The Nonlinear Dynamics Of Cable Systems

by

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Abstract

A method is developed for calculating the dynamic response of towed-cable systems. This method, which is designed to provide realistic computational times, retains important nonlinearities but uses simplified equations of longitudinal motion.

General equations of motion including terms due to elasticity, bending and internal damping are derived. The method of characteristics, which is an especially attractive method for solving these equations, can be used only if the equations are hyperbolic. It is shown that the equations are hyperbolic only if elasticity is included and bending and internal damping are neglected, and further, that bending and internal damping can be safely neglected for most cases. The excessive computational times required by the large longitudinal characteristic velocity (cable material sonic velocity) can be avoided if only the transverse equations of motion are solved by the method of characteristics.

The equations of motion are simplified by a procedure similar to that used in deriving the boundary layer equations. When higher order terms are neglected, the longitudinal equations of motion reduce to linear equations which are essentially uncoupled from the nonlinear transverse equations of motions. These linear longitudinal equations of motion can be solved analytically for suitable linear boundary conditions. Solutions for two sets of boundary conditions are derived. The computational method is based on these analytical solutions and on solution of the transverse equations of motion using the method of characteristics and finite-difference integration.

A number of computed examples are presented. These serve to demonstrate the accuracy of the computer programs and to illustrate the effect of various parameters such as external damping and towing velocity on cable motions and tensions. The effect of two sensitive parameters, grid spacing for numerical integration and number of terms used to represent infinite series, are illustrated.

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INTRODUCTION

The dynamic analysis of cable and cable-towed body systems is currently receiving considerable attention, due to the wide range of applications of such systems. Important applications exist in oceanography, geology and underwater detection. The dynamics of cable systems is also closely related to dynamics of drill strings and single-leg moorings.

Dynamic motions of cable systems result from maneuvering or changes in speed of a towing vehicle, or from the effect of the ocean environment on the cable or towing vehicle. The analysis of such motions is difficult. The equations of motion are nonlinear, partial differential equations. The unrestricted motions of the two ends of the cable make the problem a two-point boundary-value problem. Nonlinear, two-point boundary-value problems can rarely be solved in closed form.

Many methods have been and are being used to predict the dynamic behavior of cable systems. Casarella and Parsons (1) and Wang (2) present good summaries of various methods which have been used. These methods can be divided into two classes; those in which the cable is treated as a continuum and those in which the cable is represented by a series of discrete, rigid links. Each of these approaches has attractive features.

The present research is motivated by a desire to provide a method which is suitable for the analysis of nonlinear or large motions and which requires reasonable computational times.

Existing methods can handle highly nonlinear motions, but required computational times are generally large (see Reference 2). A second motivation is to develop a method which can be used to study the possibility of large, "shock-like" disturbances,which can result with hyperbolic equations. The present research is aimed at developing a method which includes all, or most, of the inherent nonlinearities and which gives solutions in reasonable computational times.

In order to study nonlinear behavior, it is important to consider all external and internal forces and moments. Bending rigidity, rotational acceleration and internal damping, which are usually neglected must be included in the equations of motion or shown to be negligible. In the present investigation each of these factors is considered in detail.

For the analysis of highly nonlinear motions, a rigid-link or finite-element representation of the cable may not be suitable unless a very large number of links are considered, and the compatibility relations between elements are fairly sophisticated. As the numbers of elements and their sophistication increases, the computational advantages of the finite-element method disappear. The treatment of the cable as a continuum offers the advantage of greater flexibility and this representation is used in the present investigation.

In the present investigation only two-dimensional (planar) cable-body systems are considered. The methods can be readily extended to arbitrary, three-dimensional systems, although computations will be considerably more difficult and time consuming.

FORMULATION OF THE EQUATIONS OF MOTION

In this section all forces and moments which can act on a cable system are discussed and described mathematically. The complete equations of motion for two-dimensional cable systems are derived including all of these forces and moments. Simplified equations of motion in which cable extensibility and other factors are neglected are also considered. In the next sections methods for solving these equations are considered.

Equations of Motion for Two-Dimensional Cable Systems

The motions in two-dimensions of a cable or cable-towed body are governed by three Newtonian equations of motion. These equations are, for an element of differential length, Δs:

$$m a_j \Delta s - \sum X'_{ej} \Delta s - \sum X'_{ij} \Delta s = 0 \quad j = 1,2 \tag{1}$$

$$K_\phi a_r - \sum M_e' \Delta s - \sum M_i' \Delta s = 0$$

where m is the mass per unit length of cable,

a_j is the acceleration in the j'th direction,

K_ϕ is the mass moment of inertia per unit length,

a_r is the rotational acceleration,

X_e' and X_i' are the external and internal forces per unit length,

M_e' and M_i' are the external and internal moments per unit length,

Δs is a differential length of cable,

j=1,2 denotes two orthogonal directions in the plane of the cable.

The various forces and moments acting on an element of length are shown in Figure 1.

The problem is most easily formulated for extensible cables in Lagrangian coordinates. For inextensible cables the formulations in Lagrangian and Eulerian coordinates are clearly identical. The equations of motion for inextensible cables are easily derived from those for extensible cables. Equations of motion are first derived in a fixed, orthogonal (x,y) coordinate system. The equations are then transformed to the "natural" (n,s) coordinate system having coordinates which are everywhere normal to and tangent to the cable. These coordinates are shown in Figures 1 and 2.

Derivation of Equations of Motion with Extensibility

Equations of motion with extensibility have been given by Craggs (3), Critescu (4) and others. Completely general equations of motion in the x,y and ϕ or $\bar{s},\bar{n}$ and ϕ directions can be derived from equations (1) with the aid of Figures 1, 2 and 3. The equations in the x,y and ϕ directions are as follows:

x-direction:

$$m_0 \frac{\partial^2 \delta_x}{\partial t^2} \Delta s_0 + T\cos\phi - \left(T + \frac{\partial T}{\partial s_0}\Delta s_0\right)\cos\left(\phi + \frac{\partial \phi}{\partial s_0}\Delta s_0\right) -$$
$$- Q\sin\phi + \left(Q + \frac{\partial Q}{\partial s_0}\Delta s_0\right)\sin\left(\phi + \frac{\partial \phi}{\partial s_0}\Delta s_0\right) - S\cos\left(\phi + \frac{1}{2}\frac{\partial \phi}{\partial s_0}\Delta s_0\right)\Delta s_0 +$$
$$+ N\sin\left(\phi + \frac{1}{2}\frac{\partial \phi}{\partial s_0}\Delta s_0\right)\Delta s_0 + F_{i_d}\cos\phi - \left(F_{i_d} + \frac{\partial F_{i_d}}{\partial s_0}\Delta s_0\right)\cos\left(\phi + \frac{\partial \phi}{\partial s_0}\Delta s_0\right) = 0$$

y-direction:

$$m_o \frac{\partial^2 \delta_y}{\partial t^2} \Delta s_o + T \sin\phi - \left(T + \frac{\partial T}{\partial s_o} \Delta s_o\right) \sin\left(\phi + \frac{\partial \phi}{\partial s_o} \Delta s_o\right) +$$

$$+ Q \cos\phi - \left(Q + \frac{\partial Q}{\partial s_o} \Delta s_o\right) \cos\left(\phi + \frac{\partial \phi}{\partial s_o} \Delta s_o\right) - S \Delta s_o \times$$

$$\times \sin\left(\phi + \frac{1}{2} \frac{\partial \phi}{\partial s_o} \Delta s_o\right) + N \Delta s_o \cos\left(\phi + \frac{1}{2} \frac{\partial \phi}{\partial s_o} \Delta s_o\right) +$$

$$+ F_{i_d} \sin\phi - \left(F_{i_d} + \frac{\partial F_{i_d}}{\partial s_o} \Delta s_o\right) \sin\left(\phi + \frac{\partial \phi}{\partial s_o} \Delta s_o\right) +$$

$$+ w_m \Delta s_o = 0$$

ϕ-direction:

$$K_\phi \frac{\partial^2 \phi}{\partial t^2} \Delta s_o + M_b - \left(M_b + \frac{\partial M_b}{\partial s_o} \Delta s_o\right) - Q \frac{\Delta s_o}{2} + \left(Q + \frac{\partial Q}{\partial s_o} \Delta s_o\right) \frac{\Delta s_o}{2} +$$

$$+ \frac{1}{2}\left(T + \frac{\partial T}{\partial s_o} \Delta s_o\right) \frac{\partial \phi}{\partial s_o} \frac{\Delta s_o^2}{2} - \frac{1}{2} T \frac{\partial \phi}{\partial s_o} \frac{\Delta s_o^2}{2} + M_{i_d} -$$

$$- \left(M_{i_d} + \frac{\partial M_{i_d}}{\partial s_o} \Delta s_o\right) = 0$$

Where T is the tension in the cable,

Q is the shear in the cable,

S and N are the tangential and normal components of the external forces,

F_{i_d} is the internal damping force,

δ_x and δ_y are the initial (Lagrangian) displacements inthe x and y directions,

m_o is the mass per unit length of the cable,

w_n is the net weight per unit length, equal to cable weight less fluid buoyancy per unit length,

ϕ is the angle of the cable to the x axis,

M is the bending moment in the cable,

M_{i_d} is the internal damping moment,

K_ϕ is the mass moment of inertia per unit length.

Dividing these equations by Δs_o, taking the limit as $\Delta s_o \to o$, and simplifying:

$$m_o \frac{\partial^2 \delta_x}{\partial t^2} - \frac{\partial}{\partial s_o}(T \cos\phi) + \frac{\partial}{\partial s_o}(Q \sin\phi) - S\cos\phi - N\sin\phi - \frac{\partial}{\partial s_o}(F_{i_d}\cos\phi) = 0 \tag{2}$$

$$m_o \frac{\partial^2 \delta_y}{\partial t^2} - \frac{\partial}{\partial s_o}(T \sin\phi) - \frac{\partial}{\partial s_o}(Q \cos\phi) - S\sin\phi + N\cos\phi - \frac{\partial}{\partial s_o}(F_{i_d}\sin\phi) + \omega_n = 0 \tag{3}$$

$$K_\phi \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial M_b}{\partial s_o} - Q - \frac{\partial M_{i_d}}{\partial s_o} = 0 \tag{4}$$

The displacements δ_x and δ_y in equations (1) and (2) can be eliminated by the method proposed by Craggs (3). The displacements can be defined, with the aid of Figure 3, as

$$\frac{\partial \delta_x}{\partial s_o} = r\cos\phi - \cos\phi_o \tag{5}$$

$$\frac{\partial \delta_y}{\partial s_o} = r\sin\phi - \sin\phi_o \tag{6}$$

where ϕ_o is the initial value of the angle ϕ, and r is the ratio of strained to unstrained length, $r = \varepsilon + 1 = \partial s / \partial s_o$. Suitable differentiation of equations (2), (3), (5), and (6) leads to two equations which do not contain

δ_x and δ_y. This method is not suitable if *N* or *S* contains nonlinear terms in δ_x and δ_y or their time derivatives. As quadratic, hydrodynamic damping terms are of this type, this method cannot be used here.

The displacements in equations (2) and (3) can be replaced by cable velocities *u* and *v*, and *u* and *v* retained as primary variables. Equations (2) and (3) can be resolved into equations of motion in the *n* and *s* directions:

<u>*n*-direction</u>: (2) $\sin\phi$ - (3) $\cos\phi \Rightarrow$

$$m_o\left(\frac{\partial u}{\partial t}\sin\phi + \frac{\partial v}{\partial t}\cos\phi\right) + T\frac{\partial\phi}{\partial s_o} + \frac{\partial Q}{\partial s_o} - N - w_n\cos\phi = 0 \qquad (7)$$

<u>*s*-direction</u>:

$$m_o\left(\frac{\partial u}{\partial t}\cos\phi + \frac{\partial v}{\partial t}\sin\phi\right) - \frac{\partial T}{\partial s_o} + Q\frac{\partial\phi}{\partial s_o} - \frac{\partial F_{4d}}{\partial s_o} - S + w_n\sin\phi = 0 \qquad (8)$$

where $\Upsilon = \partial r/\partial t$

The velocities in the normal and tangential direction, *U* and *V*, as defined in Figure 3 are related to *u* and *v* by:

$$\begin{aligned} U &= u\sin\phi - v\cos\phi \\ V &= u\cos\phi + v\sin\phi \end{aligned} \qquad (9)$$

where *U* and *V* are the cable velocities in the normal *(n)* and tangential *(s)* directions.

Differentiating these expressions with respect to time and substituting:

$$\frac{\partial u}{\partial t}\cos\phi + \frac{\partial v}{\partial t}\sin\phi = U\frac{\partial\phi}{\partial t} + \frac{\partial V}{\partial t} = \frac{\partial V_s}{\partial t} \qquad (10)$$

$$\frac{\partial u}{\partial t}\sin\phi - \frac{\partial v}{\partial t}\cos\phi = \frac{\partial U}{\partial t} - V\frac{\partial\phi}{\partial t} = \frac{\partial V_n}{\partial t} \qquad (11)$$

Substituting these values into equations (6) and (7), we obtain:

$$m_o\frac{\partial U}{\partial t} - m_o V\frac{\partial\phi}{\partial t} + T\frac{\partial\phi}{\partial s_o} + \frac{\partial Q}{\partial s_o} - N - w_m\cos\phi = 0 \qquad (12)$$

$$m_o\frac{\partial V}{\partial t} + m_o U\frac{\partial\phi}{\partial t} - \frac{\partial T}{\partial s_o} + Q\frac{\partial\phi}{\partial s_o} - \frac{\partial F_{id}}{\partial s_o} - S + w_m\sin\phi = 0 \qquad (13)$$

The necessary compatibility relationships for U, V, ϕ and r can be obtained by differentiating equations (5) and (6) with respect to time, rectifying to the n and s directions, differentiating equations (9) with respect to s_o and eliminating u and v:

$$\frac{\partial U}{\partial s_o} - V\frac{\partial\phi}{\partial s_o} + r\frac{\partial\phi}{\partial t} = 0 \qquad (14)$$

$$\frac{\partial V}{\partial s_o} + U\frac{\partial\phi}{\partial s_o} - \frac{\partial r}{\partial t} = 0 \qquad (15)$$

Equations (3) and (12) through (15), with a suitable constitutive equation for the cable material, are the basic equations of motion for the cable system. The six variables in these equations are U, V, T, Q, r and ϕ (the moment M_b in equation (4) is a function of ϕ). These equations, together with the equations of motion of the towed body and the specified motions of the towed point, describe the complete problem.

In order to use these equations it is necessary to prescribe the external forces S and N, the internal damping and moment M_{i_d} and F_{i_d} and a constitutive equation for the cable material.

Definition of Component Forces and Moments

The external forces S and N are composed of external damping and added mass forces. These forces must be described in terms of known coefficients and velocities and accelerations, U, V, $\partial u/\partial t$ and $\partial v/\partial t$. The appropriate terms are derived in this section.

The internal damping force and moment must be described in terms of extension, r, angle, ϕ, and their derivatives. Appropriate terms, based on a Maxwell (strain rate) model are given in this section.

The relationship between bending moment, M_b, and angle ϕ, based on simple beam bending is given. Finally, a linear constitutive equations is given.

External Hydrodynamic Forces. An oscillating cable is subject to external forces due to the added mass and drag of the cable. These forces are defined to be 180 degrees out of phase with the acceleration and velocity respectively. These forces are most easily described by components normal to and tangent to the cable. The cable is also subject to a gravitational force due to its weight.

The external hydrodynamic forces acting normal and tangent to an element of cable of length Δs_o can be written in general form:

$$\sum (F_e)_n = -m_{an}\frac{\partial V_n}{\partial t} - C'_{dn} f(U) = N \qquad (16)$$

$$\sum (F_e)_s = -m_{as}\frac{\partial V_s}{\partial t} - C'_{ds} f(V) = S \qquad (17)$$

where $(Fe)_n$ and $(Fe)_s$ are the external forces per unit length in the normal (n) and tangential (s) directions,

m_a is the added mass per unit length,

C_d is the drag coefficient,

$\partial V_n/\partial t$ and $\partial V_s/\partial t$ are the accelerations given by equations (10) and (11),

N is the total normal, external force per unit length,

S is the total tangential, external force per unit length.

and the subscripts n and s denote the normal and tangential directions. The functional representation of the velocity-dependent terms is discussed below.

An external force is exerted on the cable by the changes in the values of $(Fe)_n$ and $(Fe)_s$ along the length of the cable. It is shown later that such moments are of higher order and can be neglected.

Many different formulations for the damping coefficients have been proposed. Casarella and Parsons (1) give a good summary of these and show comparisons between a number of formulations and data. Figures 4 and 5 showing normal and tangential coefficients for cables of circular cross-section are taken from Reference (1). The normal force and force coefficient are given, with very good accuracy, by

$$C_{d_n} = C_R \sin^2\phi$$
$$D_n = \tfrac{1}{2}\rho\, d_c\, C|C|\, C_{d_n} = [\tfrac{1}{2}\rho\, d_c\, C_R]\, C|C| \sin^2\phi \qquad (18)$$
$$D_n = C_R'\, U|U|$$

where C_{d_n} is the coefficient of normal direction drag for oblique flow,

C_R is the drag coefficient for flow normal to the cable,

ϕ is the angle between the cable and the flow, (Figure 3),

D_n is the normal drag force per unit length of cable,

ρ is the fluid mass density,

C is the steady towing velocity,

d_c is the cable diameter,

U is the normal velocity component = $C \sin\phi$,

C_R' is the modified drag coefficient = $(\rho/2) d_c C_R$.

None of the expressions for tangential-drag coefficient are in very good agreement with available data, although those of Whicker (5) and Springston (6) give the best fits to available data. A simple expression:

$$C_{d_s} = k\, C_R \cos\phi \qquad (19)$$
$$D_s = k\, C_R'\, V|C|$$

where C_{d_s} is the coefficient of tangential drag for oblique flow,

D_s is the tangential drag force per unit length,

V is the tangential velocity component = $C \cos \phi$,

k' is a coefficient between 0.04 and 0.05,

gives a good average fit of the data in Figure 5 and leads to a purely linear external longitudinal damping. Equation (17) is therefore used in this investigation, although the expression of Whicker or Springston could also be used.

Added-mass forces are not considered by Casarella and Parsons. Based on added-mass data for simple bodies such as cylinders (7), the normal added-mass will be given by:

$$m_{an} = \rho a_c$$

where m_{a_n} is the added mass per unit length,

ρ is the mass density of water,

a_c is the cross-sectional area of the cable.

For cables of uniform cross-section, the tangential added-mass will be zero:

$$m_{as} = 0$$

The effect of extension on cable cross-sectional area will be minimal and can be neglected, so that the area a_c and diameter d_c can be taken as those for the unstrained cable.

All of the coefficients discussed above are based on tests in steady flow. Tests of oscillating cylinders, such as those of Keulegan and Carpenter (8), indicate that force coefficients can be much larger in unsteady flow. There is evidence that cable oscillations can cause larger damping than predicted from steady-flow data (9). In this investigation, however, steady-flow data are used.

Internal Damping. Internal damping and visco-elastic effects are not considered in the formulation of most physical problems. Most investigators of cable systems have neglected internal damping. Because internal damping is fundamentally different from external damping, it cannot be assumed a priori that the effect of internal damping will be negligible if external damping is included. Even small amounts of internal damping may be important. In this section suitable terms for modeling internal damping are considered.

There are many mathematical models for visco-elastic behavior (see, for example, Flügge (10)). A simple Maxwell model (10) seems appropriate for the present investigation. While the Maxwell model has been questioned since the time of Love (11), this model has proven adequate in the analysis of complex structures (see, for example, Penzien and Wilson (12)). The Maxwell model, which is assumed valid as long as the deformations of the cable remain elastic, gives a linear variation of damping force with time rate of strain:

$$f_{id} = k_i \frac{\partial \epsilon}{\partial t}$$

where f_{id} is the internal damping force per unit area,

k_i is an internal damping coefficient,

ε is the material strain = $(r-1)$.

The total internal damping force for a given cable cross-section is given by

$$F_{id} = -k_i\, a_c \frac{\partial r}{\partial t} = -k_i' \frac{\partial r}{\partial t} \tag{20}$$

where a_c is the cross-sectional area of the cable,

k_i' is the modified damping coefficient = $k_i a_c$,

r is the longitudinal-extension ratio.

The negative sign denotes that the damping force is 180 degrees out of phase with the strain rate. An internal damping moment due to bending strains will also occur. The bending strain is

$$\varepsilon_b = y\left(\frac{\partial \phi}{\partial s}\right) = \left(\frac{y}{r}\right)\frac{\partial \phi}{\partial s_o}$$

where ε_b is the bending strain at distance y from the cable cross-section neutral axis,

ϕ is the local cable inclination,

s is the Eulerian distance along the cable.

The internal damping moment per unit area is given by

$$m_{id} = -\left(K_I \frac{\partial^2 \phi}{\partial s \partial t}\right) y$$

and the total internal damping moment is given by

$$M_{id} = -\left(\frac{K_I}{r}\int_{a_c} y^2 da\right)\frac{\partial^2 \phi}{\partial s_o \partial t} = \frac{K_I}{r} I_c \frac{\partial^2 \phi}{\partial s_o \partial t}$$

and

$$M_{id} = -K_I' \frac{\partial^2 \phi}{\partial s_o \partial t} \tag{21}$$

where M_{i_d} is the internal damping moment,

K_I is the internal damping coefficient,

K_I' is the modified damping coefficient = $K_I I_c / r$,

I_c is the moment of inertial of the cable cross-section,

r is the longitudinal extension ratio = $1+\varepsilon$.

For realistic strains ($\varepsilon = 10^{-3}$ for steel) the effect of strain on the cable cross-section characteristics (a_c and I_c) and on the internal damping force and moment can be neglected. The value of r can be assumed equal to one in calculating cable geometric properties.

Cable Bending and Rotational Motions. Two-dimensional (planar) cable systems have three degrees of freedom, two translational and one rotational. These are represented by the variables U, V and ϕ in equations (3) and (12) through (15). Previous investigations of cable systems have neglected the bending mode, although Paidoussis (13) and Pao (14) have included bending in the formulation of a closely related problem. While changes in the angle ϕ are considered by all investigators, the cable is assumed to have no bending rigidity and shear.

Bending moments must be accompanied by transverse shear forces. The assumption of finite bending rigidity thus leads not only to an additional equation for rotational motions (equation (3)) but also to additional terms due to shear ($\partial Q/\partial s_o$ and $Q\partial\phi/\partial s_o$) in the equation of transverse and longitudinal motion. Assuming Navier bending, which is reasonable for cables which are like very slender beams, the moment is given by

$$M_b = EI_c \frac{\partial\phi}{\partial s_o} = K_E \frac{\partial\phi}{\partial s_o} \tag{22}$$

where M_b is the moment due to bending rigidity,
E is the modulus of elasticity of the cable material,
I_c is the moment of inertia,
K_E is the bending moment coefficient = EI_c/r^2.

It seems reasonable, as in the case of the internal damping, to neglect the effect of extensibility on bending moment and to approximate the coefficient K_E by

$$K_E = EI_c$$

where I_c is calculated for the unstrained cable cross-section.

Cable Extensibility. Elasticity or extensibility has been considered in the analysis of strings by Craggs (3) and Critescu (4) and of cable systems by Schram (15) and Nath (16), although it has been neglected in most analyses of cable-towed body systems. While Schram (15) considers extensibility, his calculations and subsequent calculations reported by Schram and Reyle (17) neglect extensibility. While physical extensions (strains) are very small (10^{-3} or less, typically, for steel cables), the omission of extensibility has a fundamental effect on the behavior of the system.

If extensibility is neglected $(r \equiv 1)$, the equations governing longitudinal motions (equations (13) and (15)) are not hyperbolic and the method of characteristics cannot be used to solve these equations (2). Whicker (5), who was the first to apply the method of characteristics to cable systems, and many others did not appreciate this fact, although Whicker found no characteristic values corresponding to the longitudinal equations. In order to use the method of characteristics, it is essential to include extensibility in the formulation of the equations.

For elastic cable materials, a linear compatibility relationship can be used:

$$\varepsilon = (r-1) = \sigma_t / E = T / a_c E$$

where ε is the tensile strain,

r is the extension ratio, $r = \varepsilon + 1$,

σ_t is the tensile (compressive) stress,

a_c is the cable cross-sectional area,

T is the tension (or compression).

For a given material, the tension is a function only of the strain or extension

$$T = T(r) = E\, a_c (r-1) \quad . \tag{23}$$

Partial derivatives of tension can thus be replaced by partial derivatives of r:

$$\frac{\partial T}{\partial s_o} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial s_o} = \left(\frac{dT}{dr}\right)\frac{\partial r}{\partial s_o} \quad . \tag{24}$$

Equations (23) and (24) are both valid forms of the required material constitutive relationship.

Complete Equations of Motion

The equations of motion are put in usable form by combining equations (12) and (13) with equations (16) through (20) and combining equations (3), (21) and (22). The resulting equations can be reduced to a set of first-order equations by suitable changes of variable.

Substituting equations (16) and (18) into equation (12) yields:

$$m_m \frac{\partial U}{\partial t} - m_m V \frac{\partial \phi}{\partial t} + T \frac{\partial \phi}{\partial s_o} + \frac{\partial Q}{\partial s_o} + C_R' U|U| - w_m \cos\phi = 0 \qquad (25)$$

where m_n is the total mass, $m_n = m_o + m_{a_n}$.

Substituting equations (17) and (19) into equation (13) yields:

$$m_o \frac{\partial V}{\partial t} + m_o U \frac{\partial \phi}{\partial t} - \frac{\partial T}{\partial s_o} + Q \frac{\partial \phi}{\partial s_o} + k C_R' V|C| + w_m \sin\phi - K_i' \frac{\partial^2 r}{\partial s_o \partial t} = 0 \qquad (26)$$

Combining equations (3), (21) and (22) yields

$$K_\phi \frac{\partial^2 \phi}{\partial t^2} - K_E \frac{\partial^2 \phi}{\partial s_o^2} + Q - K_I' \frac{\partial^3 \phi}{\partial s_o^2 \partial t} = 0 \qquad (27)$$

The negative signs for the internal damping in equations (26) and (27) are required because these terms involve derivatives whose order is greater by one than the order of the derivatives for the acceleration terms.

Equations (26) and (27) contain higher-order derivatives. These equations can be reduced to first-order equations by the following changes of variable:

$$\begin{aligned} \alpha &= \frac{\partial \phi}{\partial s_0} \\ \beta &= \frac{\partial \phi}{\partial t} \\ \gamma &= \frac{\partial r}{\partial t} \\ \delta &= \frac{\partial \alpha}{\partial t} = \frac{\partial^2 \phi}{\partial s_0 \partial t} \end{aligned} \qquad (28)$$

Substituting these values only in the higher-order terms in equations (26) and (27) yields:

$$m_o \frac{\partial V}{\partial t} + m_o U \frac{\partial \phi}{\partial t} - \frac{\partial T}{\partial s_0} + Q \frac{\partial \phi}{\partial s_0} + k C_R' V|C| + w_n \sin\phi - K_i' \frac{\partial \gamma}{\partial s_0} = 0 \qquad (29)$$

$$K_\phi \frac{\partial \beta}{\partial t} - K_E \frac{\partial \alpha}{\partial s_0} + Q - K_I' \frac{\partial \delta}{\partial s_0} = 0 \qquad (30)$$

The complete formulation of the equations of motion is now given by equations (14), (15), (24), (25) and (28) to (30):

$$m_n \frac{\partial U}{\partial t} - m_n V \frac{\partial \phi}{\partial t} + T \frac{\partial \phi}{\partial s_0} + \frac{\partial Q}{\partial s_0} + C_R' U|U| - w_n \cos\phi = 0 \qquad \text{(a)}$$

$$m_o \frac{\partial V}{\partial t} + m_o U \frac{\partial \phi}{\partial t} - \frac{\partial T}{\partial s_0} + Q \frac{\partial \phi}{\partial s_0} + k C_R' V|C| + w_n \sin\phi - K_i' \frac{\partial \gamma}{\partial s_0} = 0 \qquad \text{(b)} \qquad (31)$$

$$\frac{\partial U}{\partial s_0} - V \frac{\partial \phi}{\partial s_0} + r \frac{\partial \phi}{\partial t} = 0 \qquad \text{(c)}$$

$$\frac{\partial V}{\partial s_0} + U \frac{\partial \phi}{\partial s_0} - \frac{\partial r}{\partial t} = 0 \qquad \text{(d)}$$

$$\frac{\partial T}{\partial s_o} - \left(\frac{dT}{dr}\right)\frac{\partial r}{\partial s_o} = 0 \qquad \text{(e)}$$

$$K \frac{\partial \beta}{\partial t} - K_E \frac{\partial \alpha}{\partial s_o} + Q - K_I' \frac{\partial \delta}{\partial s_o} = 0 \qquad \text{(f)}$$

$$\alpha - \frac{\partial \phi}{\partial s_o} = 0 \qquad \text{(g)}$$

$$\frac{\partial \alpha}{\partial t} - \frac{\partial \beta}{\partial s_o} = 0 \qquad \text{(h)} \qquad (31)$$

$$\gamma - \frac{\partial r}{\partial t} = 0 \qquad \text{(i)}$$

$$\delta - \frac{\partial \alpha}{\partial t} = 0 \qquad \text{(j)}$$

This is a set of 10 equations in the ten dependent variables U, V, T, r, Q, ϕ, α, β, γ and δ, and the two independent variables t and s_o. T or r can be eliminated by using equation (23) or equation (31e). Equations (31a) to (31d) are first-order, quasi-linear equations. All of the other equations are first-order, linear equations.

Equations (31) represent the complete equations of motion for the most general formulation of the problem. The equations of motion for a less general formulation can be obtained by eliminating the appropriate terms and equations. If, for example, bending rigidity (and thus shear) and internal-damping terms are set equal to zero, the equations reduce to:

$$m_n \frac{\partial U}{\partial t} - m_n V \frac{\partial \phi}{\partial t} + T \frac{\partial \phi}{\partial s_o} + C_R' U|U| - w_n \cos \phi = 0$$

$$m_o \frac{\partial V}{\partial t} + m_o U \frac{\partial \phi}{\partial t} - \frac{\partial T}{\partial s_o} + k C_R' V|C| + w_n \sin \phi = 0$$

$$\frac{\partial U}{\partial s_o} - V\frac{\partial \phi}{\partial s_o} + r\frac{\partial \phi}{\partial t} = 0$$

$$\frac{\partial V}{\partial s_o} + U\frac{\partial \phi}{\partial s_o} - \frac{\partial r}{\partial t} = 0$$

$$\frac{\partial T}{\partial s_o} - \left(\frac{dT}{dr}\right)\frac{\partial r}{\partial s_o} = 0$$

If, in addition, the cable material is assumed inelastic ($r \equiv 1$) the equations reduce, except for the added-mass term, m_n, and the details of the damping terms, to exactly the four equations obtained by Whicker (5).

Methods of Solving the Equations of Motion

There are at least three methods of solving the equations of motion of a cable-towed body system when a continuous representation of the cable is used: direct numerical integration of the partial differential equations of motion; linearization of the equations of motion and solution of the linearized equations; and reduction, by the method of characteristics, of the equations of motion to ordinary differential equations which are solved by finite-difference methods.

Direct numerical integration of the partial differential equations of motion (equations (31)) is inherently rather slow. Computational times will be greatly increased by iterative procedures necessary to satisfy the two-point boundary values, although these times may be reduced by the use of variational methods. It seems likely that direct integration will require the largest computational times of any of the methods considered. This fact, together with the sensitivity of such numerical procedures, makes this the least attractive method.

Linearization of the equations of motion have been carried out by several investigators. Kerney (18) has recently presented an analysis based on a small-perturbation parameter method. He considers equations of motion with no internal damping and bending rigidity. The method employed by Kerney can be extended to the complete equations. Kerney reduces the linearized, partial differential equations of motion to ordinary differential equations by assuming simple harmonic motions of fixed frequency. This method can be extended to complex harmonic motions by linear superposition. Alternatively the partial differential equations can be reduced to ordinary differential equations using Laplace transforms, but the difficulty in finding the inverse transform is a formidable one. Such a method can lead, potentially, to solutions in closed form. Computational times required for numerical solution of the ordinary differential equations should be moderate. The primary disadvantage of such methods is that inherent in linearization.

Solution of the equations of motions using the method of characteristics is attractive because the two-point boundary values are automatically satisfied and because no linearization of the equations is required. Computational times are dependent on the characteristic values or velocities; as characteristic velocity increases, computational times increase (17). Because of its attractive features, the method of characteristics has been selected for this investigation. To treat steel wire rope cables, which are rather inelastic, other methods may be used for the longitudinal equations to reduce computational times.

SOLUTION BY THE METHOD OF CHARACTERISTICS

The cable system is governed by quasi-linear hyperbolic partial differential equations. Such equations can be solved by direct numerical integration or can be solved numerically after reduction to ordinary differential equations by the method of characteristics. Whicker (5) apparently was the first to apply this method to cable problems. This method is widely used in supersonic aerodynamics. The method of characteristics is used to decrease the sensitivity, and hopefully, increase the speed of the numerical integration.

The method of characteristics is applicable only to quasi-linear hyperbolic equations, but there is no guarantee that the method can be applied to all such equations. The method consists of determining characteristic values or velocities which, when substituted into the partial differential equations, affect a reduction of those equations to ordinary differential equations. When no characteristic values exist, this method cannot be employed.

Courant and Friedrichs (19) present a good introduction to the method of characteristics. A more complete mathematical treatment is given by Courant and Hilbert (20). These discussions are primarily concerned with systems of one (spatial) dimension, but the method can be applied to systems of more than one spatial dimension. The two-dimensional cable is a system of one spatial dimension (measured along the cable). A cable free in all six degrees of freedom leads to a system of more than one spatial dimension. A brief introduction to the method of characteristics is given below.

Consider a system of n first-order, quasi-linear, partial differential equations in n dependent variables, U_n, and two independent variables t and s:

$$
\begin{aligned}
L_1 &= A_{11} X_{1t} + A_{12} X_{1s} + \cdots\cdots + A_{1m} X_{ms} + B_1 = 0 \\
&\;\;\vdots \\
&\;\;\vdots \\
L_n &= A_{n1} X_{nt} + A_{m2} X_{ns} + \cdots\cdots + A_{nm} X_{ns} + B_n = 0
\end{aligned}
\qquad (32)
$$

or in matrix form

$$\{L\} = A_1 \frac{\partial}{\partial t}\{X\} + A_2 \frac{\partial}{\partial s_o}\{X\} + \{B\} = 0$$

where $L_1, \ldots L_n$ are quasi-linear, partial differential operators

$A_{11}, A_{12}, \ldots A_{nm}, B_n$ are known functions of U_n, t and s

$m = 2n$

and where all functions are continuous and possess as many continuous derivatives as required. We try to find an operator

$$L = \sum_n \lambda_n L_n \qquad (33)$$

such that in L, the derivatives of $X_1, X_2, \ldots, X_n$ are in the same direction, the so-called characteristic direction or value. The condition for this to be true is:

$$(\lambda_1 A_{11} + \cdots + \lambda_n A_{n1})/(\lambda_1 A_{12} + \cdots + \lambda_n A_{n2}) = \\ = \cdots = (\lambda_1 A_{1(m-1)} + \cdots + \lambda_n A_{n(m-1)})/(\lambda_1 A_{1m} + \cdots + \lambda_n A_{nm}) = \\ = C \tag{34}$$

The characteristic value C can be defined in terms of a parameter σ,

$$C = \frac{\partial t/\partial\sigma}{\partial s_o/\partial\sigma}$$

Multiplying equation (33) by $dt/d\sigma$ and $ds/d\sigma$ and making use of equation (34) and the differential relationship

$$\frac{dX}{d\sigma} = \frac{\partial X}{\partial t}\frac{dt}{d\sigma} + \frac{\partial X}{\partial s}\frac{ds}{d\sigma} = X_t \frac{dt}{d\sigma} + X_s \frac{ds}{d\sigma}$$

we obtain:

$$\sum_{k=1}^{n} \sum_{n} \lambda_n A_{n(2k-1)} \frac{dX_k}{d\sigma} + \sum_{n} \lambda_n B_n \frac{dt}{d\sigma} = \frac{dt}{d\sigma} L \\ \sum_{k=1}^{n} \sum_{n} \lambda_n A_{n(2k-1)} \frac{dX_k}{d\sigma} + \sum_{n} \lambda_n B_n \frac{ds}{d\sigma} = \frac{ds}{d\sigma} L \tag{35}$$

If at any point (t,s) the functions X_n satisfy equation (32), then we obtain a set of n homogeneous, linear equations for $\lambda_1, \ldots, \lambda_n$:

$$\sum_{k=1}^{n} \lambda_k \left(A_{k1}\frac{ds}{d\sigma} - A_{k2}\frac{dt}{d\sigma}\right) = 0 \\ \vdots \\ \sum_{k=1}^{n} \lambda_k \left(A_{k(n-1)}\frac{ds}{d\sigma} - A_{kn}\frac{dt}{d\sigma}\right) = 0 \tag{36}$$

For these equations to be satisfied, the determinant of the coefficients of the λ's must be identically zero. The solution of the resulting n'th-order determinant leads to n solutions or n characteristic values. If some of the coefficients of the λ's in equations (36) are zero, some, and possibly all, of the solutions will be trivial or equal to zero. If no characteristic values occur, the method of characteristics cannot be used.

Calculation of Characteristic Values

Characteristic values can be calculated from equations (31) using the method described in the previous section. These equations are all in the general, quasi-linear operator form:

$$\begin{aligned} L_n = {} & A_n \frac{\partial U}{\partial t} + B_n \frac{\partial U}{\partial s_0} + C_n \frac{\partial V}{\partial t} + D_n \frac{\partial V}{\partial s_0} + E_n \frac{\partial \phi}{\partial t} + F_n \frac{\partial \phi}{\partial s_0} + \\ & + G_n \frac{\partial T}{\partial t} + H_n \frac{\partial T}{\partial s_0} + I_n \frac{\partial Q}{\partial t} + J_n \frac{\partial Q}{\partial s_0} + K_n \frac{\partial r}{\partial t} + M_n \frac{\partial r}{\partial s_0} + \\ & + N_n \frac{\partial \alpha}{\partial t} + O_n \frac{\partial \alpha}{\partial s_0} + P_n \frac{\partial \beta}{\partial t} + R_n \frac{\partial \beta}{\partial s_0} + T_n \frac{\partial \gamma}{\partial t} + U_n \frac{\partial \gamma}{\partial s_0} + \\ & + V_n \frac{\partial \sigma}{\partial t} + W_n \frac{\partial \sigma}{\partial s_0} + S_n = 0 \end{aligned} \tag{37}$$

where the coefficients A_n, ..., S_n can be functions of the dependent and independent variables, but not of their derivatives. The ten equations are:

$$L_1 = m_n \frac{\partial U}{\partial t} - m_n V \frac{\partial \phi}{\partial t} + T \frac{\partial \phi}{\partial s_0} + \frac{\partial Q}{\partial s_0} + C_R' U|U| - w_n \cos\phi = 0 \qquad \text{(a)}$$

$$L_2 = m_o \frac{\partial V}{\partial t} + m_o U \frac{\partial \phi}{\partial t} - \frac{\partial T}{\partial s_0} + Q \frac{\partial \phi}{\partial s_0} + k C_R' V|C| + w_m \sin\phi - K_i' \frac{\partial \gamma}{\partial s_0} = 0 \qquad \text{(b)}$$

$$L_3 = \frac{\partial U}{\partial s_0} - V \frac{\partial \phi}{\partial s_0} + r \frac{\partial \phi}{\partial t} = 0 \qquad \text{(c)}$$

$$L_4 = \frac{\partial V}{\partial s_0} + U \frac{\partial \phi}{\partial s_0} - \frac{\partial r}{\partial t} = 0 \qquad \text{(d)} \qquad (31)$$

$$L_5 = \frac{\partial T}{\partial s_0} - \left(\frac{dT}{dr}\right) \frac{\partial r}{\partial s_0} = 0 \qquad \text{(e)}$$

$$L_6 = K_\phi \frac{\partial \beta}{\partial t} - K_E \frac{\partial \alpha}{\partial s_0} - Q - K_I' \frac{\partial \delta}{\partial s_0} = 0 \qquad \text{(f)}$$

$$L_7 = \alpha - \frac{\partial \phi}{\partial s_0} = 0 \qquad \text{(g)}$$

$$L_8 = \frac{\partial \alpha}{\partial t} - \frac{\partial \beta}{\partial s_0} = 0 \qquad \text{(h)}$$

$$L_9 = \gamma - \frac{\partial r}{\partial t} = 0 \qquad \text{(i)}$$

$$L_{10} = \delta - \frac{\partial \alpha}{\partial t} = 0 \qquad \text{(j)}$$

The characteristic values can now be determined by setting the determinant of the coefficients of λ_k in equation (36) equal to zero. The values of $A_{k\ell}(k,\ell=1, \ldots, n)$ in equation (32) and A_n, B_n, etc. in equation (37) are defined by equation (31). The characteristic values are thus defined by:

$$\begin{vmatrix}
A_1 s_\sigma & 0 & -B_3 t_\sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_2 s_\sigma & 0 & -D_4 t_\sigma & 0 & 0 & 0 & 0 & 0 & 0 \\
E_1 s_\sigma - F_1 t_\sigma & E_2 s_\sigma - F_2 t_\sigma & E_3 s_\sigma - F_3 t_\sigma & -F_4 t_\sigma & 0 & 0 & -F_7 t_\sigma & 0 & 0 & 0 \\
0 & -H_2 t_\sigma & 0 & 0 & -H_5 t_\sigma & 0 & 0 & 0 & 0 & 0 \\
-J_1 t_\sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & K_4 s_\sigma & -M_5 t_\sigma & 0 & 0 & 0 & K_9 s_\sigma & 0 \\
0 & 0 & 0 & 0 & 0 & -O_6 t_\sigma & 0 & N_8 s_\sigma & 0 & N_{10} s_\sigma \\
0 & 0 & 0 & 0 & 0 & P_6 s_\sigma & 0 & -R_8 t_\sigma & 0 & 0 \\
0 & -U_2 t_\sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -W_6 t_\sigma & 0 & 0 & 0 & 0
\end{vmatrix} = 0$$

This determinant can be readily reduced by minors and cofactors and shown to be identically zero, or it can be shown that $s_\sigma \equiv 0$ and $t_\sigma \equiv 0$. Thus, no characteristics values exist, a clear consequence of the higher-order internal damping terms. The higher-order damping terms in equations L_2 and L_6 of equation (31) make these equations non-hyperbolic.

If the damping terms in L_2 and L_6 and equations L_9 and L_{10} are dropped, a new set of equations for the case of zero internal damping results. Setting the determinant of the coefficients for this problem equal to zero, we obtain:

$$\begin{vmatrix} A_1 s_\sigma & 0 & -B_3 t_\sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & C_2 s_\sigma & 0 & -D_4 t_\sigma & 0 & 0 & 0 & 0 \\ E_1 s_\sigma - F_1 t_\sigma & E_2 s_\sigma - F_2 t_\sigma & E_3 s_\sigma - F_3 t_\sigma & -F_4 t_\sigma & 0 & 0 & -F_7 t_\sigma & 0 \\ 0 & -H_2 t_\sigma & 0 & 0 & -H_5 t_\sigma & 0 & 0 & 0 \\ -J_1 t_\sigma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_4 s_\sigma & -M_5 t_\sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -O_6 t_\sigma & 0 & N_8 s_\sigma \\ 0 & 0 & 0 & 0 & 0 & P_6 s_\sigma & 0 & -R_8 t_\sigma \end{vmatrix} = 0$$

This reduces by minors and co-factors to

$$(O_6 R_8 t_\sigma^2 - P_6 N_8 s_\sigma^2) \begin{vmatrix} C_2 s_\sigma & -D_4 t_\sigma & 0 \\ -H_2 t_\sigma & 0 & -H_5 t_\sigma \\ 0 & K_4 s_\sigma & -M_5 t_\sigma \end{vmatrix} = 0$$

Setting both the expression in parentheses and the third-order determinant equal to zero yields two characteristic values:

$$\text{(a)} \left(\frac{s_\sigma}{t_\sigma}\right)^2 = \left(\frac{ds}{dt}\right)^2 = \frac{O_6 R_8}{P_6 N_8} = \frac{K_\varepsilon}{K_\phi}$$

$$\frac{ds}{dt} = \pm\sqrt{K_\varepsilon / K_\phi} \tag{38a}$$

$$\text{(b)}\ \left(\frac{s_\sigma}{t_\sigma}\right)^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{D_4 H_2 M_5}{C_2 K_4 H_5} = \frac{1}{m_o}\frac{dT}{dr}$$

$$\frac{ds}{dt} = \pm\sqrt{\frac{1}{m_o}\frac{dT}{dr}} \tag{38b}$$

The first characteristic is clearly associated with bending while the second is clearly associated with longitudinal motions.

Assuming purely elastic behavior of the cable material and using equation (23) and (24), we obtain

$$\left(\frac{ds}{dt}\right)_2 = \pm\sqrt{\frac{1}{m_o}\frac{dT}{dr}} = \pm\sqrt{\frac{a_c E}{a_c \rho_c}} = \pm\sqrt{E/\rho} = \pm a$$

where a is the sonic velocity in the cable material. The value of the bending characteristic is identical:

$$\left(\frac{ds}{t}\right)_2 = \pm\sqrt{K_E/K_\phi} = \pm\sqrt{\frac{E I_c}{\rho I_c}} = \pm a$$

Thus all disturbances are propagated at sonic velocity.

For the case of zero bending rigidity (and shear) and zero internal damping, the characteristic values, determined by Critescu (4) and others are:

$$\left(\frac{ds}{dt}\right)_1 = \pm\sqrt{\frac{T}{m_n r}} \tag{39a}$$

$$\left(\frac{ds}{dt}\right)_2 = \pm\sqrt{\frac{1}{m_o}\frac{dT}{dr}} \tag{39b}$$

Assuming purely elastic behavior, these become

$$\left(\frac{ds}{dt}\right)_1 = \pm \sqrt{\frac{m_o(r-1)}{m_n\, r}}\; a \;=\; \pm k_r\, a$$

$$\left(\frac{ds}{dt}\right)_2 = \pm\, a$$

It should be noted that the term k_r will usually be much smaller than one, indicating a fundamental difference in the characteristics obtained with and without bending rigidity.

The influence of various combinations of elasticity, bending rigidity, rotational acceleration term and internal damping on the characteristic values have been examined. The results are summarized in Table I. The effect of rotational acceleration (the term $K_\phi \partial\beta / \partial t$ in equation (31e)) was considered because this term is often neglected in treatments of the motions of slender bars, etc. (see, for example Paidoussis (13)). It can be seen from Table I that the assumption of finite bending rigidity and negligible bending acceleration causes the characteristic value due to bending motions (equation (38a)) to disappear. The only characteristic value which differs from the values discussed above is the value $\sqrt{T/m_o r}$ which is obtained when elasticity, bending rigidity and internal damping are neglected.

CASE	TERMS INCLUDED IN FORMULATION				CHARACTERISTIC VALUES	
	Elasticity	Bending Rigidity	Rotational Acceleration	Internal Damping	Transverse or Bending	Longitudinal
1	X	X	X	X	—	—
2	X	X	X	—	$\pm\sqrt{K_E/K_\phi}$	$\pm\sqrt{\frac{1}{m_o}\, dT/dr}$
3	X	X	—	—	—	$\pm\sqrt{\frac{1}{m_o}\, dT/dr}$
4	X	—	—	X	$\pm\sqrt{T/m_m r}$	—
5	X	—	—	—	$\pm\sqrt{T/m_m r}$	$\pm\sqrt{\frac{1}{m_o}\, dT/dr}$
6	—	X	X	X	—	—
7	—	X	X	—	$\pm\sqrt{K_E/K_\phi}$	—
8	—	X	—	—	—	—
9	—	—	—	—	$\pm\sqrt{T/m_m r}$	—

Table I - Summary of Calculated Characteristic Values

Table I shows that, even if internal damping is neglected, three distinct characteristic values corresponding to the three modes of motion do not occur. If bending is also neglected, however, two characteristic values corresponding to the two remaining modes (longitudinal and transverse) occur. The method of characteristics can thus be applied only if internal damping and bending are neglected, although it may be possible to treat these terms approximately. Nath has considered the internal damping term as a forcing term, but difficulties in the calculations have occurred (21). In the next section the case with no internal damping and no bending rigidity is considered.

Reduction of the Equations of Motion to Ordinary Differential Equations

If internal damping and bending are neglected, the equations of motion, equations(31), reduce to four equations describing the transverse and longitudinal motions. These partial differential equations can be reduced to ordinary differential equations by introduction of the appropriate characteristic values, equation (39). The resulting ordinary differential equations can be solved by the method of characteristics.

Neglecting internal damping and bending, equations (31) become:

$$m_m \frac{\partial U}{\partial t} - m_m V \frac{\partial \phi}{\partial t} + T \frac{\partial \phi}{\partial s_o} + C_R' U|U| + w_m \cos\phi = 0 \qquad \text{(a)}$$

$$m_o \frac{\partial V}{\partial t} + m_o U \frac{\partial \phi}{\partial t} - \left(\frac{dT}{dr}\right) \frac{\partial r}{\partial s_o} + k C_R' V|C| - w_m \sin\phi = 0 \qquad \text{(b)}$$

(40)

$$\frac{\partial U}{\partial s_o} - V \frac{\partial \phi}{\partial s_o} + r \frac{\partial \phi}{\partial t} = 0 \qquad \text{(c)}$$

$$\frac{\partial V}{\partial s_o} + U \frac{\partial \phi}{\partial s_o} - \frac{\partial r}{\partial t} = 0 \qquad \text{(d)}$$

Equations (a) and (c) describe transverse motions, equations (b) and (d) longitudinal motions. The ordinary differential equation governing longitudinal motions can be obtained by forming the sum

$$\frac{(40b) \times dt}{m_o} + (40d) \times ds_o = 0$$

Upon substitution, we obtain:

$$\left(\frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial s_o} ds_o\right) + U\left(\frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial s_o} ds_o\right) - \frac{\partial r}{\partial t} ds_o - \\ - \frac{1}{m_o}\left(\frac{dT}{dr}\right)\frac{\partial r}{\partial s_o} dt + \frac{k C_B'}{m_o} V|C| dt - \frac{w_o}{m_o} \sin\phi \, dt = 0 \tag{41}$$

Introducing the characteristic value for longitudinal motions,

$$\frac{ds_o}{dt} = \pm\sqrt{\frac{1}{m_o}\frac{dT}{dr}}$$

and making use of the chain rule of differentiation, equation (41) becomes after simplification

$$dV \pm U d\phi \mp \sqrt{\frac{1}{m_o}\frac{dT}{dr}}\, dr + \frac{k C_B'}{m_o} V|C| dt - \\ - \frac{w_o}{m_o} \sin\phi \, dt = 0 \tag{42}$$

The minus sign in the third term refers to the characteristic path with positive slope (*p* characteristic), the negative sign to the characteristic with negative slope (*q* characteristic).

The ordinary differential equation of transverse motion is obtained in a similar manner. Forming the sum

$$\frac{(40a) \times dt}{m_n} + (40c) \times ds_o = 0$$

and substituting as before, one easily obtains the ordinary differential equations

$$dU - \left[V \mp r\sqrt{\frac{I}{m_m r}}\right] d\phi + \frac{C_R'}{m_n} U|U| dt + \frac{w_o}{m_n} \cos\phi \, dt = 0 \qquad (43)$$

The plus and minus signs of the third term apply along the *p* and *q* characteristic paths respectively.

Equations (42) and (43) can be solved in finite-difference form along the appropriate characteristic lines.

Numerical Solutions of Equations of Motion Along Characteristic Paths

Equations (42) and (43) can be solved numerically along characteristic paths or lines using finite-difference methods. A good discussion of finite-difference methods suitable for solving characteristic equations is given by Ames (22). Several "hybrid" methods which make use of a predetermined time-space grid are described by Ames. These methods are attractive because they do not require double interpolation in both the time and space coordinates. Ames gives two methods, one having a first order truncation error, the other a second order truncation error. The method of Hartree, which has the second-order error should be used because of the improved accuracy.

Figure 6 illustrates the time-space grid used to obtain values of the independent variables at a new time. The right *(p)* and left *(q)* running characteristics passing through the point *R*, where it is desired to obtain values of the independent variables, are used. Values of the independent variables at the points *P* and *Q* are determined by interpolation of known values at the points *A*, *B*, *D*, etc. The time increment Δt is determined by the spatial increment. Δs_o and the characteristic values, f_p and f_q. For the finite-difference process to be stable, the time increment must be such that the points *P* and *Q* lie in the intervals *(A,B)* and *(A,D)* respectively (22). The time increment must therefore satisfy

$$\Delta t \leq \Delta s_o / |f_p|$$
$$\Delta t \leq \Delta s_o / |f_q| \qquad (44)$$

where f_p is the characteristic velocity on p, $+ \sqrt{T/m_n r}$,
f_q is the characteristic velocity on q, $- \sqrt{T/m_n r}$,
T and r are the average values on the paths $\overline{RP}$ and $\overline{RQ}$.

The spatial increment, Δs_o, will be selected to insure an adequate description of the cable motions and shape. The maximum allowable time increment will thus depend almost entirely on the characteristic velocities.

The motions for non-end points (points in Region I of Figure 7) are found by simultaneous solution of the equations along the *p* and *q* characteristics intersecting at the point *R*. For end points *L* and *U*, the motions are found by simultaneous solution of the equation on the characteristic lines passing through *L* and *U* and the equations describing the appropriate boundary condition. Numerical methods of solution are described in later sections.

Limitation on the Use of the Method Characteristics

The only serious limitation of the method of characteristics is the large computational times required with large characteristic velocities. This limitation is particularly acute for relatively inelastic cables, which have very large longitudinal (sonic) characteristic velocities. For a steel cable with an average tensile stress of 10,000 psi, the allowable time increments for a typical spatial increment of 10 feet are approximately 0.03 and 0.0006 second for transverse and longitudinal waves respectively. All calculations will be limited by the smaller value (0.0006 seconds). Thus, for a typical steel wire-rope cable, approximately 10,000 time increments will be required for six seconds of real time, or about one period of oscillation. The resulting computational time will certainly be excessive. The allowable time increment can be increased only by increasing the spatial increment, but a relatively small spatial increment is required to insure good definition of motions and cable shape.

If only the transverse motion are computed by the method of characteristics, the allowable time increments and resulting computational times will be much more realistic. For steel cables, computational times may be reduced by a factor of up to 50, if longitudinal motions can be calculated by a more efficient method. Because of the great difference, or disjointness, of the transverse and longitudinal characteristic velocities, or velocities of propagation of disturbances, it seems quite probable that one should be able, with only small approximation, to decouple one or both sets of equations. In particular, if the longitudinal equations of motion can be decoupled from transverse motions, a more efficient computational method may be found.

SIMPLIFICATION OF THE EQUATIONS OF MOTION

As noted in the last section, the method of characteristics is not well suited to the computation of longitudinal motions, particularly for steel cables. An alternative method is needed for calculating longitudinal motions at realistic time intervals. If the longitudinal equations of motion can be linearized, or decoupled from the transverse equations, they can probably be solved by more rapid computational methods. In order to retain the ability to study large motions, a careful evaluation of the magnitude of non-linearities is required. Such an analysis is discussed in this section.

The equations of motion can be simplified by a method analogous to that used to derive the boundary-layer equations in fluid mechanics: The equations are non-dimensionalized in such a way that each term is the product of non-dimensional, order-one variables, and one or more small non-dimensional parameters. These parameters can be evaluated for typical values of dimensional variables and higher-order terms then identified and eliminated.

Non-Dimensionalization of the Equations

In order to non-dimensionalize the equations of motion it is necessary to represent all the independent variables as sums of steady-state components and components due to transverse, longitudinal and rotational motions. The variables in equations (31) can be written:

$$
\begin{aligned}
U &= U_o + U_{t_1} + U_{t_2} \\
V &= V_o + V_\ell \\
\phi &= \phi_o + \phi_{t_1} + \phi_{t_2} \\
r &= r_o + (r_{t_1} - 1) + (r_{t_2} - 1) + (r_\ell - 1) \\
T &= T_o + T_{t_1} + T_{t_2} + T_\ell \\
Q &= Q_o + Q_{t_1} + Q_{t_2} \\
&\text{etc.}
\end{aligned}
\tag{45}
$$

where the subscripts denote the following:

o is the steady-state (uniform velocity) value,

t_1 is the value due to transverse motions,

t_2 is the value due to rotational motions,

ℓ is the value due to longitudinal motions.

The component variables can be non-dimensionalized and made of order unity as follows:

$$
\begin{aligned}
(U_o / C U_*) &= \bar{U}_o = o(1) \\
(V_o / C V_*) &= \bar{V}_o = o(1) \\
(U_{t_1} / \omega \delta_{t_1}) &= \bar{U}_{t_1} = o(1) \\
(U_{t_2} / \omega \delta_{t_2}) &= \bar{U}_{t_2} = o(1) \\
(V_\ell / \omega \delta_\ell) &= \bar{V}_\ell = o(1) \\
(\phi_o / \phi_*) &= \bar{\phi}_o = o(1)
\end{aligned}
$$

$$\Phi_{t_1}/(\delta_{t_1}\omega/V_1) = \bar{\Phi}_{t_1} = o(1)$$

$$\Phi_{t_2}/(\delta_{t_2}\omega/V_2) = \bar{\Phi}_{t_2} = o(1)$$

$$(r_o - 1)/\varepsilon_* = \bar{r}_o = o(1)$$

$$(r_{t_1} - 1)/(\delta_{t_1}\omega/V_1)^2 = \bar{r}_{t_1} = o(1)$$

$$(r_{t_2} - 1)/(\delta_{t_2}\omega/V_2)^2 = \bar{r}_{t_2} = o(1)$$

$$(r_\ell - 1)/(\delta_\ell\omega/V_2) = \bar{r}_\ell = o(1)$$

$$Q_o/(EI\partial^2\Phi_o/\partial s_o^2) = \bar{Q}_o = o(1)$$

$$Q_{t_1}/(EI\partial^2\Phi_{t_1}/\partial s_o^2) = \bar{Q}_{t_1} = o(1)$$

$$Q_{t_2}/(EI\partial^2\Phi_{t_2}/\partial s_o^2) = \bar{Q}_{t_2} = o(1)$$

$$t\omega = \bar{t} = o(1)$$

and the length scales for various components are:

$$[s_o/(L s_*)]_o = \bar{s}_o = o(1)$$

$$[s_o/(V_1/\omega)]_{t_1} = (\bar{s}_o)_{t_1} = o(1)$$

$$[s_o/(V_2/\omega)]_{t_2} = (\bar{s}_o)_{t_2} = o(1)$$

$$[s_o/(V_2/\omega)]_\ell = (\bar{s}_o)_\ell = o(1)$$

where C is the steady-state towing velocity,

ω is the fundamental frequency of excitation,

δ_{t1} is the input amplitude of transverse excitation,

δ_{t2} is the input amplitude of rotational excitation,

δ_ℓ is the input amplitude of longitudinal excitation,

L is the cable length,

V_1 is the transverse characteristic velocity = $(T/m_n r)$,

V_2 is the longitudinal and rotational characteristic velocity

I_c is moment of inertia of the cable cross-section,

ρ_c is mass density of the cable material,

ϕ_* is the reference, steady-state cable angle,

ε_* is the reference, steady-state strain.

Equations (45) are substituted in equations (31) which are non-dimensionalized so as to obtain all variables in the above form. To simplify the non-dimensionalization process, the variables α, β, δ and γ are eliminated by substituting equations (31g) to (31j) into equations (31a) to (31f).

The buoyancy, external and internal damping terms require special attention. In order to carry out the non-dimensionalization, the terms sin ϕ and cos ϕ in the buoyancy terms must be simplified. We note that in most cases:

$$|\phi_o| \gg |\phi_{t_1} + \phi_{t_2}|$$

$$\phi_o = o(1)$$

and thus

$$\sin\phi \cong \sin\phi_o + (\phi_{t_1} + \phi_{t_2})\cos\phi_o \cong 1 + \phi_{t_1} + \phi_{t_2}$$

$$\cos\phi \cong \cos\phi_o + (\phi_{t_1} + \phi_{t_2})\sin\phi_o \cong 1 + \phi_{t_1} + \phi_{t_2}$$

The transverse, external damping term, which is nonlinear, can be written as

$$C_R U|U| = C_R (U_o^2 + U_{t_1}^2 + U_{t_2}^2 + 2U_oU_{t_1} + 2U_oU_{t_2} + 2U_{t_1}U_{t_2}) \times$$
$$\times \operatorname{sgn}(U_o + U_{t_1} + U_{t_2})$$

The *sgn* function is not significant in the non-dimensionalization process and can be neglected. The internal-damping coefficients, K_i' and K_I' can be expressed as a percentage, κ, of the critical internal damping which is shown later to be approximately $2V_2L$. Thus

$$K_i' = \kappa K_{i_{CR}}' \cong \kappa 2 V_2 L$$

$$K_I' = \kappa K_{I_{CR}}' \cong \kappa 2 V_2 L$$

where V_2 is the characteristic (sonic) velocity.

Equations (31a) becomes upon substitution of non-dimensional variables, division by $m_n\omega^2L$, and simplification:

$$\begin{aligned}
&\Big[\frac{\varepsilon_* \phi_*}{s_*}\left(\frac{V_2}{\omega L}\right)^2 \bar r_0 \frac{\partial\bar\Phi_0}{\partial\xi_0} + \frac{\phi_*}{s_*^3}\left(\frac{V_2}{\omega L}\right)^2\left(\frac{I}{a_c L^2}\right)\left(\frac{\partial^2\bar\Phi_0}{\partial\xi_0^2}\frac{\partial\bar Q_0}{\partial\xi_0} + \frac{\partial^3\bar\Phi_0}{\partial\xi_0^3}\bar Q_0\right) - \left(\frac{g'}{\omega^2 L}\right) + \\
&+ U_*^2 C_{R*} \rho' \bar C_R \left(\frac{c}{\omega L}\right)^2\left(\frac{L}{\pi r_c}\right)\bar U_0|\bar U_0| \Big] + \left(\frac{\delta_{t_1}}{L}\right)\frac{\partial \bar U_{t_1}}{\partial\tau} + \left(\frac{\delta_{t_2}}{L}\right)\frac{\partial\bar U_{t_2}}{\partial\tau} - \\
&- V_*\left(\frac{c}{V_1}\right)\left(\frac{\delta_{t_1}}{L}\right)\bar V_0 \frac{\partial\bar\Phi_{t_1}}{\partial\tau} - V_*\left(\frac{c}{V_2}\right)\left(\frac{\delta_{t_2}}{L}\right)\bar V_0\frac{\partial\bar\Phi_{t_2}}{\partial\tau} - \left(\frac{\omega L}{V_1}\right)\left(\frac{\delta_1}{L}\right)\left(\frac{\delta_{t_1}}{L}\right)\bar V_1 \frac{\partial\bar\Phi_{t_1}}{\partial\tau} - \left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_1}{L}\right)\left(\frac{\delta_{t_2}}{L}\right)\bar V_1\frac{\partial\bar\Phi_{t_2}}{\partial\tau} + \\
&+ \varepsilon_*\left(\frac{V_2}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right)\bar r_0\frac{\partial\bar\Phi_{t_1}}{\partial\xi_0} + \varepsilon_*\left(\frac{V_2}{\omega L}\right)\left(\frac{\delta_{t_2}}{L}\right)\bar r_0\frac{\partial\bar\Phi_{t_2}}{\partial\xi_0} + \frac{\phi_*}{s_*}\left(\frac{V_2}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right)^2\bar r_{t_1}\frac{\partial\bar\Phi_0}{\partial\xi_0} + \\
&+ \left(\frac{\omega L}{V_1}\right)\left(\frac{V_2}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right)^3\bar r_{t_1}\frac{\partial\bar\Phi_{t_1}}{\partial\xi_0} + \left(\frac{\omega L}{V_1}\right)\left(\frac{\delta_{t_1}}{L}\right)^2\left(\frac{\delta_{t_2}}{L}\right)\bar r_{t_1}\frac{\partial\bar\Phi_{t_2}}{\partial\xi_0} + \frac{\phi_*}{s_*}\left(\frac{\delta_{t_2}}{L}\right)^2\bar r_{t_2}\frac{\partial\bar\Phi_0}{\partial\xi_0} + \\
&+ \left(\frac{\omega L}{V_1}\right)\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{\delta_{t_2}}{L}\right)^2\bar r_{t_2}\frac{\partial\bar\Phi_{t_1}}{\partial\xi_0} + \left(\frac{\omega L}{V_2}\right)^2\left(\frac{\delta_{t_2}}{L}\right)^3\bar r_{t_2}\frac{\partial\bar\Phi_{t_2}}{\partial\xi_0} + \frac{\phi_*}{s_*}\left(\frac{V_2}{\omega L}\right)\left(\frac{\delta_1}{L}\right)\bar r_1\frac{\partial\bar\Phi_0}{\partial\xi_0} + \\
&+ \left(\frac{\delta_{t_2}}{L}\right)\left(\frac{\delta_1}{L}\right)\bar r_1\frac{\partial\bar\Phi_{t_2}}{\partial\xi_0} + \left(\frac{V_2}{V_1}\right)\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{\delta_1}{L}\right)\bar r_1\frac{\partial\bar\Phi_{t_1}}{\partial\xi_0} + \left(\frac{\omega L}{V_1}\right)\left(\frac{V_2}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{I}{a_cL^2}\right)\left(\frac{\partial^2\bar\Phi_{t_1}}{\partial\xi_0^2}\frac{\partial\bar Q_{t_1}}{\partial\xi_0} + \right. \\
&\left. + \frac{\partial^3\bar\Phi_{t_1}}{\partial\xi_0^3}\bar Q_{t_1}\right) + \left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_{t_2}}{L}\right)\left(\frac{I}{a_cL^2}\right)\left(\frac{\partial^2\bar\Phi_{t_2}}{\partial\xi_0^2}\frac{\partial\bar Q_{t_2}}{\partial\xi_0} + \frac{\partial^3\bar\Phi_{t_2}}{\partial\xi_0^3}\bar Q_{t_2}\right) + \left(\frac{\omega L}{V_1}\right)\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{g'}{\omega^2L}\right)\bar\Phi_{t_1} + \\
&+ \left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_{t_2}}{L}\right)\left(\frac{g'}{\omega^2 L}\right)\bar\Phi_{t_2} + 2\rho' U_* C_{R*}\bar C_R\left(\frac{c}{\omega L}\right)\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{L}{\pi r_c}\right)\bar U_{t_1}\bar U_0 + \qquad\qquad (47) \\
&+ 2\rho'U_*C_{R*}\bar C_R\left(\frac{c}{\omega L}\right)\left(\frac{\delta_{t_2}}{L}\right)\left(\frac{L}{\pi r_c}\right)\bar U_{t_2}\bar U_0 + 2\rho' C_{R*}\bar C_R\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{\delta_{t_2}}{L}\right)\left(\frac{L}{\pi r_c}\right)\bar U_{t_1}\bar U_{t_2} + \\
&+ \rho' C_{R*}\bar C_R\left(\frac{\delta_{t_1}}{L}\right)^2\left(\frac{L}{\pi r_c}\right)\bar U_{t_1}^2 + \rho' C_{R*}\bar C_R\left(\frac{\delta_{t_2}}{L}\right)^2\left(\frac{L}{\pi r_c}\right)\bar U_{t_2}^2 = 0
\end{aligned}$$

where $\rho' = \rho/(\rho + \rho_c)$

$g' = g(\rho_c - \rho)/(\rho_c + \rho)$.

The terms in square brackets represent the complete steady-state equation which must be identically zero. Throughout this section, square brackets will be used to denote the steady-state equation.

Equation (31c) becomes upon substitution, division by ω and simplification:

$$
\begin{aligned}
&\left[\frac{U_*}{s_*}\left(\frac{C}{\omega L}\right)\frac{\partial \bar{U}_0}{\partial \xi_0} - \frac{V_* \Phi_*}{s_*}\left(\frac{C}{\omega L}\right)\bar{V}_0 \frac{\partial \bar{\Phi}_0}{\partial \xi_0} \right] + \left(\frac{\omega L}{V_1}\right)\left(\frac{\delta_{t_1}}{L}\right)\frac{\partial \bar{U}_{t_1}}{\partial \tau} + \\
&+ \left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_{t_2}}{L}\right)\frac{\partial \bar{U}_{t_2}}{\partial \tau} - V_*\left(\frac{C}{V_1}\right)\left(\frac{\omega L}{V_1}\right)\left(\frac{\delta_{t_1}}{L}\right)\bar{V}_0 \frac{\partial \bar{\Phi}_{t_1}}{\partial \xi_0} - V_*\left(\frac{C}{V_2}\right)\left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_{t_2}}{L}\right)\bar{V}_0 \frac{\partial \bar{\Phi}_{t_2}}{\partial \xi_0} - \\
&- \frac{\Phi_*}{s_*}\left(\frac{\delta_\ell}{L}\right)\bar{V}_\ell \frac{\partial \bar{\Phi}_0}{\partial \xi_0} - \left(\frac{\omega L}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{\delta_\ell}{L}\right)\bar{V}_\ell \frac{\partial \bar{\Phi}_{t_1}}{\partial \xi_0} - \left(\frac{\omega L}{V_2}\right)^2\left(\frac{\delta_{t_2}}{L}\right)\left(\frac{\delta_\ell}{L}\right)\bar{V}_\ell \frac{\partial \bar{\Phi}_{t_2}}{\partial \xi_0} - \\
&+ \left(\frac{\omega L}{V_1}\right)\left(\frac{\delta_{t_1}}{L}\right)\frac{\partial \bar{\Phi}_{t_1}}{\partial \tau} + \left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_{t_2}}{L}\right)\frac{\partial \bar{\Phi}_{t_2}}{\partial \tau} + \varepsilon_*\left(\frac{\omega L}{V_1}\right)\left(\frac{\delta_{t_1}}{L}\right)\bar{r}_0 \frac{\partial \bar{\Phi}_{t_1}}{\partial \tau} + \qquad (48) \\
&+ \varepsilon_*\left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_{t_2}}{L}\right)\bar{r}_0 \frac{\partial \bar{\Phi}_{t_2}}{\partial \tau} + \left(\frac{\omega L}{V_1}\right)^3\left(\frac{\delta_{t_1}}{L}\right)^3 \bar{r}_{t_1}\frac{\partial \bar{\Phi}_{t_1}}{\partial \tau} + \left(\frac{\omega L}{V_2}\right)\left(\frac{\omega L}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right)^2\left(\frac{\delta_{t_2}}{L}\right)\bar{r}_{t_1}\frac{\partial \bar{\Phi}_{t_2}}{\partial \tau} + \\
&+ \left(\frac{\omega L}{V_1}\right)\left(\frac{\omega L}{V_2}\right)^2\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{\delta_{t_2}}{L}\right)^2 \bar{r}_{t_2}\frac{\partial \bar{\Phi}_{t_1}}{\partial \tau} + \left(\frac{\omega L}{V_2}\right)^3\left(\frac{\delta_{t_2}}{L}\right)^3 \bar{r}_{t_2}\frac{\partial \bar{\Phi}_{t_2}}{\partial \tau} + \\
&+ \left(\frac{\omega L}{V_2}\right)\left(\frac{\omega L}{V_1}\right)\left(\frac{\delta_\ell}{L}\right)\left(\frac{\delta_{t_1}}{L}\right)\bar{r}_\ell \frac{\partial \bar{\Phi}_{t_1}}{\partial \tau} + \left(\frac{\omega L}{V_2}\right)^2\left(\frac{\delta_\ell}{L}\right)\left(\frac{\delta_{t_2}}{L}\right)\bar{r}_\ell \frac{\partial \bar{\Phi}_{t_2}}{\partial \tau} = 0 \, .
\end{aligned}
$$

Equation (31b) becomes, upon substitution, division by $m_o \omega^2 L$ and simplification:

$$
\begin{aligned}
&\Big[-\frac{\epsilon_*}{S_*}\left(\frac{V_2}{\omega L}\right)^2 \frac{\partial \bar{\Gamma}_0}{\partial \xi_0} + \frac{\Phi_*^2}{S_*^3}\left(\frac{V_2}{\omega L}\right)^2 I' \frac{\partial^2 \bar{\Phi}_0}{\partial \xi_0^2}\frac{\partial \bar{\Phi}_0}{\partial \xi_0}\bar{Q}_0 + \left(\frac{g'}{\omega^2 L}\right) + \\
&+ k C_{R_*} \bar{C}_R V_* \left(\frac{C}{\omega L}\right)^2 \left(\frac{L}{\pi r_c}\right) \bar{V}_0 \Big] + \left(\frac{\delta_\ell}{L}\right)\frac{\partial \bar{V}_\ell}{\partial \tau} + U_*\left(\frac{C}{V_1}\right)\left(\frac{\delta_{t_1}}{L}\right)\bar{U}_0 \frac{\partial \bar{\Phi}_{t_1}}{\partial \tau} + \\
&+ U_*\left(\frac{C}{V_2}\right)\left(\frac{\delta_{t_2}}{L}\right)\bar{U}_0 \frac{\partial \bar{\Phi}_{t_2}}{\partial \tau} + \left(\frac{\omega L}{V_1}\right)\left(\frac{\delta_{t_1}}{L}\right)^2 \bar{U}_{t_1}\frac{\partial \bar{\Phi}_{t_1}}{\partial \tau} + \left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{\delta_{t_2}}{L}\right)\bar{U}_{t_1}\frac{\partial \bar{\Phi}_{t_2}}{\partial \tau} + \\
&+ \left(\frac{\omega L}{V_1}\right)\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{\delta_{t_2}}{L}\right)\bar{U}_{t_2}\frac{\partial \bar{\Phi}_{t_1}}{\partial \tau} + \left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_{t_2}}{L}\right)^2 \bar{U}_{t_2}\frac{\partial \bar{\Phi}_{t_2}}{\partial \tau} - \left(\frac{\omega L}{V_1}\right)\left(\frac{V_2}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right)^2 \frac{\partial \bar{\Gamma}_{t_1}}{\partial \xi_0} - \\
&- \left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_{t_2}}{L}\right)^2 \frac{\partial \bar{\Gamma}_{t_2}}{\partial \xi_0} - \left(\frac{\delta_\ell}{L}\right)\frac{\partial \bar{\Gamma}_\ell}{\partial \xi_0} + k C_{R_*}\bar{C}_R \left(\frac{C}{\omega L}\right)\left(\frac{\delta_\ell}{L}\right)\left(\frac{L}{\pi r_c}\right)\bar{V}_\ell + \\
&+ \left(\frac{\omega L}{V_1}\right)\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{g'}{\omega^2 L}\right)\bar{\Phi}_{t_1} + \left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_{t_2}}{L}\right)\left(\frac{g'}{\omega^2 L}\right)\bar{\Phi}_{t_2} + \frac{\Phi_*}{S_*^2}\left(\frac{V_2}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right) I' \frac{\partial^2 \bar{\Phi}_0}{\partial \xi_0^2}\frac{\partial \bar{\Phi}_{t_1}}{\partial \xi_0}\bar{Q}_0 + \\
&+ \frac{\Phi_*}{S_*^2}\left(\frac{\delta_{t_2}}{L}\right) I' \frac{\partial^2 \bar{\Phi}_0}{\partial \xi_0^2}\frac{\partial \bar{\Phi}_{t_2}}{\partial \xi_0}\bar{Q}_0 + \left(\frac{\omega L}{V_1}\right)^3\left(\frac{V_2}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right)^2 I' \frac{\partial^2 \bar{\Phi}_{t_1}}{\partial \xi_0^2}\frac{\partial \bar{\Phi}_{t_1}}{\partial \xi_0}\bar{Q}_{t_1} + \\
&+ \left(\frac{\omega L}{V_1}\right)^3\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{\delta_{t_2}}{L}\right) I' \frac{\partial^2 \bar{\Phi}_{t_1}}{\partial \xi_0^2}\frac{\partial \bar{\Phi}_{t_2}}{\partial \xi_0}\bar{Q}_{t_1} + \frac{\Phi_*}{S_*}\left(\frac{\omega L}{V_1}\right)\left(\frac{V_2}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right) I' \frac{\partial^2 \bar{\Phi}_{t_1}}{\partial \xi_0^2}\frac{\partial \bar{\Phi}_0}{\partial \xi_0}\bar{Q}_{t_1} + \\
&+ \frac{\Phi_*}{S_*}\left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_{t_2}}{L}\right) I' \frac{\partial^2 \bar{\Phi}_{t_2}}{\partial \xi_0^2}\frac{\partial \bar{\Phi}_0}{\partial \xi_0}\bar{Q}_{t_2} + \left(\frac{\omega L}{V_1}\right)^2\left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{\delta_{t_2}}{L}\right) I' \frac{\partial^2 \bar{\Phi}_{t_1}}{\partial \xi_0^2}\frac{\partial \bar{\Phi}_{t_1}}{\partial \xi_0}\bar{Q}_{t_2} + \\
&+ \left(\frac{\omega L}{V_2}\right)^3\left(\frac{\delta_{t_2}}{L}\right)^2 I' \frac{\partial^2 \bar{\Phi}_{t_2}}{\partial \xi_0^2}\frac{\partial \bar{\Phi}_{t_2}}{\partial \xi_0}\bar{Q}_{t_2} = 0
\end{aligned}
\tag{49}
$$

where $I' = (I_c / a_c L^2)$

Equation (32d) becomes, upon substitution, division by ω and simplification:

$$
\begin{aligned}
&\Big[\frac{V_*}{S_*}\left(\frac{C}{\omega L}\right)\frac{\partial \bar{V}_0}{\partial \xi_0} + \frac{U_* \Phi_*}{S_*}\left(\frac{C}{\omega L}\right)\bar{U}_0 \frac{\partial \bar{\Phi}_0}{\partial \xi_0} \Big] + \left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_\ell}{L}\right)\frac{\partial \bar{V}_\ell}{\partial \xi_0} + \\
&+ U_*\left(\frac{C}{V_1}\right)\left(\frac{\delta_{t_1}}{L}\right)\bar{U}_0 \frac{\partial \bar{\Phi}_{t_1}}{\partial \xi_0} + U_*\left(\frac{C}{V_2}\right)\left(\frac{\delta_{t_2}}{L}\right)\bar{U}_0 \frac{\partial \bar{\Phi}_{t_2}}{\partial \xi_0} + \frac{\Phi_*}{S_*}\left(\frac{\delta_{t_1}}{L}\right)\bar{U}_{t_1}\frac{\partial \bar{\Phi}_0}{\partial \xi_0} + \\
&+ \left(\frac{\omega L}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right)^2 \bar{U}_{t_1}\frac{\partial \bar{\Phi}_{t_1}}{\partial \xi_0} + \left(\frac{\omega L}{V_2}\right)^2\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{\delta_{t_2}}{L}\right)\bar{U}_{t_1}\frac{\partial \bar{\Phi}_{t_2}}{\partial \xi_0} + \frac{\Phi_*}{S_*}\left(\frac{\delta_{t_2}}{L}\right)\bar{U}_{t_2}\frac{\partial \bar{\Phi}_0}{\partial \xi_0} + \\
&+ \left(\frac{\omega L}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right)\left(\frac{\delta_{t_2}}{L}\right)\bar{U}_{t_2}\frac{\partial \bar{\Phi}_{t_1}}{\partial \xi_0} + \left(\frac{\omega L}{V_2}\right)^2\left(\frac{\delta_{t_2}}{L}\right)^2 \bar{U}_{t_2}\frac{\partial \bar{\Phi}_{t_2}}{\partial \xi_0} - \left(\frac{\omega L}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right)^2 \frac{\partial \bar{\Gamma}_{t_1}}{\partial \tau} \\
&- \left(\frac{\omega L}{V_2}\right)^2\left(\frac{\delta_{t_2}}{L}\right)^2 \frac{\partial \bar{\Gamma}_{t_2}}{\partial \tau} - \left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_\ell}{L}\right)\frac{\partial \bar{\Gamma}_\ell}{\partial \tau} = 0
\end{aligned}
\tag{50}
$$

Equation (31f) becomes, upon substitution, division by K_E/L^2 and simplification:

$$\left[\Phi_*/S_*^2 \frac{\partial^2 \bar{\Phi}_0}{\partial \xi_0^2} \bar{Q}_0 - \Phi_*/S_*^2 \frac{\partial^2 \bar{\Phi}_0}{\partial \xi_0} \right] + \left(\frac{\omega L}{V_1}\right)\left(\frac{\omega L}{V_2}\right)^2\left(\frac{\delta_{t_1}}{L}\right)\frac{\partial^2 \bar{\Phi}_{t_1}}{\partial \tau^2} +$$
$$+ \left(\frac{\omega L}{V_2}\right)^3\left(\frac{\delta_{t_2}}{L}\right)\frac{\partial^2 \bar{\Phi}_{t_2}}{\partial \tau^2} - \left(\frac{\omega L}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right)\frac{\partial^2 \bar{\Phi}_{t_1}}{\partial \xi_0} - \left(\frac{\omega L}{V_2}\right)\left(\frac{\delta_{t_2}}{L}\right)\frac{\partial^2 \bar{\Phi}_{t_2}}{\partial \xi_0} +$$
$$+ \left(\frac{\omega L}{V_1}\right)^2\left(\frac{\delta_{t_1}}{L}\right)\frac{\partial^2 \bar{\Phi}_{t_1}}{\partial \xi_0^2} \bar{Q}_{t_1} + \left(\frac{\omega L}{V_2}\right)^2\left(\frac{\delta_{t_2}}{L}\right)\frac{\partial^2 \bar{\Phi}_{t_2}}{\partial \xi_0} \bar{Q}_{t_2} = 0 \qquad (51)$$

Order-of-Magnitude Analysis

To evaluate the coefficients of each term of equations (47) to (51) and thus the relative importance of these terms, a consistent set of values must be assumed for all dimensional variables. In order to facilitate the analysis, all dimensional variables will be expressed as powers of a small variable, $\varepsilon = 10^{-1}$. Each coefficient in equations (47) to (51) will thus become a power of ε, and it will be easy to identify and neglect higher-order terms.

As the purpose of this simplification procedure is to find a computational method suitable for relatively inelastic cables, typical geometric properties for a cable system with round, non-faired, steel cable will be assumed. Typical properties are:

$$\pi r_c / L = 10^{-4} = \varepsilon^{4}$$
$$I_c / a_c L^2 = 10^{-8} = \varepsilon^{8}$$
$$\rho_c / \rho = 10 = \varepsilon^{-1}$$

$$V_1/\omega L = 1 = \varepsilon^0$$

$$V_2/\omega L = 10 = \varepsilon^{-1}$$

The towing velocity and frequency of oscillation are assumed to be given by:

$$c/\omega L = 10^{-2} = \varepsilon^2$$

$$g/\omega^2 L = 10^{-2} = \varepsilon^2$$

where ω is assumed to be the fundamental frequency of excitation or motion.

The cable damping coefficients can be defined by the typical values

$$C_R = 1.0 = \varepsilon^0$$

$$k = 0.04 \cong \varepsilon$$

$$\kappa = 10^{-2} = \varepsilon^2$$

It should be noted that the value k, which defines the magnitude of longitudinal damping, lies between ε and ε^2. The use of the larger value insures that the effect of longitudinal damping will not be overlooked.

The cable critical angle of free-streaming angle, ϕ_{CR}, can be taken as a typical cable steady-state angle. This angle is given, for a steel cable, by:

$$\frac{\sin^2 \phi_{CR}}{\cos \phi_{CR}} = \frac{\pi}{C_R} \frac{\rho_c - \rho}{\rho} \frac{g r_c}{C^2} = \frac{\pi}{C_R} \frac{\rho_c - \rho}{\rho} \frac{g}{\omega^2 L} \left(\frac{\omega L}{C}\right)^2 \left(\frac{r_c}{L}\right)$$

and thus:

$$\phi_{CR} = 32^\circ$$

A typical cable steady-state reference angle can thus be taken to be:

$$\phi_* = 1 = \varepsilon^0$$

The corresponding steady-state velocity components are thus defined to be:

$$U_* = \sin \phi_* \cong \varepsilon^0$$

$$V_* = \cos \phi_* \cong \varepsilon^0$$

A typical length-scale for the steady-state variables can be assumed to be:

$$S_* = 10 = \varepsilon^{-1}$$

A typical, steady-state strain can be defined by:

$$\varepsilon_* = 10^{-3} = \varepsilon^3$$

Typical amplitudes of transverse and longitudinal motions are:

$$\delta_l / L = \varepsilon^3$$

$$\delta_{t_1} / L = \varepsilon^3$$

$$\delta_{t_2} / L = \varepsilon^4$$

The amplitude of the rotational displacement, δ_{t_2}, is assumed one order-of-magnitude smaller than the other displacements because this is almost certainly a secondary motion.

Substituting all of the above values into equation (47) and expressing all coefficients in powers of ϵ, we obtain:

$$\Big[\Big\{ \bar{r}_0 \frac{\partial \bar{\Phi}_0}{\partial \bar{s}_0} + \bar{U}_0^{\,2} + \phi_* \Big\} \varepsilon^2 + \Big\{ \frac{\partial^2 \bar{\Phi}_0}{\partial \bar{s}_0^{\,2}} \frac{\partial \bar{Q}_0}{\partial \bar{s}_0} + \frac{\partial^3 \bar{\Phi}_0}{\partial \bar{s}_0^{\,3}} Q_0 \Big\} \varepsilon^9 \Big] +$$
$$+ \Big\{ \frac{\partial \bar{U}_{t_1}}{\partial \bar{t}} - \bar{r}_\ell \frac{\partial \bar{\Phi}_0}{\partial \bar{s}_0} + \bar{U}_{t_1} \bar{U}_0 \Big\} \varepsilon^3 + \Big\{ \bar{U}_{t_1}^{\,2} + \frac{\partial \bar{U}_{t_2}}{\partial \bar{t}} - \bar{r}_0 \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{s}_0} + \bar{U}_{t_2} \bar{U}_0 \Big\} \varepsilon^4 +$$
$$+ \Big\{ - \bar{r}_{t_1} \frac{\partial \bar{\Phi}_0}{\partial \bar{s}_0} - \bar{r}_\ell \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{s}_0} + \bar{U}_{t_1} \bar{U}_{t_2} + \bar{\Phi}_{t_1} + \bar{V}_0 \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{t}} \Big\} \varepsilon^5 + \Big\{ \bar{V}_\ell \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{t}} + \bar{U}_{t_1}^{\,2} \Big\} \varepsilon^6 +$$
$$+ \Big\{ \bar{V}_0 \frac{\partial \bar{\Phi}_{t_2}}{\partial \bar{t}} - \bar{r}_0 \frac{\partial \bar{\Phi}_{t_2}}{\partial \bar{s}_0} - \bar{r}_{t_1} \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{s}_0} - \bar{r}_\ell \frac{\partial \bar{\Phi}_{t_2}}{\partial \bar{s}_0} + \bar{\Phi}_{t_2} \Big\} \varepsilon^7 + \bar{V}_\ell \frac{\partial \bar{\Phi}_{t_2}}{\partial \bar{t}} \varepsilon^8 + \qquad (52)$$
$$+ \Big\{ - \bar{r}_{t_1} \frac{\partial \bar{\Phi}_{t_2}}{\partial \bar{s}_0} - \bar{r}_{t_2} \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{s}_0} + \frac{\partial^2 \bar{\Phi}_{t_1}}{\partial \bar{s}_0^{\,2}} \frac{\partial \bar{Q}_{t_1}}{\partial \bar{s}_0} + \frac{\partial^3 \bar{\Phi}}{\partial \bar{s}_0^{\,3}} Q_{t_1} \Big\} \varepsilon^9 - \bar{r}_{t_2} \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{s}_0} \varepsilon^{11}$$
$$+ \Big\{ \frac{\partial^2 \bar{\Phi}_{t_2}}{\partial \bar{s}_0^{\,2}} \frac{\partial \bar{Q}_{t_2}}{\partial \bar{s}_0} + \frac{\partial^3 \bar{\Phi}_{t_2}}{\partial \bar{s}_0^{\,3}} \bar{Q}_{t_2} \Big\} \varepsilon^{13} - \bar{r}_{t_2} \frac{\partial \bar{\Phi}_{t_2}}{\partial \bar{s}_0} \varepsilon^{14} = 0$$

All higher-order terms have been retained for illustrative purposes. The terms in brackets are the steady-state terms. It is clear that all shear terms are negligible. The predominant terms are:

$$\frac{\partial U}{\partial t}$$

and

$$r \frac{\partial \Phi}{\partial s_0}$$

Although the nonlinear, centrifugal acceleration term

$$V \frac{\partial \phi}{\partial t}$$

is of higher order and can be neglected, this equation cannot be decoupled from the longitudinal motions because of the term $r_{\ell} \partial \phi_o / \partial s_o$. The equation cannot be linearized because of the external-damping terms. Since the equation cannot be linearized and because the centrifugal acceleration term does not increase computational difficulties, this term will be retained. The shear term, $\partial Q_o / \partial s_o$ in equation (31a) can be neglected, however, and

$$m_m \frac{\partial U}{\partial t} - m_m V \frac{\partial \phi}{\partial t} + T \frac{\partial \phi}{\partial s_o} + C_R' U|U| - w_m \cos \phi = 0$$

can be taken as the equation of motion.

Substituting the dimensional values into equation (48), we obtain:

$$\left[\left\{ \frac{\partial \bar{U}_o}{\partial \bar{s}_o} - \bar{V}_o \frac{\partial \bar{\Phi}_o}{\partial \bar{s}_o} \right\} \varepsilon^3 \right] + \left\{ \frac{\partial \bar{U}_{t_1}}{\partial \bar{s}_o} - \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{s}_o} \right\} \varepsilon^3 -$$
$$- \bar{V}_{\ell} \frac{\partial \bar{\Phi}_o}{\partial \bar{s}_o} \varepsilon^4 + \left\{ \frac{\partial \bar{U}_{t_2}}{\partial \bar{s}_o} - \bar{V}_o \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{s}_o} + \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{t}} \right\} \varepsilon^5 - \tag{53}$$
$$- \bar{V}_{\ell} \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{s}_o} \varepsilon^6 + \bar{r}_{\ell} \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{t}} \varepsilon^7 + O(\varepsilon^8) = 0$$

where the terms in square brackets again represent the steady-state solution. The nonlinear terms of the form

$$\bar{V} \frac{\partial \bar{\Phi}}{\partial \bar{s}_o}$$

are again of higher order and could be neglected, but are retained for the reasons noted above. The lowest-order terms involve only transverse variables, U_{t_1} and ϕ_{t_1}, but the equation cannot be decoupled from longitudinal variables since equation (31a) cannot be decoupled. Thus, for the purpose of this investigation equation (31c) will not be simplified.

Substituting the dimensional values into equation (49), we obtain:

$$\left[\left\{ -\frac{\partial \bar{r}_0}{\partial \bar{s}_0} + \phi_* \right\} \varepsilon^2 + \bar{V}_0 \, \varepsilon^3 + \frac{\partial^2 \bar{\Phi}_0}{\partial \bar{s}_0} \frac{\partial \bar{\Phi}_0}{\partial \bar{s}_0} \bar{Q}_0 \, \varepsilon^9 \right] + \left\{ \frac{\partial \bar{V}_\ell}{\partial \bar{t}} - \frac{\partial \bar{r}_\ell}{\partial \bar{s}_0} \right\} \varepsilon^3 - \left\{ \frac{\partial \bar{r}_{t_1}}{\partial \bar{s}_0} - V_\ell \right\} \varepsilon^4 + \left\{ \bar{\Phi}_{t_1} + \bar{U}_0 \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{t}} \right\} \varepsilon^5 + o(\varepsilon^6) = 0 \qquad (54)$$

The internal damping and shear terms are clearly of higher-order and can be neglected. The only nonlinear term, the centrifugal acceleration terms of the form

$$\bar{U} \frac{\partial \bar{\Phi}}{\partial \bar{t}}$$

are of higher-order and can be neglected. The external damping term represented by $\bar{V}_\ell$, is one order higher than the lowest-order term and can probably be neglected. The lowest-order term involving non-longitudinal variables, is the term

$$\frac{\partial \bar{r}_{t_1}}{\partial \bar{s}_0}$$

which is also of one order higher than the lowest-order term. Reference to equation (49) shows that the power of ε of this term is proportional to the amplitude, δ_{t_1}, squared, while the power of ε of the lowest-order terms

$$\frac{\partial \bar{V}_1}{\partial \bar{t}} \quad \text{and} \quad \frac{\partial \bar{r}_1}{\partial \bar{s}_o}$$

are proportional to the first power of amplitude, δ_ℓ. Thus, if amplitude ratios δ_{t_1}/L and δ_ℓ/L, of 10^{-2}, rather than 10^{-3}, are assumed, the leading terms of equation (54) become:

$$\left\{ \frac{\partial \bar{V}_1}{\partial \bar{t}} - \frac{\partial \bar{r}_1}{\partial \bar{s}_o} + \frac{\partial \bar{r}_{t_1}}{\partial \bar{s}_o} \right\} \varepsilon^2 + \bar{V}_1 \varepsilon^3 + \cdots\cdots \tag{54'}$$

In this case the term $\partial \bar{r}_{t_1}/\partial \bar{s}_o$ is not of higher order and cannot be neglected. For most cases, where the amplitude ratios should be expected to be 10^{-3} or less, this term can almost certainly be neglected. Equation (31b) can then be satisfactorily approximated, after equating steady-state terms to zero, by:

$$\frac{\partial V_1}{\partial t} - V_2^{\,2} \frac{\partial r_1}{\partial s_o}$$

which is shown later to be the wave equation.

Substituting the dimensional values into equation (50), we obtain:

$$\left[\left\{ \frac{\partial \bar{V}_o}{\partial \bar{s}_o} + \bar{U}_o \frac{\partial \bar{\Phi}_o}{\partial \bar{s}_o} \right\} \varepsilon^3 \right] + \left\{ \frac{\partial \bar{V}_1}{\partial \bar{s}_o} - \frac{\partial \bar{r}_1}{\partial \bar{t}} + \bar{U}_{t_1} \frac{\partial \bar{\Phi}_o}{\partial \bar{s}_o} \right\} \varepsilon \; +$$
$$+ \left\{ \bar{U}_o \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{s}_o} + \bar{U}_{t_2} \frac{\partial \bar{\Phi}_o}{\partial \bar{s}_o} \right\} \varepsilon^5 + \left\{ \bar{U}_{t_1} \frac{\partial \bar{\Phi}_{t_1}}{\partial \bar{s}_o} - \frac{\partial \bar{r}_{t_1}}{\partial \bar{t}} \right\} \varepsilon^6 + o(\varepsilon^7) = 0 \tag{55}$$

The nonlinear terms of the form

$$\bar{U}\frac{\partial \Phi}{\partial \xi_0}$$

are higher order and can be neglected. The only term containing a transverse variable which is of lowest-order is the linear term

$$\bar{U}_{t_1}\frac{\partial \Phi_0}{\partial \xi_0}$$

Reference to equation (50) shows that the exponents of ε for all three lowest-order terms are proportional to amplitude, so that the relative importance of these terms is independent of assumed amplitudes. Equation (31d) can thus be approximated by:

$$\frac{\partial V_\ell}{\partial s_0} - \frac{\partial r_1}{\partial t} + U_{t_1}\frac{\partial \Phi_0}{\partial \xi_0} = 0$$

The value of the transverse-velocity component can probably be approximated to facilitate solution of the longitudinal equations of motion.

Substituting the dimensional values into equation (51), we obtain:

$$\left[\left\{\frac{\partial^2 \Phi_0}{\partial \xi_0^2}\bar{Q}_0 - \frac{\partial^2 \Phi_0}{\partial \xi_0^2}\right\}\varepsilon^2\right] + \left\{\frac{\partial^2 \Phi_{t_1}}{\partial \xi_0^2}\bar{Q}_{t_1} - \frac{\partial^2 \Phi_{t_1}}{\partial \xi_0^2}\right\}\varepsilon^3 + \qquad (56)$$
$$+ \frac{\partial^2 \Phi_{t_1}}{\partial t^2}\varepsilon^5 + \left\{\frac{\partial^2 \Phi_{t_2}}{\partial \xi_0^2}\bar{Q}_{t_2} - \frac{\partial^2 \Phi_{t_2}}{\partial \xi_0^2}\right\}\varepsilon^6 + o(\varepsilon^7) = 0$$

All dynamic, or time dependent terms are clearly of higher order and can be neglected. The resulting equation becomes

$$Q = K_B \frac{\partial^2 \phi}{\partial s_o^2}$$

the well known beam equation. It is clear that bending dynamics are not important and can be neglected. The consequence of this conclusion is that equations (31f) to (31h) and (31j) can be dropped. Since the internal damping has been eliminated from all equations, equation (31i) can also be dropped.

For a highly elastic material, such as polypropylene, the relative importance of some of the terms in equations (52) to (56) will change. Differences will arise from the lower characteristic velocity for such materials. The primary effect of this lower characteristic velocity will be to increase the relative importance of terms involving longitudinal variables, and particularly spatial derivatives of longitudinal variables.

Summary of Simplified Equations of Motion

Based on the preceding analysis, bending and internal damping terms can be neglected in all equations, and the original set of equations, equations (31), reduces to four equations, two governing transverse and two longitudinal motions. The analysis indicates that the nonlinear terms $V\partial\phi/\partial t$, $U\partial\phi/\partial t$ and $V\partial\phi/\partial s_o$ in equations (31a), (31b) and

(31c) can be neglected, and the term $U\partial\phi/\partial s_o$ in equation (31b) replaced by the linear term:

$$U_t \frac{\partial\phi_o}{\partial s_o} = (U - U_o)\frac{\partial\phi_o}{\partial s_o}$$

The external-damping term in equation (31c) may be neglected, but the nonlinear, external-damping term in equation (31a) cannot be neglected. Equation (31a) is thus inherently nonlinear and cannot be solved analytically. Equations (31b) and (31d) are essentially linear, however, and can be solved analytically if the term $U_t\,\partial\phi/\partial s_o$ is treated approximately. These equations can thus be solved analytically. Since equation (31a) is nonlinear, the terms $V\partial\phi/\partial t$ and $V\partial\phi/\partial s_o$ can be retained with little increases in computation difficulty, and this will be done.

The simplified equations of motion which will be used to compute cable system dynamics are:

$$m_m \frac{\partial U}{\partial t} - m_m V \frac{\partial\phi}{\partial t} + E a_c (r-1) \frac{\partial\phi}{\partial s_o} + C_R' U|U| - w_m \cos\phi = 0 \quad \text{(a)}$$

$$\frac{\partial U}{\partial s_o} - V\frac{\partial\phi}{\partial s_o} + r\frac{\partial\phi}{\partial t} = 0 \quad \text{(b)}$$

$$\frac{\partial V_1}{\partial t} - V_2^2 \frac{\partial r_1}{\partial s_o} = 0 \quad \text{(c)} \qquad (57)$$

$$\frac{\partial V_1}{\partial s_o} - \frac{\partial r_1}{\partial t} + U_t(\bar{s})\frac{\partial\phi_o(\bar{s})}{\partial s_o} = 0 \quad \text{(d)}$$

where U is the total transverse velocity

ϕ is the cable angle of inclination

V_ℓ is the unsteady longitudinal velocity, $V_\ell = V - V_o$,

r_ℓ is the unsteady strain due to longitudinal motions, $r_\ell = r - r_o$,

U_t is the unsteady transverse velocity, $U_t = U - U_o$,

ϕ_o is the steady-state angle of inclination.

Equations (57a) and (57b) reduce by the method of characteristics to two ordinary differential equations, equations (43). These equations can be solved for U and ϕ. Equations (57c) and (57d) can be solved analytically for V_ℓ and r_ℓ, and thus for V and r, using Laplace transform methods. The calculated values of V and r are then used in the solution of equations (43). In the next section suitable boundary conditions are discussed. In the following sections, solutions of equations (57) are discussed.

BOUNDARY CONDITIONS

Suitable boundary conditions at the two ends of the cable are required to calculate cable-system motions. Because the primary purpose of this investigation is to study methods of solution rather than to study cable and body hydrodynamics, only simplified boundary conditions will be considered for the towed body. In general most body end boundary conditions which can be linearized can be used in the computational methods considered here.

For any towed-cable system, the boundary conditions at the upper (towing) end will usually be specified, time-dependent displacements or velocities of the towing point. Since it is reasonable to assume that the cable will be unrestricted in bending at the towing point, it is necessary to specify only two translational components of displacement in the $\bar{x}$ and $\bar{y}$ (horizontal and vertical) directions. Required displacements or velocities in cable coordinate directions ($\bar{n}$ and $\bar{s}$) can be determined from specified values in the $\bar{x}$ and $\bar{y}$ directions:

$$\left.\begin{aligned} \delta_n(L,t) &= \delta_n(t) = \delta_x \sin\phi(L) - \delta_y \cos\phi(L) \\ \text{or} \quad U(L,t) &= U_L(t) = u_L \sin\phi(L) - v_L \cos\phi(L) \end{aligned}\right\} \tag{58}$$

and

$$\left.\begin{aligned} \delta_s(L,t) &= \delta_s(t) = \delta_x \cos\phi(L) + \delta_y \sin\phi(L) \\ V(L,t) &= V_L(t) = u_L \cos\phi(L) + v_L \sin\phi(L) \end{aligned}\right\} \tag{59}$$

where

δ_n and δ_s are towing-point displacements in the cable (n,s) coordinate system,

U_L and V_L are towing-point velocities in the cable coordinate system,

δ_x and δ_y are displacements in the (x,y) coordinate system,

u_L and v_L are velocities in the (x,y) coordinate system,

$\phi(L)$ is the cable inclination at the towing point.

Similar relations can be derived for towing-point motions specified in other coordinate systems.

Boundary conditions corresponding to a towed body are typically very complicated. For all but the simplest body geometries, these boundary conditions are highly nonlinear. While such boundary conditions can be approximated by finite-difference equations, iterative solutions will usually be required. Even the simple boundary conditions proposed by Whicker (5) require approximate finite-difference equations.

Simple body boundary conditions can be derived by considering limiting cases of the boundary condition propose by Whicker (5):

$$\left.\begin{aligned} T_b \cos\phi(0) - D_b &= m_x \frac{\partial u_b}{\partial t} \\ T_b \sin\phi(0) - L_b &= m_y \frac{\partial v_b}{\partial t} \end{aligned}\right\} \qquad (60)$$

where T_b is the cable tension at the body,

D_b is the body drag,

L_b is the net body lift (dynamic lift plus buoyancy less weight),

m_x and m_y are the body masses (including added mass) in the x and y directions,

u_b and v_b are the body velocities in the x and y directions,

$\phi(o)$ is the cable inclination at the body.

Equations (60) are nonlinear, even if the body lift and drag are assumed constant. If the body mass and forces are considered very large compared with cable mass and forces, the body accelerations can be neglected and the boundary conditions become:

$$\begin{aligned} U_o(t) &= -C \sin\phi(0) \\ V_o(t) &= C \cos\phi(0) \end{aligned} \qquad (61)$$

Neglecting the acceleration terms in equations (61) leads to the conditions:

$$T_b = D_b \cos\phi(0) + L_b \sin\phi(0)$$

$$\phi(0) = \tan^{-1}(L_b/D_b)$$

Since the body is assumed to have zero acceleration, the lift and drag must remain constant and equal to their steady-state values so that

$$\phi(0) = \tan^{-1}(L_b/D_b) \cong \tan^{-1}(L_{b_o}/D_{b_o}) = \phi_o(0)$$

where the subscripts o denote the steady-state values

Equations (61) therefore become:

$$U_o(t) \cong -C \sin \phi_o(0)$$
$$V_o(t) \cong C \cos \phi_o(0) \qquad (62)$$

where $\phi_o(0)$ is the steady-state angle at $s_o=0$. Equations (62) are linear boundary conditions.

If the body mass and forces are assumed negligible, the cable acts like a free-ended cable and the boundary conditions become:

$$T_b = 0$$
$$\frac{\partial \phi}{\partial s_o} = 0 \qquad (63)$$

The latter condition guarantees that there is no moment at the end of the cable. These boundary conditions are linear.

In order to solve analytically, the simplified, linear equations of longitudinal motion, equations (57), linear boundary conditions in the longitudinal direction are required. The longitudinal boundary condition discussed in this section can, for most cable systems, be linearized with only minor approximations. The boundary conditions for transverse motions can also be linearized, although this is not essential. Linearizing these boundary conditions will usually lead to reduced computational times.

For most towed-cable systems, and particularly those with steel wire rope cables, which have a large critical angle:

$$\phi_o(L) = o(1) \quad \text{and} \quad \phi'(L) \ll 1$$

and hence

$$\phi_o(L) \gg \phi'(L)$$

where $\phi_o(L)$ is the steady-state component of $\phi(L)$

$\phi'(L)$ is the unsteady component of $\phi(L)$,

$\phi(L) = \phi_o(L) + \phi'(L)$

Making use of this approximation, the boundary condition in the longitudinal direction at the cable upper end can be approximated:

$$\begin{aligned} \delta_s(t) &= \delta_x \cos\phi(L) + \delta_y \sin\phi(L) \\ &= \delta_x\left[\cos\phi_o(L)\cos\phi'(L) - \sin\phi_o(L)\sin\phi'(L)\right] \\ &\quad + \delta_y\left[\sin\phi_o(L)\cos\phi'(L) + \cos\phi_o(L)\sin\phi'(L)\right] \end{aligned}$$

$$\delta_s(t) \cong \delta_x \cos\phi_o(L) + \delta_y \sin\phi_o(L)$$

Equation (64) should be sufficiently accurate, even for buoyant cables where

$$\phi_o(L) \rightarrow 0$$

for in this case the displacement δ_x makes the only important contribution.

SOLUTION OF LONGITUDINAL EQUATIONS OF MOTIONS

The simplified equations of longitudinal motion derived earlier can be solved analytically using Laplace transform methods, if boundary conditions are linear and of suitable form. The resulting solutions, which are essentially independent of the transverse solution, give the longitudinal velocity and strain at any time and position. In this section analytical solutions are derived for the linear boundary conditions considered in the previous section.

The simplified equations of longitudinal motion are:

$$\frac{\partial V'}{\partial t} - V_2^2 \frac{\partial r'}{\partial s_o} = 0 \qquad (57c)$$

$$\frac{\partial V'}{\partial s_o} - \frac{\partial r'}{\partial t} + U_t \frac{\partial \phi_o}{\partial s_o} = 0 \qquad (57d)$$

where V' is the unsteady component of longitudinal velocity

r' is the unsteady component of longitudinal strain

V_2 is the longitudinal velocity of propagation $= \sqrt{E/\rho_c}$

U_t is the transverse velocity of the cable

ϕ_o is the static cable angle

These equations are satisfied by the following changes in variable:

$$V' = \frac{\partial Y}{\partial t}$$

$$r' = \frac{\partial Y}{\partial s_o} + f(\bar{s},t) \qquad (65)$$

where Y is the longitudinal displacement.

Taking derivatives and substituting in equations (57c) and (57d), we obtain:

$$\frac{\partial^2 Y}{\partial t^2} - V_2^2 \frac{\partial^2 Y}{\partial s_o^2} = 0 \tag{66}$$

$$f(\bar{s},t) = \frac{\partial \phi_o(\bar{s})}{\partial s_o} \int_t U(\bar{s},t)\,dt \tag{67}$$

Equation (66) is the well known wave equation, which can be solved for the displacement Y. The longitudinal velocity and strain can be determined from Y using equations (65) and (67). The term $f(\bar{s},t)$ represents the contribution of transverse motions to the strain. It has been shown that this contribution of transverse motions to the strain. It has been shown that this contribution can be significant.

Cables with Free Lower-End

For a cable with a free lower-end, oscillated at the towing point by a harmonic longitudinal velocity, the boundary and initial conditions are

$$Y(L,t) = \sum_{m=1}^{N} a_m \sin m\omega t \; ; \quad \frac{\partial Y(0,t)}{\partial s_o} = 0 \tag{68}$$

$$Y(s_o,0) = \frac{\partial Y(s_o,t)}{\partial s_o} = 0, \text{all } s_o \; ; \; \frac{\partial Y(s_o,0)}{\partial t} = 0 \,, \; 0 \le s_o < L$$

Taking the Laplace transform with respect to time of equation (66), and making use of the initial conditions, we obtain:

$$\frac{d^2 y(s_o,\alpha)}{d s_o^2} - \frac{\alpha^2}{V_2^2}\, y(s_o,\alpha) = 0$$

where α is the transformed variable

$y(s_o, \alpha)$ is the transform of $Y(s_o, t)$.

The solution of this equation is

$$y(s_o, \alpha) = A \sinh\left(\frac{\alpha s_o}{V_2}\right) + B \cosh\left(\frac{\alpha s_o}{V_2}\right)$$

Taking the Laplace transform of the boundary conditions and solving for A and B, the transformed equation becomes

$$y(s_o, \alpha) = \frac{\cosh\left(\frac{\alpha s}{V_2}\right)}{\cosh\left(\frac{\alpha s_o}{V_2}\right)} \sum_{m=1}^{N} a_m \frac{m\omega}{(m\omega)^2 + \alpha^2} \qquad (69)$$

The inverse transform of this equation can be found using residues, as outlined in Chapter 6 of Churchill (23).

The inverse Laplace transform of any function $f(\alpha)$ is given by the inversion integral:

$$F(t) = \mathcal{L}^{-1}[f(\alpha)] = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{\alpha t} f(\alpha)\, d\alpha$$

where α is now considered to be a complex variable = $\alpha_r + i\alpha_i$

and where the function $f(\alpha)$ is analytic everywhere in the half plane $\alpha_r \geq \delta$. For functions which cannot be inverted using tabulated inverse transforms, it is often possible to evaluate this inversion integral using the theory of residues.

By the residue theorem the integral of $e^{\alpha t} f(\alpha)$ around the closed path $C_I + C_O$ in Figure 8, enclosing N poles, S_1, $S_2, .., S_N$, is given by:

$$\int_{C_I} e^{\alpha t} f(\alpha)\,d\alpha + \int_{C_o} e^{\alpha t} f(\alpha)\,d\alpha = 2\pi i \sum_{m=1}^{N} \rho_m(t)$$

where ρ_n is the residue of the pole at S_n

As $\beta_n \to \infty$ and hence $R_o \to \infty$, the integral C_I tends to the inversion integral. Assuming that $\beta_N \to \infty$ as $N \to \infty$, then:

$$\lim_{\substack{N \to \infty \\ \beta \to \infty}} \frac{1}{2\pi i} \int_{C_I} e^{\alpha t} f(\alpha)\,d\alpha \to \mathcal{L}^{-1}[f(\alpha)]$$

and

$$\mathcal{L}^{-1}[f(\alpha)] = \sum_{m=1}^{\infty} \rho_m(t) - \frac{1}{2\pi i} \lim_{R_o \to \infty} \int_{C_o} e^{\alpha t} f(\alpha)\,d\alpha$$

For suitable functions $y(S_o, \alpha)$, the integral over $C_o \to 0$ as $R_o \to \infty$ and hence:

$$\mathcal{L}^{-1}[f(\alpha)] = \sum_{m=1}^{\infty} \rho_m(t) \tag{70}$$

The exact condition for the validity of equation (70) is given by Theorem 10, Section 67 of Churchill (23). This representation is valid for all times $t>0$ if, on suitable (circular arc or parabolic) paths C_o,

$$|f(\alpha)| < \frac{M}{|\alpha|^k} \tag{71}$$

where k is positive and M and k are constants independent of N. If, in addition, $k>1$, equation (70) is valid for all $t \geq 0$. The essential uniqueness of the inverse transform and of equation (70) is demonstrated by a theorem of Lerch (23).

The function $y(s_0,\alpha)e^{\alpha t}$ has simple poles at the values of α:

$$\alpha = \pm i m\omega \quad , m = 1, \cdots, N$$

$$\alpha = \frac{\pi V_2}{L}(m+\tfrac{1}{2})i \quad , m = 0, \pm 1, \pm 2, \cdots$$

The residues of these poles are most easily evaluated using formula (8), Section 66, of Churchill (23). The residues of the poles at $+in\omega$ and $-in\omega$ are:

$$\rho_{\alpha=+im\omega} = a_m \frac{\cos(m\omega s_0/V_2)}{\cos(m\omega L/V_2)} \frac{e^{im\omega t}}{2i}$$

$$\rho_{\alpha=-im\omega} = -a_m \frac{\cos(m\omega s_0/V_2)}{\cos(m\omega L/V_2)} \frac{e^{-im\omega t}}{2i}$$

Adding these values, combining the exponentials to form a sine, and summing over all values of n, we obtain

$$\sum_{m=1}^{N} \rho_{im\omega t} + \rho_{-im\omega t} = \sum_{m=1}^{N} a_m \frac{\cos(m\omega s_0/V_2)}{\cos(m\omega L/V_2)} \sin m\omega t \qquad (72)$$

The residues of the infinity of poles corresponding to zero of cosh $(\alpha L/V_2)$ are obtained in the same way. The residues for the poles corresponding to m=0 and m=-1 are:

$$\rho_{m=0} = \frac{V_2}{L}\cos(\tfrac{1}{2}\pi s_0/L)\frac{e^{\frac{1}{2}i\pi V_2 t/L}}{i}\sum_{m=1}^{N} a_m \frac{m\omega}{(m\omega)^2-(\frac{1}{2}\pi V_2/L)^2}$$

$$\rho_{m=-1} = -\frac{V_2}{L}\cos(\tfrac{1}{2}\pi s_0/L)\frac{e^{-\frac{1}{2}i\pi V_2 t/L}}{i}\sum_{m=1}^{N} a_m \frac{m\omega}{(m\omega)^2-(\frac{1}{2}\pi V_2/L)^2}$$

Adding and simplifying, we obtain:

$$\sum \rho_{m=0} + \rho_{m=-1} = 2\frac{V_2}{L}\cos(\tfrac{1}{2}\pi s_o/L)\sin(\tfrac{1}{2}\pi V_2 t/L) \times$$
$$\times \sum_{m=1}^{N} a_m \frac{m\omega}{(m\omega)^2 - (\tfrac{1}{2}\pi V_2/L)^2}$$

Forming the same sum for other sets of values of $m(1,-2;3,-3;\ldots.)$, the general sum is obtained:

$$\sum \rho_m + \rho_{-(m+1)} = 2\left(\frac{V_2}{L}\right)(-1)^m \cos[(m+\tfrac{1}{2})\pi\ s_o/L]\sin[(m+\tfrac{1}{2})\pi V_2 t/L] \times$$
$$\times \sum_{m=1}^{N} a_m \frac{m\omega}{(m\omega)^2 - (\tfrac{1}{2}\pi V_2/L)^2}$$

Summing for all values of m yields

$$\sum_m \rho_m = 2\left(\frac{V_2}{L}\right)\sum_{m=0}^{\infty}(-1)^m \cos[(m+\tfrac{1}{2})\pi s_o/L]\sin[(m+\tfrac{1}{2})\pi V_2 t/L] \times$$
$$\times \sum_{m=1}^{N} a_m \frac{m\omega}{(m\omega)^2 - [(m+\tfrac{1}{2})\pi V_2/L]^2} \quad (73)$$

The longitudinal displacement is now given by the sum of equation (72) and (73) if the integral over the path C_o is zero. It can be shown that the function $y(s_o,\alpha)$ satisfies the condition of equation (71). First it can be noted that:

$$\left|\cosh\left(\frac{\alpha L}{V_2}\right)\right|^2 = \cosh^2(rL\cos\theta/V_2) + \cos^2(rL\sin\theta/V_2)$$

where $\alpha = re^{i\theta}$

Now if the argument of the cosine is taken to be

$$r\frac{L}{V_2}\sin\theta = r_m L/V_2 = m\pi \quad \text{where } m = 1, 2, \ldots\ldots$$

then:

$$\left|\cosh\left(\frac{\alpha L}{V_2}\right)\right|^2 = \cosh^2(rL\cos\theta/V_2) + 1$$

We now assume that C_o is a circular arc of radius r_n, with center at the origin (α=0). This circular arc cuts the imaginary axis at the points

$$\alpha = \pm i m \pi V_2 / L$$

These points lie midway between the m poles, $\alpha=\pm i(m+\frac{1}{2})\pi V_2/\ell$, and do not correspond to the N poles at $\alpha=\pm in\omega$ unless

$$m\omega = \pi(m+\tfrac{1}{2})V_2/L$$

for any value of n and m. This condition corresponds to resonance, and must be excluded for any bounded solution to exist. Now for α on C_o:

$$\left| y(s_o,\alpha)\sum_m \frac{(m\omega)^2+\alpha^2}{a_m m\omega}\right|^2 = \frac{\cosh^2(rs_o\cos\theta/V_2)+\cos^2(rs_o\sin\theta/L)}{\cosh^2(rL\cos\theta/V_2)+1} \leq 1$$

The limit exists because $0\leq s_o/L\leq 1$ and because $\cosh^2 y$ increases when $|y|$ increases. As $|\alpha|\to\infty$, the term $(n\omega)^2$ in the numerator can be neglected and

$$\left|\frac{y(s_o,\alpha)\,\alpha^2}{\sum_m a_m m\omega}\right| \leq 1$$

Thus $|y(s_o,\alpha)|$ is of order

$$|y(s_o,\alpha)| = O(\alpha^{-2})$$

and the condition of equation (71) is satisfied and the integral over the circular arc C_o is zero.

The longitudinal displacement is thus given by:

$$Y(s_o,t) = \sum_{m=1}^{N} \rho_{imw} + \rho_{-imw} + \sum_{m=0}^{\infty} \rho_m$$

$$Y(s_o,t) = \sum_{m=1}^{N} a_m \frac{\cos(m\omega s_o/V_2)}{\cos(m\omega L/V_2)} \sin m\omega t + \tag{74}$$

$$+2\left(\frac{V_2}{L}\right)\sum_{m=0}^{\infty}(-1)^m \cos\left[(m+\tfrac{1}{2})\pi s_o/L\right] \sin\left[(m+\tfrac{1}{2})\pi V_2 t/L\right] \sum_{m=1}^{\infty} a_m \frac{m\omega}{(m\omega)^2-\left[(m+\frac{1}{2})\pi V_2/L\right]^2}$$

The summation over m must be taken for a sufficient number of terms to insure suitable accuracy. The validity and uniqueness of this solution is insured by the theory of Laplace transforms, as noted earlier. It can be easily shown that equation (74) satisfies all boundary and initial conditions. It can be shown by direct substitution that equation (74) satisfies equation (57c) and is thus the required solution.

The longitudinal velocity and the longitudinal component of strain can be obtained by differentiating equation (74) with respect to t:

$$\begin{aligned} V(s_o,t) &= \omega \sum_{m=1}^{N} a_m m \frac{\cos(m\omega s_o/V_2)}{\cos(m\omega L/V_2)} \cos m\omega t + \\ &+ 2\pi\left(\frac{V_2}{L}\right)^2 \sum_{m=0}^{\infty}(-1)^m (m+\tfrac{1}{2}) \cos\left[(m+\tfrac{1}{2})\pi s_o/L\right] \times \\ &\times \cos\left[(m+\tfrac{1}{2})\pi V_2 t/L\right] \times \sum_{m=1}^{N} a_m \frac{m\omega}{(m\omega)^2-\left[(m+\frac{1}{2})\pi V_2/L\right]^2} \end{aligned} \tag{75}$$

$$\begin{aligned} r(s_o,t) &= \frac{\partial \Phi_o}{\partial s_o} \int_t U(s_o,t)\,dt - \left(\frac{\omega}{V_2}\right) \sum_{m=1}^{N} a_m \frac{\sin(m\omega s_o/V_2)}{\cos(m\omega L/V_2)} \sin m\omega t - \\ &- 2\pi\left(\frac{V_2}{L^2}\right) \sum_{m=0}^{\infty}(-1)^m (m+\tfrac{1}{2}) \sin\left[(m+\tfrac{1}{2})\pi s_o/L\right] \times \\ &\times \cos\left[(m+\tfrac{1}{2})\pi V_2 t/L\right] \times \sum_{m=1}^{N} a_m \frac{m\omega}{(m\omega)^2-\left[(m+\frac{1}{2})\pi V_2/L\right]^2} \end{aligned} \tag{76}$$

The longitudinal velocity, V, can be calculated at any time and position using equation (75). The strain can be calculated at any time and position only if the transverse velocity is known. As noted earlier, the motions become unbounded if:

$$m\omega = \pi(m+\tfrac{1}{2})\pi V_2/L$$

for any values of n and m. This condition corresponds to undamped resonance. It is assumed that this condition does not exist.

Cables with Free Lower-End and External Damping

As noted earlier, external damping can have a significant influence on the longitudinal motions of some cable systems. It is useful to consider the effect of such damping on calculated longitudinal motions. It is necessary to make some approximations in order to obtain a solution using Laplace transforms, but the results clearly indicate the magnitude of the effect of external damping.

The longitudinal equation of motion, including external, hydrodynamic damping is:

$$\frac{\partial^2 Y}{\partial t^2} - V_2^2 \frac{\partial^2 Y}{\partial s_0^2} + C_d' \frac{\partial Y}{\partial t} = 0 \qquad (77)$$

where

$$C_d' = k C_R |C| / m_0$$

Taking the Laplace transform of this equation, introducing the initial conditions, and solving the resulting ordinary differential equation for the transformed variable, y:

$$y(s_o,\alpha) = A \sinh\left(\frac{\alpha^2 + C_d'\alpha}{V_2^2}\right)^{1/2} s_o + B \cosh\left(\frac{\alpha^2 + C_d'\alpha}{V_2^2}\right)^{1/2} s_o$$

Introducing the boundary conditions for a cable with free lower end, equation (68), we obtain

$$y(s_o,\alpha) = \frac{\cosh(\alpha^2 + C_d'\alpha)^{1/2}(s_o/V_2)}{\cosh(\alpha^2 + C_d'\alpha)^{1/2}(L/V_2)} \sum_{m=1}^{N} \frac{a_m m\omega}{(m\omega)^2 + \alpha^2}$$

The inverse transform is found by evaluating the residues of the poles of the function $y(s_o,\alpha)e^{\alpha t}$, as outlined earlier. The poles of this function are:

$$\alpha = \pm i m\omega \qquad m = 1, 2, \cdots N \tag{78a}$$

$$(\alpha^2 + C_d'\alpha)^{1/2}(L/V_2) = (m+\tfrac{1}{2})\pi i \qquad m = 0, \pm 1, \pm 2, \cdots \tag{78b}$$

The position of the latter poles is given by

$$\alpha^2 + C_d'\alpha + [(m+\tfrac{1}{2})\pi L/V_2]^2 = 0$$

and

$$\alpha = \frac{C_d'}{2} \pm \left\{ \left(\frac{C_d'}{2}\right)^2 + [(m+\tfrac{1}{2})\pi i L/V_2]^2 \right\}^{1/2}$$

It can be shown that all of these are simple poles.

The residues of the poles given by (78a) are given by:

$$\rho_{\pm i m\omega} = \pm\sum_{m=1}^{N} a_m \frac{\cosh[(m\omega)^2 - i m\omega C_d']^{1/2}(s_o/V_2)i}{\cosh[(m\omega)^2 - i m\omega C_d']^{1/2}(L/V_2)i} \; \frac{e^{\pm i m\omega t}}{2i}$$

which reduces by the procedures outlined earlier to

$$\sum_{m=1}^{N} p_{\pm imw} = \sum_{m=1}^{N} a_m \frac{\cos[(m\omega)^2 - im\omega C_d']^{1/2}(s_o/V_2)}{\cos[(m\omega)^2 - im\omega C_d']^{1/2}(L/V_2)} \sin m\omega t \tag{79}$$

The complex argument of the cosines can be simplified by rewriting these in exponential form:

$$[(m\omega)^2 - im\omega C_d']^{1/2}(s_o/L) = (\bar{r}\, e^{i\bar{\theta}})^{1/2}(s_o/V_2)$$
$$= (\bar{r}^{1/2} s_o/V_2)\, e^{i\bar{\theta}/2}$$

where

$$\bar{r} = +[(m\omega)^4 + (m\omega C_d')^2]^{1/2} = (m\omega)^2 [1 + \frac{C_d'}{(m\omega)^2}]^{1/2}$$

$$\bar{\theta} = -\tan^{-1}(C_d'/m\omega)$$

The argument is thus given by

$$[(m\omega)^2 - im\omega C_d']^{1/2}(s_o/V_2) = \frac{s_o m\omega}{V_2}[1 + \frac{C_d'}{(m\omega)^2}]^{1/4} e^{\frac{1}{2} i \tan^{-1}(C_d'/m\omega)}$$

For typical steel cable systems:

$$\omega = o(1)$$
$$C_d' = o(10^{-2}) - o(10^{-1})$$
$$s_o/V_2 \leq o(1)$$
$$L/V_2 \leq o(1)$$

so that, for all values of n,

$$C_d'/m\omega \ll 1$$
$$C_d'/(m\omega)^2 \ll 1$$

and, to a very good approximation

$$[(m\omega)^2 - im\omega C_d']^{1/2}(s_o/V_2) \cong m\omega s_o/V_2$$

$$[(m\omega)^2 - im\omega C_d']^{1/2}(L/V_2) \cong m\omega L/V_2$$

The values of the residues, equation (79), are then given by

$$\sum_{m=1}^{N} \rho_{\pm im\omega} \cong \sum_{m=1}^{N} a_m \frac{\cos(m\omega s_0/V_2)}{\cos(m\omega L/V_2)} \sin m\omega t \tag{80}$$

which is the same as the value obtained with no damping.

The residues of the poles corresponding to (78b) are given by:

$$\rho_m = \frac{V_2}{L} \frac{(\alpha^2 + C_d'\alpha)^{1/2}}{(2\alpha + C_d')} \frac{\cosh(\alpha^2 + C_d'\alpha)^{1/2}(s_0/V_2)}{\sinh(\alpha^2 + C_d'\alpha)^{1/2}(L/V_2)} e^{\alpha t} \sum_{m=1}^{N} \frac{a_m m\omega}{(m\omega)^2 + \alpha^2} \tag{81}$$

where

$$(\alpha^2 + C_d'\alpha)^{1/2} = (m + \tfrac{1}{2})\pi V_2 i/L$$

$$2\alpha + C_d' = \pm\left\{C_d' - [(2m+1)\pi V_2/L]^{1/2}\right\}^{1/2}$$

and

$$\alpha = -\frac{C_d'}{2} \pm \left\{\left(\frac{C_d'}{2}\right)^2 - [(m+\tfrac{1}{2})\pi V_2/L]^2\right\}^{1/2}$$

Now, for the typical values noted above:

$$\frac{C_d'}{(2m+1)\pi V_2/L} \ll 1$$

for all values of m, and to a very good approximation

$$2\alpha + C_d' \cong i\,[(2m+1)\pi V_2/L]$$

and

$$\alpha \cong -\frac{C_d'}{2} + i\,[(m+\tfrac{1}{2})\pi V_2/L]$$

Substituting these values into equation (81), we get

$$\rho_m \cong 2\frac{V_2}{L} \frac{\cos[(m+\frac{1}{2})\pi s_0/L]}{(-1)^m} \frac{e^{(m+\frac{1}{2})\pi a t i/L}}{2i} e^{-C_d' t} \times$$

$$\times \sum_{m=1}^{N} \frac{a_n m\omega}{(m\omega)^2 - [(m+\frac{1}{2})\pi V_2/L]^2}$$

Combining the values for $m=-1$, and 0, etc. this reduces to

$$\sum_m \rho_m \cong 2\left(\frac{V_2}{L}\right) e^{-C_d' t} \sum_{m=0}^{\infty} (-1)^m \cos\left[(m+\tfrac{1}{2})\pi s_o/L\right] \times$$

$$\times \sin\left[(m+\tfrac{1}{2})\pi V_2 t/L\right] \times \sum_{m=1}^{N} a_m \frac{m\omega}{(m\omega)^2 - \left[(m+\tfrac{1}{2})\pi V_2/L\right]^2}. \qquad (82)$$

Except for the term $e^{-C_d' t}$, this result is identical with the result for zero damping, equation (72). It is interesting to note that the damping decay term, $e^{-C_d' t}$, is independent of the input mode, n. The magnitude of the damping term can be calculated for the typical properties of a steel-cable system chosen earlier. Assuming a time corresponding to the time for a given disturbance to reach the lower end of the cable, L/V_2:

$$C_d' t = (\rho/\rho_c)\left(\frac{C}{V_2}\right)\left(\frac{L}{\pi r_c}\right) \cong 0.25$$

and

$$e^{-C_d' t} = 0.78$$

which represents an amplitude reduction of 22 percent. This result confirms the conclusion of the earlier analysis that external damping can have a significant effect on longitudinal motions.

The solution for $Y(s_o, t)$ is given by the sum of equations (79) and (82). It can be shown, by the same process used for the equation without damping, that the condition of equation (71) is satisfied because

$$y(s_o, \alpha) = O(\alpha^{-2})$$

This solution cannot be verified exactly by substitution into equation (70) because of the approximations made.

Cables with Fixed Lower-End

For the case where the towed body has a finite drag but such a large mass that its accelerations are negligible, the boundary condition at the cable lower-end is

$$Y(0,t) = 0$$

The other boundary and initial conditions are assumed the same as those of equation (68). Neglecting damping, the differential equation of motion and its Laplace transform are the same as those for the free-end case. Taking the inverse Laplace transform of the boundary conditions and introducing these in the transformed equation, we obtain

$$y(s_0,\alpha) = \frac{\sinh\left(\frac{\alpha s_0}{V_2}\right)}{\sinh\left(\frac{\alpha s_0}{V_2}\right)} \sum_{m=1}^{N} a_m \frac{m\omega}{(m\omega)^2 + \alpha^2}$$

The procedure for finding the inverse transform is very similar to that used for the free-end case, equation (69). The poles of $y(s_0,\alpha)$ are

$$\alpha = \pm i m\omega \ , \quad m = 1, 2, \cdots N$$

$$\alpha = i\pi V_2 m/L \quad , \quad m = 0, \pm 1, \pm 2, \cdots\cdots$$

The first set of poles yieldsthe contribution to the inverse transform

$$\sum_{m=1}^{N} \rho_{im\omega} + \rho_{-im\omega} = \sum_{m=1}^{N} a_m \frac{\sin(m\omega s_o/V_2)}{\sin(m\omega L/V_2)} \sin m\omega t \tag{83}$$

The second set of poles yields the contribution:

$$\sum_{m} \rho_{m} = 2\frac{V_2}{L} \sum_{m=1}^{\infty} \sin(m\pi s_o/L) \sin(m\pi V_2 t/L) \sum_{m=1}^{N} \frac{a_m m\omega}{(m\omega)^2 - (m\pi V_2/L)^2} \tag{84}$$

The longitudinal displacement of the cable is given by the sum of equations (83) and (84):

$$\begin{aligned} Y(s_o,t) = &\sum_{m=1}^{N} a_m \frac{\sin(m\omega s_o/V_2)}{\sin(m\omega L/V_2)} \sin m\omega t + \\ &+ 2\frac{V_2}{L} \sum_{m=1}^{\infty} \sin(m\pi s_o/L) \sin(m\pi V_2 t/L) \sum_{m=1}^{N} \frac{a_m m\omega}{(m\omega)^2 - (m\pi V_2/L)^2} \end{aligned} \tag{85}$$

It can be easily seen that this solution satisfies all boundary and initial conditions. The longitudinal velocity and strain component are obtained by differentiating equation (85) with respect to t and s_o respectively:

$$\begin{aligned} V(s_o,t) = \omega &\sum_{m=1}^{N} m a_m \frac{\sin(m\omega s_o/V_2)}{\sin(m\omega L/V_2)} \cos m\omega t + \\ &+ 2\pi\left(\frac{V_2}{L}\right)^2 \sum_{m=1}^{\infty} m \sin(m\pi s_o/L) \cos(m\pi V_2 t/L) \sum_{m=1}^{N} \frac{a_m m\omega}{(m\omega)^2 - (m\pi V_2/L)^2} \end{aligned} \tag{86}$$

and

$$\begin{aligned} r(s_o,t) = &\frac{\partial \Phi_o}{\partial s_o} \int_t U(s_o,t)\,dt + \left(\frac{\omega}{V_2}\right) \sum_{m=1}^{N} m a_m \frac{\cos(m\omega s_o/V_2)}{\sin(m\omega L/V_2)} \sin m\omega t + \\ &+ 2\pi \frac{V_2}{L^2} \sum_{m=1}^{\infty} m \cos(m\pi s_o/L) \sin(m\pi V_2 t/L) \sum_{m=1}^{N} \frac{a_m m\omega}{(m\omega)^2 - (m\pi V_2/L)^2} \end{aligned} \tag{87}$$

Equation (86) can be used to calculate the longitudinal velocity, V, at any time and position. Equation (87) can be used to calculate the strain at any time and position, if the term dependent on the transverse velocity, U, can be estimated.

It can be shown by the same methods used for the free-end case, that the contribution of the integral over C_o is zero. The finite limit,

$$\left| y(s_o,\alpha) \sum \frac{(m\omega)^2 + \alpha^2}{a_m m\omega} \right|^2 \leq 1$$

occurs in this case because sinh y like cosh y, increases when $|y|$ increases. In this case $y(s_o,\alpha)$ is again of order

$$\left| y(s_o,\alpha) \right| = O(\alpha^{-2})$$

assuring the validity of equation (85) for all times, $t \geq 0$.

SOLUTION OF THE TRANSVERSE EQUATIONS OF MOTION

The ordinary differential equations of transverse motion, equation (43), can be solved using finite-difference methods, as outlined earlier. According to the method of Hartree, equations (39a) and (43), which govern transverse motions are, in finite-difference form:

$$s_o(R) - s_o(P) = \tfrac{1}{2}\Delta t\left[f_p(R) + f_p(P)\right] \qquad \text{(a)}$$

$$s_o(R) - s_o(Q) = \tfrac{1}{2}\Delta t\left[f_q(R) + f_q(Q)\right] \qquad \text{(b)}$$

$$\begin{aligned} U(R) - U(P) = {} & \tfrac{1}{2}\left[V(R) + V(P) - r(R)f_p(R) - r(P)f_p(P)\right] \times \\ & \times\left[\phi(R) - \phi(P)\right] - \tfrac{1}{2}\frac{C_R'}{m_m}\left[U(R)|U(R)| + U(P)|U(P)|\right]\Delta t + \\ & + \tfrac{1}{2}\frac{w_m}{m_m}\left[\cos\phi(R) + \cos\phi(P)\right]\Delta t \qquad \text{(c)} \end{aligned} \qquad (88)$$

$$\begin{aligned} U(Q) - U(Q) = {} & \tfrac{1}{2}\left[V(R) + V(Q) - r(R)f_q(R) - r(Q)f_q(Q)\right] \times \\ & \times\left[\phi(R) - \phi(Q)\right] - \tfrac{1}{2}\frac{C_R'}{m_m}\left[U(R)|U(R)| + U(Q)|U(Q)|\right]\Delta t + \\ & + \tfrac{1}{2}\frac{w_m}{m_m}\left[\cos\phi(R) + \cos\phi(Q)\right]\Delta t \qquad \text{(d)} \end{aligned}$$

where f_p and f_q are the characteristic velocities on the p p and q characteristics; $f_p = (T/m_n r)^{\frac{1}{2}}$, $f_q = -(T/m_n r)^{\frac{1}{2}}$ and where *(R)*, *(P)* and *(Q)* denote the values of the variables at the points *R*, *P* and *Q* in Figure 6. The use of average values for the non-differential damping and cable weight terms is consistent with the accuracy of Hartree's method.

The values of the variables at the points *P* and *Q* must be obtained by interpolation of known values at the points *A*, *B* and *D* in Figure 6. Parabolic interpolation must be used to maintain a second-order truncation error. Using parabolic interpolation, the values of all variables at points P and Q are calculated using equations of the form:

$$f_p(P) = f_p(B) + \frac{\xi}{2}\left[4f_p(A) - 3f_p(B) - f_p(D)\right] + \frac{\xi^2}{2}\left[f_p(B) + f_p(D) - 2f_p(A)\right] \quad \text{(a)}$$

$$f_q(Q) = f_q(D) + \frac{\xi'}{2}\left[4f_q(A) - 3f_q(D) - f_q(B)\right] + \frac{\xi'^2}{2}\left[f_q(D) + f_q(B) - 2f_q(A)\right] \quad \text{(b)} \qquad (89)$$

where the ratios $\xi = \overline{BP}/\overline{BA}$ and $\xi' = \overline{DQ}/\overline{DA}$ are defined in Figure 6. If the characteristic velocities are known at the point R, as well as at the points A, B and D, the values of ξ and ξ' can be determined directly from equations (88a) and (88b) by noting that

$$1-\xi = \frac{s_o(R) - s_o(P)}{\Delta s_o} = \frac{1}{2}\frac{\Delta t}{\Delta s_o}\left[f_p(R) + f_p(P)\right] \quad \text{(a)}$$

$$1-\xi' = \frac{s_o(R) - s_o(Q)}{\Delta s_o} = \frac{1}{2}\frac{\Delta t}{\Delta s_o}\left[f_q(R) + f_q(Q)\right] \qquad (90)$$

If the characteristic velocities are not known at point R, the values of ξ and ξ' must be determined by an iterative process.

Consider calculation of the value ξ for the p characteristic. Combining equations (89a) and (90a) yields:

$$(1-\xi)^2 - (1-\xi)\frac{\left[f_p(D) - f_p(B) + 4\Delta s_o/\Delta t\right]}{\left[f_p(D) + f_p(B) - 2f_p(A)\right]} + \frac{2\left[f_p(A) + f_p(R)\right]}{\left[f_p(D) + f_p(B) - 2f_p(A)\right]} = 0$$

which can be readily solved for $1 - \xi$ or ξ:

$$\xi = 1 - \frac{[f_p(D) - f_p(B) + 4\Delta s_o/\Delta t]}{2[f_p(D) + f_p(B) - 2f_p(A)]} \pm \pm \left\{ \left[\frac{f_p(D) - f_p(B) + 4\Delta s_o/\Delta t}{f_p(D) + f_p(B) - 2f_p(A)} \right]^2 - 2[f_p(A) + f_p(R)] \right\}^{1/2} \tag{91}$$

If the characteristic velocity is not known initially at point R, the value of ξ can be estimated by assuming

$$f_p(P) = \frac{[1 - \xi_{(1)}]\Delta s_o}{\Delta t} \tag{92}$$

where $\xi_{(1)}$ denotes the first approximation of ξ .

Combining equations (89a) and (92) and solving the resulting quadratic equation in ξ yield:

$$\xi_{(1)} = 1 - \frac{[f_p(D) - f_p(B) + 2\Delta s_o/\Delta t]}{2[f_p(D) + f_p(B) - 2f_p(A)]} \pm \pm \left\{ \left[\frac{f_p(D) - f_p(B) + 2\Delta s_o/\Delta t}{f_p(D) + f_p(B) - 2f_p(A)} \right]^2 - 2f_p(A) \right\}^{1/2} \tag{91'}$$

Using an initial estimate, $\xi_{(1)}$, initial estimates of the variables at the point P (and also at point Q) can be made and the values of variables at point R calculated. The calculated value of $f_p(R)$ can be used in equation (90a) to calculate a second estimate of ξ, $\xi_{(2)}$. This process can be repeated until the value of ξ is determined with sufficient accuracy. In the present case, where the longitudinal equation can be solved, and values of f_p and f_q obtained for any time and position, such an iterative procedure will not be required.

The values of the independent variables can be calculated from equations (88) once the values of ξ and ξ'

have been calculated or estimated. The procedure for the end points R_U and R_L and for non-end points, R, Figure 7, are different and will be considered separately.

Calculation for Non-End Points

For non-end points, or points lying within region I in Figure 7, the values of $U(R)$ and $\phi(R)$ are obtained by simultaneous solution of equations (88c) and (88d). Subtracting equation (88d) from equation (88c) and solving for $\phi(R)$, we obtain

$$\begin{aligned}\phi(R) = \Big\{ & 2\left[U(Q)-U(P)\right] + \left[V(R)+V(P)-r(R)f_p(R)-r(P)f_p(P)\right]\times \\ & \times\phi(P) - \left[V(R)+V(Q)-r(R)f_q(R)-r(Q)f_q(Q)\right]\phi(Q) \\ & + \frac{C_R'}{m_m}\left[U(P)\left|U(P)\right| - U(Q)\left|U(Q)\right|\right]\Delta t - g'\left[\cos\phi(Q) - \cos\phi(P)\right]\Big\}\times \\ & \times 1\Big/\left[V(P)-V(Q)-2r(R)f_p(R)-r(P)f_p(P)+r(Q)f_q(Q)\right]\end{aligned} \tag{93}$$

where $g' = \omega_m/m_m = (\rho_c-\rho)\,g/(\rho_c+\rho)$

Adding equations (88c) and (88d) and solving for $U(R)$ yield

$$\begin{aligned}U(R) = \tfrac{1}{4}\Big\{ & 2\left[U(P)+U(Q)\right] + \left[2V(R)+V(P)+V(Q)+r(P)f_p(P)+\right. \\ & \left.+\, r(Q)f_q(Q)\right]\phi(R) - \left[V(R)+V(P)-r(R)f_p(R)-r(P)f_p(P)\right]\times \\ & \times\phi(P) - \left[V(R)+V(Q)-r(R)f_q(R)-r(Q)f_q(Q)\right]\phi(Q) + \\ & + g'\left[2\cos\phi(R)+\cos\phi(P)+\cos\phi(Q)\right] \\ & - \frac{C_R'}{m_m}\left[2U(R)\left|U(R)\right| + U(P)\left|U(P)\right| + U(Q)\left|U(Q)\right|\right]\Big\}\end{aligned} \tag{94}$$

All of the values in equation (93), except perhaps $r(R)$ and $f_p(R)$, are known when ξ and ξ' are known. This equation can therefore be solved, at least to a first approximation, for $\phi(R)$. This value of $\phi(R)$ can then be used in the solution of equation (94) for $U(R)$. The damping term in $U(R)$ makes equation (94) highly nonlinear, so that this equation must be solved by an iterative process. A first estimate of $U(R)$ might be:

$$U(R)_{(1)} = \frac{1}{2}\left[U(P) + U(Q)\right]$$

Two or three iterations should be sufficient to determine $U(R)$ to sufficient accuracy.

Calculation for Upper-End Point

Only a p characteristic passes through the upper end-point, point R_U in Figure 7. The location of the point P_U and the values of the independent variables at this point must be determined using the values of these variables at the points A_U, B_U and C_U, as no point D_U exists. The parabolic interpolation formulas for the upper end point are of the form:

$$f_p(P) = f_p(B) + \frac{\xi}{2}\left[f_p(A) - f_p(C)\right] + \frac{\xi^2}{2}\left[f_p(A) + f_p(C) - 2f_p(B)\right]$$

etc.

Combining this equation and equation (90) and solving the resulting quadratic equation for ξ yield:

$$\xi = -\frac{[f_p(A)-f_p(C)+4\Delta s_o/\Delta t]}{2[f_p(A)+f_p(C)-2f_p(B)]} \pm$$

$$\pm\left\{\left[\frac{f_p(A)-f_p(C)+4\Delta s_o/\Delta t}{2(f_p(A)+f_p(C)-2f_p(B))}\right]^2 - 2\left[f_p(B)+f_p(R)-2\Delta s_o/\Delta t\right]\right\}^{1/2} \quad (96)$$

This equation is used when $f_p(R)$ is known. When $f_p(R)$ is not known initially, an iterative scheme similar to that for the non-end points can be used.

At the upper end point the velocity $U(R)$ is specified for all times. Equation (87c) can thus be solved directly for $\phi(R)$, with no approximations, once the point P is defined, and the values of the variables at P are calculated.

Calculation for Lower-End Point

Only a q characteristic passes through the lower-end point, point R_L in Figure 7. The location of the point Q_L and the values of independent variables at this point can be determined using values at the points A_L, D_L, E_L and R_L. Equations (95) and (96) can be used to determine ξ' and the values at point Q_L, if the values at points D and E are used in place of the values at points B and C respectively and f_p is replaced by f_q:

$$f_q(Q) = f_q(D) + \frac{\xi'}{2}[f_q(A)-f_q(E)] + \frac{\xi'^2}{2}[f_q(A)+f_q(E)-2f_q(D)] \quad (97)$$

and similarly for ξ'.

For the fixed-end case, the transverse velocity at the cable lower end is:

$$U(R_L) = C \sin \phi(R_L);$$

Substituting this value into equation (88d) gives a nonlinear equation for $\phi(R_L)$. This equation can be readily solved by an iterative process. A good first approximation for $\phi(R_L)$ is:

$$\phi(R_L) \cong \phi(A_L)$$

Two or three iterations should be sufficient to determine $U(R_L)$ and $\phi(R_L)$ with sufficient accuracy.

For the free-end case, the lower boundary condition is given by equation (63):

$$\frac{\partial \phi(R_L)}{\partial s_o} = 0$$

The value of $\phi(R_L)$ must be such that this condition is satisfied. Using a parabolic interpolation formula such as equation (97) it can be shown that

$$\frac{\partial \phi(R_L)}{\partial s_o} = \frac{1}{2\Delta s_o}\left[4\phi(R_1) - 3\phi(R_L) - \phi(R_2)\right] \quad \text{(a)}$$

$$\frac{\partial U(R_L)}{\partial s_o} = \frac{1}{2\Delta s_o}\left[4U(R_1) - 3U(R_L) - U(R_2)\right] \quad \text{(b)} \qquad (98)$$

where R_1 and R_2 are points defined in Figure 7. Since the values at R_1 and R_2 are calculated before the value of R_L, equation (98a) can be solved for $\phi(R_L)$. This value can then be substituted into equation (88d) which can be solved for $U(R_L)$.

An alternative method for determining $\phi(R_L)$ and $U(R_L)$, which may be more accurate is available. Noting that everywhere on $\overline{A_L R_L}$, $\partial\phi/\partial s_o \equiv 0$, equation (57b) becomes:

$$\frac{\partial U}{\partial s_o} = -r\frac{\partial \Phi}{\partial t}$$

or, in finite-difference form:

$$\Phi(R_L) = \Phi(A_L) - \tfrac{1}{2}\left[\frac{1}{r(A_L)}\frac{\partial U(A_L)}{\partial s_o} + \frac{1}{r(R_L)}\frac{\partial U(R_L)}{\partial s_o}\right]\Delta t \qquad (99)$$

The values of $\delta U(A_L)/\delta s_o$ can be determined using equation (98b). As an initial estimate

$$\frac{\partial U(R_L)}{\partial s_o} \cong \frac{\partial U(A_L)}{\partial s_o}$$

can be assumed. Equation (99) can be solved for (R_L). This value can be substituted into equation (88d) which is solved, by iteration, for $U(R_L)$. The value of $\partial U(R_L)/\partial s_o$ can then be calculated using equation (98b) and the process repeated. This process is much less direct than that using only equation (98a), but has the advantage that variations of the variables are considered along the characteristic path.

CALCULATION OF CABLE SHAPE

In the preceding sections, methods are presented for calculating the longitudinal and transverse velocities of the cable. The displacements or shape of the cable is generally of greater interest than the cable velocities. In this section, methods of calculating the cable shape as a function of time are considered.

Two methods are available for calculating the cable shape and thus the displacements of any point on the cable. The first is by numerical integration of the velocities starting at time zero. The second is by numerical integration along the cable using the calculated strains and cable angles. The first method admits the possibility of an error which increases monotonically with time due to the cumulative effect of the errors inherent in the numerical integration. Such difficulties have been found by Nath (21). The errors in the second method should be inherently smaller since, at each time increment, the integration is re-initialized from a known position, that at the upper end of the cable. For this reason the second method is used here.

The displacements of the towing point (cable upper end) relative to its steady-state position is assumed to be specified by Fourier series. For convenience, the cable shape will be calculated in the α - β coordinate system of Figure 9. The displacements of the towing point in the α - β system can be calculated from specified displacements in the n-s coordinate system of Figure 2:

$$\delta_\alpha = -\left[\delta_s \cos\phi(L) + \delta_m \sin\phi(L)\right]$$

$$\delta_\beta = \delta_m \cos\phi(L) - \delta_s \sin\phi(L)$$

The relative position of two points on the cable are determined by the average cable strain and average cable inclination between the two points. It is convenient to calculate the relative cable positions at the specified grid points where cable strain and inclination are known. Thus for two adjacent grid-points P_n and P_{n-1}, as shown in Figure 9:

$$\Delta\alpha = \alpha_m - \alpha_{m-1} = \bar{c}\cos\left[(\phi_m + \phi_{m-1})/2\right]$$

$$\Delta\beta = \beta_m - \beta_{m-1} = \bar{c}\sin\left[(\phi_m + \phi_{m-1})/2\right] \qquad (100)$$

where $\bar{c}$ is the chord length shown in Figure 9. The location of the N'th grid-point is given by:

$$\alpha_N = \delta_\alpha + \sum_{m=1}^{N} \Delta\alpha_m$$

$$\beta_N = \delta_\beta + \sum_{m=1}^{N} \Delta\beta_m \qquad (101)$$

where the towing point is defined by $n=0$. The arc length $\bar{a}$ in Figure 9 can be calculated using the average strain at the two end points:

$$\bar{a}_m = \frac{\Delta s_o}{2}\left(r_m + r_{m-1}\right)$$

where r_n and r_{n-1} are the elongations, $1+\varepsilon$, at the points p_n and p_{n-1}, respectively.

The chord length $\bar{c}$ is defined by the arc length and the difference in angle between the two end points

$$\Delta\phi = \phi_m - \phi_{m-1}$$

If the cable shape between any two points is approximated by a segment of a circular arc and the change in angle $\Delta\phi$ is assumed small, i.e.,

$$\Delta\phi \ll 1$$

The chord length is given to a very good approximation by:

$$\bar{c} \cong \bar{a} \; \frac{1 - \tan^2(\Delta\phi/2)}{1 + \tan^2(\Delta\phi/2)} \tag{102}$$

Equations (100), (101) and (102) can be used to calculate the offsets of every grid-point. These offsets are:

$$\alpha_N = \delta_\alpha + \frac{\Delta s_0}{2} \sum_{m=1}^{N} (r_m + r_{m-1}) \frac{1 - \tan^2(\Delta\phi_m)}{1 + \tan^2(\Delta\phi_m)} \cos\bar{\phi}_m$$

$$\beta_N = \delta_\beta + \frac{\Delta s_0}{2} \sum_{m=1}^{N} (r_m + r_{m-1}) \frac{1 - \tan^2(\Delta\phi_m)}{1 + \tan^2(\Delta\phi_m)} \sin\bar{\phi}_m \tag{103}$$

where $\Delta\phi_m = (\phi_m - \phi_{m-1})/2$

$\bar{\phi}_m = (\phi_m + \phi_{m-1})/2$

Equations (103) are solved for each time increment using the calculated angles and strains at that time.

CALCULATION OF STEADY-STATE VARIABLES

In order to calculate the dynamic response of a cable system, it is first necessary to determine the steady-state solution. A number of methods are available for calculating steady-state variables. The method described by Cuthill (24) appears to be the best available method. The computer program given by Cuthill is sufficiently flexible to handle almost any cable or cable-towed body system. Using this program, the cable tension and cable inclination can be calculated at any desired spacing along the cable length.

A detailed description of Cuthill's program is given in Reference (24). The required inputs are cable geometry (length, diameter, density) and any specified tension and cable angle at the lower (body) end of the cable. If there is no body at the lower end of the cable, the cable will be straight and have an inclination equal to the cable critical angle. The critical angle is given by:

$$\frac{\sin^2 \phi_{CR}}{\cos \phi_{CR}} = \frac{\pi}{C_R} \frac{\rho_c}{\rho} \frac{g r_c}{C^2}$$

When there is a towed body, the cable angle at the body is determined by the body lift-drag ratio:

$$\phi_b = \tan^{-1}(L_b/D_b)$$

The tension at the body is given by:

$$T_b = D_b \cos \phi(0) + L_b \sin \phi(0)$$

In using Cuthill's program, care must be taken to define the value of ϕ_b in the correct quadrant.

There are several options in Cuthill's program for specifying the external drag coefficients of the cable. One option uses the drag coefficients given by Whicker (5). As these are very close to those used in the present analysis, this option can be used with negligible error.

It should be noted that Cuthill's program does not take the effect of cable elasticity into effect. For typical tensions and strains $(\varepsilon = o(10^{-3}))$ the effect of elasticity on cable steady-state inclination and tensions will be very small and can be neglected.

COMPUTER PROGRAMS

Two Fortran IV programs have been developed for calculating the dynamic motions and shape of cable systems having a free lower-end and having a lower-end fixed relative to the steady-state towing point. Source listing for these programs, and descriptions and examples of program input and output are given in Appendices I and II. The steady-state cable shape is calculated using a Fortran IV program developed and described in detail by Cuthill.

The two programs for calculating the dynamic motions and shape of the cable are identical except for the equations and terms describing the lower-end boundary conditions. A common flow chart for both programs is given in Appendix III. Each program is divided into two main parts, one for calculating longitudinal motions, the other for calculating transverse motions. The longitudinal motions are calculated and the results used to calculate the transverse motions. Because of the contribution of transverse motions to strain and tension, an iterative process is required. Iteration is also required in the solution of the transverse equations of motion because of the large magnitude of the nonlinear external damping term.

These computer programs have been used to calculate the dynamic response of a number of cable system having free and fixed lower ends and appropriate steady-state solutions. These calculations were carried out not only to illustrate typical response, but also to show the effect of various parameters such as external damping. Calculated results are discussed in the next section.

DISCUSSION OF CALCULATED CABLE-SYSTEM RESPONSE

The computer programs described in Appendices I and II have been used to calculate dynamic response of typical cable systems for a number of assumed operating conditions. The calculations are primarily designed to illustrate the effect of important and critical parameters on response and to verify the accuracy of the computer programs. Calculations have been carried out for non-faired cables of steel and a highly elastic material (such as polypropylene). The assumed cable-system geometry and the calculated results are discussed in the next sections.

Assumed Cable-System Characteristics

Since the primary purpose of this investigation is to develop a computational technique, no attempt has been made to make systematic calculations. All calculations have been carried out for a typical non-faired cable having the following geometric properties:

	Steel Cable	Plastic Cable
Cable Length-L	1000 feet	1000
Cable diameter-d_c	0.2 feet	0.2 feet
Cable density-ρ_c	15	2
Modulus of Elasticity-E	30×10^6 psi	0.2×10^6 psi

A water density of two, corresponding to salt water, has been used. The following values have been used for cable external-drag coefficients:

$$C_R = 0, 1.0$$

$$kC_R = 0, 0.04$$

The calculations designed to check the accuracy of the computer programs have been carried out for cables which are initially horizontal $(\phi_o \equiv 0)$ and which have zero towing velocity $(C \equiv 0)$. The steady-state tensions assumed for these calculations are as follows:

Cables with free ends: $T = 20 \times s_o$ pounds (0-20,000 pounds)

Cables with fixed ends: $T = 10{,}000$ pounds

In order to check the calculated longitudinal response, the following motions of the towing point have been used:

$$\delta_s(L,t) = 10 \sin t \quad (a_1 = 10, \omega_1 = 1)$$

$$\delta_n(L,t) \equiv 0.$$

In order to check the calculated transverse response, the following motions of the towing point have been used:

$$\delta_s(L,t) \equiv 0$$

$$\delta_n(L,t) = -\cos t \quad (a_1 = 1, \omega = 1)$$

or

$$U(L,t) = \sin t$$

In both cases zero external damping has been assumed.

Additional calculations have been carried out to study the response to combined longitudinal and transverse motions of typical cable systems having more varied and more realistic operating conditions. Calculations have been carried out for two towing velocities (10 and 20 feet per second), two amplitudes of towing point motion (1 and 10 feet) and for steady-state solutions corresponding to different assumed towed-body characteristics.

Verification of Numerical Calculations

The numerical calculations have been checked for the simplified cases noted earlier. Both the calculated longitudinal and transverse motions have been checked by hand using appropriate analytical solutions. Calculated longitudinal displacements and dynamic tension for cables with fixed ends and free ends have been checked using Equations (73), (75), (85) and (87). In all cases results are found to be in very good agreement. Comparisons of calculated dynamic tensions are shown in Figures 10, 11 and 12. Calculated transverse velocities have been checked for cables with fixed ends using the analytical solution discussed below. Again the results are in good agreement, as seen in Figure 13.

An analytical solution for transverse motions in a cable with a fixed lower end can be obtained from equations (57a) and (57b) if the problem is suitably restricted. In particular, if the cable is assumed to be neutrally buoyant and thus initially horizontal, is assumed to have uniform tension and is assumed to have no longitudinal motion $(V \equiv 0)$, equations (57a) and (57b) reduce to:

$$\frac{\partial U}{\partial t} + \frac{T}{m_n} \frac{\partial \phi}{\partial s_o} = 0 \qquad \text{(a)}$$

$$\frac{\partial U}{\partial s_o} + r \frac{\partial \phi}{\partial t} = 0 \qquad \text{(b)} \qquad (104)$$

Assuming that the velocity and inclination can be defined in terms of a transverse displacement X:

$$U = \frac{\partial X}{\partial t}$$

$$\phi = -\frac{1}{r} \frac{\partial X}{\partial s_o}$$

equation (104b) is identically zero and equation (104a) becomes:

$$\frac{\partial^2 X}{\partial t^2} - \frac{T}{m_n r} \frac{\partial^2 X}{\partial s_o^2} = 0 \qquad (105)$$

Since T and r are assumed uniform and constant, this is the wave equation with velocity of propagation

$$V_1 = \pm \sqrt{\frac{1}{m_n r} T}$$

which is, as expected, the characteristic velocity for transverse motions. Equation (105) can be solved by the same method used to solve equation (66). The assumed boundary conditions are:

$$X(o,t) = 0$$

$$U(L,t) = \sum_{m=1}^{N} a_n \sin m\omega t$$

The transverse displacement is given by:

$$X(s_o,t) = \int_0^t \Big[\sum_{m=1}^{N} a_m \frac{\sin(m\omega s_o/V_2)}{\sin(m\omega L/V_2)} \sin m\omega t \; +$$

$$+ 2\left(\frac{V_2}{L}\right) \sum_{m=1}^{\infty} \sin(m\pi s_o/L) \sin(m\pi V_2 t/L) \sum_{m=1}^{N} a_m \frac{n\omega}{(m\omega)^2 - (m\pi V_2/L)^2} \Big] dt \quad (106)$$

The transverse velocity is given by:

$$U(s_o,t) = \sum_{m=1}^{N} a_m \frac{\sin(m\omega s_o/V_2)}{\sin(m\omega L/V_2)} \sin m\omega t \; +$$

$$+ 2\left(\frac{V_2}{L}\right) \sum_{m=1}^{\infty} \sin(m\pi s_o/L) \sin(m\pi V_2 t/L) \sum_{m=1}^{N} a_m \frac{m\omega}{(m\omega)^2 - (m\pi V_2/L)^2} \quad (107)$$

Equation (107) has been used to obtain the hand-calculated values shown in Figure 13.

Effect of Sensitive Parameters

The accuracy of numerical solutions is sensitive to two parameters or values. The solution of the longitudinal equations of motion (equations (73), etc.) is sensitive to the number of terms, M, used to approximate the infinite series in these solutions. The solution of the transverse equations of motion (equations (93), (94), etc.) is sensitive to the assumed spatial increment Δs_o.

The infinite series in the solution of the longitudinal equations of motion are convergent, but the number of terms required to approximate the infinite series with sufficient accuracy must be determined. Several studies have been carried out to determine suitable values of M, the number of terms in the series. Typical results of these studies are shown in Figures 10, 11 and 12. Values of M of 10, 20 and 40 have been considered, although computed results are only shown for 10 and 40 terms in most cases. Figure 10 shows calculated dynamic tensions for a plastic cable with free lower end. Figure 11 shows calculated dynamic tensions for a steel cable with free lower-end. Figure 12 shows dynamic tensions for a plastic cable with fixed lower-end. Tensions calculated by hand from the analytical solutions are also shown.

It is clear from Figures 10-12 that at least 40 terms ($M=40$) are generally required to give a good approximation of the initial tension wave front. Maximum tensions calculated with 10 terms, differ considerably from maximum tensions calculated with 40 terms. The difference is particularly noticeable for the case of the steel cable at 0.116 seconds. Based on these calculations, 40 terms have been used in all subsequent calculations.

The accuracy of the methods used for solving the transverse equations of motion will increase as the spatial increment Δs_o is decreased. Because computational times increase as the inverse square of Δs_o, it is desirable to use the largest suitable value of Δs_o. It was decided that

50 feet was a maximum suitable spatial increment for a 1000 foot long cable. Calculations of transverse motions of plastic cables were carried out with 50 and 20 foot increments. Originally it was intended also to carry out calculations for 10 foot increments, but this appeared unnecessary. Because of the small differences in the calculated values, it is not feasible to show the results in graphic form. Typical results are therefore shown in Table II.

Long. Position s_o	t = 3.21 seconds				t = 6.095 seconds				t = 8.02 seconds			
	U		100ϕ		U		100ϕ		U		100ϕ	
$\Delta s_o \rightarrow$	50	20	50	20	50	20	50	20	50	20	50	20
1000	.07	.07	-.02	-.02	.19	.19	-.07	-.07	.49	.99	.87	.88
900	.28	.29	.10	.10	.52	.52	-.18	-.18	.58	.55	.84	.85
800	.60	.60	.21	.21	.80	.78	-.27	-.27	.15	.08	.72	.72
700	.83	.84	.29	.30	1.09	1.05	-.28	-.30	-.19	-.26	.55	.55
600	.97	.97	.34	.34	1.38	1.37	-.21	-.22	-.38	-.41	.35	.36
500	.83	.84	.34	.35	1.56	1.58	-.09	-.09	-.47	-.49	.14	.14
400	.86	.87	.30	.31	1.57	1.60	.05	.05	-.48	-.50	.10	.10
300	.64	.65	.23	.23	1.39	1.42	.18	.19	-.43	-.44	-.32	-.32
200	.35	.35	.12	.12	1.03	1.06	.29	.30	-.33	-.33	-.50	-.51
100	.09	.06	.03	.02	.55	.56	.36	.37	-.17	-.18	-.62	-.63
0	0	0	0	0	0	0	0	.39	0	0	-.66	-.67

Table II - Effect of Spatial Increment Δs_o on Calculated Transverse Motions of a Plastic Cable with Fixed Lower End (C_R=0, C =0)

The minimum acceptable value of the spatial increment Δs_o is a function of the cable geometry, the velocity of propagation of transverse waves and the frequencies of towing-point motions. As the velocity of propagation decreases or the frequency of motions increases, smaller values of Δs_o will be required. As the cable length decreases, a smaller value of Δs_o should be used. It should be noted, however, that computational times will decrease with decreasing cable length.

Illustrative Calculations

Calculations of combined longitudinal and transverse motions have been carried out for a number of cases to illustrate the effect of various parameters such as damping, towing velocity and initial cable shape on cable motions. All calculations have been carried out for simple harmonic towing point motions of frequency one radian per second (T = 6.28 seconds). Most calculations have been carried out for the case of neutrally-buoyant plastic cables which are initially horizontal. Two sets of calculations have been carried out for cases where the cable is not initially horizontal. Typical calculated results are presented in Figures 14-19 and Tables III and IV and are discussed below.

Figure 14 shows the effect of longitudinal damping, as described by equation (76), on the dynamic tensions in a plastic cable with a free end. It can be seen that the effect of damping is large. The unusual behavior of the decay curves is due to the nature of the damping term which is not really correct at zero cable inclination (see Figure 5). It can be seen that cable dynamic tensions are largely damped after a very short time. The effect of longitudinal damping on cables with fixed lower ends or with large towed bodies will, however, be much smaller. The effect of damping on steel cables will also be much smaller because of the larger cable mass.

Figure 15 shows calculated mid-length (s_o=500 feet) transverse velocities and cable inclinations for a plastic cable with free lower-end and zero towing velocity. These results indicate clearly the large effect of damping on transverse velocity. Figure 16 shows calculated transverse velocities and cable inclinations at s_o=750 and s_o=500 for a plastic cable with fixed lower-end and zero towing velocity. Damping again has a large effect on transverse velocity. Figure 16 illustrates that at distance from the towing point, damping causes an increase in cable inclination. Tables III and IV show the effect of towing velocities of 10 and 20 feet per second on motions. Increasing towing velocity causes an increase in transverse velocity and cable inclination. The effect of towing velocity is moderate and is much smaller than the effect of damping.

Figure 17 illustrates the effect of towing point motion amplitudes on transverse motions of a plastic cable with fixed lower-end and constant steady-state tension of 10,000 pounds. The steady-state inclination is assumed to be zero

Long. Position s_o	t = 1.66 seconds						t = 2.36 seconds					
	C = 0		C = 10		C = 20		C = 0		C = 10		C = 20	
	U	100ϕ	U	100ϕ	U	100ϕ	U	100ϕ	U	100ϕ	U	100ϕ
1000	9.96	10.64	9.96	11.15	9.96	11.68	7.05	11.42	7.05	12.19	7.05	13.01
900	5.65	4.49	5.81	4.82	5.98	5.18	6.86	6.75	7.27	7.37	7.74	8.05
800	3.31	1.98	3.42	2.14	3.54	2.31	4.78	3.42	5.03	3.76	5.29	4.15
700	2.04	.98	2.11	1.05	2.17	1.13	2.97	1.78	3.12	1.96	3.29	2.15
600	1.26	.52	1.30	.55	1.33	.59	1.86	1.00	1.96	1.09	2.06	1.19
500	.73	.28	.74	.30	.76	.32	1.21	.60	1.26	.65	1.31	.70
400	.23	.09	.23	.10	.23	.10	.78	.36	.80	.39	.83	.42
300	.01	0	.01	0	.01	0	.44	.19	.45	.21	.46	.22
200	-	-	-	-	-	-	.12	.05	.12	.06	.12	.06
100	-	-	-	-	-	-	.01	0	.01	0	.01	0
0	-	-	-	-	-	-	0	0	0	0	0	0

Table III - Effect of Towing Velocity on Transverse Motions of a Plastic Cable with Free Lower End (C_R=1.0)

Long. Position s_o	t = 1.63 seconds				t = 2.32 seconds			
	C = 0		C = 10		C = 0		C = 10	
	U	100ϕ	U	100ϕ	U	100ϕ	U	100ϕ
1000	9.98	11.25	9.98	11.88	7.30	11.9	7.30	12.85
900	5.95	4.20	6.14	4.54	4.88	6.22	4.66	6.83
800	3.51	1.73	3.66	1.88	4.17	3.09	4.31	3.42
700	2.20	.85	2.29	.91	2.92	1.61	3.08	1.78
600	1.41	.45	1.46	.48	1.99	.92	2.09	1.00
500	.98	.26	1.01	.28	1.39	.54	1.45	.59
400	.25	.06	.25	.07	1.02	.35	1.06	.37
300	-	-	-	-	.77	.23	.80	.25
200	-	-	-	-	.54	.15	.56	.16
100	-	-	-	-	.09	.03	.09	.03
0	-	-	-	-	0	0	0	0

Table IV - Effect of Towing Velocity on Transverse Motions of a Plastic Cable with Fixed Lower End (C_R=1.0) everywhere. As the motion amplitudes increase, and damping becomes relatively more important, transverse velocities decrease markedly. For an increase in amplitude from one to 10 feet, a decrease in transverse velocity of 50 percent or more occurs. A phase shift is also evident.

Figure 18 shows the transverse motions of a similar plastic cable having a steady-state inclination of 30 degrees at the lower-end. This corresponds to a towed body having a lift-drag ratio of -0.58. The tension at the lower-end is assumed equal to 10,000 pounds. Towing point motion amplitudes of one-foot have been assumed. The steady-state inclination and tensions were calculated using the program of Cuthill (24).

Figure 19 shows calculated transverse motions of a steel cable with fixed lower-end which is being towed at 10 feet per second. The steady-state cable inclination is assumed to be equal everywhere to the critical angle of 43.5 degrees. The assumed tensions correspond to a lower-end (or towed body) tension of 2000 pounds. Towing point motion amplitudes of 0.1 foot have been assumed; if larger amplitudes are assumed, negative tension (compression) occurs after a fraction of a second (real time) and calculations are terminated.

Required computational times are inversely proportional to the square of the spatial increment Δs_o and to the square root of the maximum tension. Computational times are also influenced by the amplitudes of towing point motion and the cable geometry. Increasing input amplitude from one to 10 feet can cause an increase of 50 percent or more in computational time because of the increase in the number of iterations required. Computational times will also increase as the steady-state curvature $(\partial\phi_o/\partial s_o)$ increases because of the increasing significance of the transverse displacement term on strain and tension. Computational times on a CDC 6400 computer are typically 5 to 20 times real time for 1000 foot long cables with spatial increments of 50 feet and average tensions of 10,000 pounds.

CONCLUSIONS

A number of interesting conclusions can be drawn from this investigation. Significant conclusions include:

1. The magnitude of internal damping and bending (shear) forces are small and these forces can be neglected in almost all cases. The inclusion of one or both of these terms in the equations of motion has a profound effect, however. These terms are of higher order than other terms in the equations and make the equations of motion parabolic rather than hyperbolic. Parabolic equations cannot be solved using the method of characteristics.

2. Assuming that the cable is inelastic makes the longitudinal equations of motion parabolic so that the method of characteristics cannot be used for these equations. It also prevents solution of the equations of motion by the analytical methods employed here.

3. If internal damping and bending (shear) forces are neglected,cable system motions are governed by a set of four quasi-linear hyperbolic partial differential equations. If the cable is assumed inelastic these equations become identical with the equations derived by Whicker, except for the transverse added mass which Whicker erroneously neglected.

4. The set of four equations including elasticity effects can be solved using the method of characteristics, but computational times will be excessive due to the large longitudinal characteristic velocity which is equal to cable material sonic velocity and is typically 4000 to

20,000 feet per second. The transverse characteristic velocity, on the other hand, is typically only a few hundred feet per second.

5. An analysis based on a non-dimensionalization process such as used to derive the well-known boundary layer equations in fluid mechanics, can be used to simplify the equations of motion, particularly the longitudinal equations of motion. These equations can be essentially uncoupled from the transverse equations and then solved analytically for reasonable, linearized boundary conditions. Calculations can then be carried out on a reasonable time scale determined by the transverse characteristic velocities. As a result, computational times can be decreased by a factor of 10 or more.

6. Longitudinal damping has a significant effect on dynamic tension of cables with free ends. Dynamic tension may be largely damped after a relatively short time. Longitudinal damping will have a lesser effect on dynamic tension in cables with fixed ends or with large towed bodies.

7. Damping has a significant effect on transverse motions. Transverse motions calculated with typical transverse damping (C_R=1) are much smaller that those calculated with no transverse damping. The amplitude of towing point motion also has a highly nonlinear effect on the transverse motions. Towing velocity has only a moderate effect on transverse motions.

8. Using a CDC 6400 computer, computation times of approximately 5 to 20 times real time are indicated for a cable length of 1000 feet, a spatial increment of 50 feet

and an average tension of 10,000 pounds. Computational times do not vary significantly with cable material.

REFERENCES

1. Casarella, Mario J.; Parsons, Michael.

"Cable Systems Under Hydrodynamic Loading," Journal of the Marine Technology Society, vol. 4, no. 4, July-August 1970.

2. Wang, Henry J.

"A Two-Degree-of-Freedom Model for the Two-Dimensional Dynamic Motions of Suspended Extensible Cable Systems," Naval Ship Research and Development Center Report 3663, October 1971.

3. Craggs, J. W.

"Wave Motions in Plastic-Elastic Strings," Journal of the Mechanics and Physics of Solids, vol. 2, 1954.

4. Critescu, N.

"Spatial Motions of Elastic-Plastic Strings," Journal of the Mechanics and Physics of Solids, vol. 9, 1961.

5. Whicker, Lester Folger.

"The Oscillatory Motions of Cable-Towed Bodies," University of California, Berkeley, Engineering Report, Series 82, Issue 2, May 1957.

6. Springston, G. B., Jr.

"Generalized Hydrodynamic Loading Functions for Bare and Faired Cables in Two-Dimensional Steady-State Cable Configuration," Naval Ship Research and Development Center Report 2424, June 1967.

7. Rouse, Hunter.

Advanced Mechanics of Fluids, John Wiley, New York, 1959.

8. Keulegan, G. H.; Carpenter, L. H.

"Forces on Cylinders and Plates in an Oscillating Fluid," Journal of Research of the National Bureau of Standards, vol. 60, no. 5, May 1958, pp. 423-440.

9. Ivers, William D.; Mudie, John D.

"Towing a Long Cable at Slow Speeds: A Three-Dimensional Dynamic Model," Scripps Institution of Oceanography Report MPL-U-33/71, 1971.

10. Flügge, Wilhelm.

Viscoelasticity, Blaisdell Publishing Co., Waltham, Mass., 1967.

11. Love, Augustus E. H.

A Treatise on the Mathematical Theory of Elasticity, Cambridge University Press, 1934. Also Dover Publications, New York.

12. Wilson, E. L.; Penzien, J.

"Evaluation of Orthogonal Damping Matrices," International Journal for Numerical Methods in Engineering, vol. 4, 1972.

13. Paidoissis, M. P.

"Dynamics of Flexible Slender Cylinders in Axial Flow: Part I, Theory," Journal of Fluid Mechanics, vol. 26, part 4, 1966.

14. Pao, Hsien-Pao.

"Dynamic Stability of a Towed Thin Flexible Cylinder," Catholic University of America, Themis Report 69-11, December 1969.

15. Schram, J. W.

"A Three-Dimensional Analysis of a Towed System," Ph.D. Thesis, Rutgers University, 1968.

16. Nath, John H.

"Dynamics of Single Point Ocean Moorings of a Buoy - A Numerical Model for Solution by Computer," Oregon State University, Department of Oceanography, Reference 69-10, July 1969.

17. Schram, J. W.; Reyle, S. P.

"A Three-Dimensional Dynamic Analysis of a Towed System," AIAA Journal of Hydronautics, vol. 2, no. 4, October 1968.

18. Kerney, Keith P.

"Small-Perturbation Analysis of Oscillatory Tow-Cable Motion," Naval Ship Research and Development Center Report 3430, November 1971.

19. Courant, R.; Friedrichs, K. O.

Supersonic Flow and Shock Waves, Interscience Publishers, Inc., New York, 1948.

20. Courant, R.; Hilbert, D.

Methods of Mathematical Physics - Volume II, Interscience Publishers, Inc., New York, 1966.

21. Nath, John H.

Personal Communications to the author, August 1972.

22. Ames, William F.

Nonlinear Partial Differential Equations in Engineering, Academic Press, New York, 1965.

23. Churchill, Ruel V.

Operational Mathematics, McGraw-Hill Book Company, Inc., New York, 1958.

24. Cuthill, Elizabeth

"A Fortran IV Program for the Calculation of the Equilibrium Configuration of a Flexible Cable in a Uniform Stream," Naval Ship Research and Development Center Report 2531, February 1968.

APPENDIX I

COMPUTER PROGRAM CABL1 FOR CABLES WITH FREE LOWER ENDS

This Fortran IV program which is designed CABL1, calculates the dynamic response of a cable having an arbitrary steady-state shape and a free lower-end. The free lower-end corresponds to a cable with no towed-body or a cable with a very small towed-body whose effect on cable lower-end motions can be neglected. The cable is assumed to have a circular cross-section and uniform properties along its entire length. The cable is assumed to be fully submerged in a fluid of uniform density and current. Cable motions are assumed to be excited by prescribed harmonic motions of the towing point. A source listing for this program is given in this section.

The program is divided into four basic parts. These are: calculation of the time increment; calculation of longitudinal variables and motions; calculation of transverse motions and variables; and calculation of cable offsets or shape. The longitudinal variables are calculated using analytical solutions, equations (73)-(75). The transverse variables are calculated using equations (88). The effect of transverse motions on cable elongation and tension is calculated using equation (67). The cable offsets are calculated using equation (103). Iteration is used to account for the highly nonlinear transverse damping and the effect of transverse motions on cable elongation. A flow chart for the program is given in Appendix III.

The basic program inputs are cable and fluid physical characteristics, towing velocity, cable steady-state shape and tension, desired spatial increment for calculated motions and characteristics of the towing point motion. The spatial increment must be constant. If the cable operates in a uniform current, the towing velocity can be interpreted as the sum of towing and current velocities.

The towing-point motions are specified by two orthogonal, time-dependent motions in the x and y direction of Figure 1. The motions are described by Fourier series:

$$\delta_x(t) = \sum_{m=1}^{m=NX} a_m \sin(m\omega t)$$

$$\delta_y(t) = \sum_{m=1}^{m=NY} a_m' \sin(m\omega t)$$

The numbers of terms NX and NY are independent, and can have any value from one to eight. The fundamental frequency ω is specified.

The allowable time increment is calculated from the prescribed spatial increment and the maximum characteristic velocity for transverse motion at the initial time t_n:

$$\Delta t = 0.9\,\Delta s_o / (f_p)_{MAX}$$

The program determines the maximum tension at each initial time. If the calculated values of ξ or ξ' are less than zero, the program increases the time increment and goes back to the beginning of calculations.

The values of all variables at the points P and Q in Figure 6 are calculated using linear interpolation rather than parabolic interpolation.

Program Mnemonics

A list of mnemonics used in the programs is given below. Mnemonics for input variables are given in the source listing and are not repeated here. The mnemonics are given for control values, calculated constants and working values and variables.

Control Values

NS Grid point identification (NS = 1 at towing-point, NS=NDEL at lower end).

NDEL Number of grid points = number of spatial increments plus one.

NX Number of terms in the series for horizontal towing-point motion.

NY Number of terms in the series for vertical towing-point motion.

NT Maximum number of terms in the series for $\delta_x(t)$ and $\delta_y(t)$.

Calculated Constants and Working Values

A Cable cross-section area a_c.

AN(K) Coefficient of the k'th harmonic in the series for longitudinal towing-point motion, a_n.

ANP(K) Coefficient of the k'th harmonic in the series for transverse towing-point motion, a_n' .

VA Velocity of propagation of longitudinal motions V_2.

VTM Velocity of propagation of transverse motions, V_1 or f.

CMN Total transverse cable mass per unit length - m_n.

TMAX Maximum tension at an initial time, t_n.

RMAX Corresponding maximum extension at an initial time, t_n.

TM Real time, t_n.

DELTM Increment in time Δt.

RATO, RATP Constants ξ and ξ' locating points P and Q in Figure 6.

TP,TQ Tensions at points P and Q.

RP,RQ Extensions at points P and Q.

VP,VQ Longitudinal velocities at points P and Q.

UP,UQ Transverse velocities at points P and Q.

FALFP, FALFQ Transverse characteristic velocities, V_1, at points P and Q.

FALF(N) Transverse characteristic velocity V_1, at the point N where N=1 is point A, N=2 is point B, N=3 is point B, N=4 is point D.

DUDS1 Value of the derivative $\partial U/\partial s_o$ at the point A_L.

DUDS2 Value of the derivative $\partial U/\partial s_o$ at the point R_L.

DFEDS Value of the derivative $\partial \phi_o/\partial s_o$.

Variables

R(1,NS) Extension r at given grid point and at initial time, t_n.

R(2,NS) Extension r at end of time increment, $t=t_n+\Delta t$.

T(2,NS) Tension T at given grid point and at initial time, t_n.

T(3,NS) Tension T at end of time increment, $t=t_n+\Delta t$.

U(1,NS) Transverse velocity U at initial time, t_n.

U(2,NS) Transverse velocity U at end of time increment, $t=t_n+\Delta t$.

V(1,NS) Longitudinal velocity V at initial time, t_n.

V(2,NS) Longitudinal velocity V at end of time increment, $t=t_n+\Delta t$.

X(1,NS) Transverse displacement X at initial time, t_n.

X(2,NS) Transverse displacement X at end of time increment, $t=t_n+\Delta t$.

Y(1,NS) Longitudinal displacement Y at initial time, t_n.

Y(2,NS) Longitudinal displacement Y at end of time increment, $t=t_n+\Delta t$.

PHI(2,NS) Inclination angle ϕ at initial time, t_n.

PHI(3,NS) Inclination angle ϕ at end of time increment, $t=t_n+\Delta t$.

Program Inputs

The required program input variables and corresponding formats are described in detail by comment statements in the program source listing given below. The variable mnemonics and dimensions are also described by the comment statements. The formats are designed to accommodate any normal cable geometry. It should be noted that cable modulus of elasticity divided by 10^6 is used in the input. The total number of data cards required is 4 plus NDEL where NDEL is the specified number of grid points on the cable.

Program Outputs

The program output variables are described in detail by comment statements in the program source listing. The variable mnemonics and dimensions are described by the comment statements. The calculated longitudinal and transverse velocity, cable tension, cable inclination and cable offsets in the α-β coordinate system of Figure 9 are printed out for each specified grid point (distance from the cable lower-end) and for each calculated time increment.

Sample Problem

A printout of a sample run is given following the program source listing. This printout is for a steel cable having a constant inclination equal to the cable critical angle of 43.5 degrees. The assumed cable characteristics and operating variables are:

CL	=	1000 feet
CD	=	0.2 feet
CRHO	=	15 slugs/foot^3
CMOD	=	30 pounds/inch^2 (E = 30 x 10^6 pounds/inch^2)
WRHO	=	2 slugs/foot^3
CR	=	1.0

DELSO	=	50 feet
VT	=	10 feet/second
OM	=	1 radian/second
TF	=	10 seconds
NX	=	1
NY	=	1
NM	=	40
AX(1)	=	2, all other AX(N)=0
AY(1)	=	-1, all other AY(N)=0
PHI(1,NS)	=	0.762 radians
T(1,NS)	=	(10,000 + 0.6 s_o) pounds

Calculated results for the first two time increments are given.

PROGRAM CABL1

SOURCE LISTING

```
      DIMENSION SO(110),V(2,110),T(3,110),R(2,110),Y(2,110),U(2,110),PHI
     1(3,110),X(2,110),XC(110),YC(110)
      DIMENSION AN(8),AX(8),AY(8),ANP(8),FALF(4),DFEDS(1,110),DELR(110)
C
C
C     CALCULATION OF CABLE MOTIONS FOR A CABLE WITH FREE LOWER END
C     COMBINED LONGITUDINAL AND TRANSVERSE MOTIONS
C
C     LONGITUDINAL MOTIONS CALCULATED FROM ANALYTICAL SOLUTION WITH
C     CORRECTION FOR THE EFFECT OF TRANSVERSE DISPLACEMENTS
C
C     TRANSVERSE MOTIONS CALCULATED USING THE METHOD OF CHARACTERISTICS
C     AND COMPLETELY COUPLED TRANSVERSE MOTIONS
C
C
C          REQUIRED PROGRAM INPUT
C
C     THE TOTAL NUMBER OF INPUT CARDS REQUIRED IS 5 PLUS NDEL
C     WHERE NDEL IS THE NUMBER OF SPATIAL INCREMENTS ALONG THE CABLE
C
C
C     THE FIRST CARD CONTAINS THE FOLLOWING DATA AT THE FOLLOWING FORMAT
C
C     CL = CABLE LENGTH IN FEET  F10.5
C     CD = CABLE DIAMETER IN FEET  F10.5
C     CRHO = CABLE MASS DENSITY IN SLUGS PER CUBIC FOOT  F10.5
C     CMOL = CABLE MODULUS OF ELASTICITY IN PSI DIVIDED BY 1000000.
C     F10.5
C     WRHO = WATER MASS DENSITY IN SLUGS PER CUBIC FOOT  F10.5
C     CR = NORMAL DRAG COEFFICIENT FOR STEADY FLOW  F10.5
C     DELSO = INCREMENT IN CABLE LENGTH IN FEET  F10.5
C     VT = TOWING VELOCITY IN FEET PER SECOND  F10.5
C
C     THE SECOND CARD CONTAINS THE FOLLOWING DATA
C
C     CM = FUNDEMENTAL FREQUENCY OF EXCITATION IN RADIANS PER SECOND
C     F10.5
C     TF = REAL TIME AT WHICH CALCULATIONS ARE TERMINATED IN SECUNDS
C     F10.5
C     NX = NUMBER OF TERMS IN FOURIER SERIES FOR DELTA( X )  I5
C     NY = NUMBER OF TERMS IN FOURIER SERIES FOR DELTA ( Y )  I5
C     NM = NUMBER OF TERMS USED TO APPROXIMATE THE SERIES IN THE
C     LONGITUDINAL SOLUTIONS  I5
C
C     THE THIRD THROUGH THIRD PLUS NDEL CARDS CONTAIN THE FOLLOWING DATA
C
C     SO ( NS ) = LONGITUDINAL POSITION ( NS=1 FOR SO=L, NS=NDEL+1 FOR
```

```
C       SC = 0 )  F10.2
C       T ( 1,NS ) = STEADY-STATE TENSION AT SO ( NS )  F10.2
C       PHI ( 1,NS ) = STEADY-STATE INCLINATION AT SO ( NS )  F10.2
C
C       THE FOURTH PLUS NDEL CARD CONTAINS THE FOLLOWING DATA
C
C       AX ( 1 ),.....,AX ( 8 ) WHERE AX ( N ) IS THE COEFFICIENT OF THE N'
C       TH TERM IN THE FOURIER SERIES FOR DELTA SUB X
C
C       THE FIFTH PLUS NDEL CARD CONTAINS THE FOLLOWING DATA
C
C       AY(1), .....,AY(8)  WHERE AY(N) IS THE COEFFICIENT OF THE N'TH
C       TERM IN THE FOURIER SERIES FOR DELTA SUB Y
C
C
C                     PROGRAM OUTPUT
C
C
C      THE FOLLOWING OUTPUT IS PROVIDED FOR EACH TIME INCREMENT
C
C      TIME =    . THE REAL TIME FOR WHICH THE CALCULATED RESULTS APPLY
C
C      SO =    , XC =    , YC =    .
C      WHERE   SO IS THE LONGITUDINAL POSITION ( SO=L IS THE TOWING POINT
C              XC IS THE COORDINATE DELTA SUB ALPHA FOR SO
C              YC IS THE COORDINATE DELTA SUB BETA FOR SO
C
C       SO =    , U =    , PHI =    , T =    , V =    .
C      WHERE   SC IS THE LONGITUDINAL POSITION
C              U IS THE TRANSVERSE VELOCITY AT SO
C              PHI IS THE CABLE INCLINATION AT SC
C              T IS THE CABLE TOTAL TENSION AT SO
C              V IS THE LONGITUDINAL VELOCITY AT SO
C
C
      READ (2,1410) CL,CD,CRHO,CMOD,WRHO,CR,DELSO,VT
      CMOD=CMOD*144000000.
      DELTA=1.+CL/DELSO
      NDEL=DELTA
      READ (2,1415) OM,TF,NX,NY,NM
      PI=3.1416
      G=32.2
      A=PI*CD**2/4.
      DO 20  NS=1,NDEL
      READ (2,1420) SO(NS),T(1,NS),PHI(1,NS)

      U(1,NS)=VT*SIN(PHI(1,NS))

      PHI(2,NS)=PHI(1,NS)
      R(1,NS)=T(1,NS)/(A*CMOD)+1.
      V(1,NS)=VT*COS(PHI(1,NS))
      X(1,NS)=0.
   20 T(2,NS)=T(1,NS)
C
C     CALCULATION OF CABLE STEADY STATE CURVATURE
C
      NLIM=NDEL-1
      DFEDS(1,1)=-(4.*PHI(1,2)-3.*PHI(1,1)-PHI(1,3))/(2.*DELSO)
      DFEDS(1,NDEL)=(4.*PHI(1,NLIM)-3.*PHI(1,NDEL)-PHI(1,NDEL-2))/(2.*
```

```
      NT=NX
      IF (NT-NY) 110,120,120
  110 NT=NY
  120 CONTINUE
      READ (2,1410) AX(1),AX(2),AX(3),AX(4),AX(5),AX(6),AX(7),AX(8)
      READ (2,1410) AY(1),AY(2),AY(3),AY(4),AY(5),AY(6),AY(7),AY(8)
      DO 130 K=1,NT
      AN(K)=AX(K)*COS(PHI(1,1))+AY(K)*SIN(PHI(1,1))
  130 ANP(K)=-AY(K)*COS(PHI(1,1))+AX(K)*SIN(PHI(1,1))
C     CALCULATION OF CABLE COEFFICIENTS AND CONSTANTS
      VA=(CMOD/CRHO)**.5
      CMN=(CRHO+WRHO)*A
C
C     CALCULATION OF MAXIMUM TENSION AND TIME INCREMENT
C
      TM=0.
      DO 140 NS=1,NDEL
  140 T(2,NS)=T(1,NS)
  200 TMAX=T(2,1)
      KTRCL=0
      DO 400 NS=1,NDEL
      IF (TMAX-T(2,NS)) 350,400,400
  350 TMAX=T(2,NS)
  400 CONTINUE
      RMAX=1.+TMAX/(A*CMOD)
      VTM=(TMAX/(CMN*RMAX))**.5
      DELTM=.9*DELSO/VTM
      TM=TM+DELTM
      WRITE(5,1460) TM
C
C     CALCULATION OF LONGITUDINAL MOTIONS FOR A GIVEN TIME
C
      DO 530 NS=1,NDEL
      S1=0.
      S2=0.
      S3=0.
      DO 510 K=1,NT
      CA=K
      CSS=COS(CA*OM*SO(NS)/VA)
      SSS=SIN(CA*OM*SO(NS)/VA)
      CST=COS(CA*OM*TM)
      SST=SIN(CA*OM*TM)
      CSL=COS(CA*OM*CL/VA)
      SSL=SIN(CA*OM*CL/VA)
      S1=S1+AN(K)*CSS*SST/CSL
      S2=S2+AN(K)*CA*SSS*SST/CSL
      S3=S3+AN(K)*CA*CSS*CST/CSL
  510 CONTINUE
      SM1=0.
      SM2=0.
      SM3=0.
      DO 520 M=1,NM
      MX=M+1
      EM=M-1
      S4=0.
      DO 515 K=1,NT
      CA=K
      S4=S4+CA*OM*AN(K)/((CA*OM)**2-((EM+.5)*PI*VA/CL)**2)
  515 CONTINUE
```

```
      SM1=SM1+SIN((EM+.5)*PI*VA*TM/CL)*COS((EM+.5)*PI*SO(NS)/CL)*S4
     1*(-1.)**MX
      SM2=SM2+SIN((EM+.5)*PI*VA*TM/CL)*SIN((EM+.5)*PI*SO(NS)/CL)*S4
     1*(-1.)**MX*(2.*EM+1.)
      SM3=SM3+COS((EM+.5)*PI*VA*TM/CL)*COS((EM+.5)*PI*SO(NS)/CL)*S4
     1*(-1.)**MX*(2.*EM+1.)
  520 CONTINUE
      Y(2,NS)=S1+2.*VA*SM1/CL
      V(2,NS)=CM*S3+PI*VA**2*SM3/CL**2
      V(2,NS)=V(2,NS)+VT*COS(PHI(2,NS))
      R(2,NS)=1.-OM*S2/VA-PI*VA*SM2/CL**2+T(1,NS)/(A*CMOD)
      T(3,NS)=A*CMOD*(R(2,NS)-1.)
  530 CONTINUE
C
C     CALCULATION OF MOTIONS OF UPPER END POINT
C
  700 U(2,1)=VI*SIN(PHI(1,1))
      DO 705 K=1,NT
      CAY=K
  705 U(2,1)=U(2,1)+ANP(K)*SIN(CAY*OM*TM)
      FALF(1)=(T(2,1)/(R(1,1)*CMN))**.5
      FALF(3)=(T(2,2)/(R(1,2)*CMN))**.5
      FALF(2)=(T(3,1)/(R(2,1)*CMN))**.5
      RATO=1.-(FALF(1)/((DELSO/DELTM)+FALF(1)-FALF(3)))
      IF (RATO) 710,715,715
  710 TM=TM-DELTM*(1.-.9/(1.-RATO))
      DELTM=DELTM*.9/(1.-RATO)
      GO TO 700
  715 TP=T(2,1)+(1.-RATO)*(T(2,2)-T(2,1))
      RP=1.+TP/(A*CMOD)
      VP=V(1,1)+(1.-RATO)*(V(1,2)-V(1,1))
      UP=U(1,1)+(1.-RATO)*(U(1,2)-U(1,1))
      FALFP=(TP/(RP*CMN))**.5
      PHIP=PHI(2,1)+(1.-RATO)*(PHI(2,2)-PHI(2,1))
      PHI(3,1)=PHIP+(2.*(U(2,1)-UP)+.5*DELTM*(UP*ABS(UP)+U(2,1)*ABS(U(2,
     11)))*WRHO*CR*CD/CMN-2.*G*COS(PHIP)*(CRHO-WRHO)/(CRHO+WRHO)*DELTM)/
     2(V(2,1)+VP-R(2,1)*FALF(2)-RP*FALFP)
C
C     CALCULATION OF MOTIONS AT NON END POINTS
C
      DO 1090 NS=2,NLIM
      FALF(1)=(T(2,NS)/(R(1,NS)*CMN))**.5
      FALF(2)=(T(3,NS)/(R(2,NS)*CMN))**.5
      FALF(3)=(T(2,NS+1)/(R(1,NS+1)*CMN))**.5
      FALF(4)=(T(2,NS-1)/(R(1,NS-1)*CMN))**.5
      RATO=1.-(FALF(1)/((DELSO/DELTM)+FALF(1)-FALF(3)))
      IF (RATO) 810,820,820
  810 TM=TM-DELTM*(1.-.9/(1.-RATO))
      DELTM=DELTM*.9/(1.-RATO)
      GO TO 700
  820 TP=T(2,NS)+(1.-RATO)*(T(2,NS+1)-T(2,NS))
      RP=1.+TP/(A*CMOD)
      FALFP=(TP/(RP*CMN))**.5
      UP=U(1,NS)+(1.-RATO)*(U(1,NS+1)-U(1,NS))
      VP=V(1,NS)+(1.-RATO)*(V(1,NS+1)-V(1,NS))
      PHIP=PHI(2,NS)+(1.-RATO)*(PHI(2,NS+1)-PHI(2,NS))
      RATP=1.-(FALF(1)/((DELSO/DELTM)+FALF(1)-FALF(4)))
      IF (RATP) 910,920,920
  910 TM=TM-DELTM*(1.-.9/(1.-RATP))
```

```
      DELTM=DELTM*.9/(1.-RATP)                                          CABL1221
      GO TO 700                                                         CABL1222
  920 TQ=T(2,NS)+(1.-RATP)*(T(2,NS-1)-T(2,NS))                          CABL1223
      RQ=1.+TQ/(A*CMCD)                                                 CABL1224
      FALFQ=(TQ/(RQ*CMN))**.5                                           CABL1225
      UQ=U(1,NS)+(1.-RATP)*(U(1,NS-1)-U(1,NS))                          CABL1226
      VQ=V(1,NS)+(1.-RATP)*(V(1,NS-1)-V(1,NS))                          CABL1227
      PHIQ=PHI(2,NS)+(1.-RATP)*(PHI(2,NS-1)-PHI(2,NS))                  CABL1228
      UBAR=.5*(UP+UQ)                                                   CABL1229
 1050 PHI(3,NS)=(2.*(UQ-UP)+(V(2,NS)+VP-R(2,NS)*FALF(2)-RP*FALFP)*PHIP  CABL1230
     1-(V(2,NS)+VQ+R(2,NS)*FALF(2)+RQ*FALFQ)*PHIQ-G*(COS(PHIP)-COS(PHIQ)CABL1231
     2)*(CRHO-WRHO)*DELTM/(CRHO+WRHO)+DELTM*(UP*ABS(UP)-UQ*ABS(UQ)      CABL1232
     3+(UP+UBAR)*ABS(UP+UBAR)-(UQ+UBAR)*ABS(UQ+UBAR))*CR*CD*WRHO*.5/    CABL1233
     4(3.*CMN))                                                         CABL1234
      PHI(3,NS)=PHI(3,NS)/(VP-VQ-RP*FALFP-RQ*FALFQ-2.*FALF(2)*R(2,NS))  CABL1235
      U(2,NS)=0.25*(2.*(UP+UQ)+(2.*V(2,NS)+VP+VQ-RP*FALFP+RQ*FALFQ)     CABL1236
     1*PHI(3,NS)-(V(2,NS)+VP-R(2,NS)*FALF(2)-RP*FALFP)*PHIP-            CABL1237
     2(V(2,NS)+VQ+R(2,NS)*FALF(2)+RQ*FALFQ)*PHIQ+G*(2.*COS(PHI(3,NS))+  CABL1238
     3COS(PHIP)+COS(PHIQ))*(CRHO-WRHO)*DELTM/(CRHO+WRHO)-DELTM*(UP*ABS  CABL1239
     4(UP)+UQ*ABS(UQ)+(UP+UBAR)*ABS(UP+UBAR)+(UQ+UBAR)*ABS(UQ+UBAR)     CABL1240
     5+2.*UBAR*ABS(UBAR))*CR*CD*WRHO*.5/(3.*CMN))                       CABL1241
      TEST=0.01-U(2,NS)                                                 CABL1242
      IF (TEST) 1070,1070,1090                                          CABL1243
 1070 CONT=1.-20.*ABS(UBAR-U(2,NS))/ABS(U(2,NS))                        CABL1244
      IF (CONT) 1080,1090,1090                                          CABL1245
 1080 UBAR=.5*(UBAR+U(2,NS))                                            CABL1246
      GO TO 1050                                                        CABL1247
 1090 CONTINUE                                                          CABL1248
C                                                                       CABL1249
C     CALCULATION OF MOTIONS FOR LOWER END POINT                        CABL1250
C                                                                       CABL1251
      FALF(1)=(T(2,NDEL)/(R(1,NDEL)*CMN))**.5                           CABL1252
      FALF(2)=(T(3,NDEL)/(R(2,NDEL)*CMN))**.5                           CABL1253
      FALF(4)=(T(2,NDEL-1)/(R(1,NDEL-1)*CMN))**.5                       CABL1254
      RATQ=1.-(FALF(1)/((DELSO/DELTM)+FALF(1)-FALF(4)))                 CABL1255
      IF (RATQ) 1100,1110,1110                                          CABL1256
 1100 TM=TM-DELTM*(1.-.9/(1.-RATQ))                                     CABL1257
      DELTM=DELTM*.9/(1.-RATQ)                                          CABL1258
      GO TO 700                                                         CABL1259
 1110 TQ=T(2,NDEL)+(1.-RATQ)*(T(2,NDEL-1)-T(2,NDEL))                    CABL1260
      RQ=1.+TQ/(A*CMCD)                                                 CABL1261
      FALFQ=(TQ/(RQ*CMN))**.5                                           CABL1262
      UQ=U(1,NDEL)+(1.-RATQ)*(U(1,NDEL-1)-U(1,NDEL))                    CABL1263
      VQ=V(1,NDEL)+(1.-RATQ)*(V(1,NDEL-1)-V(1,NDEL))                    CABL1264
      PHIQ=PHI(2,NDEL)+(1.-RATQ)*(PHI(2,NDEL-1)-PHI(2,NDEL))            CABL1265
      CP=.5*CR*CD*WRHO                                                  CABL1266
      DUDS1=(4.*U(1,NDEL-1)-3.*U(1,NDEL)-U(1,NDEL-2))/(2.*DELSO)        CABL1167
      PHI(3,NDEL)=PHI(2,NDEL)-DUDS1*DELTM/R(1,NDEL)                     CABL1168
      U(2,NDEL)=UQ+.5*(V(2,NDEL)+VQ+R(2,NDEL)*FALF(2)+RQ*FALFQ)*        CABL1169
     1(PHI(3,NDEL)-PHIQ)-CP*UQ*ABS(UQ)*DELTM/CMN                        CABL1170
     2+.5*G*(COS(PHIQ)+COS(PHI(3,NDEL)))*DELTM*(CRHO-WRHO)/(CRHO+WRHO)  CABL1171
      DUDS2=(4.*U(2,NDEL-1)-3.*U(2,NDEL)-U(2,NDEL-2))/(2.*DELSO)        CABL1172
      PHI(3,NDEL)=PHI(2,NDEL)-.5*(DUDS1/R(1,NDEL)+DUDS2/R(2,NDEL))*DELTMCABL1173
      U(2,NDEL)=UQ+.5*(V(2,NDEL)+VQ+R(2,NDEL)*FALF(2)+RQ*FALFQ)*        CABL1174
     1(PHI(3,NDEL)-PHIQ)                                                CABL1175
     2-.5*CP*DELTM*(UQ*ABS(UQ)+U(2,NDEL)*ABS(U(2,NDEL)))/CMN            CABL1176
     3+.5*G*(COS(PHIQ)+COS(PHI(3,NDEL)))*DELTM*(CRHO-WRHO)/(CRHO+WRHO)  CABL1277
C                                                                       CABL1277
C     CALCULATION OF TRANSVERSE CONTRIBUTION TO STRAIN                  CABL1278
```

```
C
      DO 1120 NS=1,NDEL
      X(2,NS)=X(1,NS)+(U(2,NS)-U(1,NS))*DELTM
      DELR(NS)=X(2,NS)*DFEDS(1,NS)
 1120 X(1,NS)=X(2,NS)
      DO 1130 NS=1,NDEL
      ERR=ABS (DELR(NS)/(R(2,NS)-1.))
      IF (.01-ERR) 1140,1140,1130
 1130 CONTINUE
      GO TO 1200
 1140 DO 1150 NS=1,NDEL
      R(2,NS)=R(2,NS)+DELR(NS)
 1150 T(3,NS)=A*CMOD*(R(2,NS)-1.)
      GO TO 700
C
C     CALCULATION OF CABLE COORDINATES RELATIVE TO FIXED TOWING  POINT
C
 1200 XC(1)=0.
      YC(1)=0.
      DO 1310 NREF=1,NT
      XC(1)=XC(1)-AX(NREF)*SIN(NREF*OM*TM)/OM
 1310 YC(1)=YC(1)-AY(NREF)*(1.-COS(NREF*OM*TM))/OM
      XCREF=XC(1)
      YCREF=YC(1)
      DO 1330 NO=2,NDEL
      NI=NO-1
      PHIAV=(PHI(3,NI)+PHI(3,NO))/2.
      DLPHI=ABS(PHI(3,NI)-PHI(3,NO))/2.
      ARCL=DELSO*(R(2,NO)+R(2,NI))/2.
      TANF=SIN(DLPHI)/COS(DLPHI)
      CHORD=ARCL*(1.-TANF**2)/(1.+TANF**2)
      XC(NO)=XCREF+CHORD*COS(PHIAV)
      YC(NO)=YCREF+CHORD*SIN(PHIAV)
      XCREF=XC(NO)
 1330 YCREF=YC(NO)
      WRITE(5,1480)
      DO 1340 NO=1,NDEL
 1340 WRITE (5,1470) SO(NO),XC(NO),YC(NO)
C
C     CALCULATION OF OUTPUT
C
      DO 1350 NS=1,NDEL
      WRITE (5,1450) SO(NS),U(2,NS),PHI(3,NS),T(3,NS),V(2,NS)
      T(2,NS)=T(3,NS)
      R(1,NS)=R(2,NS)
      Y(1,NS)=Y(2,NS)
      V(1,NS)=V(2,NS)
      U(1,NS)=U(2,NS)
 1350 PHI(2,NS)=PHI(3,NS)
      IF (TM-TF) 200,1500,1500
 1410 FORMAT (6F10.5)
 1415 FORMAT (2F10.5,3I5)
 1420 FORMAT (8F10.2)
 1450 FORMAT (5H SO= ,F8.2,4H U= ,F8.2,6H PHI= ,F7.4,4H T= ,F8.2,4H V= ,
     1F8.2)
 1460 FORMAT (10H   TIME=   ,F8.3)
 1470 FORMAT (7H  SO=   ,F10.2,8H   XC=   ,F10.2,8H   YC=   ,F10.2)
 1480 FORMAT (43H  XC=DELTA SUB ALPHA ,  YC=DELTA SUB BETA   )
 1500 STOP
```

SAMPLE OUTPUT - PROGRAM CABL1

```
 TIME=      0.319
 XC=DELTA SUB ALPHA ,  YC=DELTA SUB BETA
 SC=       1000.00     XC=          -0.62    YC=            0.05
 SC=        950.00     XC=          36.12    YC=           33.91
 SC=        900.00     XC=          72.30    YC=           68.43
 SC=        850.00     XC=         108.48    YC=          102.95
 SC=        800.00     XC=         144.65    YC=          137.47
 SC=        750.00     XC=         180.83    YC=          172.00
 SC=        700.00     XC=         217.01    YC=          206.52
 SC=        650.00     XC=         253.19    YC=          241.04
 SC=        600.00     XC=         289.36    YC=          275.57
 SC=        550.00     XC=         325.54    YC=          310.09
 SC=        500.00     XC=         361.72    YC=          344.61
 SC=        450.00     XC=         397.89    YC=          379.13
 SC=        400.00     XC=         434.07    YC=          413.66
 SC=        350.00     XC=         470.25    YC=          448.18
 SC=        300.00     XC=         506.42    YC=          482.70
 SC=        250.00     XC=         542.60    YC=          517.22
 SC=        200.00     XC=         578.77    YC=          551.74
 SC=        150.00     XC=         614.95    YC=          586.26
 SC=        100.00     XC=         651.12    YC=          620.78
 SC=         50.00     XC=         687.30    YC=          655.30
 SC=          0.00     XC=         723.40    YC=          689.90
SC=  1000.00 U=       7.56 PHI=   0.7268 T= 16567.02 V=        7.95
SC=   950.00 U=       4.80 PHI=   0.7620 T= 16534.66 V=        7.95
SC=   900.00 U=       4.80 PHI=   0.7619 T= 16502.30 V=        7.95
SC=   850.00 U=       4.80 PHI=   0.7619 T= 16502.30 V=        7.95
SC=   800.00 U=       4.80 PHI=   0.7619 T= 16469.95 V=        7.95
SC=   750.00 U=       4.80 PHI=   0.7619 T= 16437.59 V=        7.95
SC=   700.00 U=       4.80 PHI=   0.7619 T= 16437.59 V=        7.95
SC=   650.00 U=       4.80 PHI=   0.7619 T= 16437.59 V=        7.94
SC=   600.00 U=       4.80 PHI=   0.7619 T= 16405.23 V=        7.94
SC=   550.00 U=       4.80 PHI=   0.7619 T= 16437.59 V=        7.94
SC=   500.00 U=       4.80 PHI=   0.7619 T= 16469.95 V=        7.93
SC=   450.00 U=       4.80 PHI=   0.7619 T= 16728.81 V=        7.89
SC=   400.00 U=       4.80 PHI=   0.7619 T=  9739.59 V=        8.77
SC=   350.00 U=       4.80 PHI=   0.7619 T=  9998.45 V=        8.73
SC=   300.00 U=       4.80 PHI=   0.7619 T= 10030.81 V=        8.72
SC=   250.00 U=       4.80 PHI=   0.7619 T= 10030.81 V=        8.72
SC=   200.00 U=       4.80 PHI=   0.7619 T= 10030.81 V=        8.72
SC=   150.00 U=       4.80 PHI=   0.7619 T=  9998.45 V=        8.72
SC=   100.00 U=       4.80 PHI=   0.7619 T=  9998.45 V=        8.72
SC=    50.00 U=       4.80 PHI=   0.7619 T=  9966.10 V=        8.72
SC=     0.00 U=       4.35 PHI=   0.7662 T=  9998.45 V=        8.71
 TIME=      0.573
 XC=DELTA SUB ALPHA ,  YC=DELTA SUB BETA
 SC=       1000.00     XC=          -1.08    YC=            0.16
 SC=        950.00     XC=          35.75    YC=           33.96
 SC=        900.00     XC=          72.25    YC=           68.13
 SC=        850.00     XC=         108.43    YC=          102.66
 SC=        800.00     XC=         144.61    YC=          137.18
 SC=        750.00     XC=         180.78    YC=          171.70
 SC=        700.00     XC=         216.96    YC=          206.22
 SC=        650.00     XC=         253.14    YC=          240.74
 SC=        600.00     XC=         289.31    YC=          275.27
 SC=        550.00     XC=         325.49    YC=          309.79
 SC=        500.00     XC=         361.66    YC=          344.31
```

```
SC=      450.00   XC=      397.84   YC=       378.83
SC=      400.00   XC=      434.01   YC=       413.35
SC=      350.00   XC=      470.19   YC=       447.87
SC=      300.00   XC=      506.36   YC=       482.39
SC=      250.00   XC=      542.54   YC=       516.91
SC=      200.00   XC=      578.72   YC=       551.43
SC=      150.00   XC=      614.89   YC=       585.95
SC=      100.00   XC=      651.07   YC=       620.48
SC=       50.00   XC=      687.23   YC=       655.01
SC=        0.00   XC=      723.31   YC=       689.63
SC=  1000.00 U=      8.04 PHI=  0.7420 T= 16437.59 V=        8.10
SC=   950.00 U=      8.39 PHI=  0.7428 T= 16405.23 V=        7.86
SC=   900.00 U=      6.40 PHI=  0.7619 T= 16405.23 V=        7.86
SC=   850.00 U=      6.40 PHI=  0.7619 T= 16372.87 V=        7.86
SC=   800.00 U=      6.40 PHI=  0.7619 T= 16340.52 V=        7.86
SC=   750.00 U=      6.40 PHI=  0.7619 T= 16114.01 V=        7.89
SC=   700.00 U=      6.40 PHI=  0.7619 T= 10127.88 V=        8.64
SC=   650.00 U=      6.40 PHI=  0.7619 T= 10192.60 V=        8.63
SC=   600.00 U=      6.40 PHI=  0.7619 T= 10192.60 V=        8.63
SC=   550.00 U=      6.40 PHI=  0.7619 T= 10192.60 V=        8.63
SC=   500.00 U=      6.40 PHI=  0.7619 T= 10160.24 V=        8.63
SC=   450.00 U=      6.40 PHI=  0.7619 T= 10127.88 V=        8.63
SC=   400.00 U=      6.40 PHI=  0.7619 T= 10127.88 V=        8.63
SC=   350.00 U=      6.40 PHI=  0.7619 T= 10095.53 V=        8.63
SC=   300.00 U=      6.40 PHI=  0.7619 T= 10095.53 V=        8.63
SC=   250.00 U=      6.40 PHI=  0.7619 T= 10063.17 V=        8.63
SC=   200.00 U=      6.40 PHI=  0.7619 T= 10030.81 V=        8.63
SC=   150.00 U=      6.40 PHI=  0.7619 T= 10030.81 V=        8.63
SC=   100.00 U=      6.40 PHI=  0.7619 T=  9998.45 V=        8.63
SC=    50.00 U=      6.48 PHI=  0.7627 T=  9966.10 V=        8.63
SC=     0.00 U=      6.26 PHI=  0.7665 T=  9998.45 V=        8.60
 TIME=      0.830
XC=DELTA SUB ALPHA ,  YC=DELTA SUB BETA
SC=      1000.00   XC=        -1.47   YC=         0.32
SC=       950.00   XC=        35.54   YC=        33.91
SC=       900.00   XC=        72.13   YC=        67.99
SC=       850.00   XC=       108.48   YC=       102.32
SC=       800.00   XC=       144.66   YC=       136.84
SC=       750.00   XC=       180.84   YC=       171.36
SC=       700.00   XC=       217.01   YC=       205.88
SC=       650.00   XC=       253.19   YC=       240.40
SC=       600.00   XC=       289.36   YC=       274.92
SC=       550.00   XC=       325.54   YC=       309.44
SC=       500.00   XC=       361.71   YC=       343.96
SC=       450.00   XC=       397.89   YC=       378.48
SC=       400.00   XC=       434.06   YC=       413.01
SC=       350.00   XC=       470.24   YC=       447.53
SC=       300.00   XC=       506.42   YC=       482.05
SC=       250.00   XC=       542.59   YC=       516.57
SC=       200.00   XC=       578.77   YC=       551.09
SC=       150.00   XC=       614.94   YC=       585.61
SC=       100.00   XC=       651.11   YC=       620.14
SC=        50.00   XC=       687.25   YC=       654.69
SC=         0.00   XC=       723.33   YC=       689.32
SC=  1000.00 U=      8.45 PHI=  0.7246 T=  4303.54 V=        7.88
SC=   950.00 U=      7.72 PHI=  0.7486 T=  4271.18 V=        7.87
SC=   900.00 U=      7.79 PHI=  0.7511 T=  9577.81 V=        8.41
SC=   850.00 U=      6.83 PHI=  0.7619 T= 10192.60 V=        8.48
SC=   800.00 U=      6.83 PHI=  0.7619 T= 10192.60 V=        8.49
SC=   750.00 U=      6.83 PHI=  0.7619 T= 10192.60 V=        8.49
SC=   700.00 U=      6.83 PHI=  0.7619 T= 10192.60 V=        8.49
```

APPENDIX II

COMPUTER PROGRAM CABL2 FOR CABLES WITH FIXED LOWER ENDS

This Fortran IV program, which is designated CABL2, calculates the dynamic response of a cable having an arbitrary steady-state shape and a lower-end whose position is fixed relative to the steady-state towing point. The fixed lower-end, which has zero acceleration, corresponds to a towed-body whose mass is so large that its acceleration is negligible. Except for the lower-end boundary conditions this program is identical with Program CABL1. Again the cable is assumed to be of uniform, circular cross-section and to move in a fluid of uniform properties. The flow chart for this program is identical with that of CABL1, and is given by Appendix III.

Program Inputs and Outputs

The required input variables and formats, the output variables and the mnemonics are identical with those of Program CABL1 and are described in detail by comment statements in the program source listing.

Sample Problem

A printout of a sample run is given after the program source listing. This printout is for a plastic cable which is neutrally buoyant and has a steady-state inclination, ϕ, which is everywhere zero. The assumed cable characteristics and operating variables are:

CL	=	1000 feet
CD	=	0.2 feet
CRHO	=	2 slugs/foot^3
CMOD	=	0.2 pounds/inch^2
WRHO	=	2 slugs/foot^3
CR	=	1.0
DELSO	=	50 feet
VT	=	10 feet/second
OM	=	1 radian/second
TF	=	10 seconds
NX	=	1
NY	=	1
NM	=	40
AX(1)	=	1, all other AX(N)=0
AY(1)	=	-1, all other AY(N)=0
PHI(1,NS)	=	0
T(s_o)	=	(20,000 + 0.6 s_o) pounds

Calculated results are given for the first two time increments.

PROGRAM CABL2

SOURCE LISTING

```
      DIMENSION SO(110),V(2,110),T(3,110),R(2,110),Y(2,110),U(2,110),PHICABL2  1
     1(3,110),X(2,110),XC(110),YC(110)                                 CABL2& 2
      DIMENSION AN(8),AX(8),AY(8),ANP(8),FALF(4),DFEDS(1,110),DELR(110) CABL2& 3
C                                                                      CABL2& 4
C           PROGRAM   CABL2
C                                                                      CABL2& 5
C     CALCULATION OF CABLE MOTIONS FOR A CABLE WITH FIXED LOWER END    CABL2& 6
C     COMBINED LONGITUDINAL AND TRANSVERSE MOTIONS                     CABL2& 7
C                                                                      CABL2& 8
C     LONGITUDINAL MOTIONS CALCULATED FTOM ANALYTICAL SOLUTION WITH    CABL2& 9
C     CORRECTION FOR THE EFFECT OF TRANSVERSE DISPLACEMENTS            CABL2 10
C                                                                      CABL2 11
C     TRANSVERSE MOTIONS CALCULATED USING THE METHOD OF CHARACTERISTICS CABL2 12
C     AND COMPLETELY COUPLED TRANSVERSE MOTIONS                        CABL2 13
C                                                                      CABL2 14
C                                                                      CABL2 15
C           REQUIRED PROGRAM INPUT                                     CABL2 16
C                                                                      CABL2 17
C     THE TOTAL NUMBER OF INPUT CARDS REQUIRED IS 5 PLUS NDEL          CABL2 18
C     WHERE NDEL IS THE NUMBER OF SPATIAL INCREMENTS ALONG THE CABLE   CABL2 19
C                                                                      CABL2 20
C                                                                      CABL2 21
C     THE FIRST CARD CONTAINS THE FOLLOWING DATA AT THE FOLLOWING FORMATCABL2 22
C                                                                      CABL2 23
C     CL = CABLE LENGTH IN FEET  F10.5                                 CABL2 24
C     CD = CABLE DIAMETER IN FEET  F10.5                               CABL2 25
C     CRHO = CABLE MASS DENSITY IN SLUGS PER CUBIC FOOT  F10.5         CABL2 26
C     CMOD = CABLE MODULUS OF ELASTICITY IN PSI DIVIDED BY 1000000.    CABL2 27
C     F10.5
C     WRHO = WATER MASS DENSITY IN SLUGS PER CUBIC FOOT  F10.5         CABL2 28
C     CR = NORMAL DRAG COEFFICIENT FOR STEADY FLOW  F10.5              CABL2 29
C     DELSO = INCREMENT IN CABLE LENGTH IN FEET  F10.5                 CABL2 30
C     VT = TOWING VELOCITY IN FEET PER SECOND  F10.5                   CABL2 31
C                                                                      CABL2 32
C     THE SECOND CARD CONTAINS THE FOLLOWING DATA                      CABL2 33
C                                                                      CABL2 34
C     OM = FUNDEMENTAL FREQUENCY OF EXCITATION IN RADIANS PER SECOND   CABL2 35
C     F10.5
C     TF = REAL TIME AT WHICH CALCULATIONS ARE TERMINATED IN SECONDS   CABL2 36
C     F10.5
C     NX = NUMBER OF TERMS IN FOURIER SERIES FOR DELTA( X )  I5        CABL2 37
C     NY = NUMBER OF TERMS IN FOURIER SERIES FOR DELTA ( Y )  I5       CABL2 38
C     NM = NUMBER OF TERMS USED TO APPROXIMATE THE SERIES IN THE       CABL2 39
C     LONGITUDINAL SOLUTIONS  I5                                       CABL2 40
C                                                                      CABL2 41
C     THE THIRD THROUGH THIRD PLUS NDEL CARDS CONTAIN THE FOLLOWING DATACABL2 42
C                                                                      CABL2 43
C     SO ( NS ) = LONGITUDINAL POSITION ( NS=1 FOR SO=L, NS=NDEL+1 FOR CABL2 44
```

```
C       SO=0 )  F10.2
C       T ( 1,NS ) = STEADY-STATE TENSION AT SO ( NS )  F10.2
C       PHI ( 1,NS ) = STEADY-STATE INCLINATION AT SO ( NS )  F10.2
C
C       THE FOURTH PLUS NDEL CARD CONTAINS THE FOLLOWING DATA
C
C       AX ( 1 ),.....,AX ( 8 ) WHERE AX ( N ) IS THE COEFFICIENT OF THE N'
C       TH TERM IN THE FOURIER SERIES FOR DELTA SUB X
C
C       THE FIFTH PLUS NDEL CARD CONTAINS THE FOLLOWING DATA
C
C       AY(1), ....,AY(8)  WHERE AY(N) IS THE COEFFICIENT OF THE N'TH
C       TERM IN THE FOURIER SERIES FOR DELTA SUB Y
C
C
C                    PROGRAM OUTPUT
C
C
C       THE FOLLOWING OUTPUT IS PROVIDED FOR EACH TIME INCREMENT
C
C       TIME =    . THE REAL TIME FOR WHICH THE CALCULATED RESULTS APPLY
C
C       SO =    , XC =    , YC =    .
C       WHERE  SO IS THE LONGITUDINAL POSITION ( SO=L IS THE TOWING POINT)
C              XC IS THE COORDINATE DELTA SUB ALPHA FOR SO
C              YC IS THE COORDINATE DELTA SUB BETA FOR SO
C
C        SO =    , U =    , PHI =    , T =    , V =    .
C       WHERE  SO IS THE LONGITUDINAL POSITION
C              U IS THE TRANSVERSE VELOCITY AT SO
C              PHI IS THE CABLE INCLINATION AT SO
C              T IS THE CABLE TOTAL TENSION AT SO
C              V IS THE LONGITUDINAL VELOCITY AT SO
C
C
      READ (2,1410) CL,CD,CRHO,CMOD,WRHO,CR,DELSO,VT
      CMOD=CMOD*144000000.
      DELTA=1.+CL/DELSO
      NDEL=DELTA
      READ (2,1415) OM,TF,NX,NY,NM
      PI=3.1416
      G=32.2
      A=PI*CD**2/4.
      DO 20  NS=1,NDEL
      READ (2,1420) SO(NS),T(1,NS),PHI(1,NS)
      U(1,NS)=VT*SIN(PHI(1,NS))
      PHI(2,NS)=PHI(1,NS)
      R(1,NS)=T(1,NS)/(A*CMOD)+1.
      V(1,NS)=VT*COS(PHI(1,NS))
      X(1,NS)=0.
   20 T(2,NS)=T(1,NS)
C
C     CALCULATION OF CABLE STEADY STATE CURVATURE
C
      NLIM=NDEL-1
      DFEDS(1,1)=-(4.*PHI(1,2)-3.*PHI(1,1)-PHI(1,3))/(2.*DELSO)
      DFEDS(1,NDEL)=(4.*PHI(1,NLIM)-3.*PHI(1,NDEL)-PHI(1,NDEL-2))/(2.*
     1DELSO)
      DO 100 NS=2,NLIM
```

```
  100 DFEDS(1,NS)=(PHI(1,NS-1)-PHI(1,NS+1))/(2.*DELSO)
      NT=NX
      IF (NT-NY) 110,120,120
  110 NT=NY
  120 CONTINUE
      READ (2,1410) AX(1),AX(2),AX(3),AX(4),AX(5),AX(6),AX(7),AX(8)
      READ (2,1410) AY(1),AY(2),AY(3),AY(4),AY(5),AY(6),AY(7),AY(8)
      DO 130 K=1,NT
      AN(K)=AX(K)*COS(PHI(1,1))+AY(K)*SIN(PHI(1,1))
  130 ANP(K)=-AY(K)*COS(PHI(1,1))+AX(K)*SIN(PHI(1,1))
C     CALCULATION OF CABLE COEFFICIENTS AND CONSTANTS
      VA=(CMOD/CRHO)**.5
      CMN=(CRHO+WRHO)*A
C
C     CALCULATION OF MAXIMUM TENSION AND TIME INCREMENT
C
      TM=0.
      DO 140 NS=1,NDEL
  140 T(2,NS)=T(1,NS)
  200 TMAX=T(2,1)
      KTROL=0
      DO 400 NS=1,NDEL
      IF (TMAX-T(2,NS)) 350,400,400
  350 TMAX=T(2,NS)
  400 CONTINUE
      RMAX=1.+TMAX/(A*CMOD)
      VTM=(TMAX/(CMN*RMAX))**.5
      DELTM=.9*DELSO/VTM
      TM=TM+DELTM
      WRITE(5,1460) TM
C
C     CALCULATION OF LONGITUDINAL MOTIONS FOR A GIVEN TIME
C
      DO 530 NS=1,NDEL
      S1=0.
      S2=0.
      S3=0.
      DO 510 K=1,NT
      CA=K
      CSS=COS(CA*OM*SO(NS)/VA)
      SSS=SIN(CA*OM*SO(NS)/VA)
      CST=COS(CA*OM*TM)
      SST=SIN(CA*OM*TM)
      CSL=COS(CA*OM*CL/VA)
      SSL=SIN(CA*OM*CL/VA)
      S1=S1+AN(K)*SSS*SST/SSL
      S2=S2+AN(K)*CSS*SST*CA/SSL
      S3=S3+AN(K)*SSS*CST*CA/SSL
  510 CONTINUE
      SM1=0.
      SM2=0.
      SM3=0.
      DO 520 M=1,NM
      MX=M+1
      EM=M
      S4=0.
      DO 515 K=1,NT
      CA=K
      S4=S4+CA*OM*AN(K)/((CA*OM)**2-(EM*PI*VA/CL)**2)
  515 CONTINUE
```

```
      SM1=SM1+SIN(EM*PI*VA*TM/CL)*SIN(EM*PI*SO(NS)/CL)*S4*(-1.)**MX      CABL2165
      SM2=SM2+SIN(EM*PI*VA*TM/CL)*COS(EM*PI*SO(NS)/CL)*S4*EM*(-1.)**MX   CABL2166
      SM3=SM3+COS(EM*PI*VA*TM/CL)*SIN(EM*PI*SO(NS)/CL)*S4*EM*(-1.)**MX   CABL2167
  520 CONTINUE                                                            CABL2168
      Y(2,NS)=S1+2.*VA*SM1/CL                                             CABL2169
      V(2,NS)=OM*S3+2.*PI*VA**2*SM3/CL**2                                 CABL2170
      V(2,NS)=V(2,NS)+VT*COS(PHI(2,NS))                                   CABL2171
      R(2,NS)=1.+OM*S2/VA+2.*PI*VA*SM2/CL**2+T(1,NS)/(A*CMOD)             CABL2172
      T(3,NS)=A*CMOD*(R(2,NS)-1.)                                         CABL2173
  530 CONTINUE                                                            CABL2174
C                                                                         CABL2175
C     CALCULATION OF MOTIONS OF UPPER END POINT                           CABL2176
C                                                                         CABL2177
  700 U(2,1)=VT*SIN(PHI(1,1))                                             CABL2178
      DO 705 K=1,NT                                                       CABL2179
      CAY=K                                                               CABL2180
  705 U(2,1)=U(2,1)+ANP(K)*SIN(CAY*OM*TM)                                 CABL2181
      FALF(1)=(T(2,1)/(R(1,1)*CMN))**.5                                   CABL2182
      FALF(3)=(T(2,2)/(R(1,2)*CMN))**.5                                   CABL2183
      FALF(2)=(T(3,1)/(R(2,1)*CMN))**.5                                   CABL2184
      RATO=1.-(FALF(1)/((DELSO/DELTM)+FALF(1)-FALF(3)))                   CABL2185
      IF (RATO) 710,715,715                                               CABL2186
  710 TM=TM-DELTM*(1.-.9/(1.-RATO))                                       CABL2187
      DELTM=DELTM*.9/(1.-RATO)                                            CABL2188
      GO TO 700                                                           CABL2189
  715 TP=T(2,1)+(1.-RATO)*(T(2,2)-T(2,1))                                 CABL2190
      RP=1.+TP/(A*CMOD)                                                   CABL2191
      VP=V(1,1)+(1.-RATO)*(V(1,2)-V(1,1))                                 CABL2192
      UP=U(1,1)+(1.-RATO)*(U(1,2)-U(1,1))                                 CABL2193
      FALFP=(TP/(RP*CMN))**.5                                             CABL2194
      PHIP=PHI(2,1)+(1.-RATO)*(PHI(2,2)-PHI(2,1))                         CABL2195
      PHI(3,1)=PHIP+(2.*(U(2,1)-UP)+.5*DELTM*(UP*ABS(UP)+U(2,1)*ABS(U(2,CABL2196
     11)))*WRHO*CR*CD/CMN-2.*G*COS(PHIP)*(CRHO-WRHO)/(CRHO+WRHO)*DELTM)/CABL2197
     2(V(2,1)+VP-R(2,1)*FALF(2)-RP*FALFP)                                 CABL2198
C                                                                         CABL2199
C     CALCULATION OF MOTIONS AT NON END POINTS                            CABL2200
C                                                                         CABL2201
      DO 1090 NS=2,NLIM                                                   CABL2202
      FALF(1)=(T(2,NS)/(R(1,NS)*CMN))**.5                                 CABL2203
      FALF(2)=(T(3,NS)/(R(2,NS)*CMN))**.5                                 CABL2204
      FALF(3)=(T(2,NS+1)/(R(1,NS+1)*CMN))**.5                             CABL2205
      FALF(4)=(T(2,NS-1)/(R(1,NS-1)*CMN))**.5                             CABL2206
      RATO=1.-(FALF(1)/((DELSO/DELTM)+FALF(1)-FALF(3)))                   CABL2207
      IF (RATO) 810,820,820                                               CABL2208
  810 TM=TM-DELTM*(1.-.9/(1.-RATO))                                       CABL2209
      DELTM=DELTM*.9/(1.-RATO)                                            CABL2210
      GO TO 700                                                           CABL2211
  820 TP=T(2,NS)+(1.-RATO)*(T(2,NS+1)-T(2,NS))                            CABL2212
      RP=1.+TP/(A*CMOD)                                                   CABL2213
      FALFP=(TP/(RP*CMN))**.5                                             CABL2214
      UP=U(1,NS)+(1.-RATO)*(U(1,NS+1)-U(1,NS))                            CABL2215
      VP=V(1,NS)+(1.-RATO)*(V(1,NS+1)-V(1,NS))                            CABL2216
      PHIP=PHI(2,NS)+(1.-RATO)*(PHI(2,NS+1)-PHI(2,NS))                    CABL2217
      RATP=1.-(FALF(1)/((DELSO/DELTM)+FALF(1)-FALF(4)))                   CABL2218
      IF (RATP) 910,920,920                                               CABL2219
  910 TM=TM-DELTM*(1.-.9/(1.-RATP))                                       CABL2220
      DELTM=DELTM*.9/(1.-RATP)                                            CABL2221
      GO TO 700                                                           CABL2222
  920 TQ=T(2,NS)+(1.-RATP)*(T(2,NS-1)-T(2,NS))                            CABL2223
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```
      RQ=1.+TQ/(A*CMOD)
      FALFQ=(TQ/(RQ*CMN))**.5
      UQ=U(1,NS)+(1.-RATP)*(U(1,NS-1)-U(1,NS))
      VQ=V(1,NS)+(1.-RATP)*(V(1,NS-1)-V(1,NS))
      PHIQ=PHI(2,NS)+(1.-RATP)*(PHI(2,NS-1)-PHI(2,NS))
      UBAR=.5*(UP+UQ)
 1050 PHI(3,NS)=(2.*(UQ-UP)+(V(2,NS)+VP-R(2,NS)*FALF(2)-RP*FALFP)*PHIP
     1-(V(2,NS)+VQ+R(2,NS)*FALF(2)+RQ*FALFQ)*PHIQ-G*(COS(PHIP)-COS(PHIQ)
     2)*(CRHO-WRHO)*DELTM/(CRHO+WRHO)+DELTM*(UP*ABS(UP)-UQ*ABS(UQ)
     3+(UP+UBAR)*ABS(UP+UBAR)-(UQ+UBAR)*ABS(UQ+UBAR))*CR*CD*WRHO*.5/
     4(3.*CMN))
      PHI(3,NS)=PHI(3,NS)/(VP-VQ-RP*FALFP-RQ*FALFQ-2.*FALF(2)*R(2,NS))
      U(2,NS)=0.25*(2.*(UP+UQ)+(2.*V(2,NS)+VP+VQ-RP*FALFP+RQ*FALFQ)
     1*PHI(3,NS)-(V(2,NS)+VP-R(2,NS)*FALF(2)-RP*FALFP)*PHIP-
     2(V(2,NS)+VQ+R(2,NS)*FALF(2)+RQ*FALFQ)*PHIQ+G*(2.*COS(PHI(3,NS))+
     3COS(PHIP)+COS(PHIQ))*(CRHO-WRHO)*DELTM/(CRHO+WRHO)-DELTM*(UP*ABS
     4(UP)+UQ*ABS(UQ)+(UP+UBAR)*ABS(UP+UBAR)+(UQ+UBAR)*ABS(UQ+UBAR)
     5+2.*UBAR*ABS(UBAR))*CR*CD*WRHO*.5/(3.*CMN))
      TEST=0.01-U(2,NS)
      IF (TEST) 1070,1070,1090
 1070 CONT=1.-20.*ABS(UBAR-U(2,NS))/ABS(U(2,NS))
      IF (CONT) 1080,1090,1090
 1080 UBAR=.5*(UBAR+U(2,NS))
      GO TO 1050
 1090 CONTINUE
C
C     CALCULATION OF MOTIONS FOR LOWER END POINT
C
      FALF(1)=(T(2,NDEL)/(R(1,NDEL)*CMN))**.5
      FALF(2)=(T(3,NDEL)/(R(2,NDEL)*CMN))**.5
      FALF(4)=(T(2,NDEL-1)/(R(1,NDEL-1)*CMN))**.5
      RATO=1.-(FALF(1)/((DELSO/DELTM)+FALF(1)-FALF(4)))
      IF (RATO) 1100,1110,1110
 1100 TM=TM-DELTM*(1.-.9/(1.-RATO))
      DELTM=DELTM*.9/(1.-RATO)
      GO TO 700
 1110 TQ=T(2,NDEL)+(1.-RATO)*(T(2,NDEL-1)-T(2,NDEL))
      RQ=1.+TQ/(A*CMOD)
      FALFQ=(TQ/(RQ*CMN))**.5
      UQ=U(1,NDEL)+(1.-RATO)*(U(1,NDEL-1)-U(1,NDEL))
      VQ=V(1,NDEL)+(1.-RATO)*(V(1,NDEL-1)-V(1,NDEL))
      PHIQ=PHI(2,NDEL)+(1.-RATO)*(PHI(2,NDEL-1)-PHI(2,NDEL))
      CP=.5*CR*CD*WRHO
      U(2,NDEL)=VT*SIN(PHIQ)
      PHI(3,NDEL)=PHIQ+(U(2,NDEL)-UQ-(G*(CRHO-WRHO)*COS(PHIQ)*DELTM/
     1(CRHO+WRHO))+.25*CR*CD*WRHO*(UQ*ABS(UQ)+U(2,NDEL)*ABS(U(2,NDEL))/
     2CMN)*DELTM)/(.5*(VQ+V(2,NDEL)+RQ*FALFQ+R(2,NDEL)*FALF(2)))
      U(2,NDEL)=VT*SIN(PHI(3,NDEL))
      PHI(3,NDEL)=PHIQ+(U(2,NDEL)-UQ-.5*G*(CRHO-WRHO)*DELTM*(COS(PHIQ)+
     1COS(PHI(3,NDEL)))/(CRHO+WRHO)+.25*CR*CD*WRHO*(UQ*ABS(UQ)+U(2,NDEL)
     2*ABS(U(2,NDEL)))*DELTM/CMN)/(.5*(VQ+V(2,NDEL)+RQ*FALFQ+R(2,NDEL)*
     3FALF(2)))
      U(2,NDEL)=VT*SIN(PHI(3,NDEL))
C
C     CALCULATION OF TRANSVERSE CONTRIBUTION TO STRAIN
C
      DO 1120 NS=1,NDEL
      X(2,NS)=X(1,NS)+(U(2,NS)-U(1,NS))*DELTM
      DELR(NS)=X(2,NS)*DFEDS(1,NS)
```

```
 1120 X(1,NS)=X(2,NS)
      DO 1130 NS=1,NDEL
      ERR=ABS (DELR(NS)/(R(2,NS)-1.))
      IF (.01-ERR) 1140,1140,1130
 1130 CONTINUE
      GO TO 1200
 1140 DO 1150 NS=1,NDEL
      R(2,NS)=R(2,NS)+DELR(NS)
 1150 T(3,NS)=A*CMOD*(R(2,NS)-1.)
      GO TO 700
C
C     CALCULATION OF CABLE COORDINATES RELATIVE TO FIXED TOWING  POINT
C
 1200 XC(1)=0.
      YC(1)=0.
      DO 1310 NREF=1,NT
      XC(1)=XC(1)-AX(NREF)*SIN(NREF*OM*TM)/OM
 1310 YC(1)=YC(1)-AY(NREF)*(1.-COS(NREF*OM*TM))/OM
      XCREF=XC(1)
      YCREF=YC(1)
      DO 1330 NO=2,NDEL
      NI=NO-1
      PHIAV=(PHI(3,NI)+PHI(3,NO))/2.
      DLPHI=ABS(PHI(3,NI)-PHI(3,NO))/2.
      ARCL=DELSO*(R(2,NO)+R(2,NI))/2.
      TANF=SIN(DLPHI)/COS(DLPHI)
      CHORD=ARCL*(1.-TANF**2)/(1.+TANF**2)
      XC(NO)=XCREF+CHORD*COS(PHIAV)
      YC(NO)=YCREF+CHORD*SIN(PHIAV)
      XCREF=XC(NO)
 1330 YCREF=YC(NO)
      WRITE(5,1480)
      DO 1340 NO=1,NDEL
 1340 WRITE (5,1470) SO(NO),XC(NO),YC(NO)
C
C     CALCULATION OF OUTPUT
C
      DO 1350 NS=1,NDEL
      WRITE (5,1450) SO(NS),U(2,NS),PHI(3,NS),T(3,NS),V(2,NS)
      T(2,NS)=T(3,NS)
      R(1,NS)=R(2,NS)
      Y(1,NS)=Y(2,NS)
      V(1,NS)=V(2,NS)
      U(1,NS)=U(2,NS)
 1350 PHI(2,NS)=PHI(3,NS)
      IF (TM-TF) 200,1500,1500
 1410 FORMAT (8F10.5)
 1415 FORMAT (2F10.5,3I5)
 1420 FORMAT (8F10.2)
 1450 FORMAT (5H SO= ,F8.2,4H U= ,F8.2,6H PHI= ,F7.4,4H T= ,F8.2,4H V= ,
     1F8.2)
 1460 FORMAT (10H   TIME=  ,F8.3)
 1470 FORMAT (7H  SO=  ,F10.2,8H   XC=  ,F10.2,8H   YC=  ,F10.2)
 1480 FORMAT (43H  XC=DELTA SUB ALPHA ,  YC=DELTA SUB BETA  )
 1500 STOP
      END
VARIABLE ALLOCATIONS
    SO(R )=00DA-0000     V(R )=0292-00DC     T(R )=0526-0294     R(R )=06DE-0528
   PHI(R )=0CE2-0A50     X(R )=0E9A-0CE4    XC(R )=0F76-0E9C    YC(R )=1052-0F78
```

SAMPLE OUTPUT - PROGRAM CABL2

```
 TIME=      0.112
XC=DELTA SUB ALPHA ,  YC=DELTA SUB BETA
SC=      1000.00    XC=         -0.22    YC=          0.00
SC=       950.00    XC=         50.93    YC=         -0.00
SC=       900.00    XC=        102.10    YC=         -0.00
SC=       850.00    XC=        153.26    YC=         -0.00
SC=       800.00    XC=        204.42    YC=         -0.00
SC=       750.00    XC=        255.57    YC=         -0.00
SC=       700.00    XC=        306.73    YC=         -0.00
SC=       650.00    XC=        357.89    YC=         -0.00
SC=       600.00    XC=        409.04    YC=         -0.00
SC=       550.00    XC=        460.18    YC=         -0.00
SC=       500.00    XC=        511.30    YC=         -0.00
SC=       450.00    XC=        562.42    YC=         -0.00
SC=       400.00    XC=        613.54    YC=         -0.00
SC=       350.00    XC=        664.65    YC=         -0.00
SC=       300.00    XC=        715.77    YC=         -0.00
SC=       250.00    XC=        766.88    YC=         -0.00
SC=       200.00    XC=        818.00    YC=         -0.00
SC=       150.00    XC=        869.11    YC=         -0.00
SC=       100.00    XC=        920.22    YC=         -0.00
SC=        50.00    XC=        971.32    YC=         -0.00
SC=         0.00    XC=       1022.43    YC=         -0.00
SC=  1000.00 U=      0.11 PHI= -0.0002 T= 21077.65 V=      11.98
SC=   950.00 U=      0.00 PHI=  0.0000 T= 21048.32 V=      11.99
SC=   900.00 U=      0.00 PHI=  0.0000 T= 21019.19 V=      11.99
SC=   850.00 U=      0.00 PHI=  0.0000 T= 20990.07 V=      12.00
SC=   800.00 U=      0.00 PHI=  0.0000 T= 20961.16 V=      12.01
SC=   750.00 U=      0.00 PHI=  0.0000 T= 20932.91 V=      12.02
SC=   700.00 U=      0.00 PHI=  0.0000 T= 20905.73 V=      12.03
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SC=   400.00 U=      0.00 PHI=  0.0000 T= 20232.26 V=       9.97
SC=   350.00 U=      0.00 PHI=  0.0000 T= 20203.57 V=       9.97
SC=   300.00 U=      0.00 PHI=  0.0000 T= 20174.45 V=       9.98
SC=   250.00 U=      0.00 PHI=  0.0000 T= 20145.11 V=       9.98
SC=   200.00 U=      0.00 PHI=  0.0000 T= 20115.56 V=       9.99
SC=   150.00 U=      0.00 PHI=  0.0000 T= 20086.00 V=       9.99
SC=   100.00 U=      0.00 PHI=  0.0000 T= 20055.80 V=       9.99
SC=    50.00 U=      0.00 PHI=  0.0000 T= 20026.03 V=       9.99
SC=     0.00 U=      0.00 PHI=  0.0000 T= 19996.05 V=      10.00
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SC=       950.00    XC=         50.71    YC=          0.00
SC=       900.00    XC=        101.88    YC=         -0.00
SC=       850.00    XC=        153.04    YC=         -0.00
SC=       800.00    XC=        204.19    YC=         -0.00
SC=       750.00    XC=        255.35    YC=         -0.00
SC=       700.00    XC=        306.51    YC=         -0.00
SC=       650.00    XC=        357.66    YC=         -0.00
SC=       600.00    XC=        408.81    YC=         -0.00
SC=       550.00    XC=        459.96    YC=         -0.00
SC=       500.00    XC=        511.11    YC=         -0.00
SC=       450.00    XC=        562.26    YC=         -0.00
SC=       400.00    XC=        613.40    YC=         -0.00
SC=       350.00    XC=        664.55    YC=         -0.00
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 SC=       300.00   XC=        715.69   YC=         -0.00
 SC=       250.00   XC=        766.83   YC=         -0.00
 SC=       200.00   XC=        817.97   YC=         -0.00
 SC=       150.00   XC=        869.10   YC=         -0.00
 SC=       100.00   XC=        920.21   YC=         -0.00
 SC=        50.00   XC=        971.32   YC=         -0.00
 SC=         0.00   XC=       1022.43   YC=         -0.00
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SC=   950.00 U=      0.10 PHI= -0.0002 T= 21034.29 V=      11.95
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SC=   850.00 U=      0.00 PHI=  0.0000 T= 20976.69 V=      11.96
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SC=   750.00 U=      0.00 PHI=  0.0000 T= 20918.67 V=      11.96
SC=   700.00 U=      0.00 PHI=  0.0000 T= 20889.76 V=      11.97
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SC=   500.00 U=      0.00 PHI=  0.0000 T= 20771.98 V=      11.97
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SC=   250.00 U=      0.00 PHI=  0.0000 T= 20616.67 V=      11.94
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SC=   150.00 U=      0.00 PHI=  0.0000 T= 20298.92 V=      10.85
SC=   100.00 U=      0.00 PHI=  0.0000 T= 20086.00 V=      10.07
SC=    50.00 U=      0.00 PHI=  0.0000 T= 20046.31 V=      10.02
SC=     0.00 U=      0.00 PHI=  0.0000 T= 20014.39 V=      10.00
  TIME=      0.334
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 SC=       900.00   XC=        101.66   YC=          0.00
 SC=       850.00   XC=        152.82   YC=         -0.00
 SC=       800.00   XC=        203.98   YC=         -0.00
 SC=       750.00   XC=        255.13   YC=         -0.00
 SC=       700.00   XC=        306.29   YC=         -0.00
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 SC=       550.00   XC=        459.74   YC=         -0.00
 SC=       500.00   XC=        510.89   YC=         -0.00
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 SC=       400.00   XC=        613.18   YC=         -0.00
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SC=   900.00 U=      0.08 PHI= -0.0002 T= 20994.60 V=      11.90
SC=   850.00 U=      0.00 PHI=  0.0000 T= 20966.56 V=      11.91
SC=   800.00 U=      0.00 PHI=  0.0000 T= 20938.30 V=      11.92
SC=   750.00 U=      0.00 PHI=  0.0000 T= 20909.82 V=      11.93
SC=   700.00 U=      0.00 PHI=  0.0000 T= 20881.14 V=      11.94
SC=   650.00 U=      0.00 PHI=  0.0000 T= 20852.66 V=      11.95
SC=   600.00 U=      0.00 PHI=  0.0000 T= 20823.75 V=      11.96
SC=   550.00 U=      0.00 PHI=  0.0000 T= 20794.63 V=      11.96
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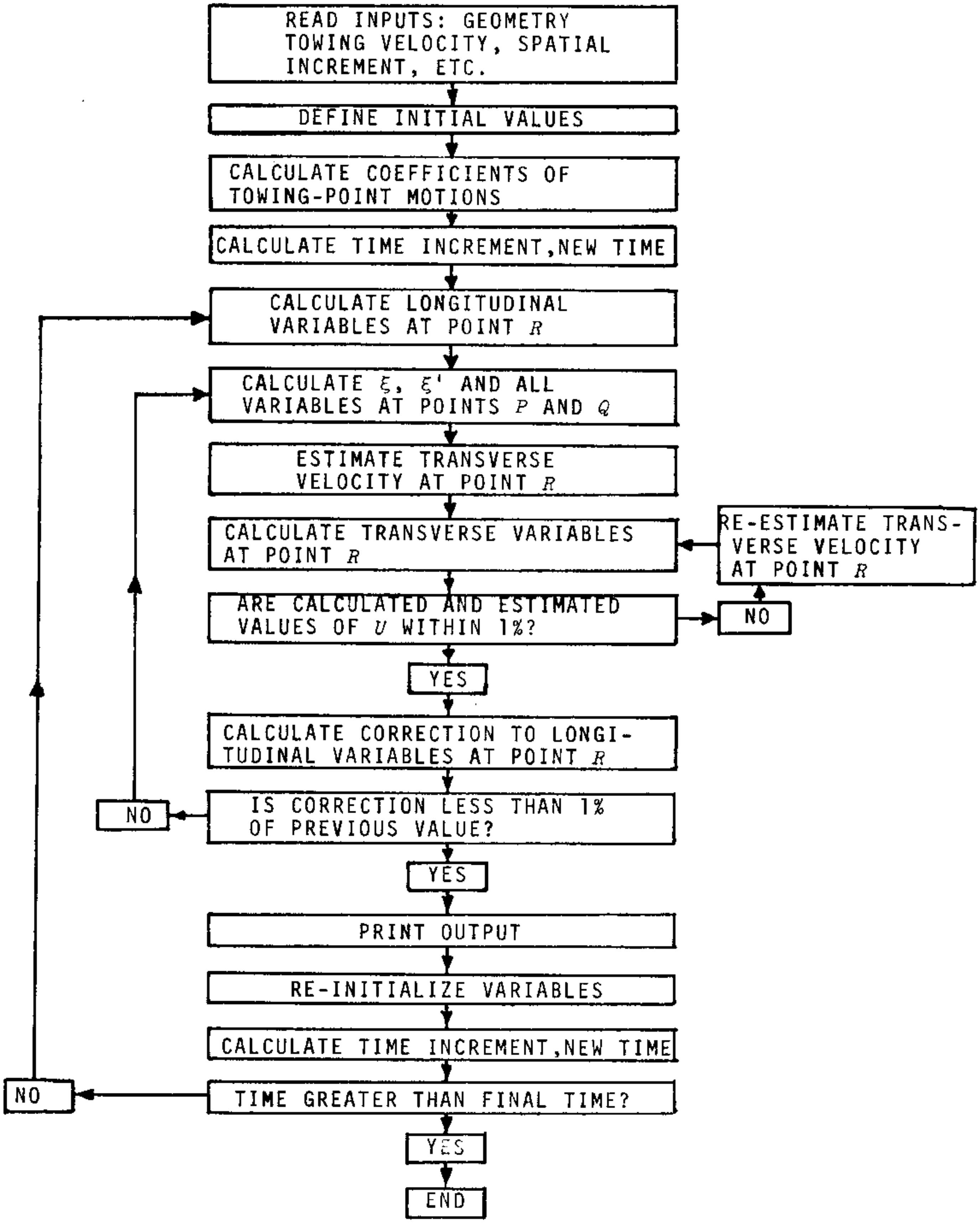
APPENDIX III



FLOW CHART FOR PROGRAMS CABL1 AND CABL2

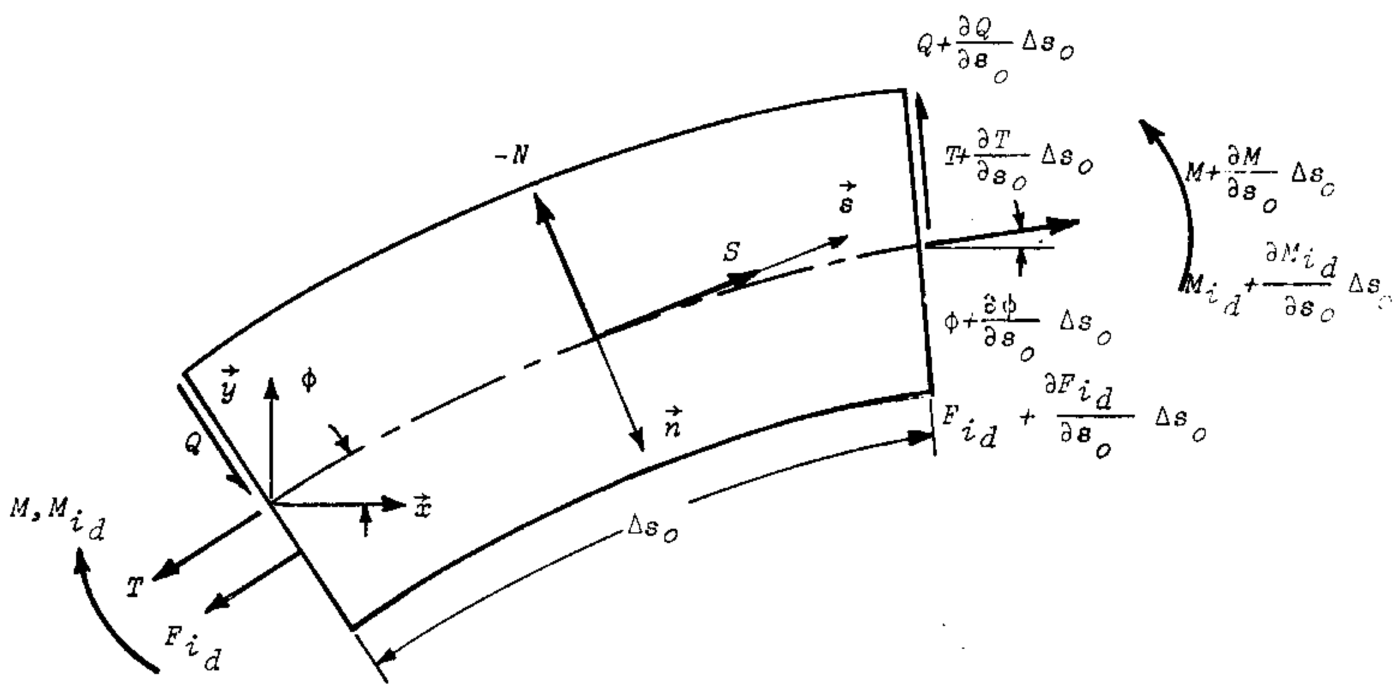


Figure 1. Forces and Moments Acting on a Cable Element of Length Δs_o

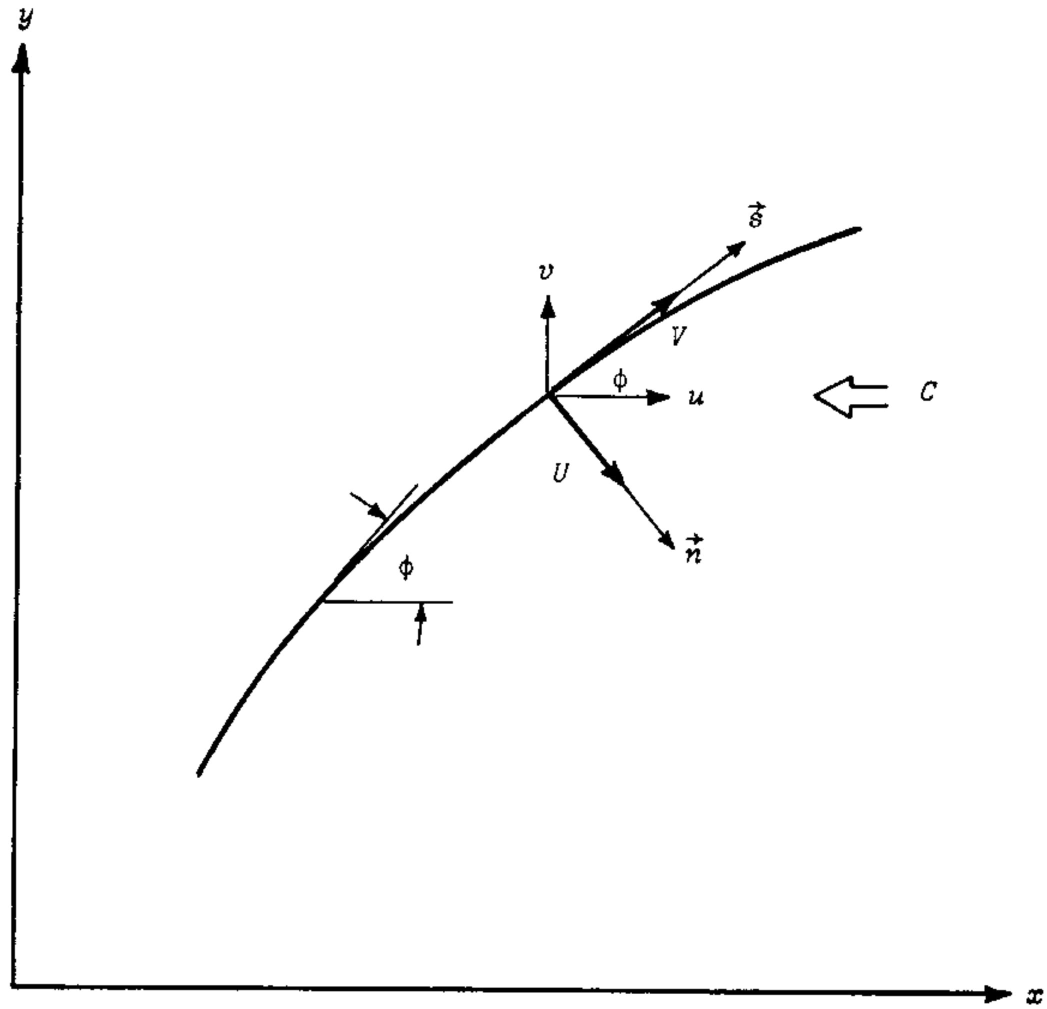


Figure 2. Definition of Coordinate Systems and Velocities

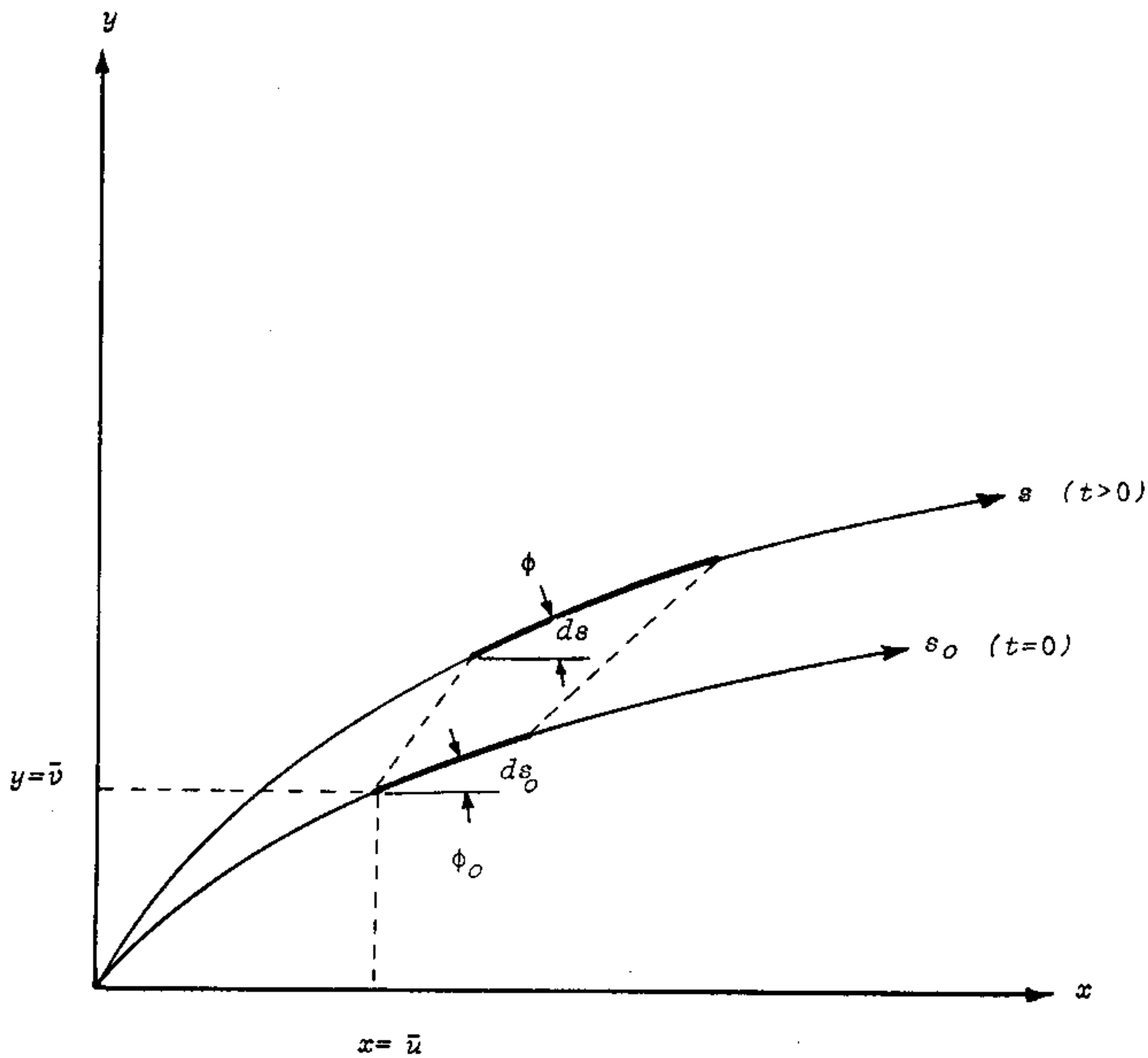


Figure 3. Displacement of a Cable Element

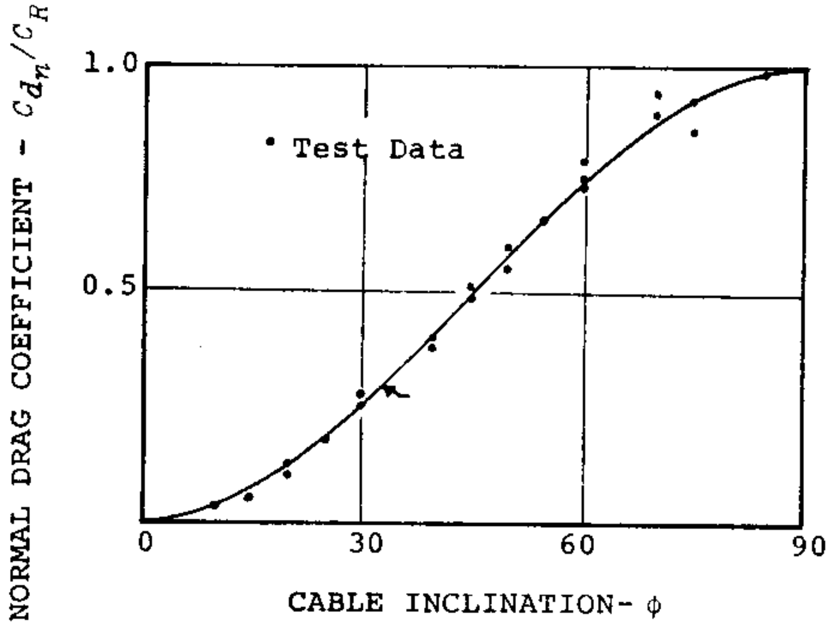


Figure 4. Summary of Available Normal Drag Force Data for Bare, Circular Cables - From Reference 1

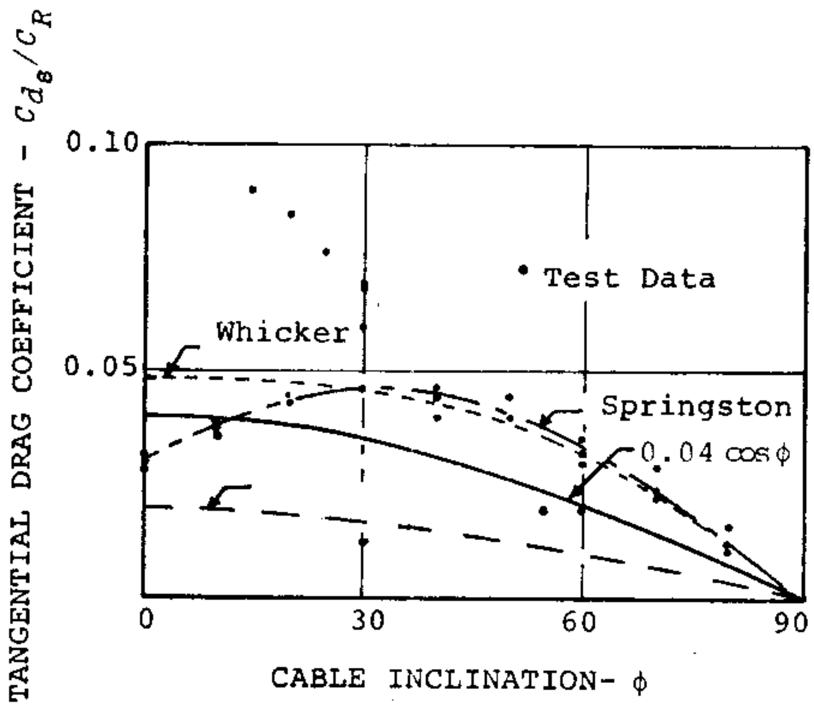


Figure 5. Summary of Available Tangential Force Data for Bare, Circular Cables - From Reference 1

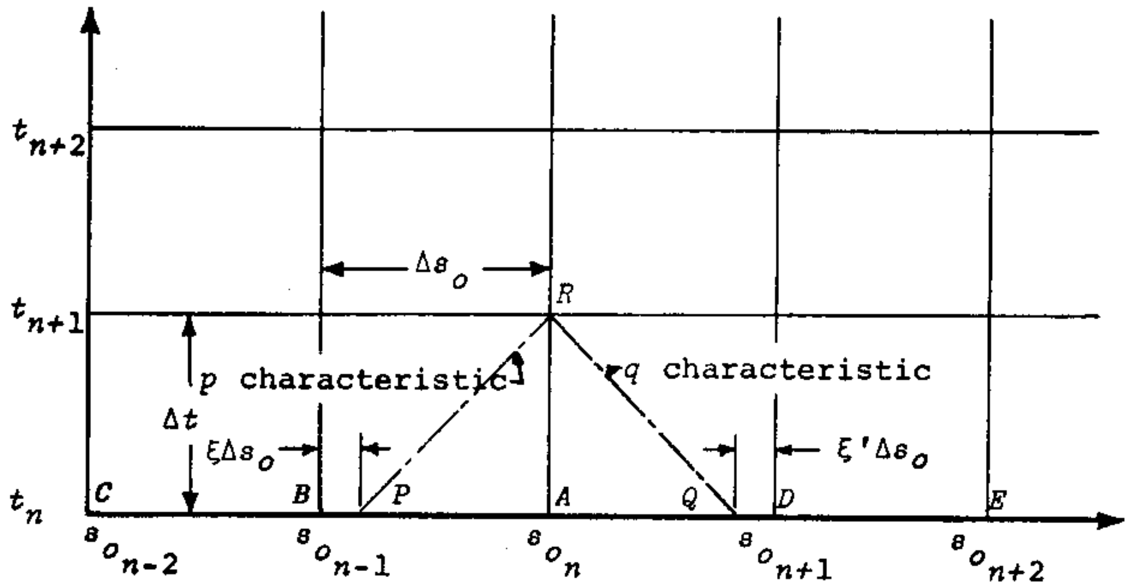


Figure 6. Fixed Time-Space Grid Used for Numerical Integration

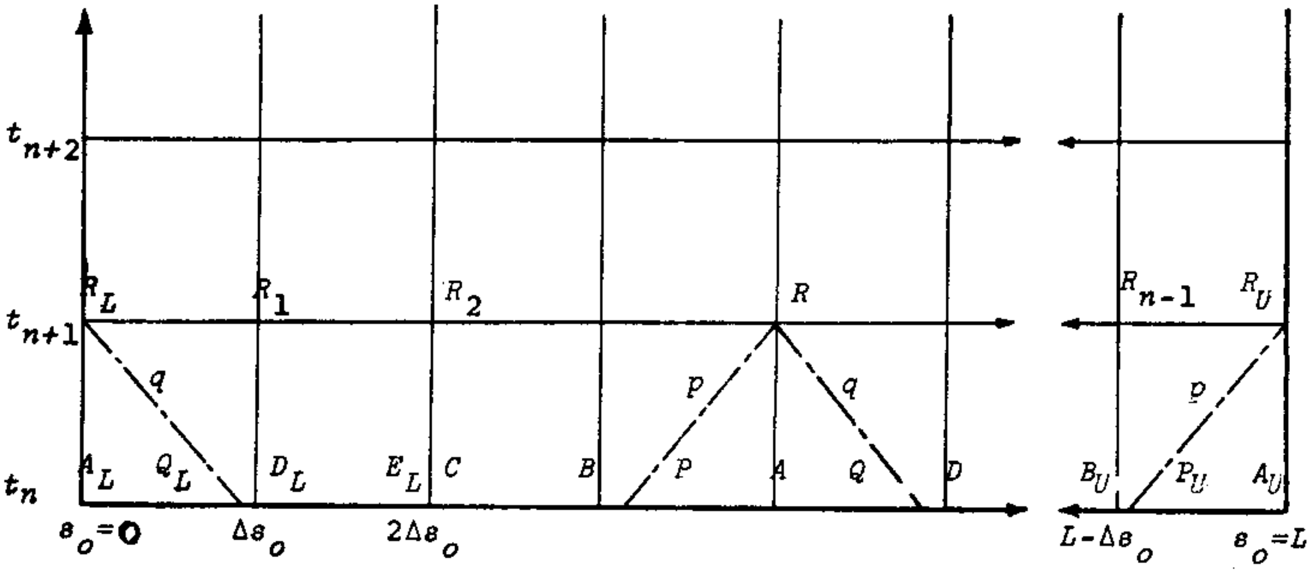


Figure 7. Time-Space Grid for End-Points

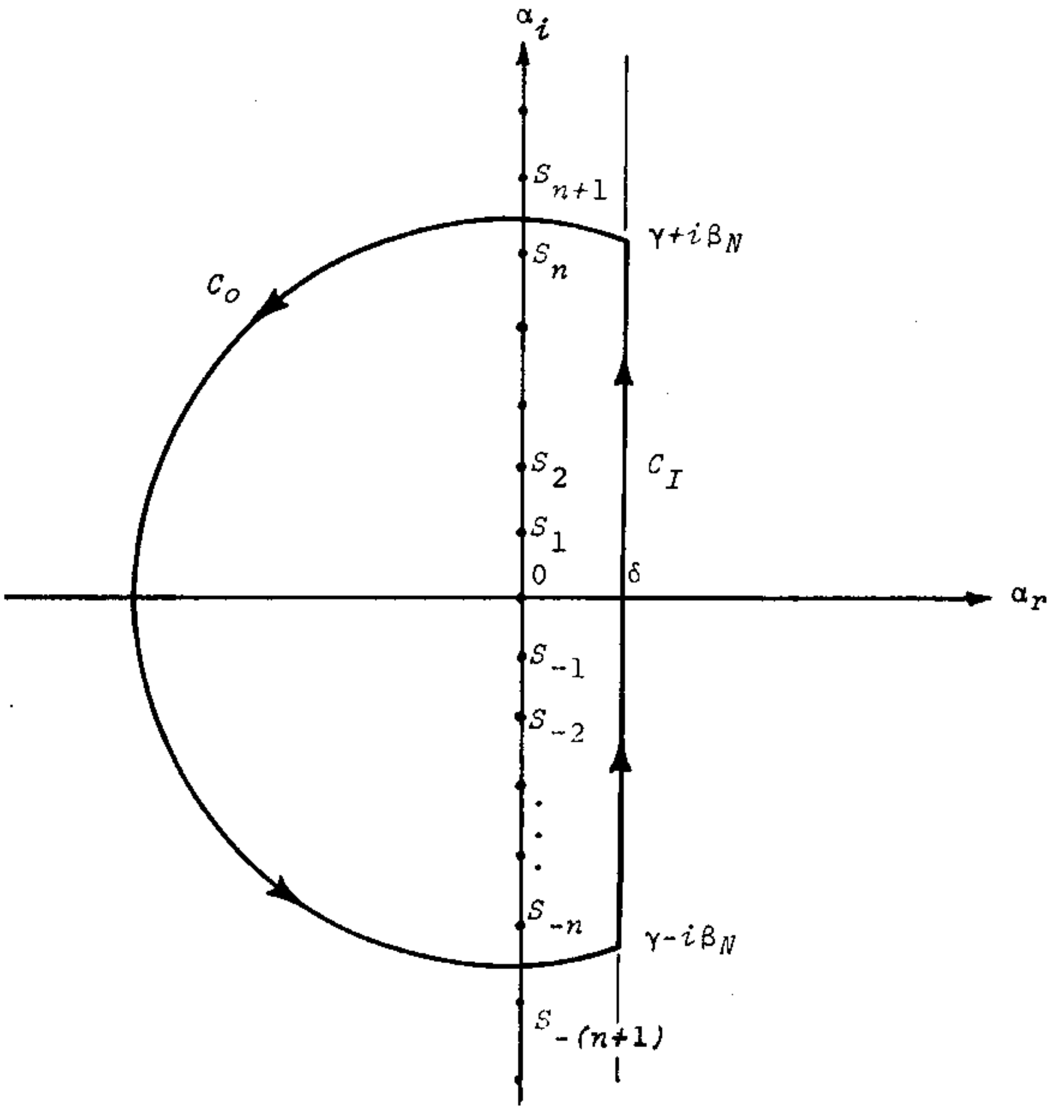


Figure 8. Path of Integration for Evaluating the Laplace Inversion Integral in the Complex α Plane, $\alpha=\alpha_r+i\alpha_i$

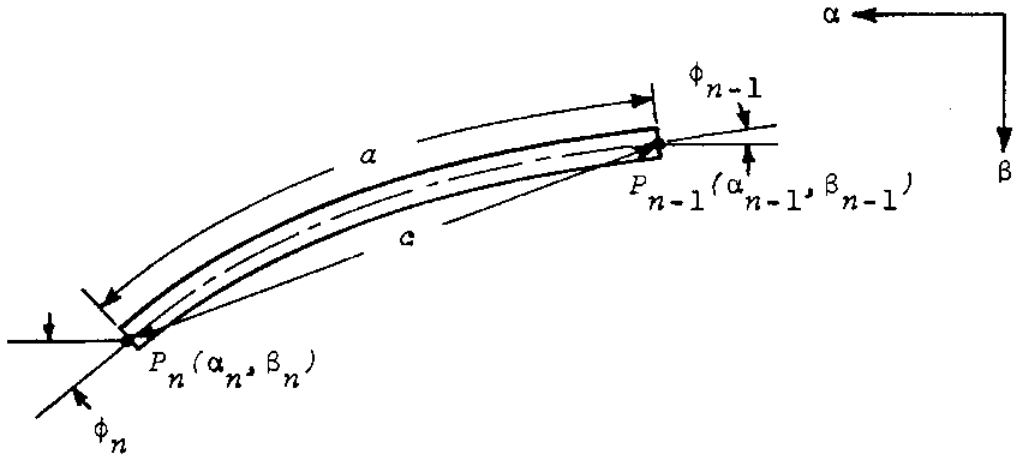


Figure 9. Definition of Lengths and Coordinates used for Calculating Cable Shape

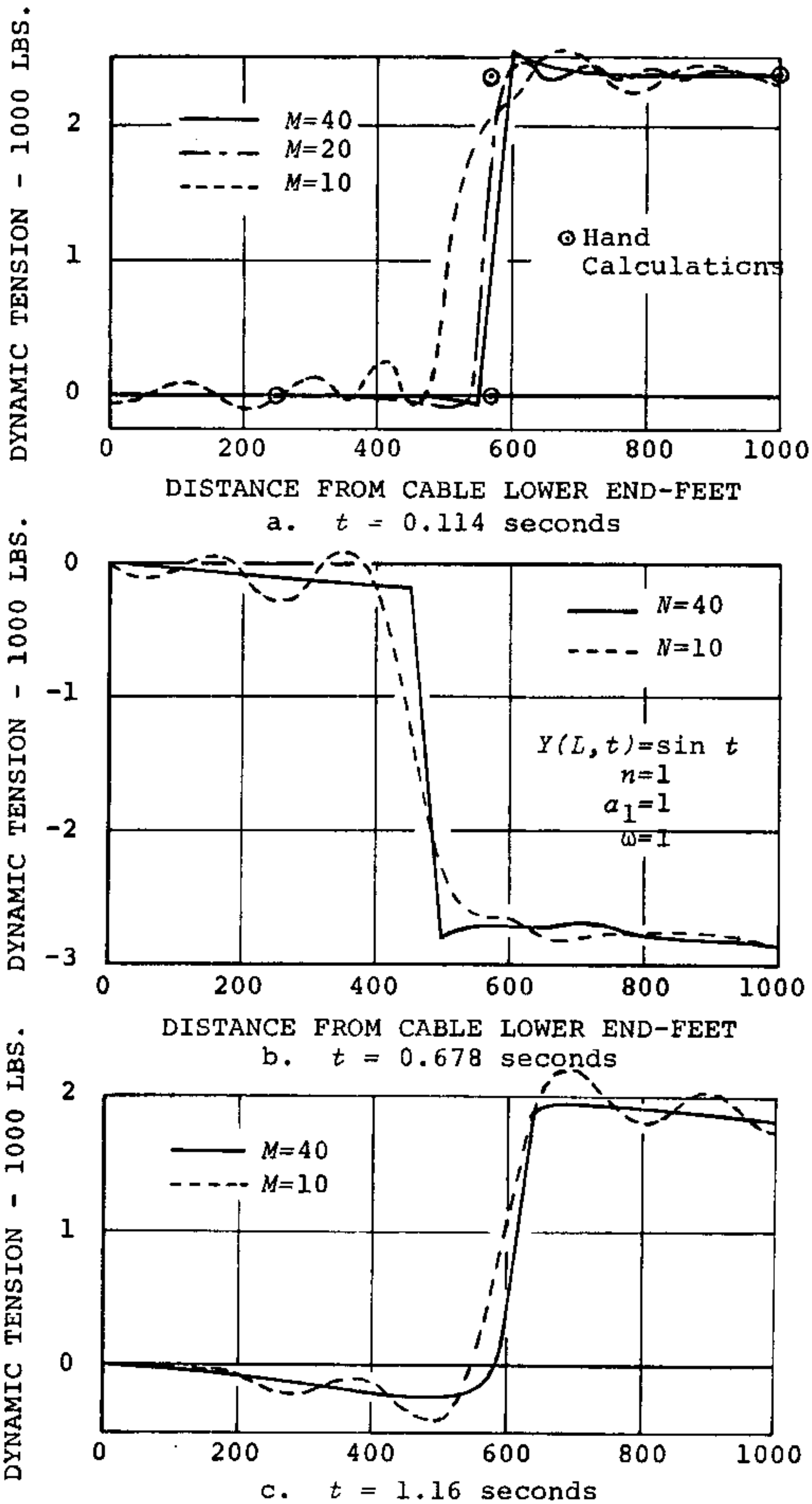
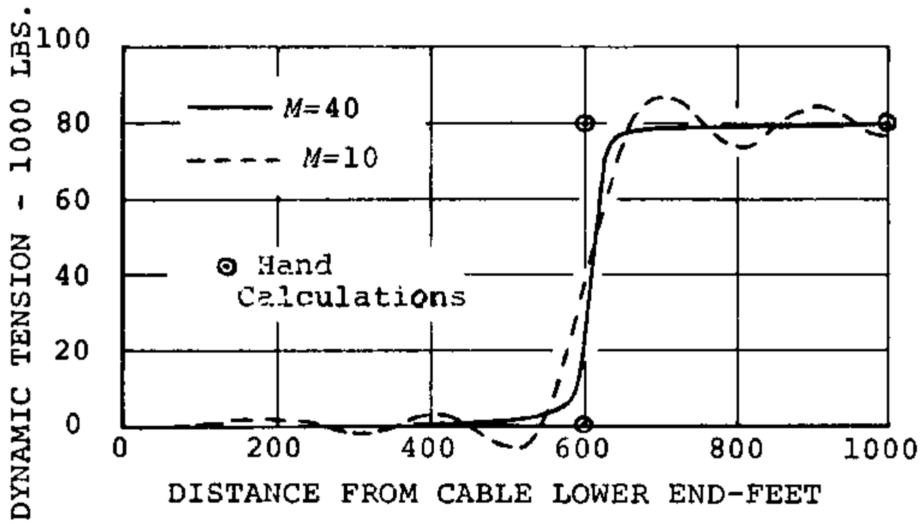
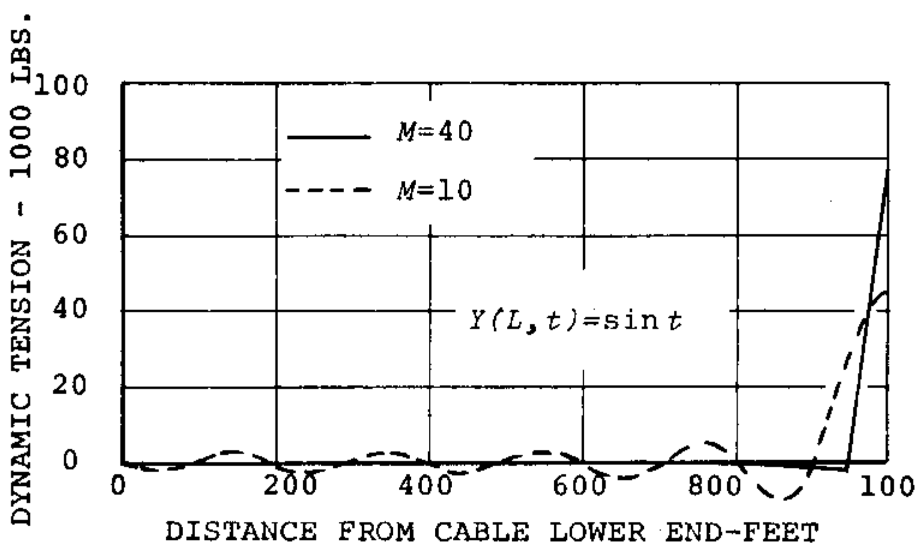


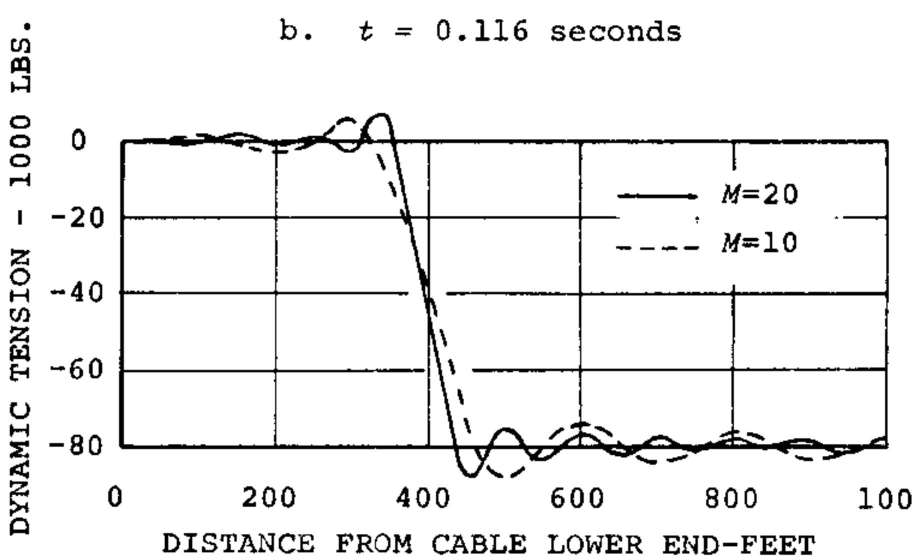
Figure 10. Calculated Dynamic Tensions Due to Cable Longitudinal Input Motion of 10 Feet - Cable Lower-End Free, Plastic Cable Material



a. t = 0.0233 seconds



b. t = 0.116 seconds



c. t = 0.153 seconds

Figure 11. Calculated Dynamic Tensions Due to Cable Longitudinal Input Motion of 10 Feet - Cable Lower-End Free, Steel Cable Material

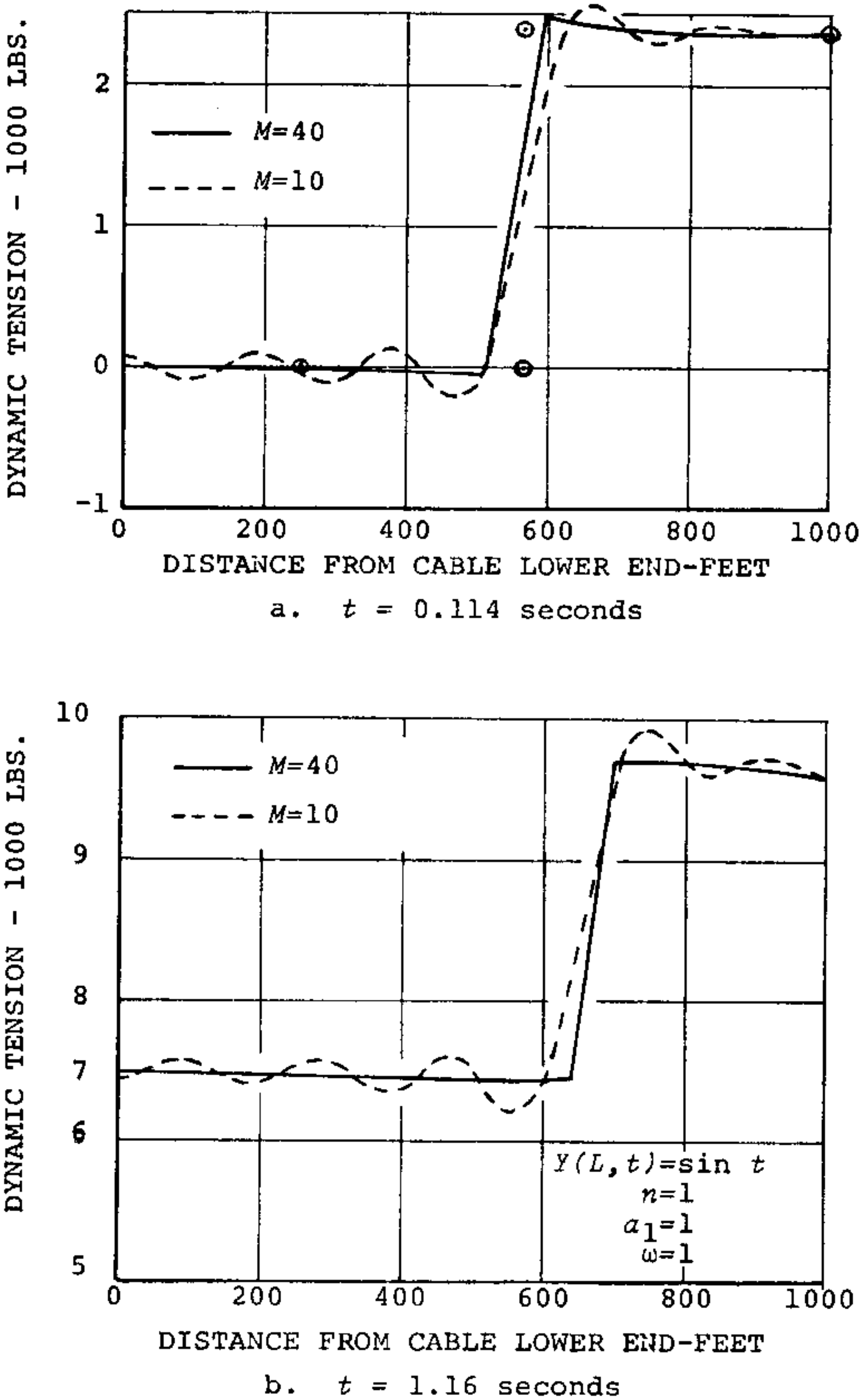


Figure 12. Calculated Dynamic Tensions Due to Cable Longitudinal Input Motion of 10 Feet - Cable Lower-End Fixed, Plastic Cable Material

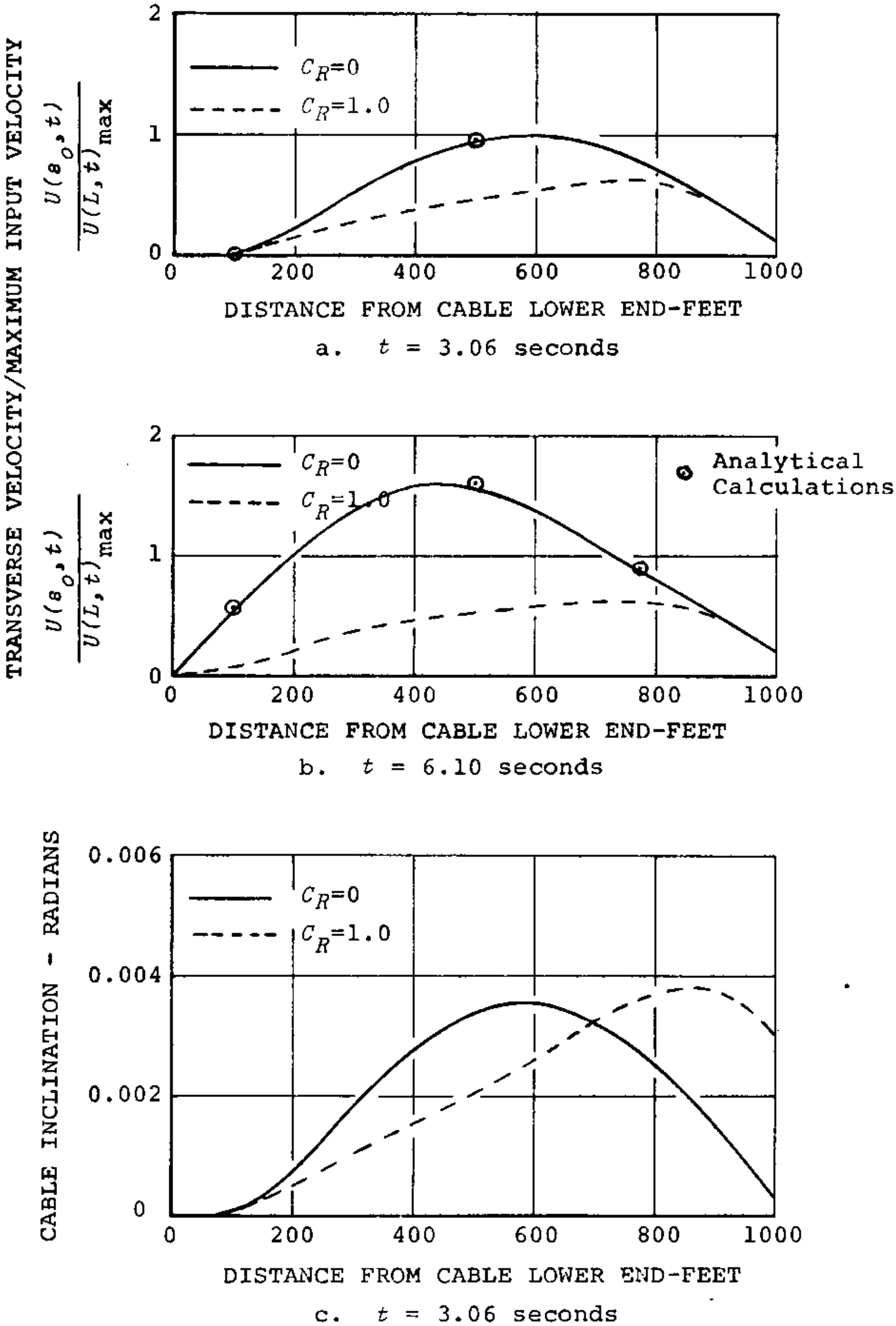


Figure 13. Calculated Transverse Motions for a Cable with no Longitudinal Motion - Cable Lower-End Fixed, Plastic Material

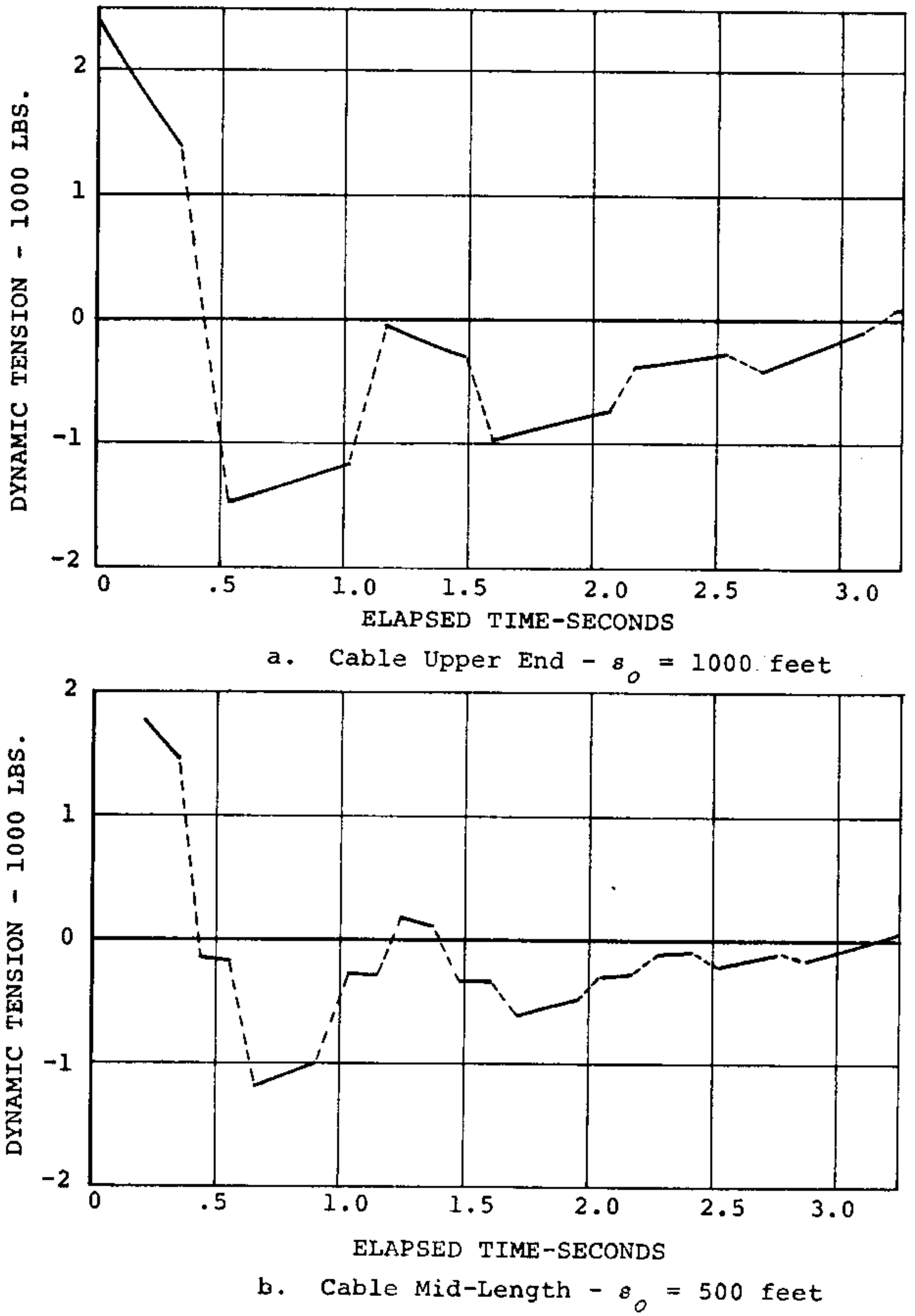
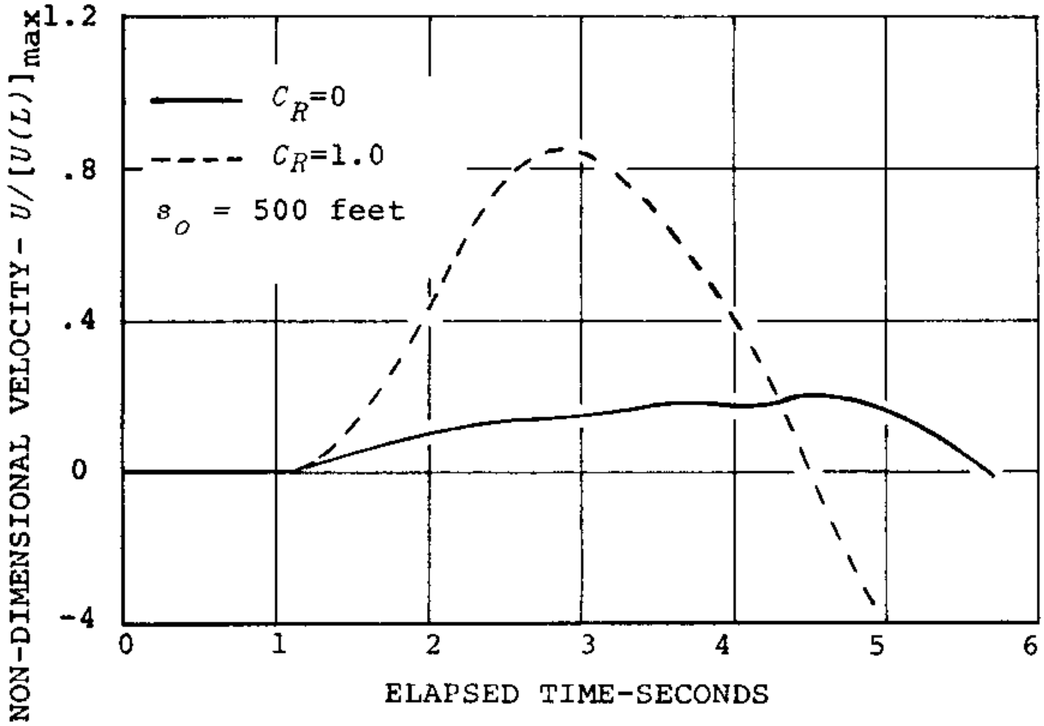
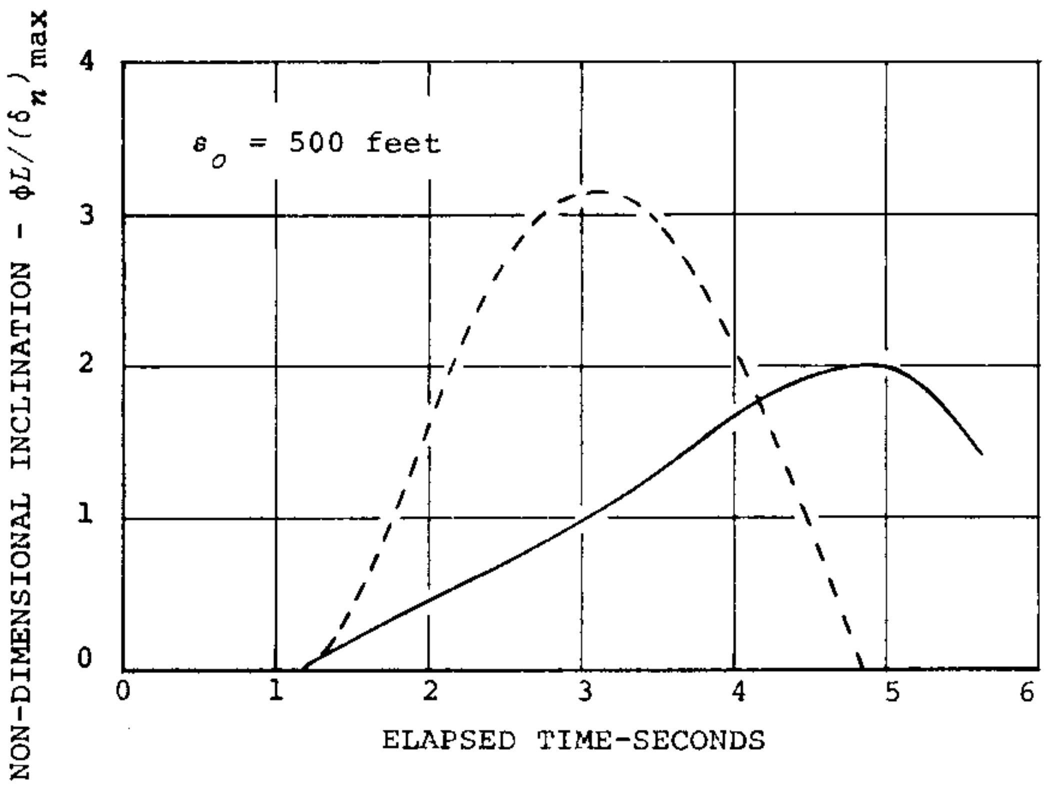


Figure 14. Calculated Dynamic Tensions Including the Effect of Approximate Longitudinal Damping (kC_R=0.04)

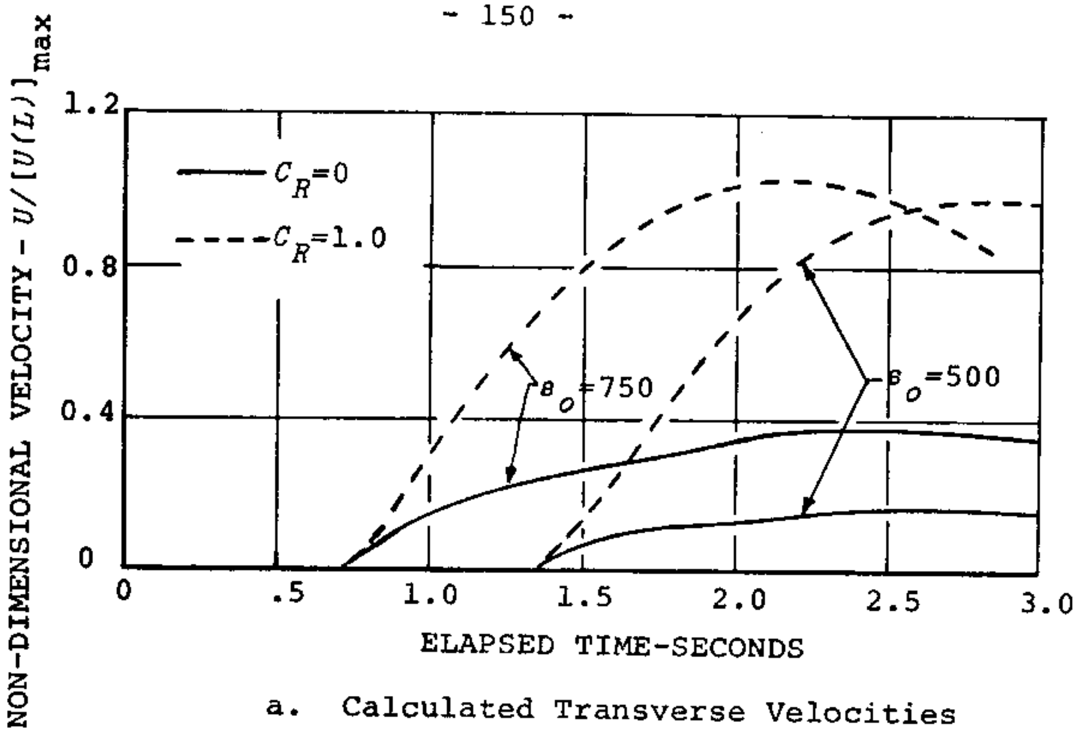


a. Calculated Transverse Velocity

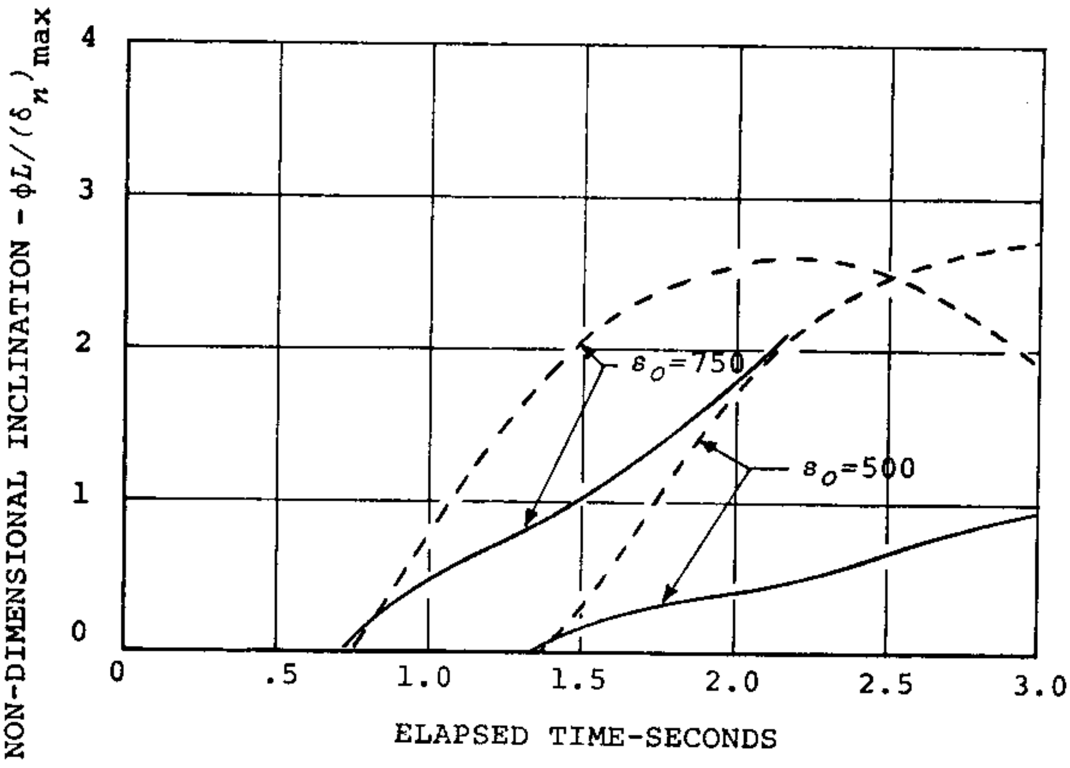


b. Calculated Cable Inclination Angles

Figure 15. Calculated Transverse Motions at s_o=500 feet for a Plastic Cable with Free Lower-End and Zero Towing Velocity

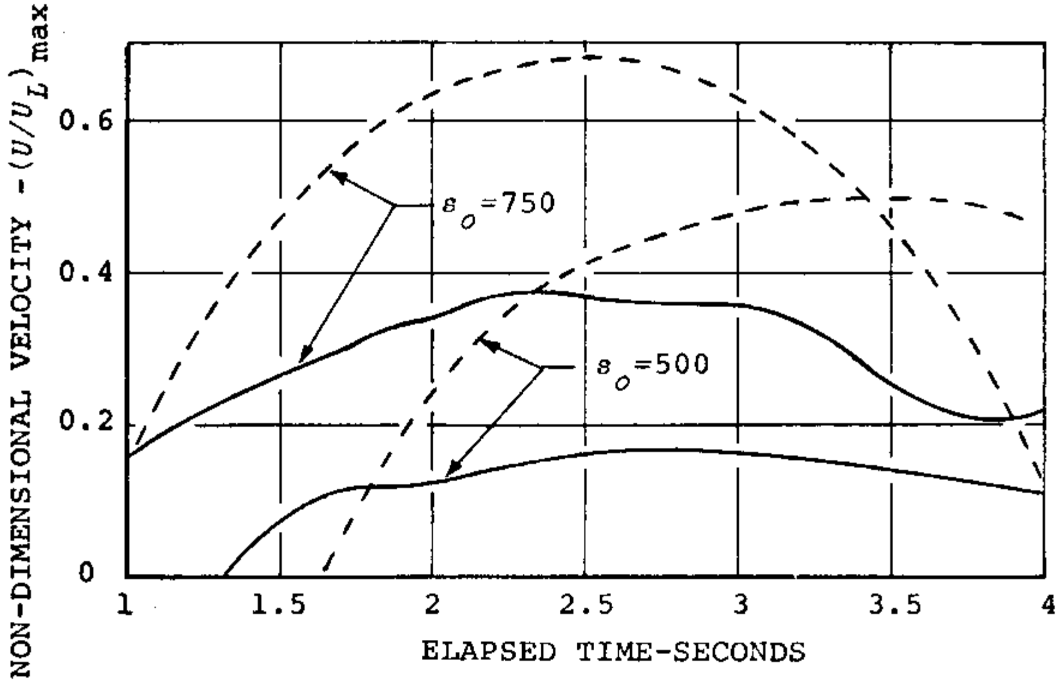


a. Calculated Transverse Velocities

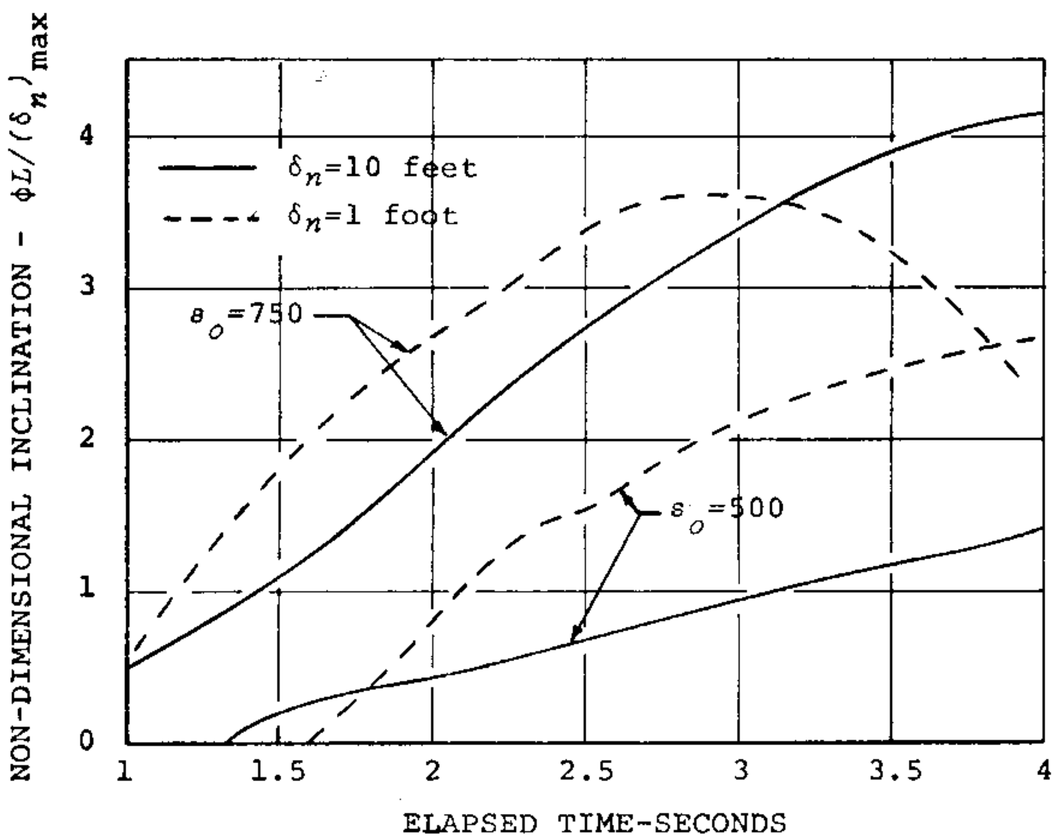


b. Calculated Cable Inclination Angles

Figure 16. Calculated Transverse Motions for a Plastic Cable with Fixed Lower-End and Zero Towing Velocity

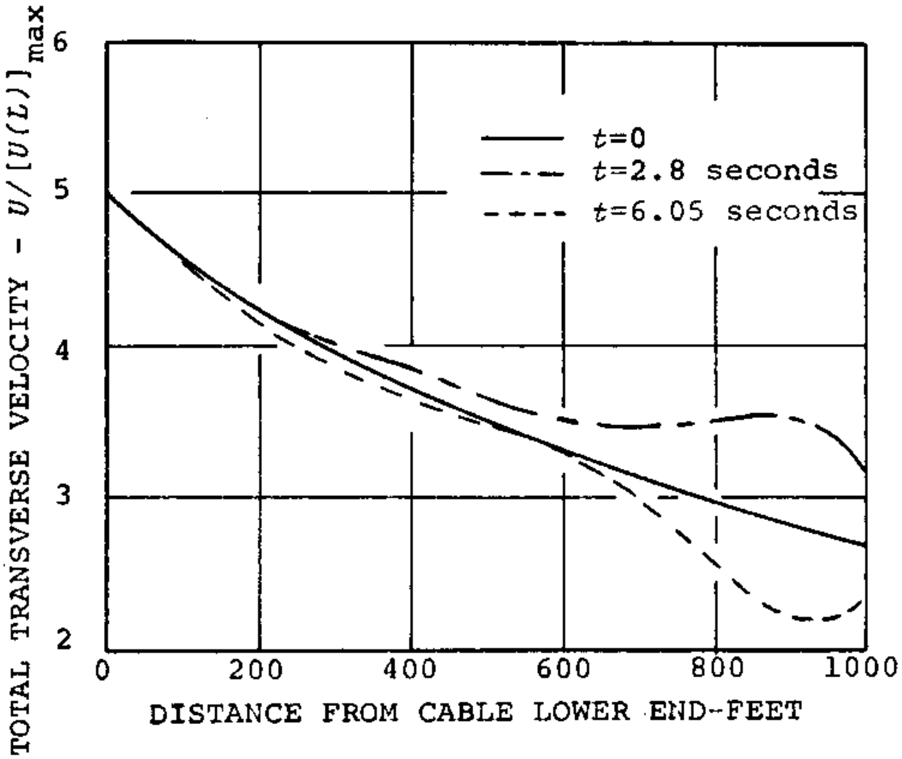


a. Calculated Transverse Velocities

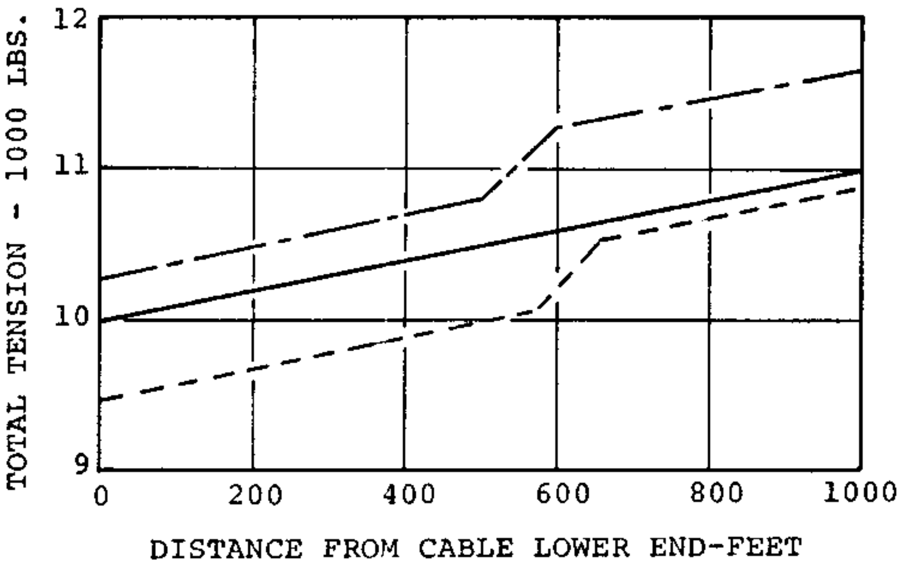


b. Calculated Cable Inclination Angles

Figure 17. Effect of Amplitude of Towing-Point Motions on Calculated Transverse Motions, Fixed End Plastic Cable (C_R=1.0, V_T=10, C_R=1.0)

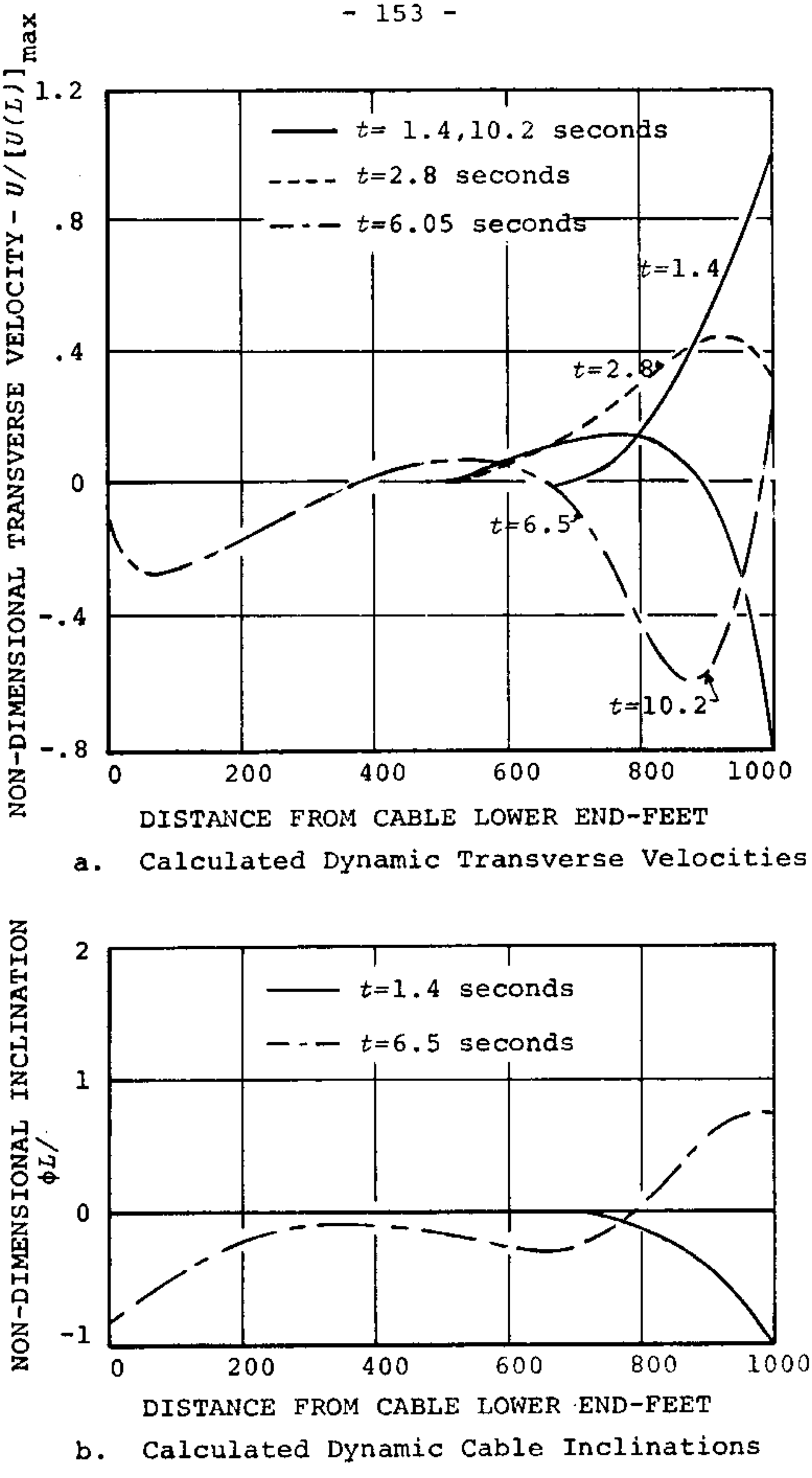


a. Calculated Transverse Velocities



b. Calculated Cable Tensions

Figure 18. Calculated Transverse Velocities and Tensions for a Plastic Cable with Fixed Lower-End with Inclination of 30 Degrees



a. Calculated Dynamic Transverse Velocities

b. Calculated Dynamic Cable Inclinations

Figure 19. Calculated Transverse Dynamic Motions of a Steel Cable with Fixed Lower-End and Steady-State Inclination Equal to Cable Critical Angle (C_R=1.0, C=10 feet/second)