.

Since m is a density ratio, each of these possibilities represents a physically impossible situation. We conclude that the zeroth order phase velocity does not exist and equation (B-7) reduces to

$$c^2 = \varepsilon_1 c''^2 \tag{B-9}$$

Substituting equations (B-4a), (B-5), and (B-9) into the modal equation (B-6) will yield:

$$0 = m(1-\varepsilon_1(1+\varepsilon_2)(\frac{c''}{c_L})^2)^{1/2}[(2-(\frac{c''}{c_L})^2)^2-4(1-\varepsilon_1(\frac{c''}{c_L})^2)^{1/2}(1-(\frac{c''}{c_L})^2)^{1/2}]$$
$$+ (\frac{c''}{c_L})^4(1-\varepsilon_1(\frac{c''}{c_L})^2)^{1/2}$$

Applying the binomial theorem to this result we obtain

$$0 = m(1-\frac{\varepsilon_1+\varepsilon_1\varepsilon_2}{2}(\frac{c''}{c_L})^2)[(2-(\frac{c''}{c_L})^2)^2-4(1-\frac{\varepsilon_1}{2}(\frac{c''}{c_L})^2)(1-(\frac{c''}{c_L})^2)^{1/2}]$$
$$+ (\frac{c''}{c_L})^4(1-\frac{\varepsilon_1}{2}(\frac{c''}{c_L})^2) \tag{B-10}$$

Equating each order in equation (B-10) to zero will give the following results:

$$m[(2-(\frac{c''}{c_L})^2)^2-4(1-(\frac{c''}{c_L})^2)^{1/2}] = -(\frac{c''}{c_L})^4 \tag{B-11a}$$

$$\varepsilon_1[-\frac{m}{2}(\frac{c''}{c_L})^2[(2-(\frac{c''}{c_L})^2)^2-4(1-(\frac{c''}{c_L})^2)^{1/2}]+2m(\frac{c''}{c_L})^2(1-(\frac{c''}{c_L})^2)^{1/2}-\frac{1}{2}(\frac{c''}{c_L})^6] = 0 \tag{B-11b}$$

Combining equations (B-11a) and (B-11b) we obtain

$$c'' = c_L \tag{B-11c}$$

Applying this result, it follows from equations (B-8c) and (B-4a) that $c = c_T$ to first order in ε .

APPENDIX C

DETERMINATION OF THE LOWEST ORDER CONSTANT FOR THE DISCONTINUITY ACROSS BRANCH LINE L_1

The discontinuity across the branch line L_1 is given by the following relation

$$F(a_o^2) = F((\alpha_L - i\eta_L)^2) = \frac{1}{a_L}\left[\frac{N(a_o^2,-a_L)}{D(a_o^2,-a_L)} - \frac{N(a_o^2,a_L)}{D(a_o^2,a_L)}\right] = A + B\eta_L + C\eta_L^2 + \cdots \tag{C-1}$$

It should be clear that the lowest order constant A is determined by setting $\eta_L = 0$. It follows that

$$F(a_o^2) = F(\alpha_L^2) = A = \frac{1}{a_L}\left[\frac{N(-a_L)}{D(-a_L)} - \frac{N(a_L)}{D(a_L)}\right] = \frac{1}{a_L}\left[\frac{N(-a_L)D(a_L) - N(a_L)D(-a_L)}{D(-a_L)D(a_L)}\right] \tag{C-2}$$

Recalling that $\xi = K_L$ at the branch point, we may write

$$N(a_L) = \alpha_L m[(2K_L^2-K_T^2)^2 - 4a_L a_T K_L^2] - K_T^4 a_L \tag{C-3a}$$

$$D(a_L) = \alpha_L m[(2K_L^2-K_T^2)^2 - 4a_L a_T K_L^2] + K_T^4 a_L \tag{C-3b}$$

$$N(-a_L) = \alpha_L m[(2K_L^2-K_T^2)^2 + 4a_L a_T K_L^2] + K_T^4 a_L \tag{C-3c}$$

$$D(-a_L) = \alpha_L m[(2K_L^2-K_T^2)^2 + 4a_L a_T K_L^2] - K_T^4 a_L \tag{C-3d}$$

It follows that

$$N(-a_L)D(a_L) = \{(\alpha_L m)^2[(2K_L^2-K_T^2)^4 - 16a_L^2 a_T^2 K_L^4] + 2\alpha_L m a_L K_T^4(2K_L^2-K_T^2)^2 + K_T^8 a_L^2\} \tag{C-4a}$$

$$N(a_L)D(-a_L) = \{(\alpha_L m)^2[(2K_L^2-K_T^2)^4 - 16a_L^2 a_T^2 K_L^4] - 2\alpha_L m a_L K_T^4(2K_L^2-K_T^2)^2 + K_T^8 a_L^2\} \tag{C-4b}$$

$$D(-a_L)D(a_L) = \{(\alpha_L m)^2[(2K_L^2-K_T^2)^4 - 16a_L^2 a_T^2 K_L^4] - 8\alpha_L m a_L^2 a_T K_L^2 K_T^4 - K_T^8 a_L^2\} \tag{C-4c}$$

Substituting equations (C-4a-c) into equation (C-2) will yield:

$$A = \frac{4\alpha_L m K_T^4 (2K_L^2 - K_T^2)^2}{(\alpha_L m)^2 [(2K_L^2 - K_T^2)^4 - 16a_L^2 a_T^2 K_L^4] - 8\alpha_L m a_L^2 a_T K_L^2 K_T^4 - K_T^8 a_L^2} \qquad \text{(C-5)}$$

We note that at the branch point

$$a_L = (-2i\eta_L a_L - \eta_L^2)^{1/2} = 0 \qquad \text{(C-6)}$$

Applying this result, equation (C-5) reduces to

$$A = A_1' = \frac{4K_T^4}{\alpha_L m (2K_L^2 - K_T^2)^2}$$

APPENDIX D

DETERMINATION OF THE LOWEST ORDER CONSTANT FOR THE DISCONTINUITY ACROSS BRANCH LINE L_3

The procedure used to determine the lowest order constant for $F(\alpha_T^2)$ is identical to that used in Appendix C. The discontinuity is given by

$$F(\alpha_T^2) = \frac{1}{a_T}\left[\frac{N(-a_T)D(a_T) - N(a_T)D(-a_T)}{D(-a_T)D(a_T)}\right] \tag{D-1}$$

Recalling that $\xi = K_T$ at the branch point, we write

$$N(a_T) = \alpha_T m[K_T^4 - 4a_L a_T K_T^2] - K_T^4 a_L \tag{D-2a}$$

$$N(-a_T) = \alpha_T m[K_T^4 + 4a_L a_T K_T^2] - K_T^4 a_L \tag{D-2b}$$

$$D(a_T) = \alpha_T m[K_T^4 - 4a_L a_T K_T^2] + K_T^4 a_L \tag{D-2c}$$

$$D(-a_T) = \alpha_T m[K_T^4 + 4a_L a_T K_T^2] + K_T^4 a_L \tag{D-2d}$$

It follows that

$$N(-a_T)D(a_T) = \{(\alpha_T m)^2 [K_T^8 - 16a_L^2 a_T^2 K_T^4] - 8\alpha_T m a_T a_L^2 K_T^6 - K_T^8 a_L^2\} \tag{D-3a}$$

$$N(a_T)D(-a_T) = \{(\alpha_T m)^2 [K_T^8 - 16a_L^2 a_T^2 K_T^4] + 8\alpha_T m a_T a_L^2 K_T^6 - K_T^8 a_L^2\} \tag{D-3b}$$

$$D(-a_T)D(a_T) = \{(\alpha_T m)^2 [K_T^8 - 16a_L^2 a_T^2 K_T^4] + 2\alpha_T m a_L K_T^8 + K_T^8 a_L^2\}$$

Substituting equations (D-3a-c) into equation (D-1), and noting that $a_T = 0$ at the branch point we obtain

$$B_1' = \frac{-16\alpha_T m a_L^2}{K_T^2(\alpha_T m + a_L)^2} \tag{D-4}$$

APPENDIX E

SEPARATION OF GREEN'S FUNCTION INTO ELASTIC AND VISCOELASTIC COMPONENTS

The Green's Function characterizing a semi-infinite liquid overlying a viscoelastic halfspace (for normal incidence) is given by the following expression:

$$G(r,z,h,\omega) = \frac{1}{4\pi}\left\{\frac{e^{-iK_o(h-z)}}{(h-z)} - \frac{e^{-iK_o(h+z)}}{h+z}\left[\frac{\rho_1 c_L - \rho_o c_o}{\rho_1 c_L + \rho_o c_o}\right] - \frac{e^{-\alpha_T(h+z)}\,16\sqrt{2}m\pi^{1/2}\alpha_T^{3/2}a_L^2}{(h+z)K_T^2(\alpha_T m + a_L)^2}\right\} \quad \text{(E-1)}$$

For small damping, we may set

$$K_L = K_{Lo} - i\epsilon K_L' \quad \text{(E-2a)}$$

$$\xi = K_T = K_{To} - i\epsilon K_T' \quad \text{(E-2b)}$$

It follows that

$$\alpha_T = (K_T^2 - K_o^2)^{1/2} \simeq (K_{To}^2 - K_o^2)^{1/2} - i\epsilon K_{To}K_T' \quad \text{(E-3a)}$$

$$a_L = (K_T^2 - K_L^2)^{1/2} \simeq (K_{To}^2 - K_{Lo}^2)^{1/2} - i\epsilon(K_{To}K_T' - K_{Lo}K_L') \quad \text{(E-3b)}$$

$$a_L^2 = (K_{To}^2 - K_{Lo}^2)^{1/2} \simeq 2i\epsilon(K_{To}^2 - K_{Lo}^2)^{1/2}(K_{To}K_T' - K_{Lo}K_L') \quad \text{(E-3c)}$$

$$K_L^2 = K_{Lo}^2 - 2i\epsilon K_{Lo}K_L' \quad \text{(E-3d)}$$

$$K_T^2 = K_{To}^2 - 2i\epsilon K_{To}K_T' \quad \text{(E-3e)}$$

To minimize algebraic complexity, we define the following functions:

$$\beta_1 = (K_{To}^2 - K_o^2)^{1/2} \tag{E-4a}$$

$$\beta_2 = (K_{To}^2 - K_{Lo}^2)^{1/2} \tag{E-4b}$$

$$\beta_3 = (K_{To}K_T' - K_{Lo}K_L') \tag{E-4c}$$

In separating the Green's Function we must analyze each term. The first term of equation (E-1) represents the direct wave contribution for which there is no damping. The plane wave reflection coefficient in the second term of equation (E-1) is separated into real and imaginary components as follows:

$$\frac{\rho_1 c_L - \rho_o c_o}{\rho_1 c_L + \rho_o c_o} = \frac{(\rho_1\omega)^2 - (\rho_o c_o K_{Lo})^2}{(\rho_1\omega + \rho_o c_o K_{Lo})^2} + \frac{2i\omega\rho_1\rho_o c_o K_L'}{(\rho_1\omega + \rho_o c_o K_{Lo})^2} \tag{E-5}$$

Using equations (E-2), (E-3), and (E-4), the last term of equation (E-1) is simplified as follows:

$$\frac{-e^{-\alpha_T(h+z)}\, 16\sqrt{2}m\pi^{1/2}\, \alpha_T^{3/2}\, a_L^2}{(h+z)K_T^2(\alpha_T m + a_L)^2}$$

$$= \frac{-e^{-\alpha_T(h+z)}\, 16\sqrt{2}m\pi^{1/2}\, (\beta_1 - i\varepsilon K_{To}K_T')^{3/2}\, (\beta_2^2 - 2i\varepsilon\beta_2\beta_3)}{(h+z)(K_{To}^2 - 2i\varepsilon K_{To}K_T')[(m\beta_1 + \beta_2) - i\varepsilon(\beta_3 + mK_{To}K_T')]^2}$$

$$= \frac{-e^{-\alpha_T(h+z)}\,16\sqrt{2}m\pi^{1/2}\,[\beta_1^{3/2}\beta_2^2 - i\epsilon(2\beta_1^{3/2}\beta_2\beta_3 - \frac{3}{2}\beta_2^2 K_{To}K_T')]}{(h+z)(K_{To}^2(m\beta_1+\beta_2)^2 - 2i\epsilon[K_{To}K_T'(m\beta_1+\beta_2)^2 + K_{To}^2(m\beta_1+\beta_2)(\beta_3+mK_{To}K_T')])}$$

$$= \frac{-e^{-\alpha_T(h+z)}\,16\sqrt{2}m\pi^{1/2}\,\beta_2\{\beta_1^{3/2}\beta_2K_{To}(m\beta_1+\beta_2) - i\epsilon[K_{To}(m\beta_1+\beta_2)(2\beta_1^{3/2}\beta_3 - \frac{3}{2}\beta_2K_{To}K_T') - 2\beta_1^{3/2}\beta_2(K_T'(m\beta_1+\beta_2)+K_{To}(\beta_3+mK_{To}K_T'))]\}}{(h+z)[K_{To}(m\beta_1+\beta_2)]^3} \quad \text{(E-6)}$$

Applying the results of equations (E-5) and (E-6) to equation (E-1), we may designate the elastic component of the Green's Function as:

$$G_E = \frac{1}{4\pi}\{\frac{e^{-iK_o(h-z)}}{(h-z)} - \frac{e^{-iK_o(h+z)}}{(h+z)}[\frac{(\rho_1\omega)^2-(\rho_oc_oK_{Lo})^2}{(\rho_1\omega+\rho_oc_oK_{Lo})^2}] - \frac{e^{-\alpha_T(h+z)}\,16\sqrt{2}m\pi^{1/2}\,\beta_2^2\beta_1^{3/2}K_{To}(m\beta_1+\beta_2)}{(h+z)[K_{To}(m\beta_1+\beta_2)]^3}\} \quad \text{(E-7)}$$

Similarly, the viscoelastic contribution is expressed by the following relationship:

$$G_V = \frac{i\epsilon}{4\pi}\{-\frac{e^{-iK_o(h+z)}}{(h+z)}[\frac{2\omega\rho_1\rho_oc_oK_L'}{(\rho_1\omega+\rho_oc_oK_{Lo})^2}]$$

$$+\frac{e^{-\alpha_T(h+z)}\,16\sqrt{2}m\pi^{1/2}\,\beta_2[K_{To}(m\beta_1+\beta_2)(2\beta_1^{3/2}\beta_3 - \frac{3}{2}\beta_2K_{To}K_T') - 2\beta_1^{3/2}\beta_2(K_T'(m\beta_1+\beta_2)+K_{To}(\beta_3+mK_{To}K_T'))]}{(h+z)[K_{To}(m\beta_1+\beta_2)]^3}\} \quad \text{(E-8)}$$

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