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A Two-Signal Primer for Fourier Analysis of a Random Access Communication System

M. NESENBERGS

BOULDER, COLO.
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A TWO-SIGNAL PRIMER FOR FOURIER ANALYSIS OF A RANDOM ACCESS COMMUNICATION SYSTEM

M. Nesenbergs

The detectability parameter, i.e., the acquisition signal-to-noise ratio, is derived for an elementary two-signal multiple access channel. The basic Fourier series approach requires no approximations, and the methodology should be useful in a forthcoming rigorous analysis of a more realistic random access satellite repeater.

Key Words: Detectability parameter, ideal hard limiter, random access, satellite repeater.

1. INTRODUCTION

A communications satellite with multiple, and in particular with random, access capability is of interest to various data collection and transmission networks. General aspects of such systems have been explored (Schwartz, Aein, and Kaiser, 1966), and specific questions dealing with synthesis and analysis of multiple access have been answered (Jones, 1963; Aein, 1964; Shaft, 1965; Sollfrey, 1969; Anderson and Wintz, 1969). Still, doubt remains about some of the statistical arguments used and the results so obtained. For instance, a new tractable and reliable derivation of the effective acquisition signal-to-noise ratio (SNR) or, as it is often called, "the detectability parameter" is needed.

Our long-range goal is to analyze a multiple (e.g., M-customer) random access repeater with an ideal hard limiter. To accomplish this in a concerted single effort, unfortunately, appears too complex a task. Consequently, a short-range goal is defined and solved in this report, limited to only two signals (M=2), using appropriate signal design and word correlation detection, and establishing useful statistical properties of the correlator outputs. The results of this study are to be used later both as a guide and as a tool to treat the far more difficult M-signal case. Because a full distribution of these random variables is too cumbersome to derive, we are content to obtain the first two moments (i.e., means and variances) without a need for so-called judicious and reasonable approximations.

All signals are of equal amplitude (see Jones, 1963 and Shaft, 1965 for effects due to unequal amplitudes), the modulation is 0 to π phase shift keying (PSK), and the individual carriers deviate in frequency and possess random phases. Some coding is likely to be used to design the modulating waveforms. This will cause no complication, as the present treatment permits arbitrary codes, be they pseudo noise (PN) sequences, orthogonal codes, or what have you.

2. STATEMENT OF PROBLEM

Consider a communication system as shown in figure 1. The channel consists ot an ideal hard limiter (IHL) plus additive white Gaussian noise, n(t). A bandpass filter (BPF) is used to reject bothersome higher harmonics. By observing its distorted and noisy input, the receiver tries to detect the message of one of the transmitters, say 0. The receiver must first decide whether transmitter 0 is actually transmitting (hypothesis H_o), or some other transmitter, say 1, transmits instead (hypothesis H_1).

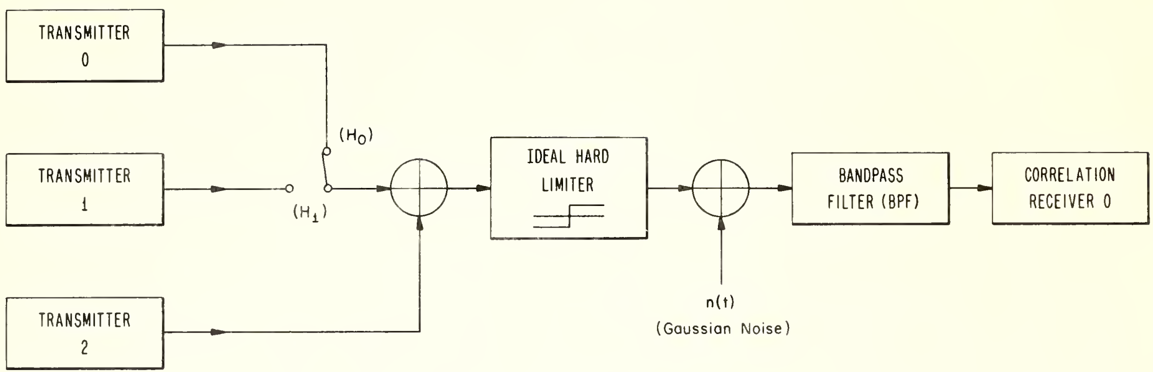

Figure 1. SYSTEM MODEL

The received signal plus noise is given by

$$x(t) = \mathrm{sgn}\left[\cos\theta_\nu(t) + \cos\theta_2(t)\right] + n(t), \quad \nu = 0, 1. \tag{1}$$

The function sgn x denotes the nonlinear IHL characteristic

$$\begin{aligned} \mathrm{sgn}\,x &= +1 \quad \text{if } x > 0, \\ &= 0 \quad \text{if } x = 0, \\ &= -1 \quad \text{if } x < 0, \end{aligned} \tag{2}$$

and n(t) is the white Gaussian noise with autocorrelation function

$$R(\tau) = N_0\,\delta(\tau), \qquad -\infty < \tau < \infty. \tag{3}$$

The other quantities in (1) are defined as follows:

$$\begin{aligned} \theta_0(t) &= \omega_0 t + \beta_0(t) + \varphi_0, \\ \theta_1(t) &= \omega_1 t + \beta_1(t-\tau_1) + \varphi_1, \\ \theta_2(t) &= \omega_2 t + \beta_2(t-\tau_2) + \varphi_2, \end{aligned} \tag{4}$$

where further natural simplifications are possible. The frequencies ω_0, ω_1, and ω_2 are slightly deviating from the center frequency ω_c,

$$\frac{\omega_0}{\omega_c} \cong \frac{\omega_1}{\omega_c} \cong \frac{\omega_2}{\omega_c} \cong 1\,. \tag{5}$$

The phases φ_0, φ_1, and φ_2 are assumed mutually independent random variables with a common uniform distribution over $(0, 2\pi)$. The binary access (or address) modulations $\beta_0(t)$, $\beta_1(t)$, $\beta_2(t)$ perform phase keying 0 to π, as shown in figure 2. The correlating (i.e., reference) signal is assumed to be perfectly matched to transmitter 0

$$r(t) = \cos\theta_0(t). \tag{6}$$

and represents the most advantageous acquisition situation. The other modulations are delayed by τ_1 and τ_2 respectively from the reference (fig. 2). We assume $0 \leq \tau_1, \tau_2 \leq T$.

The receiver correlates the stored waveform r(t) with the received waveform x(t), and eventually chooses between the two alternative hypotheses H_0 and H_1. The statistical situation is therefore as shown in figure 3, where α_ν is the correlator output if we assume H_ν to be true:

$$\begin{aligned}\alpha_\nu = {} & \frac{1}{T}\int_0^T \cos\theta_0(t)\,\mathrm{sgn}\left[\cos\theta_\nu(t) + \cos\theta_2(t)\right] dt \\ & + \frac{1}{T}\int_0^T \cos\theta_0(t)\, n(t)\, dt, \qquad \nu = 0, 1.\end{aligned} \tag{7}$$

Because of the random phases, α_ν is in general not a Gaussian random variable. The distribution of α_ν is not known, nor is it easily computed. We will be content to derive the first two moments, $E\alpha_\nu$ and $\mathrm{var}\alpha_\nu$, averaged over the independent phases and noise.

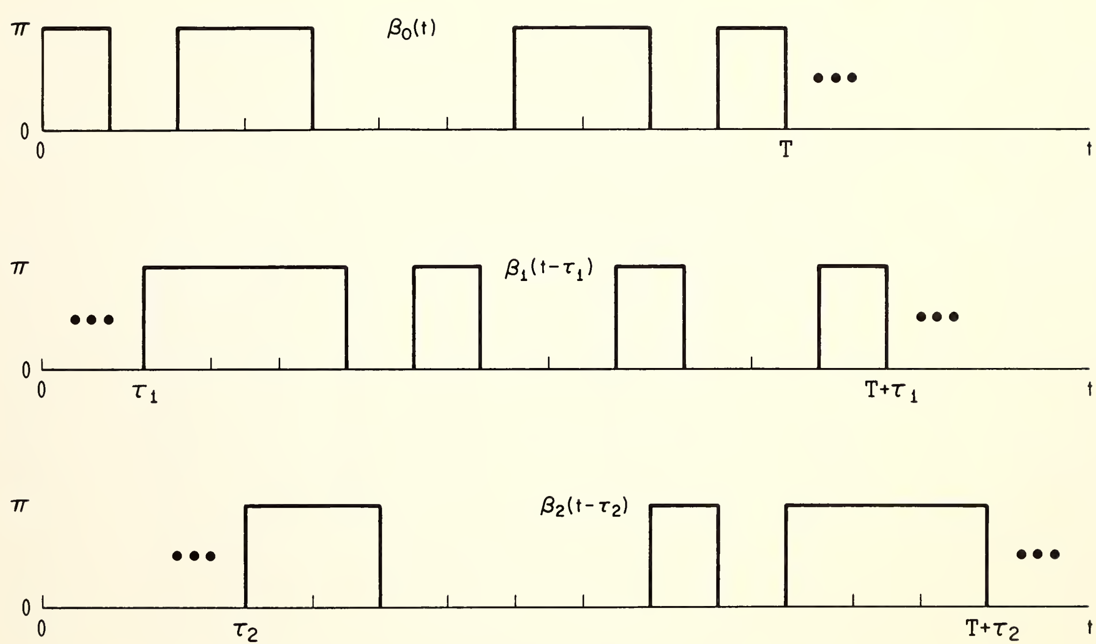

Figure 2. MODULATING SEQUENCES

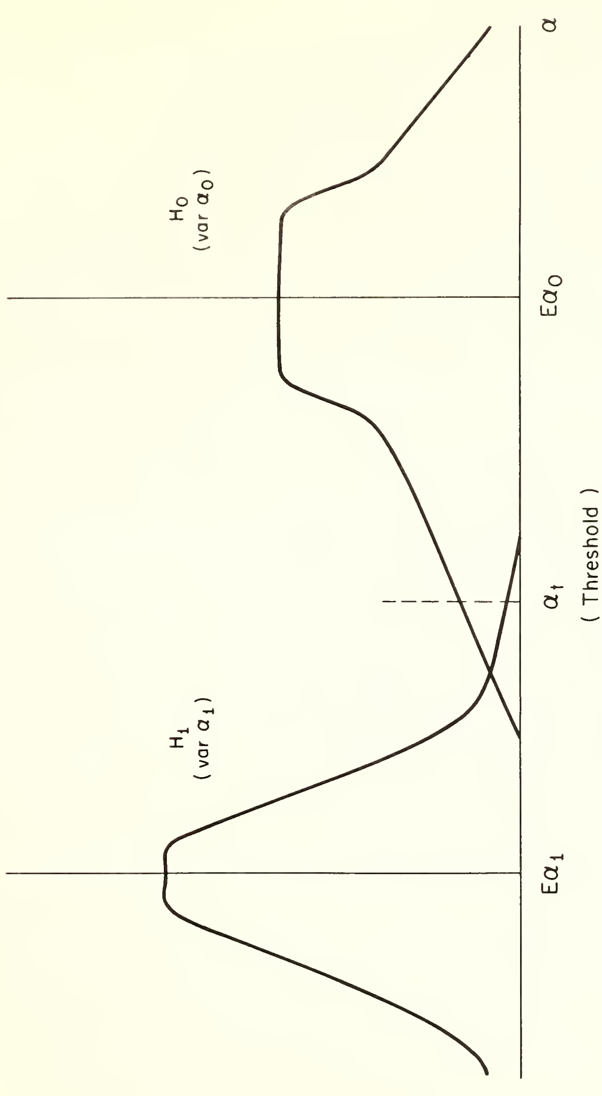

Figure 3. STATISTICAL MODEL

3. FOURIER COEFFICIENTS

The statistical moments of α_ν (7) depend on the Fourier coefficients of the function $(0 \le x \le \pi,\ 0 \le y \le \pi)$

$$\begin{aligned} \operatorname{sgn}(\cos x + \cos y) &= +1 && \text{if } 0 \le x+y < \pi \\ &= 0 && \text{if } x+y = \pi \\ &= -1 && \text{if } \pi < x+y \le 2\pi, \end{aligned} \tag{8}$$

which through even symmetry and periodicity of the arguments extends to all x and y.

The Fourier series

$$\operatorname{sgn}(\cos x + \cos y) = \sum_{m,n=0}^{\infty} C_{mn} \cos mx \cos ny \tag{9}$$

has coefficients

$$\begin{aligned} C_{oo} &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \operatorname{sgn}(\cos x + \cos y)\, dx\, dy \\ &= 0, \\ C_{mo} = C_{om} &= \frac{2}{\pi^2} \int_0^\pi \int_0^\pi \operatorname{sgn}(\cos x + \cos y) \cos mx\, dx\, dy \\ &= \frac{4}{\pi^2} \frac{1-(-1)^m}{m^2}, \qquad m = 1, 2, \ldots, \\ C_{mn} = C_{nm} &= \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \operatorname{sgn}(\cos x + \cos y) \cos mx \cos ny\, dx\, dy \\ &= \frac{8}{\pi^2} \frac{(-1)^n - (-1)^m}{m^2 - n^2}, \qquad m \ge n > 0. \end{aligned} \tag{10}$$

Two useful properties follow readily from (10).

(a) If m+n is even, the $C_{mn} = 0$. The largest coefficients are $C_{m+1,m}$, nonzero for all $m = 0, 1, 2, \ldots$.

(b) The Bessel's equality applies and through the well known identity

$$\sum_{m=1}^{\infty} m^{-4} = \frac{\pi^4}{90} \tag{11}$$

yields (see C_{mn}^2 balance sheet in fig. 4)

$$\sum_{m=1}^{\infty} C_{mo}^2 = \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} C_{mn}^2 = \frac{2}{3}. \tag{12}$$

4. DERIVATION OF MOMENTS

4.1 Means

In this section we find the average $E\alpha_\nu$ of the random variable α_ν, $\nu = 0, 1$. We substitute the Fourier series (9) into (7), and conclude that most terms must vanish in the averaging process. The

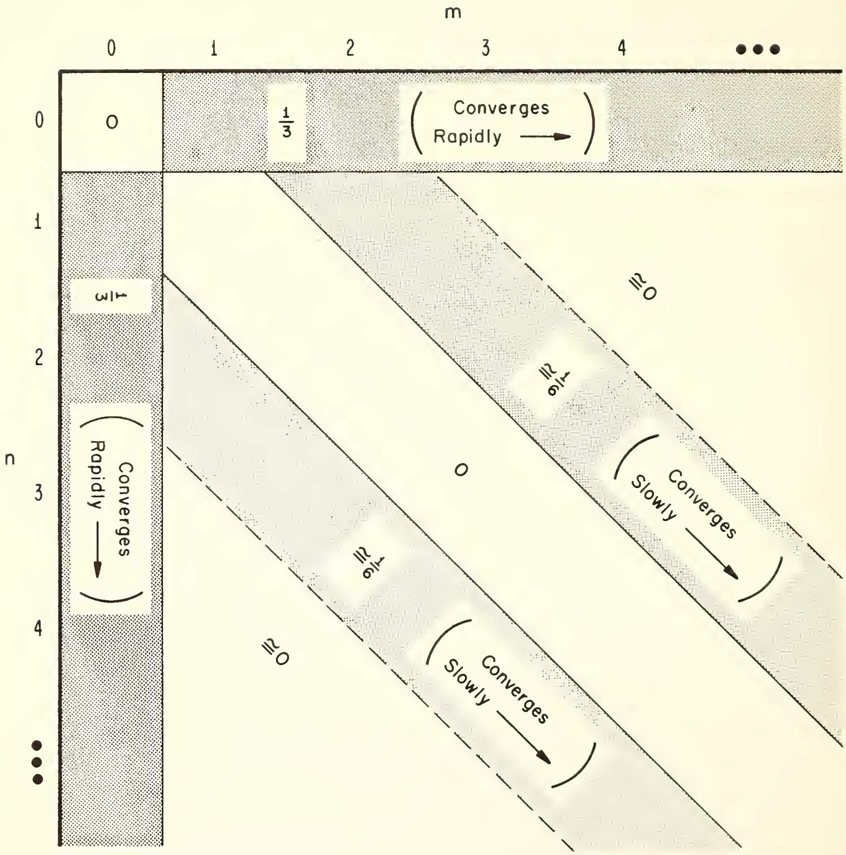

Figure 4. BALANCE SHEET FOR SUMS OF SQUARES, C^2mn.

noise term has zero mean and can be promptly ignored. Consider

$$E\alpha_\nu = \sum_{m,n=0}^{\infty} \frac{C_{mn}}{T} E \int_0^T \cos\theta_0(t) \cos m\theta_\nu(t) \cos n\theta_2(t) dt. \tag{13}$$

If $\nu = 0$, only $m = 1$ can produce a non-zero integrand. Likewise, $n = 0$ is the only way to dispose of φ_2. We conclude that $(m, n) = (1, 0)$ is the only nonvanishing term

$$\begin{aligned} E\alpha_0 &= \frac{C_{10}}{T} E \int_0^T \cos^2\theta_0(t) dt \\ &= \tfrac{1}{2} C_{10} \\ &= \frac{4}{\pi^2} \ (\cong 0.405\ 284).^* \end{aligned} \tag{14}$$

If $\nu = 1$, no choice of (m, n) disposes of φ_0 and

$$E\alpha_1 = 0. \tag{15}$$

4.2 Noise Variance

To derive var α_ν, we separate the mean square into noise and signal terms,

*Here and elsewhere, ωT need not be an integer. However, the error is of the order of ω^{-1} and can be ignored for $\omega > 10^6$ Hz.

$$
\begin{aligned}
E\alpha_\nu^2 &= E\left[\frac{1}{T}\int_0^T \cos\theta_0(t)n(t)dt\right]^2 \\
&+ E\left[\sum_{m,n=0}^{\infty} \frac{C_{mn}}{T}\int_0^T \cos\theta_0(t)\cos m\theta_\nu(t)\cos n\theta_2(t)dt\right]^2 .
\end{aligned}
\tag{16}
$$

It is clear that the first entity does not depend on ν and is by far the simpler. We consider it next (see (3)):

$$
\begin{aligned}
\sigma_n^2 &= E\left[\frac{1}{T}\int_0^T \cos\theta_0(t)\, n(t)\, dt\right]^2 \\
&= E\left[\frac{1}{T^2}\int_0^T\int_0^T \cos\theta_0(t)\cos\theta_0(\tau)n(t)n(\tau)\, dt\, d\tau\right] \\
&= \frac{1}{T^2}\int_0^T\int_0^T E[\cos\theta_0(t)\cos\theta_0(\tau)]\, E[n(t)n(\tau)]\, dt\, d\tau \\
&= \frac{N_0}{T^2}\int_0^T E[\cos^2\theta_0(t)]\, dt \\
&= \frac{N_0}{2T} .
\end{aligned}
\tag{17}
$$

4.3 Signal Variance

The second quantity in (16) is evaluated next. To make things come out neatly, i.e.,

$$\operatorname{var} \alpha_\nu = \sigma_n^2 + \sigma_\nu^2, \qquad \nu = 0,\ 1, \tag{18}$$

let us write

$$\begin{aligned}\sigma_\nu^2 + \left(E\alpha_\nu\right)^2 &= E\left[\sum_{m,\,n=0}^{\infty} \frac{C_{mn}}{T}\int_0^T \cos\theta_0(t)\cos m\theta_\nu(t)\cos n\theta_2(t)\,dt\right]^2 \\ &= \sum_{m,\,n,\,k,\,\ell=0}^{\infty} C_{mn}C_{k\ell}I_\nu(m,n;k,\ell),\end{aligned} \tag{19}$$

where

$$\begin{aligned}I_\nu(m,n;k,\ell) &= \frac{1}{T^2}\,E\int_0^T\!\!\int_0^T \cos\theta_0(t)\cos m\theta_\nu(t)\cos n\theta_2(t) \\ &\qquad\cdot\ \cos\theta_0(\tau)\cos k\theta_\nu(\tau)\cos \ell\,\theta_2(\tau)\,dt\,d\tau \\ &= \frac{1}{2^5T^2}\sum_{\pm} E\int_0^T\!\!\int_0^T \cos\Big[\theta_0(t) \pm m\theta_\nu(t) \pm n\theta_2(t) \\ &\qquad \pm \theta_0(\tau) \pm k\theta_\nu(\tau) \pm \ell\,\theta_2(\tau)\Big]\,dt\,d\tau.\end{aligned} \tag{20}$$

The second line in (20) is based on the trigonometric identity (proof by induction)

$$\prod_{j=0}^{J} \cos x_j = \frac{1}{2^J} \sum_{\pm} \cos(x_0 \pm x_1 \pm \ \ldots\ \pm x_J) \tag{21}$$

that holds for all integers $J \geq 0$ and all arguments $x_0, x_1, \ldots, x_J$, subject to proper interpretation of symbol $\sum\limits_{\pm}$. All 2^J $\pm$ combinations (distinct or not, zero or not) are included in the sum.

Let us review the conditions for a nonvanishing term $I_\nu(m, n; k, \ell)$. The indices must be such that (a) the coefficients of $\omega_c t$ and $\omega_c \tau$ vanish (to assure a spectrum passage through BPF before correlation), and (b) the coefficients of φ_0, φ_1, and φ_2 vanish (to avoid averaging to zero). These coefficient constraints are summarized as follows:

$$\begin{array}{lcc}
 & \underline{\nu = 0} & \underline{\nu = 1} \\
\omega_c t: & 1 \pm m \pm n = 0 & 1 \pm m \pm n = 0 \\
\omega_c \tau: & \pm 1 \pm k \pm \ell = 0 & \pm 1 \pm k \pm \ell = 0 \\
\varphi_0: & 1 \pm m \pm 1 \pm k = 0 & 1 \pm 1 = 0 \\
\varphi_1: & ----- & \pm m \pm k = 0 \\
\varphi_2: & \pm n \pm \ell = 0 & \pm n \pm \ell = 0
\end{array} \tag{22}$$

In both cases there are three linearly independent equations for four unknowns, and multiple solutions cannot be avoided. We proceed by treating the case $\nu = 0$ in section 4.4 and case $\nu = 1$ in section 4.5.

4.4 Hypothesis H_0

If the hypothesis H_0 is true, $\nu = 0$ may be substituted in (18) - (20). Define the quantity

$$\theta(t, \tau) = \theta_0(t) - \theta_2(t) - \theta_0(\tau) + \theta_2(\tau) \tag{23}$$

and observe that the argument of (20) that satisfies (22) for $\nu = 0$ must be a multiple of $\theta(t, \tau)$. Therefore, only (m, n, k, ℓ) values giving rise to

$$I(j) = \frac{1}{T^2}\int_0^T\int_0^T \cos j\,\theta(t,\tau)\,dt\,d\tau, \qquad j = 0, 1, 2, \ldots \tag{24}$$

can contribute to (20).

The $I_0(m, n, ;k, \ell)$'s that do not vanish are

$$\begin{aligned} I_0(1,0;1,0) &= 2^{-2}\,I(0), \\ I_0(0,1;0,1) &= 2^{-3}\,I(1), \\ I_0(0,1;2,1) &= I_0(2,1;0,1) = 2^{-4}\,I(1), \\ I_0(2,1;2,1) &= 2^{-5}\,I(1), \end{aligned} \tag{25a}$$

and four equal terms for all $j \geq 2$,

$$I_0(j \pm 1, j;\, j \pm 1, j) = 2^{-5}\,I(j). \tag{25b}$$

It remains to evaluate the integral $I(j)$ in (24). Let j be even, $j = 2k$ $(k = 0, 1, 2, \ldots)$,

$$\begin{aligned} \cos 2k\theta(t,\tau) &= \cos 2k(\omega_0 - \omega_2)\,(t - \tau) \\ &= \cos 2k(\omega_0 - \omega_2)t \,\cos 2k(\omega_0 - \omega_2)\,\tau \\ &\quad + \sin 2k(\omega_0 - \omega_2)t\, \sin 2k(\omega_0 - \omega_2)\,\tau, \end{aligned} \tag{26}$$

and

$$\begin{aligned} I(2k) &= \left[\frac{1}{T}\int_0^T \cos 2k(\omega_0 - \omega_2)t\,dt\right]^2 + \left[\frac{1}{T}\int_0^T \sin 2k(\omega_0 - \omega_2)t\,dt\right]^2 \\ &= \left[\frac{\sin 2k(\omega_0 - \omega_2)T}{2k(\omega_0 - \omega_2)T}\right]^2 + \left[\frac{1 - \cos 2k(\omega_0 - \omega_2)T}{2k(\omega_0 - \omega_2)T}\right]^2 \\ &= \left[\frac{\sin k(\omega_0 - \omega_2)T}{k(\omega_0 - \omega_2)T}\right]^2 \end{aligned} \tag{27}$$

For odd $j = 2k - 1$ $(k = 1, 2, \ldots)$ we introduce the following notation. For $\mu = 1,\ 2,$ let

$$\beta(\mu, t) = \beta_0(t) - \beta_\mu(t - \tau_\mu), \tag{28}$$

a function that is either 0 or π. Consequently $\cos \beta(\mu, t)$ must equal ± 1 for all $0 \le t \le T$. We expand this function in Fourier series (with $i = \sqrt{-1}$),

$$\cos \beta(\mu, t) = \sum_{n=-\infty}^{\infty} \gamma_\mu\left(n\frac{2\pi}{T}\right) e^{in\frac{2\pi}{T}t}$$

(29)

$$\gamma_\mu\left(n\frac{2\pi}{T}\right) = \frac{1}{T}\int_0^T \cos \beta(\mu, t) e^{-in\frac{2\pi}{T}t}\, dt\,.$$

This is a line spectrum with line separation $2\pi/T$. The power in the spectrum can be summarized as

$$\left|\gamma_\mu(\omega)\right|^2 = \left[\frac{1}{T}\int_0^T \cos \omega t \cos \beta(\mu, t)\, dt\right]^2 + \left[\frac{1}{T}\int_0^T \sin \omega t \cos \beta(\mu, t)\, dt\right]^2, \tag{30}$$

where

$$\gamma_\mu(\omega) = \gamma_\mu\left(n\frac{2\pi}{T}\right) \qquad \text{if } \omega = n\frac{2\pi}{T}$$

(31)

$$= 0 \qquad \text{otherwise.}$$

Return to the evaluation of $I(2k - 1)$, $k = 1, 2, \ldots.$ As in (26),

$\cos(2k - 1)\ \theta(t, \tau)$

$$= \cos[(2k-1)(\omega_0 - \omega_2)(t-\tau) + \beta(2, t) - \beta(2, \tau)] \qquad (32)$$

$$= \cos(2k-1)(\omega_0 - \omega_2)\ t \cos(2k-1)(\omega_0 - \omega_2)\ \tau \cos\ \beta(2, t) \cos\ \beta(2, \tau)$$

$$+ \sin(2k-1)(\omega_0 - \omega_2)\ t \sin(2k-1)(\omega_0 - \omega_2)\ \tau \cos\ \beta(2, t) \cos\ \beta(2, \tau),$$

and

$$I(2k-1) = \left[\frac{1}{T}\int_0^T \cos(2k-1)(\omega_0 - \omega_2)t \cos\beta(2, t)\, dt\right]^2$$

$$+ \left[\frac{1}{T}\int_0^T \sin(2k-1)(\omega_0 - \omega_2)t \cos\ \beta(2, t)\, dt\right]^2 \qquad (33)$$

$$= \left|\gamma_2\Big((2k-1)(\omega_0 - \omega_2)\Big)\right|^2 .$$

The major distinction between I(2k) and I(2k-1) must lie in their dependence on modulation (e. g. , coding). I(2k) does not depend on modulation at all. I(2k-1) does depend through (28)-(31). We may collect all I(2k)'s into a variance term σ_{oo}^2 that is divorced from crossmodulation effects; and all I(2k-1)'s into a variance term σ_{ox}^2 that does depend on crossmodulation between β_o and β_2(see (19)):

$$\sigma_o^2 = \sigma_{oo}^2 + \sigma_{ox}^2 . \qquad (34)$$

The component terms depend on (10), (19), (20), (27), and (33).

$$\sigma_{oo}^{2} = 2^{-5} \sum_{k=1}^{\infty} \left(C_{2k-1,\,2k} + C_{2k+1,\,2k}\right)^{2} I(2k)$$

$$= \frac{2}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{(4k^2 - \frac{1}{4})^2} \left[\frac{\sin k(\omega_0 - \omega_2)T}{k(\omega_0 - \omega_2)T}\right]^2 ,$$

$$\sigma_{ox}^{2} = 2^{-5} (2C_{01} + C_{21})^2 \, I(1) \tag{35}$$

$$+ 2^{-5} \sum_{k=2}^{\infty} \left(C_{2k-2,\,2k-1} + C_{2k,\,2k-1}\right)^{2} I(2k-1)$$

$$= \frac{2}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{[(2k-1)^2 - \frac{1}{4}]^2} \left| \gamma_2 \left((2k-1)(\omega_0 - \omega_2)\right) \right|^2 .$$

4.5 Hypothesis H_1

Let $\nu = 1$ and proceed as before. There are two arguments in (20) that contribute for all $m = 0, 1, 2, \ldots,$

$$\theta_0(t) - \theta_1(t) + m[\theta_2(t) - \theta_1(t)]$$

$$-\theta_0(\tau) + \theta_1(\tau) - m[\theta_2(\tau) - \theta_1(\tau)],$$

and (36)

$$\theta_0(t) - \theta_2(t) + m[\theta_1(t) - \theta_2(t)]$$

$$-\theta_0(\tau) + \theta_2(\tau) - m[\theta_1(\tau) - \theta_2(\tau)] .$$

Therefore in (19)

$$\sigma_1^2 = \sum_{m=0}^{\infty} C^2_{m+1,\, m}\left[I_1\,(m+1,\, m; m+1,\, m) + I_1\,(m,\, m+1; m,\, m+1)\right], \quad (37)$$

where for $j = 1, 2, \ldots$ define

$$\begin{aligned} I_1\,(1, 0; 1, 0) + I_1\,(0, 1; 0, 1) &= 2^{-3}\, J(0), \\ I_1\,(j+1, j; j+1, j) + I_1\,(j, j+1; j, j+1) &= 2^{-5}\, J(j). \end{aligned} \quad (38)$$

Each $J(j)$ is a sum of four terms. Let $j = 2k(k=0, 1, 2, \ldots)$,

$$\begin{aligned} J(2k) &= \left[\frac{1}{T}\int_0^T \cos\left(\omega_0 - (2k+1)\omega_1 + 2k\omega_2\right)t \cos\beta(1, t)dt\right]^2 \\ &+ \left[\frac{1}{T}\int_0^T \sin\left(\omega_0 - (2k+1)\omega_1 + 2k\omega_2\right) t \cos\beta(1, t)dt\right]^2 \quad (39a) \\ &+ \left[\frac{1}{T}\int_0^T \cos\left(\omega_0 + 2k\omega_1 - (2k+1)\omega_2\right) t \cos\beta(2, t)dt\right]^2 \\ &+ \left[\frac{1}{T}\int_0^T \sin\left(\omega_0 + 2k\omega_1 - (2k+1)\omega_2\right)t \cos\beta(2, t)dt\right]^2 \\ &= \left|\, \gamma_1\left(\omega_0 - (2k+1)\omega_1 + 2k\omega_2\right)\right|^2 + \left|\gamma_2\left(\omega_0 + 2k\omega_1 - (2k+1)\omega_2\right)\right|^2 . \end{aligned}$$

Next, let $j = 2k - 1$ ($k=1, 2, \ldots$),

$$J(2k-1) = \left[\frac{1}{T}\int_0^T \cos\left(\omega_0 + (2k-1)\omega_1 - 2k\omega_2\right)t \cos\beta(1,t)dt\right]^2$$
$$+ \left[\frac{1}{T}\int_0^T \sin\left(\omega_0 + (2k-1)\omega_1 - 2k\omega_2\right) t \cos\beta(1,t)dt\right]^2$$
$$+ \left[\frac{1}{T}\int_0^T \cos\left(\omega_0 - 2k\omega_1 + (2k-1)\omega_2\right)t \cos\beta(2,t)dt\right]^2 \tag{39b}$$
$$+ \left[\frac{1}{T}\int_0^T \sin\left(\omega_0 - 2k\omega_1 + (2k-1)\omega_2\right)t \cos\beta(2,t)dt\right]^2$$
$$= \left|\gamma_1\left(\omega_0 + (2k-1)\omega_1 - 2k\omega_2\right)\right|^2 + \left|\gamma_2\left(\omega_0 - 2k\omega_1 + (2k-1)\omega_2\right)\right|^2 .$$

Apparently, both $J(2k)$ and $J(2k-1)$ do depend on cross-modulation. The variance (let $\sigma_1^{\,2} = \sigma_{1x}^{\,2}$ to agree with (34) and to emphasize this dependence) follows from (10), (37), (38), and (39):

$$\sigma_{1x}^{\,2} = \frac{8}{\pi^4}\sum_{k=0}^{\infty}\frac{1}{(4k+1)^2}\left\{\left|\gamma_1\left(\omega_0 - (2k+1)\omega_1 + 2k\omega_2\right)\right|^2 + \left|\gamma_2\left(\omega_0 + 2k\omega_1 - (2k+1)\omega_2\right)\right|^2\right\}$$
$$+ \frac{8}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{(4k-1)^2}\left\{\left|\gamma_1\left(\omega_0 + (2k-1)\omega_1 - 2k\omega_2\right)\right|^2 + \left|\gamma_2\left(\omega_0 - 2k\omega_1 + (2k-1)\omega_2\right)\right|^2\right\}. \tag{40}$$

5. DETECTABILITY PARAMETER

5.1 Definitions

Aein (1964), Anderson and Wintz (1969) and others use a detectability parameter

$$d^2 = \frac{(E\alpha_0)^2}{\operatorname{var}\alpha_0} . \tag{41}$$

This definition ignores the false alarm probability completely (Helstrom, 1960), and is suited to measure the message output SNR after acquisition. The acquisition model (fig. 3) suggests an alternative definition

$$D^2 = \frac{|E\alpha_0 - E\alpha_1|^2}{\operatorname{var}\alpha_0 + \operatorname{var}\alpha_1} . \tag{42}$$

Rudnick (1962) gives Neyman-Pearson arguments for use of D^2, and in fact shows that D^2 should be maximized. Even in the present case of $E\alpha_1 = 0$ (15), the two definitions become equivalent only in the extreme $\operatorname{var}\alpha_1 \ll \operatorname{var}\alpha_0$. If the two variances are of the same order of magnitude, then $D^2 \cong \frac{1}{2} d^2$.

We have as our main result

$$D^2 = \frac{(E\alpha_0)^2}{2\sigma_n^{\ 2} + \sigma_{oo}^{\ 2} + (\sigma_{ox}^{\ 2} + \sigma_{1x}^{\ 2})} , \tag{43}$$

where $E\alpha_0$ is given in (14), $\sigma_n^{\ 2}$ in (17), $\sigma_{oo}^{\ 2}$ and $\sigma_{ox}^{\ 2}$ in (35), and $\sigma_{1x}^{\ 2}$ in (40). All the quantities are exact for the assumed system model (fig. 1); no approximations or bounds have been used so far.

To elaborate more on the σ's in (43), one needs to invoke additional properties of the channel. These properties are apt to be based partly on measurement, and partly on engineering inference.

Consider $\sigma_{oo}^{\ 2}$ in (35). The only unknown entity is the frequency offset $\omega_0 - \omega_2$. By (5), the difference should be small but not necessarily zero. The variance term can be upperbounded by setting ω_0 equal to ω_2 (see Abramowitz and Stegun, 1964),

$$\sigma_{oo}^2 \leq \frac{2}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{(4k^2 - \frac{1}{4})^2} \cong 0.001\ 567. \tag{44}$$

The crossmodulation-dependent quantities σ_{ox}^2 (35) and σ_{1x}^2 (40) also depend on the frequency spacings. The physical interpretation of the variances amounts to weighted sums of crossmodulation energy at specified spectral lines. There is a great variety of methods for estimation and bounding of spectra; the difficulty must clearly lie elsewhere. Consider our knowledge, or lack of same, about the following:

(a) frequencies ω_0, ω_1, ω_2,

(b) delays τ_1, τ_2,

(c) modulations $\beta_0(\tau)$, $\beta_1(\tau)$, $\beta_2(\tau)$.

Quantity σ_{oo}^2 (44) was upperbounded by setting all frequencies equal. This appears uncertain for σ_{ox}^2 and σ_{1x}^2. Also since message FM is certain to be much slower than address modulation, we must decide whether there is practical justification for treating the carrier frequencies ω_0, ω_1, and ω_2 as deterministic constants (functions) or as random variables (functions).

In common practice all possible delays $0 \leq \tau_\mu \leq T$ $(\mu = 1, 2)$ can occur. We may wish to single out the pair (τ_1, τ_2) that has the worst effect on $\sigma_{ox}^2 + \sigma_{1x}^2$. The final answer depends on the code (e.g., signal design) used. From figure 2 we can observe superficially that maximum variance must occur at bit sync.

The choice of code, as indicated, offers an area of concern, especially for large M. For M = 2, the three waveforms $\beta_0(t)$, $\beta_1(t)$, and $\beta_2(t)$ give little substance to a multiple access argument involving crosstalk and/or lack of orthogonality.

5.2 Example

Consider the following, atypically simple example. Set $\tau_1 = \tau_2 = 0$, $T = 2\pi$, and for $\mu = 1, 2$, let $\omega_0 - \omega_\mu = 1/M_\mu$, where $M_\mu \gg 1$ is a large integer. Pick the codewords as

$$\begin{aligned} \beta_0(t) &= (0, 0, 0, \pi, \pi, \pi), \\ \beta_1(t) &= (0, 0, \pi, 0, \pi, \pi), \\ \beta_2(t) &= (0, \pi, 0, 0, \pi, 0). \end{aligned} \tag{45}$$

By (28)

$$\begin{aligned} \cos\beta(1,t) &= (+1, +1, -1, -1, +1, +1), \\ \cos\beta(2,t) &= (+1, -1, +1, -1, +1, -1), \end{aligned} \tag{46}$$

and by (29)

$$\begin{aligned} \gamma_1(o) &= \frac{1}{3} && \text{(not orthogonal)}, \\ \gamma_2(o) &= 0 && \text{(orthogonal)}, \\ \gamma_1(n) &= (-1)^{n+1}\frac{2}{n\pi}\sin\frac{n\pi}{3} && n \neq 0, \\ \gamma_2(n) &= \frac{i}{n\pi}\left[2 + (-1)^n\right] && n \neq 0. \end{aligned} \tag{47}$$

We denote by * the constraint that $(2k-1)/M_2$ must be an integer and write (35) as

$$\sigma_{ox}^2 = \frac{2}{\pi^4} \sum_{k=1*}^{\infty} \frac{1}{[(2k-1)^2 - \frac{1}{4}]^2} \left| Y_2\left(\frac{2k-1}{M_2}\right) \right|^2$$

$$= \frac{2}{\pi^4} \left[\frac{|Y_2(1)|^2}{(M_2^2 - \frac{1}{4})^2} + \ldots \right] \tag{48}$$

$$\cong \frac{2}{\pi^6 M_2^4}$$

$$\cong 0.002\,080\,(\omega_0 - \omega_2)^4 .$$

Similarly, from (40),

$$\sigma_{1x}^2 = \frac{8}{\pi^4} \left\{ \sum_{k=0*}^{\infty} \frac{1}{(4k+1)^2} \left| Y_1\left(\frac{2k+1}{M_1} - \frac{2k}{M_2}\right) \right|^2 \right.$$

$$+ \sum_{k=0*}^{\infty} \frac{1}{(4k+1)^2} \left| Y_2\left(\frac{2k+1}{M_2} - \frac{2k}{M_1}\right) \right|^2$$

$$+ \sum_{k=1*}^{\infty} \frac{1}{(4k-1)^2} \left| Y_1\left(\frac{2k}{M_1} - \frac{2k-1}{M_2}\right) \right|^2 \tag{49}$$

$$\left. + \sum_{k=1*}^{\infty} \frac{1}{(4k-1)^2} \left| Y_2\left(\frac{2k}{M_2} - \frac{2k-1}{M_1}\right) \right|^2 \right\} ,$$

where * denotes a constraint that the argument of $\gamma_\mu(\ldots)$ must be an integer, $\mu = 1, 2$.

$$\sigma_{1x}^{2} = \frac{4}{\pi^4}\left[\frac{|\gamma_1(1)|^2}{M_1^2} + \frac{|\gamma_2(1)|^2}{M_2^2} + \frac{|\gamma_1(0)|^2}{4(M_1+M_2)^2} + \ldots\right]$$

$$\cong \frac{4}{\pi^6}\left[\frac{3}{M_1^2} + \frac{1}{M_2^2} + \frac{\pi^2}{36(M_1+M_2)^2}\right]^2 \tag{50}$$

$$\cong 0.012\,482\,(\omega_0-\omega_1)^2 + 0.004\,161\,(\omega_0-\omega_2)^2$$

$$+\,0.000\,114\,(2\omega_0-\omega_1-\omega_2)^2.$$

The total variance (denominator in (43)) contains N_o/T plus (44), (48), and (50). For $M_2 \gg 1$, σ_{ox}^{2} can be ignored in comparison with σ_{1x}^{2}. Moreover, σ_{1x}^{2} is quite likely to be negligible in respect to σ_{oo}^{2}.

5.3 Error Probability

In this brief section we introduce elementary error probabilities pertinent to the system (fig. 1). Quite typically a threshold device follows the correlator. Some level α_t is used to decide which of the hypotheses is likely to be true. Thus, if $\alpha > \alpha_t$ holds, H_0 is accepted as true; and if $\alpha \le \alpha_t$ H_1 is accepted.

The nature of such binary hypothesis testing is well understood (Cramér, 1961; Helstrom, 1960). Two types of error are possible, and occur with probabilities

$$
\begin{aligned}
P_{01} &= \Pr(\text{select } H_o | H_1 \text{ is true}) \\
&= \Pr(\alpha_1 > \alpha_t), \\
P_{10} &= \Pr(\text{select } H_1 | H_o \text{ is true}) \\
&= \Pr(\alpha_o \le \alpha_t).
\end{aligned} \tag{51}
$$

If the distributions of α_o and α_1 were accurately known, the error probabilities could be evaluated exactly. However, we know the means and variances exactly, and nothing beyond that about the distributions. We present two idealizations that are easy to use but inaccurate in practice. Nominally, the correct values will lie between them.

The Gaussian Model

Pretend that the random variable α_ν $(\nu = 0, 1)$ is Gaussian with mean and variance given above. Then,

$$
\begin{aligned}
P_{01} &= 1 - \operatorname{erf} \frac{\alpha_t}{\sqrt{\operatorname{var} \alpha_1}}, \\
P_{10} &= \operatorname{erf} \frac{\alpha_t - E\alpha_o}{\sqrt{\operatorname{var} \alpha_o}},
\end{aligned} \tag{52}
$$

where the error function is defined as in Viterbi (1966)

$$
\operatorname{erf} x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-u^2}{2}} \, du, \qquad -\infty < x < \infty. \tag{53}
$$

Improved Chebyshev Bound

A manipulation (Cramér, 1961 p. 256) of integration regions and Schwarz's inequality yields for (51)

$$P_{01} \le \frac{\operatorname{var} \alpha_1}{\operatorname{var} \alpha_1 + \alpha_t^2},$$
$$P_{10} \le \frac{\operatorname{var} \alpha_o}{\operatorname{var} \alpha_o + (E\alpha_o - \alpha_t)^2}. \tag{54}$$

The two models are far apart. The Chebyshev bound is too high for realistic distributions, and the Gaussian result is typically too low. Because of its exponential nature at high SNR, the Gaussian version is more representative.

Observe that neither in (52) nor in (54) does the detectability parameter D^2 appear in a clear-cut form. We may start by expunging the threshold α_t from its dominant role in the arguments. In the space of $\{\alpha_o, \alpha_1\}$ the set relationships

$$\{\alpha_o \le \alpha_1\} = \bigcap_{\text{all } \alpha_t} \left[\{\alpha_o \le \alpha_t\} \bigcup \{\alpha_1 > \alpha_t\}\right]$$
$$\subset \{\alpha_o \le \alpha_{to}\} \bigcup \{\alpha_1 > \alpha_{to}\} \tag{55}$$

hold for all α_{to}. Regardless of threshold setting, a union bound yields

$$P_{01} + P_{10} \geq \Pr(\alpha_o \leq \alpha_1). \tag{56}$$

We interpret $\Pr(\alpha_o \leq \alpha_1)$ as an "irreducible error probability" of sorts, and note that

$$\begin{aligned} \Pr(\alpha_o \leq \alpha_1) &= \operatorname{erf} D \qquad &\text{(Gaussian model)}, \\ &\leq \frac{1}{1 + D^2} \qquad &\text{(Chebyshev bound)}. \end{aligned} \tag{57}$$

The second line follows from the Chebyshev bound (54) applied to $\Pr\left[(\alpha_o - E\alpha_o) - (\alpha_1 - E\alpha_1) \leq E\alpha_1 - E\alpha_o\right]$.

6. CONCLUSIONS

A direct second order analysis of an extremely simplified two-signal multiple access channel has been carried out. The effects of an ideal hard limiter are worked out with the aid of standard Fourier series. This approach is conceptually quite simple (Anderson and Wintz, 1969), as it avoids the formalism of hypergeometric functions (Jones, 1963; Sollfrey, 1969).

The model, the techniques, and, in a qualitative way, the results are intended to guide us in a forthcoming computer analysis of multiple ($M \gg 2$) random access channels. To retain trust in the basic solution, the method presented does not require any approximations or bounds.

The channel model consists of an ideal hard limiter and Gaussian noise. Carrier phases are mutually independent and uniform over $(0, 2\pi)$. The relative delays and frequency deviations can be treated as deterministic or as random variables. We have derived second order statistics (means and variances) for the word correlation outputs. The underlying distributions are non-Gaussian and too cumbersome to derive.

Three properties that emerge in the above two-signal case and may very well be valid in the general M signal case, are the following. First, reasonable frequency drifts and departures from a true carrier do not seem to increase the distortion variance in a drastic fashion. The same conclusion appears to be valid for slow message FM or PM. Second, the variance of the correlator outputs contains a substantial term that is entirely independent of crossmodulations for the assumed 0 to π modulations. Third, waveform and coding departures from orthogonality do not necessarily affect the detectability parameter by robustly scaling down the signal mean.

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