U.S. DEPARTMENT OF COMMERCE
National Oceanic and Atmospheric Administration
National Weather Service

NOAA Technical Memorandum NWS NMC-50

RECENT RESEARCH IN NUMERICAL METHODS AT THE NATIONAL METEOROLOGICAL CENTER

Ronald D. McPherson

National Meteorological Center

WASHINGTON, D.C.
April 1971

National Oceanic and Atmospheric Administration

U.S. Joint Numerical Weather Prediction Unit

ERRATA NOTICE

One or more conditions of the original document may affect the quality of the image, such as:

Discolored pages
Faded or light ink
Binding intrudes into the text

This has been a co-operative project between the NOAA Central Library, National Center for Environmental Prediction and the U.S. Air Force. This project includes the imaging of the full text of each document. To view the original documents, please contact the NOAA Central Library in Silver Spring, MD at (301) 713-2607 x124 or www.reference@nodc.noaa.gov.

LASON
Imaging Contractor
12200 Kiln Court
Beltsville, MD 20704-1387
April 13, 2004

UDC 551.509.313

551.5	Meteorology
.509	Synoptic analysis and forecasting
.313	Numerical prediction

CONTENTS

The National Weather Service does not approve, recommend or endorse any proprietary product or proprietary material mentioned in this publication. No reference shall be made to the National Weather Service, or to this publication furnished by the National Weather Service, in any advertising or sales promotion which would indicate or imply that the National Weather Service approves, recommends or endorses any proprietary product or proprietary material mentioned herein, or which has as its purpose an intent to cause directly or indirectly the advertised product to be used or purchased because of this National Weather Service publication.

RECENT RESEARCH IN NUMERICAL METHODS AT THE NATIONAL METEOROLOGICAL CENTER*

ABSTRACT

An account is given of recent investigations into numerical techniques for solving the primitive meteorological equations. Although an adequate differencing system had been in use in the NMC six-layer operational model, occasional failure to complete an integration due apparently to numerical problems prompted a re-examination of the question of non-linear numerical stability.

It was noted that the model invariably exhibited an amplifying temporal computational mode prior to encountering instability. Simultaneously, experiments in primitive equation integrations over limited regions of the Northern Hemisphere revealed the development of two-grid-interval disturbances in space which had been called the *spatial computational mode*, and which are related to the lateral boundary conditions. Parallel inquiries into the nature of these two phenomena were subsequently combined into an investigation of the *lattice structure* of the difference equations. The term *lattice structure* refers to the tendency of the difference equations to yield quasi-independent numerical solutions on distinct subsets of the basic grid-point lattice. This investigation led to a theory in which the basic concept is that the interaction of high frequency/ wave number components of undifferentiated coefficients of non-linear terms with the differentiated variables can give rise to unstable growth of the numerical solutions. The theory suggested that space-time filtering of the undifferentiated coefficients would achieve relative stability, and limited experimental evidence tended to support this suggestion. An important by-product of the stability investigation was the realization that advantage could be taken of the lattice structure to effect significant economies in computation time by using only one of the quasi-independent subsets of the basic lattice.

An effort to achieve computational economy by means of semi-implicit integration techniques was also undertaken. This development followed the approach of scientists in Canada and the Soviet Union. Treating implicitly the linear terms of the primitive equations which govern gravitational oscillations results in a boundary-value problem to be solved at each time step, but permits a much longer time step to be used.

*This paper was presented at the International Conference on Meteorology held at Tel Aviv, Israel, November 30-December 4, 1970.

These investigations have culminated in a more efficient differencing system for integrating the primitive equations. The semi-implicit time integration technique is combined with a spatial-differencing system based on a staggered arrangement of dependent variables. Relative numerical stability is achieved by use of space-time filtering of coefficients of non-linear terms. In principle, this system results in an order of magnitude reduction in computing time for the same order of spatial truncation error. The results of applying this system to a primitive equation barotropic model are shown in an experimental integration.

The proposed differencing system may be extended to multi-level models in which both external and internal gravity modes must be considered. Since some of the internal modes have relatively low phase speeds, implicit treatment of all the internal modes may not be necessary, so that alternative approaches to the semi-implicit integration of baroclinic models are possible.

Two alternatives currently under development at NMC are described: one in which all gravity modes are treated implicitly, and another in which only the external and the fastest internal mode are treated implicitly. It is intended that one of these approaches will be used to develop a high-resolution short-range prediction model to be integrated over a limited portion of the Northern Hemisphere.

1. Introduction

This paper summarizes and places in perspective, investigations conducted over the past two years at the National Meteorological Center into numerical techniques for integrating the quasi-static system of equations. Originally motivating this effort was a desire to formulate an atmospheric prediction model with increased spatial resolution to be integrated over a limited portion of a hemisphere, with the intent of effecting significant improvements in forecasts for up to 36 hours. The work of British scientists (Bushby and Timpson, 1967; Benwell and Timpson, 1968; Bushby, 1968) has shown that this approach has much promise for improving short-range weather forecasts. However, in order for a technique to be of practical utility, its routine use must be economically feasible. As the spatial resolution increases, so also does the time required to perform the computations. In turn, both the cost of the computations and the delay in distributing the product to the user increase. These facts led to the realization that a relatively high-resolution model integrated over an area the size of North America would probably be doomed to non-implementable status because of the computation time required. Accordingly, a primary objective of the effort summarized in this paper has been the development of more efficient methods of numerical integration. Such methods presumably would not be restricted in applicability to high-resolution, limited-area models, but also would find use in models designed for larger-scale, longer-range forecasting. Currently under development at NMC is a hemispheric eight-layer primitive equation model formulated in terms of spherical coordinates on a latitude-longitude grid lattice. The model is destined to be the next operational model, and is designed to be easily extendible to the entire globe. A fully global model with reasonable spatial resolution could obviously profit from more efficient numerical techniques. In addition, since hemispheric and global models may be integrated for periods of several days, the problem of non-linear numerical instability (Phillips, 1959) is of more concern than in a short-range model. A second but not less important objective has therefore been to gain understanding of the problem of numerical stability of non-linear equations, and to develop devices for maintaining stability.

Subsequent sections of this paper consider four ideas which have arisen from these studies. The first two have principally to do with numerical stability; the latter two are associated with effecting computational economy. A discussion of the results of numerical integrations of a barotropic model using these four concepts occupies the penultimate section. Finally, an indication is given of alternative ways of extending the new system to multi-level baroclinic models.

2. Boundary Conditions for Limited-Area Primitive Equation Models

The tendency of centered-time-and-space difference approximations to first-order differential equations to yield quasi-independent solutions on distinct subsets of the basic lattice of grid points has been referred to as the *lattice structure* of the difference equations. This characteristic was noted by Richardson (1922), and was later studied by Platzman (1958), Smagorinsky (1958), Nitta (1962), and Matsuno (1966). One manifestation of the lattice structure is the *temporal computational mode* (Gates, 1959) which appears as a separation of numerical solutions between adjacent time steps. It results from specifying initial conditions at two successive time levels as required by centered-time (second order) differences, whereas the differential equations require only one such level. A similar phenomenon may occur on the spatial lattice when centered-space differences are used to approximate gradients. In this case, an overspecification (with respect to the requirements of the differential equations) of the boundary conditions may lead to a separation of numerical solutions at alternate grid points in space. Matsuno (1966) refers to this as the *spatial computational mode*.

To illustrate what is meant by the term lattice structure, consider a domain of integration described in one space dimension and time, subdivided into equal intervals of space Δx and of time Δt as in Figure 1. Thus horizontal position is defined by $x = j\Delta x$, where $j = 0,1,2,.....J$, and time by $t = n\Delta t$, $n = 0,1,2,....$. Averaging and differencing operators may be defined as

$$F_x = \frac{1}{\Delta x} [F(x + \frac{\Delta x}{2}) - F(x - \frac{\Delta x}{2})] \quad (1)$$

$$\overline{F}^x = \frac{1}{2} [F(x + \frac{\Delta x}{2}) + F(x - \frac{\Delta x}{2})] \quad (2)$$

The simple advection equation

$$\overline{f}^t_t + U \overline{f}^x_x = 0 \quad (3)$$

where U is, for the moment, a constant, may be seen to yield a solution on the points in Figure 1 marked by circles that is completely independent of the solution on the points marked by triangles. When U is not a constant, but is calculated as part of the system of equations, then the solutions may possess some mutual dependence. It will be noted that each set of points is characterized by the sum (j+n) either odd or even.

Renewed interest at NMC in the lattice structure of the centered-time-and-space difference equations was prompted by a series of experiments with a limited-area, higher-resolution primitive equation barotropic model (Gerrity and McPherson, 1969). In these experiments, the Euler-backward (Kurihara, 1965) time integration scheme was used, along with the semi-momentum (Shuman and Vanderman, 1966) spatial differencing system. No initial balancing was attempted, but gravitational oscillations arising from the initial imbalance were expected to be damped by the time-integration scheme. A key feature of the model was the specification of temporally-constant lateral boundary conditions.

A 12-hour 500-mb forecast from this model is shown in Figure 2. It will be observed that the fields of meteorological interest are completely obscured by very short wave length noise, which appeared initially at the boundaries and progressed into the interior with depressing regularity. Yet, beneath this crust of nonsense there remained meteorological information, as seen in Figure 3, which shows the result of passing a simple filter over the noisy field of Figure 2. Curiosity about the origin and nature of this noise led to a detailed analysis (Gerrity and McPherson, 1970), the highlights of which are presented in succeeding paragraphs, and eventually to a more general consideration of the lattice structure, discussed in the next section.

The analysis of the noise shown in Figure 2 revealed that its presence was a manifestation of a separation of numerical solutions on alternate points of the spatial grid lattice. In order to investigate the mechanism for the development of this particular spatial computational mode, a very simple one-dimensional analogue was constructed. Consider a swimming pool of infinite length but of finite width illustrated schematically in Figure 4. At either edge in the finite dimension, there is a lip and a drain and a magical spout such that if the water level exceeds the lip, the excess is removed by the drain. If the water level recedes infinitesimally below the lip, the spout is activated to add fluid. The water level is thus kept constant, and the velocity may be imagined to vanish (neglecting spillover), at the boundaries.

Ignoring the very complicated processes which occur in the zone adjacent to the boundaries, the set of linear equations governing the behavior of the fluid are

$$\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} = 0 \qquad (4)$$

$$\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0 \qquad (5)$$

where u is the velocity component normal to the boundaries, h represents the departure of the height of the fluid surface from its mean value H, and g is the acceleration of gravity. The boundary conditions are

$$u(x = 0,\ X;\ t) = 0 \qquad (6)$$

$$h(x = 0,\ X;\ t) = 0 \qquad (7)$$

where the boundaries are at $x = 0, X$. The solution of the system (4-7), with the initial conditions

$$u(x,\ 0) = U \sin \frac{2\pi x}{L} \qquad (8)$$

$$h(x,\ 0) = 0 \qquad (9)$$

where L is the wavelength of the initial disturbance, and U is an amplitude, may be obtained by the method of characteristics (von Mises, 1958; Freeman, 1951). The solutions are shown in Figure 5; the physical interpretation is as follows: As a perturbation of the free surface of this swimming pool system approaches the boundary, the wave trough will cause fluid to be added to maintain the constant height. At time $t = X/2c (c = \sqrt{gH})$, the level of the fluid is everywhere equal to or higher than its initial value. Subsequently, this excess fluid is extracted by the drain of the pool until at $t = X/c$ the perturbation has vanished. Thus, requiring conditions at the boundaries to be invariant with time should lead to absorption of incident gravity waves.

It can be demonstrated that if the region of Figure 5 is divided into intervals of Δt and Δx such that

$$\sqrt{gH}\,\frac{\Delta t}{\Delta x} = 1,$$

the analytic solutions of the system (4-9) can be written in a difference form,

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{g\Delta t}{2\Delta x}\left(h_{j+1}^n - h_{j-1}^n\right) \qquad (10)$$

$$h_j^{n+1} = \frac{1}{2}\left(h_{j+1}^n + h_{j-1}^n\right) - \frac{H\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) \qquad (11)$$

where $x = j\Delta x$, $j = 0,1,2,\ldots .J$, and $t = n\Delta t$, $n = 0,1,2,\ldots$. This form is the first step of the Lax-Wendroff method discussed by Richtmyer (1962) and others. In contrast, the conventional centered-time-and-space difference approximations to (4) and (5) are

$$u_j^{n+1} = u_j^{n-1} - \frac{g\Delta t}{\Delta x}\left(h_{j+1}^n - h_{j-1}^n\right) \qquad (12)$$

$$h_j^{n+1} = h_j^{n-1} - \frac{H\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) . \qquad (13)$$

The impact of the boundary conditions is different for the set (10-11) and for the set (12-13). In the former (with $j = 1$, the point immediately adjacent to the left boundary), the boundary conditions on both u and h enter into the solutions for each variable; whereas, in the latter, only the boundary condition on h enters the solution for u, and only the boundary for u enters the solution for h. In fact, the solution for u, for example, depends only upon the h-boundary condition at odd-numbered points and only upon the u-boundary condition at even-numbered points; the reverse is true of the solution for h. The result of this is shown in Figure 6; if lines are drawn connecting the points as in Figures 6b and 6d, the result is two-grid-interval noise, the one-dimensional analogue of Figure 2.

An effective, but not perfect, remedy can be achieved by using the system (10-11) only at the penultimate points relative to the boundaries. The imperfection arises from the non-linear nature of the equations governing an atmospheric model, as well as the fact that the preceding arguments neglect the earth's rotation. Moreover, there is a practical inability to prescribe the stability ratio $c\Delta t/\Delta x$ as exactly unity. Nevertheless, experience has shown that the use of the first step of the Lax-Wendroff method in conjunction with temporally-constant boundaries is of great assistance in suppressing the spatial computational mode.

3. The Lattice Structure and Numerical Stability

Although the influence of the spatial computational mode on the numerical stability of non-linear difference equations had been recognized by Phillips (1959), the examination of that influence was not the primary motivation for the noise analysis briefly recounted in the previous section. However, while the experiments just discussed were being conducted, concern arose over the occasional failure of the operational six-layer model (Shuman and Hovermale, 1968) to complete an integration, due apparently to numerical problems. It was observed that these occurrences were invariably associated with an amplifying temporal computational mode. Gates (1959), Lilly (1965), and Grammeltvedt (1969), among others have also noted the apparent association of the temporal computational mode and non-linear instability. Within the context of the lattice structure, the temporal and spatial computational modes are seen to possess similar characteristics, and therefore equally likely to be responsible for the development of numerical instability during the integration of non-linear equations. Accordingly, the main thrusts of the initially separate investigations of the spatial and temporal computational modes merged into an effort to understand the influence of the lattice structure on numerical stability.

The results of this investigation have been reported in a paper by Robert, Shuman, and Gerrity (1970), the salient points of which will be reviewed here. A particular linearization of equation (3) was considered, in which the advecting coefficient U is composed of four components:

$$U = U_0 + U_1 \, e^{i\pi j} + U_2 \, e^{i\pi n} + U_3 \, e^{i\pi(j+n)} \qquad (14)$$

where the subscripted U's are constants. The first term is the customary constant mean wind, the second represents a two-grid-interval wave in space (spatial computational mode), the third represents a two-grid-interval wave in time (temporal computational mode), and the fourth is a constant on each lattice, since (j+n) is odd on one lattice and even on the other. They derived the stability criterion for this difference equation,

$$0 \le \left(\frac{\Delta t}{\Delta x} \right)^2 [(U_0 \pm U_3)^2 - (U_1 \pm U_2)^2] \le 1 \qquad (15)$$

both sides of which must be satisfied for stability. If $U_1 = U_2 = U_3 = 0$, the right side of (15) reduces to the familiar Courant-Friedrichs-Lewy criterion. The duplicity of signs within the parenthetical expressions results from the two subsets of the basic lattice. It is necessary that (15) be satisfied for both lattices. The criterion indicates that, for example, the left side will be violated if the sum of the amplitudes of the temporal and spatial computational modes exceeds the sum of the mean plus U_3.

This result is helpful in explaining the relative stability exhibited by the more successful spatial differencing schemes (Shuman and Vanderman, 1966; Arakawa, 1966), since those techniques involve spatial difference approximations which effectively filter out terms such as U_1 and U_3 from (14). The stability criterion then indicates that if the amplitude of the temporal computational mode (U_2) remains small compared to the mean, the computation will be stable. An extension of this analysis to a system of equations describing gravitational oscillations also proved helpful in explaining the relative stability of such integrations. In this case, the stability criterion involves an H_0, the undisturbed height of the free surface, playing a similar role as U_0 in (15). The H_0 is usually quite large, however, and tends to overwhelm the remaining terms; hence the stability of the gravity wave equations.

Finally, the theory was subjected to a simple test in which the system of equations for an incompressible, homogeneous fluid in hydrostatic equilibrium and with slab symmetry,

$$\overline{u}^t_t + \overline{\overline{u}^x u_x}^x + g\,\overline{h}^x_x = 0 \tag{16}$$

$$\overline{h}^t_t + \overline{\overline{u}^x h_x}^x + \overline{\overline{h}^x u_x}^x = 0 \tag{17}$$

$$\overline{v}^t_t + \overline{\overline{u}^x v_x}^x = 0 \tag{18}$$

was integrated using a white spectrum for initial data. It was anticipated that the independent set (16, 17) would exhibit stability, but that (18) would eventually become unstable, on the basis of experience and with explanation provided by the theory. This was confirmed by the integration. It was then predicted that if the term $\overline{u}^x$ in (18) were replaced by

$$\tfrac{1}{2}\left(\overline{u^n}^x + \overline{u^{n-1}}^x\right),$$

thus removing both temporal and spatial computational modes from the advecting coefficient, the result would be stable. This prediction also was confirmed.

The principal conclusion of their paper is, therefore, that the interaction of high frequency/wave number components of undifferentiated coefficients with differentiated (advected) quantities is a root cause of non-linear instability, and that the suppression of these components will achieve relative stability. This is a very important concept, but the theory is incomplete. It relies on a species of linearization to explain non-linear behavior; but more importantly, the theory thus far is restricted to one-space dimension. Robert et al. (1970), therefore, stress the fact that the theory results in *necessary*, not *sufficient*, conditions for stability; and that the stability discussed is *relative* stability.

In the absence of an extension of the theory to two space dimensions, the invention of specific devices to improve the stability of atmospheric models has proceeded on an empirical basis. Gerrity and McPherson (1970) present two filters which suppress high frequency/ wave number components in a field. If, analogous to (14), an advecting wind field in two dimensions may be represented by

$$u \sim U_0 + U_1 e^{i\pi j} + U_2 e^{i\pi k} + U_3 e^{i\pi n} + U_4 e^{i\pi(j+k)} + U_5 e^{i\pi(j+n)} + U_6 e^{i\pi(k+n)} + U_7 e^{i\pi(j+k+n)} \quad (19)$$

where the indices j, k, n denote position in the x, y and t dimensions respectively, then the filter

$$\overline{U^*}^{xxyy}, \text{ where } U^* = \tfrac{1}{2}\left(U^n + U^{n-1}\right),$$

returns a completely filtered field:

$$\overline{U^*}^{xxyy} = U_0 \quad (20)$$

The use of this filter requires the presence of all of the quasi-independent lattices. Another filter, which requires only one lattice, may be written as

$$\tilde{U} = \frac{1}{4}\left[\overline{U^{n-1}}^{2x2x} + \overline{U^{n-1}}^{2y2y} + \overline{2U^n}^{2x2y}\right] \quad (21)$$

where

$$\overline{F}^{2x} \equiv \frac{1}{2}\left[F(x + \Delta x) + F(x - \Delta x)\right].$$

Eqn. (21) applied to (19) yields

$$U = U_0 + U_7 . \tag{22}$$

This filter has considerable utility in the design of an efficient differencing system, described in the succeeding sections.

4. A *Decoupled* Grid Lattice

Two of the four ideas mentioned in the introduction have been considered thus far, both related to numerical stability: the specification of temporally-constant lateral boundaries, and the suppression of non-linear instability by applying space-time filters to the undifferentiated coefficients of non-linear terms. The third is an important by-product of the investigation recounted in the preceding section: the concept of performing the calculations on a *decoupled* grid lattice. The quasi-independent solutions which result from the lattice structure are all of the same order of accuracy. It therefore appears reasonable to formulate the difference system in such a way that the lattices are completely *decoupled*. The numerical solutions then need be computed on only one lattice. This would afford a considerable potential economy in computation time.

A difference system on a *decoupled* lattice will be formulated for a model of an inviscid, homogeneous, incompressible fluid, in hydrostatic equilibrium, with a free surface and a rigid, flat lower boundary. The governing equations may be written as

$$\frac{\partial}{\partial t}\left(\frac{u}{m}\right) + \frac{\partial \Phi}{\partial x} = \left(\frac{v}{m}\right)\left(f - v\frac{\partial m}{\partial x} + u\frac{\partial m}{\partial y}\right) - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} \tag{23}$$

$$\frac{\partial}{\partial t}\left(\frac{v}{m}\right) + \frac{\partial \Phi}{\partial y} = -\left(\frac{u}{m}\right)\left(f - v\frac{\partial m}{\partial x} + u\frac{\partial m}{\partial y}\right) - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} \tag{24}$$

$$\frac{\partial \Phi}{\partial t} + m^2\left(gH + \Phi\right)\left[\frac{\partial}{\partial x}\left(\frac{u}{m}\right) + \frac{\partial}{\partial y}\left(\frac{v}{m}\right)\right] = -m\left(u\frac{\partial \Phi}{\partial x} + v\frac{\partial \Phi}{\partial y}\right) \tag{25}$$

where Φ is a geopotential departure from a constant gH_0, H_0 is the mean height of the free surface, and m is the map factor appropriate to the map projection used. The *decoupled* grid lattice is depicted schematically in Figure 7. At any time, the u-component of velocity is defined midway between adjacent values of Φ in the x-direction, and the v-component is defined midway between adjacent values of Φ

in the y-direction. Between adjacent time levels, the u- and the v-components exchange positions, while the geopotential changes to maintain the same position relative to the wind components. This arrangement has been used by Lilly (1965) and Arakawa (1966). In order to compare this with an arrangement in which the lattices are *coupled*, consider that the intersections of the dashed lines in Figure 7 represent the points of a conventional *coupled* lattice on which all dependent variables are defined at each time step. It is evident that the *decoupled*, staggered arrangement represents an array of only one-fourth the number of points of the conventional lattice.

It is convenient to define the mesh length Δx for this system as the distance between adjacent values of the same variable; the form of the basic averaging and differencing operators, as given by (11) and (12), remain unchanged. Considering at this point only the partial tendencies due to the non-linear terms, the difference equations corresponding to (23-25) are,

$$\frac{u^*}{m} = \frac{u^{n-1}}{m} + 2\Delta t \left[\left(\frac{\overline{v}^r}{m} \right) \left(\overline{u}^s m_y - \overline{v}^r m_x \right) - \overline{u}^s \, \overline{u}^y_x - \overline{v}^r \, \overline{u}^x_y \right] \tag{26}$$

$$\frac{v^*}{m} = \frac{v^{n-1}}{m} - 2\Delta t \left[\left(\frac{\overline{u}^r}{m} \right) \left(\overline{u}^r m_y - \overline{v}^s m_x \right) + \overline{u}^r \, \overline{v}^y_x + \overline{v}^s \, \overline{v}^x_y \right] \tag{27}$$

$$\Phi^* = \Phi^{n-1} - 2\Delta t m \left(\hat{u} \, \overline{\Phi}^y_x + \tilde{v} \, \overline{\Phi}^x_y \right), \tag{28}$$

The averaging operators appearing on the undifferentiated coefficients in (26-28) involve both space and time filtering and arise from the arguments of the preceding section. Analogous to (21), they are

$$\overline{F}^r = \frac{1}{4} \left(\overline{F^n}^{xx} + \overline{F^n}^{yy} + 2 \, \overline{F^{n-1}}^{xy} \right) \tag{29}$$

$$\overline{F}^s = \frac{1}{4} \left(\overline{F^{n-1}}^{xx} + \overline{F^{n-1}}^{yy} + 2 \, \overline{F^n}^{xy} \right) \tag{30}$$

where, because of the redefinition of Δx, the superscript 2 appearing in (21) is deleted. The remaining filters that appear in the continuity equation are much simpler, but still guided by the arguments presented by Robert, Shuman and Gerrity (1970). They are defined as

$$\hat{F} = \frac{1}{2} \left(\overline{F^n}^y + \overline{F^{n-1}}^x \right) \tag{31}$$

$$\tilde{F} = \frac{1}{2} \left(\overline{F^{n-1}}^y + \overline{F^n}^x \right) \tag{32}$$

The differentiated variables are evaluated at time $t = n\Delta t$. To illustrate the two more complicated averaging operators, consider (26) evaluated for point 5 in Figure 7. The quantity $\overline{v}^r$ may be written as

$$\overline{v}^r = \frac{1}{16} \left(v_1^n + v_2^n + v_3^n + v_4^n\right) + \frac{1}{4} v_5^n + \frac{1}{8} \left(v_6^{n-1} + v_7^{n-1} + v_8^{n-1} + v_9^{n-1}\right) \tag{33}$$

since the v^{n-1} occupy the same positions as the u^n. Also,

$$\overline{u}^s = \frac{1}{16} \left(u_1^{n-1} + u_2^{n-1} + u_3^{n-1} + u_4^{n-1}\right) + \frac{1}{4} u_5^{n-1} + \frac{1}{8} \left(u_6^n + u_7^n + u_8^n + u_9^n\right) \tag{34}$$

since the u^{n-1} occupy the same positions as the v^n.

The differentiated quantities are

$$\overline{u}_x^y = \frac{1}{2\Delta x} \left(u_6^n - u_9^n + u_7^n - u_8^n\right) \tag{35}$$

$$\overline{u}_y^x = \frac{1}{2\Delta x} \left(u_6^n - u_7^n + u_9^n - u_8^n\right) \tag{36}$$

That this *decoupled* differencing system is of comparable accuracy to conventional *coupled* systems for equivalent effective mesh lengths may be seen by noting that a typical gradient term in the *semi-momentum* system (Shuman and Vanderman, 1966) has leading terms of exactly the same form as (35) and (36).

5. A Semi-Implicit Time Integration Method

The last of the four significant concepts mentioned in the introduction is the semi-implicit time integration method. Although well known (Richtmyer, 1957), implicit methods did not find widespread usage in numerical weather prediction until recently. Marchuk (1965) applied an implicit method to avoid the linear stability criterion associated with the Lamb wave and the internal gravity-inertia wave solutions of the quasi-static equations. The method is partly implicit in that only those terms, e.g., the pressure gradient and divergence terms, which govern these principally linear phenomena, are approximated implicitly. The non-linear terms which principally govern the large-scale low frequency meteorological wave motions are treated explicitly. As a result, the time step may be much longer than when conventional explicit methods are used.

The method has subsequently been used by Robert (1968), Kwizak and Robert (1971), and McPherson (1971). A linear analysis presented by Kwizak and Robert (1971) confirms the fact that the gravitational modes are treated stably irrespective of the time step used. They compared quasi-hemispheric forecasts from a semi-implicit barotropic free-surface model using a one-hour time step with an integration using the conventional *leap-frog* scheme and a ten-minute time-step. Virtually identical five-day forecasts resulted,demonstrating that truncation error associated with the longer time step is of no consequence. A similar conclusion has been reported by McPherson (1971) in experiments with a limited-area, fine-mesh model, although the integrations were not extended so far in time.

The semi-implicit method is employed in the difference approximations of the linear terms of (23-25). In the previous section, the non-linear terms were approximated and partial tendencies due to those terms were formed. The complete difference equations are

$$\left(\frac{u}{m}\right)^{n+1} + \Delta t\left(\phi_x^{n+1} + \phi_x^{n-1}\right) = \frac{u^*}{m} + 2\Delta t\left(\frac{v^n}{m}\right) f \tag{37}$$

$$\left(\frac{v}{m}\right)^{n+1} + \Delta t\left(\phi_y^{n+1} + \phi_y^{n-1}\right) = \frac{v^*}{m} - 2\Delta t\left(\frac{u^n}{m}\right) f \tag{38}$$

$$\Phi^{n+1} + \Delta t m^2\left(gH_0 + \overline{\Phi^n}^{xy}\right)\left[\left(\frac{u}{m}\right)_x^{n+1} + \left(\frac{v}{m}\right)_y^{n+1} + \left(\frac{u}{m}\right)_x^{n-1} + \left(\frac{v}{m}\right)_y^{n-1}\right] = \Phi^* \tag{39}$$

where u^*, v^* and Φ^* are defined by (26-28). The pressure gradient terms of the momentum equations and the divergence term in the continuity equation are averaged over the immediately imminent and past time levels. This is the essence of the semi-implicit approximation. It will be observed that there is no spatial averaging of the pressure gradient terms or the divergence term. Care has been taken to ensure that the velocity components in the Coriolis terms are also spatially unaveraged, so as to preserve a reasonable geostrophic balance. The velocity components at $t = (n+1)\Delta t$ in the continuity equations can be eliminated using (37) and (38) to obtain a Helmholtz-type equation in Φ^{n+1}

$$\Phi^{n+1} - (m\Delta t)^2 \left(gH_0 + \overline{\Phi^n}^{xy}\right) \nabla^2 \Phi^{n+1} = \Phi' - m^2 \Delta t \left(gH_0 + \overline{\Phi^n}^{xy}\right)\left(U_x + V_y\right) \tag{40}$$

where

$$U = \frac{u^*}{m} + 2\Delta t \left(\frac{u}{m}\right)^n f - \Delta t \Phi_x^{n-1}$$

$$V = \frac{v^*}{m} - 2\Delta t \left(\frac{u}{m}\right)^n f - \Delta t \Phi_y^{n-1}$$

$$\Phi' = \Phi^* - \Delta t m^2 \left[\left(\frac{u}{m}\right)_x^{n-1} + \left(\frac{v}{m}\right)^{n-1}\right]$$

and

$$\nabla^2 \Phi^{n+1} = \Phi_{xx}^{n+1} + \Phi_{yy}^{n+1} .$$

Eqn. (40) can be solved by relaxation or other techniques for Φ^{n+1}, given appropriate specification of lateral boundary conditions. The wind components at $t = (n+1)\Delta t$ may then be obtained directly from (37) and (38). Because this boundary-value problem must be solved at each time step, the time advantage is not the simple ratio of the time steps. Kwizak and Robert (1971), using a one-hour time step in the implicit integration and a ten-minute time step in the explicit analogue, report an actual advantage of about 4:1.

Combining the *decoupled*, staggered grid arrangement, which uses one-fourth the number of points required in conventional schemes, with the semi-implicit integration technique which offers a 4:1 advantage, yields a potential reduction in computation time of 16:1.

6. Results of an Experimental Integration

The results of one integration to 48 hours with the model described in the preceding sections are presented here. The initial data were obtained from a solution of the balance equation (Shuman, 1957) on the NMC quasi-hemispheric rectangular grid. From this basic array of geopotential and non-divergent winds, the necessary variables at the locations indicated in Fig. 7 and also their values at the alternate set of points, are extracted. The integration begins with a forward time step. It should be noted that neither divergence nor its tendency are exactly zero initially, since the differencing system used in the prognostic model is different from that used in the solution of the balance equation. Lateral boundary conditions require that all dependent variables remain constant with time in the boundary zone (outside the heavy line in the schematic of the grid, (Fig. 7). The first step of the Lax-Wendroff method, as outlined in Section 2, is employed at the points immediately adjacent to the boundary zone.

The 24- and 48-hour forecasts from initial data of February 10, 1969, 1200 GMT are presented in Figs. 8 and 9. Fig. 8a shows the initial 500-mb height field, and Fig. 8b shows the 500-mb configuration 28 hours later. Comparison of the 24-hour forecast, Fig. 8c, with the observed pattern indicates that the main features of the flow have been reasonably well-treated. The deep trough near 80W is seen to rotate sharply northeastward, undergoing pronounced cyclogenesis. The forecast does not reflect cyclogenesis, of course, but gives an indication of the northeastward movement of the trough. Relatively short wavelength features appear to be forecast with approximately correct phase speed through 24 hours. However, by 48 hours, the shorter waves are noticeably slow, as seen by comparing Figs. 9a and 9b.

For comparison, mean absolute height error (δh), root-mean-square height error RMS (δh), root-mean-square vector wind error RMS (δV), and the S1 skill score (Teweles and Wobus, 1954) were computed for this case from both this barotropic model and the NMC operational six-layer baroclinic model. The data are presented in the table below. It will be noted that, for this one case, the barotropic model incorporating the new differencing system is roughly comparable to the baroclinic model.

	24 hour				48 hour			
Model	δh Meters	RMS (δh) Meters	RMS (δV) m sec^{-1}	S1	h (Meters)	RMS(δh) (Meters)	RMS(δh) m sec^{-1}	S1
Barotropic	31.3	42.2	9.5	39.7	39.2	56.7	11.4	49.0
Baroclinic	19.8	27.8	-	30.6	35.7	49.8	-	41.0

The integration shown in Figs. 8 and 9 was extended beyond 48 hours to gain some idea of the numerical stability characteristics of this differencing system. Fig. 10 displays some aspects of the extended integration. Presented are kinetic and available potential energies normalized by their initial values, and mean-square measures of vorticity and divergence, all plotted at 12-hourly intervals. The behavior of these quantities is acceptable until 15 days, but thereafter all begin to increase markedly. By 24 days, the model is well on the way to non-linear instability. This was not unexpected, for as was indicated in Section 3, the theory of non-linear stability by which the design of this model was guided is incomplete, and claims only to apply to *relative* stability. That this is achieved by use of the space-time filters on undifferentiated coefficients of non-linear terms is demonstrated by an integration of this model without the temporal aspect of the filtering; instability was encountered in 8 days.

The principal advantage of this differencing system is the efficiency with which the requisite computations are carried out. Little effort has been made to optimize the internal efficiency of the computer program of the new differencing system, especially with regard to the numerical solution of the Helmholtz equation. A strict comparison of computation time for this system with that required by a conventional explicit centered-time-and-space system has therefore not been made. However, a rough estimate of the time advantage was obtained by comparing 72-hour forecasts from the new system and a primitive equation barotropic model employing the *semi-momentum* space differencing and the *leap-frog* method in time; neither model was programmed for optimum efficiency. The former required 80 seconds on the CDC 6600 computer; the latter required slightly over 1000 seconds. This is regarded as tending to support the theoretical order-of-magnitude reduction in computing time afforded by the new system.

7. Extension to Baroclinic Models

The results of testing the proposed differencing system with a barotropic model are sufficiently encouraging to warrant its application within the framework of a multi-level baroclinic model. Extension of the staggered arrangement of dependent variables and the stability controls to a multi-level model offer no obstacles of consequence. The principal difficulty is in extending the semi-implicit time integration method; for in a multi-level model, not only the external gravity mode, but also the internal gravity modes, must be regarded as allowable high frequency modes.

There are at least two approaches to this problem. One, proposed by Robert* of Canada, treats the external gravity mode and all of the allowable internal gravity modes implicitly. Robert uses the σ-vertical coordinate suggested by Phillips (1957), which allows relatively simple treatment of the lower boundary, but is susceptible to large truncation errors over very irregular terrain (Kurihara, 1968). The essence of this approach is a linearization of the surface pressure about a constant value, and of the temperature about a specified distribution which is a function only of the vertical coordinate. This makes possible the linearization of the pressure-gradient terms in the momentum equations, divergence term in the continuity equation, and the vertical exchange term in the thermodynamic equations, which principally govern the gravitational oscillations. Once the linear parts have been separated, they may be treated implicitly, while deviations from the linearizations are treated explicitly.

Depending upon the details of the algebraic manipulations, the result is a three-dimensional Helmholtz equation, or a simultaneous system of Helmholtz equations, one per level, which must be solved at each time step. For a model with many layers, the numerical solution of this large boundary-value problem might appreciably reduce the time advantage of the method.

Another approach recognizes that many of the allowable internal gravity modes are of relatively low frequency, comparable to that of the meteorological modes. It is only the external gravity mode (Lamb wave) and those internal modes with small vertical wave numbers which have sufficiently large phase speeds as to require implicit treatments with long time steps. Accordingly, the essence of this approach is to divide the atmosphere into two *σ-domains*, one tropospheric, the other stratospheric, separated by a material surface which represents the tropopause. The linearization is carried out with respect to the specific volume about a distribution which is a constant for each *σ-domain*. In a sense, this amounts to treating certain aspects of the model atmosphere as if it were composed of two superimposed heterogeneous fluids. The linear terms which principally govern the external gravity mode and the most rapidly-moving internal mode may then be treated implicitly so that only these two modes are computed stably, irrespective of the time step used. Allowable internal modes which are computed explicitly generally have phase speeds sufficiently small that relatively long time steps might be used without violating linear stability.

---

*Personal communication

This approach results in a system of only two Helmholtz equations to be solved simultaneously at each time step, regardless of the number of layers internal to each *σ-domain*. It is therefore potentially more economical than the first one. Also, the use of two *σ-domains*, similar to that incorporated in the NMC operational six-layer (Shuman and Hovermale, 1968) model, is not as likely to incur such large truncation errors as might be the case using Phillips' "σ" coordinate.

Both of these approaches are currently being followed at NMC. Regardless of which one proves superior, it seems reasonably certain that routine numerical weather prediction can be carried out with an order-of-magnitude reduction in computer time in the near future. That this efficiency may be employed to enhance the spatial resolution of grid lattices, or to refine the modeling of physical processes, or both, augurs well for imminent improvements in global as well as short-range, detailed numerical guidance forecasts.

ACKNOWLEDGMENTS

The research summarized in this paper has been conducted in close collaboration with Dr. Joseph P. Gerrity, Jr., of the National Meteorological Center. His consistent ability to generate both ideas and enthusiasm has been essential to the progress reported here. The author also expresses his gratitude to Drs. J. A. Brown, Jr., and F. G. Shuman, of NMC for their suggestions during the course of the work and their critical review of this manuscript. Mrs. Doris Gordon performed most of the computer programming, and the typing was done by Mrs. Mary Daigle.

REFERENCES

1. Arakawa, A., 1966: "Computational Design for Long-Term Numerical Integration of the Equations of Fluid Motion: I. Two-Dimensional Incompressible Flow," *Journal of Computational Physics*, Vol. 1, pp. 119-143.

2. Benwell, G. R. R., and M. S. Timpson, 1968: "Further Work with the Bushby-Timpson 10-Level Model," *Quarterly Journal Royal Meteor. Society*, Vol. 94, No. 1, pp. 12-24.

3. Bushby, F. H., and M. S. Timpson, 1967: "A Ten-Level Atmospheric Model and Frontal Rain," *Quarterly Journal Royal Meteor. Society*, Vol. 93, pp. 1-17.

4. ______________ 1968: "Further Developments of a Model for Forecasting Rain and Weather," Proceedings of the WMO/IUGG Symposium on Numerical Weather Prediction, Tokyo, pp. II-75-84.

5. Freeman, J. C., 1951: "The Solution of Non-Linear Meteorological Problems by the Method of Characteristics," *Compendium of Meteorology*, Am. Met. Soc., Boston, 1951, pp. 421-433.

6. Gates, W. L., 1959: "On the Truncation Error, Stability, and Convergence of Difference Solutions of the Barotropic Vorticity Equations," *Journal of Meteorology*, Vol. 16, No. 5, pp. 556-558.

7. Gerrity, J. P., and R. D. McPherson, 1969: "Development of a Fine-Mesh, Limited-Area Prediction Model," *Monthly Weather Review*, Vol. 97, No. 9, pp. 665-669.

8. ____________________________ 1970: "Noise Analysis of a Limited-Area Fine-Mesh Prediction Model," ESSA Tech. Memo. WBTM NMC-46, February.

9. ____________________________ 1970: "Recent Studies in Computational Stability," ESSA Tech. Memo. WBTM NMC-48, May.

10. Grammeltvedt, A., 1969: "A Survey of Finite-Difference Schemes for the Primitive Equations for a Barotropic Fluid," *Monthly Weather Review*, Vol. 97, No. 5, pp. 384-404.

11. Kurihara, Y., 1965: "On the Use of Implicit and Iterative Methods for the Time Integration of the Wave Equation," *Monthly Weather Review*, Vol. 93, No. 1, pp. 33-46.

12. Kurihara, Y., 1968: "Note on Finite Difference Expressions for the Hydrostatic Relation and Pressure-Gradient Force," *Monthly Weather Review*, Vol. 96, No. 9, pp. 654-656.

13. Kwizak, M., and A. Robert, 1970: "Implicit Integration of a Grid-Point Model," *Monthly Weather Review*, Vol. 99, No. 1, pp. 32-36.

14. Lilly, D., 1965: "On the Computational Stability of Numerical Solutions of Time-Dependent Non-Linear Geophysical Fluid Dynamic Problems," *Monthly Weather Review*, Vol. 93, No. 1, pp. 11-26.

15. Marchuk, G., 1965: Numerical Methods for Solving Weather-Forecasting and Climate-Theory Problems - Part 1. Translated from Russian by American Meteorological Society for Air Force Cambridge Research Laboratory, Contract AF 19(628)-3880, Sept. 1965, T-R-520.

16. Matsuno, T., 1966: "False Reflection of Waves at the Boundary Due to the Use of Finite Differences," *J. Meteorol. Soc. Japan*, Vol. 44, No. 2, pp. 145-157.

17. McPherson, R. D., 1971: "Note on the Semi-Implicit Integration of a Fine-Mesh Limited-Area Prediction Model on an Offset Grid," *Monthly Weather Review*, Vol. 99, No. 3, pp. 242-247.

18. Nitta, T., 1962: "The Outflow Boundary Condition in Numerical Time Integration of Advective Equations," *J. Meteorol. Soc. Japan*, Vol. 40, No. 1, pp. 13-24.

19. Phillips, N. A., 1957: "A Coordinate System Having Some Special Advantages for Numerical Forecasting," *Journal of Meteorology*, Vol. 14, No. 2, pp. 184-185.

20. ____________ 1959: "An Example of Non-Linear Computational Instability," The Rossby Memorial Volume, B. Balin, ed., The Rockefeller Institute Press, New York, pp. 501-504.

21. Platzman, G., 1958: "The Lattice Structure of the Finite-Difference Primitive and Vorticity Equations," *Monthly Weather Review*, Vol. 86, No. 8, pp. 285-292.

22. Richardson, L. F., 1922: Weather Prediction by Numerical Process, London: Cambridge University Press, 1922. Republished in 1965 by Dover Publications, New York.

23. Richtmyer, R. D., 1957: "Difference Methods for Initial-Value Problems," 1st ed., Wiley & Sons: New York, 1957.

24. ______________ and K. W. Morton, 1967: "Difference Methods for Initial Value Problems," 2nd ed., Interscience Publishers: New York, 1967.

25. Robert, A. J., 1968: "The Integration of a Spectral Model of the Atmosphere by the Implicit Method," Proceedings of the WMO/IUGG Symposium on Numerical Weather Prediction in Tokyo, Nov. 1968, Section VII, pp. 19-24.

26. ____________, F. G. Shuman, and J. P. Gerrity, 1970: "On Partial Difference Equations in Mathematical Physics," *Monthly Weather Review*, Vol. 98, No. 1, pp. 1-6.

27. Shuman, F. G., 1957: "Numerical Methods in Weather Prediction, Part I: The Balance Equation," *Monthly Weather Review*, Vol. 85, No. 10, pp. 329-332.

28. ____________ and L. W. Vanderman, 1966: "Difference System and Boundary Conditions for the Primitive-Equation Barotropic Forecast," *Monthly Weather Review*, Vol. 94, No. 5, pp. 329-335.

29. ____________ and J. B. Hovermale, 1968: "An Operational Six-Layer Primitive Equation Model," *Journal of Applied Meteorology*, Vol. 7, No. 4, pp. 525-547.

30. Smagorinsky, J., 1958: "on the Numerical Integration of the Primitive Equation of Motion for Baroclinic Flow in a Closed Region," *Monthly Weather Review*, Vol. 86, No. 12, pp. 457-466.

31. Teweles, S., and H. Wobus, 1954: "Verification of Prognostic Charts," *Bulletin of the American Meteorological Society*, Vol. 35, No. 12, pp. 455-463.

32. Von Mises, R., 1958: Mathematical Theory of Compressible Fluid Flow, Academic Press, New York, 514 pp.

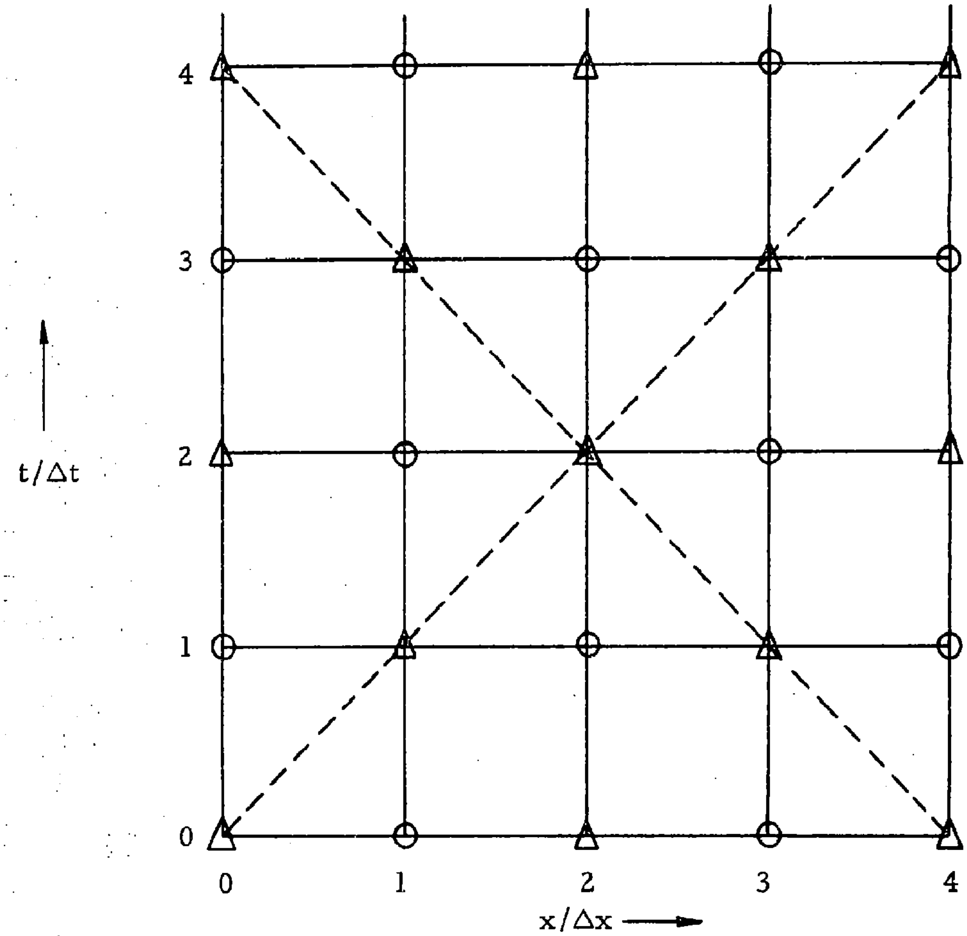

Figure 1.--Schematic illustrating the lattice structure of the grid used in centered space-time differencing. Points denoted by circles constitute one complete lattice, entirely independent of that marked by triangles. The dashed lines are characteristic lines for $(\sqrt{gH})\Delta t/\Delta x = 1$.

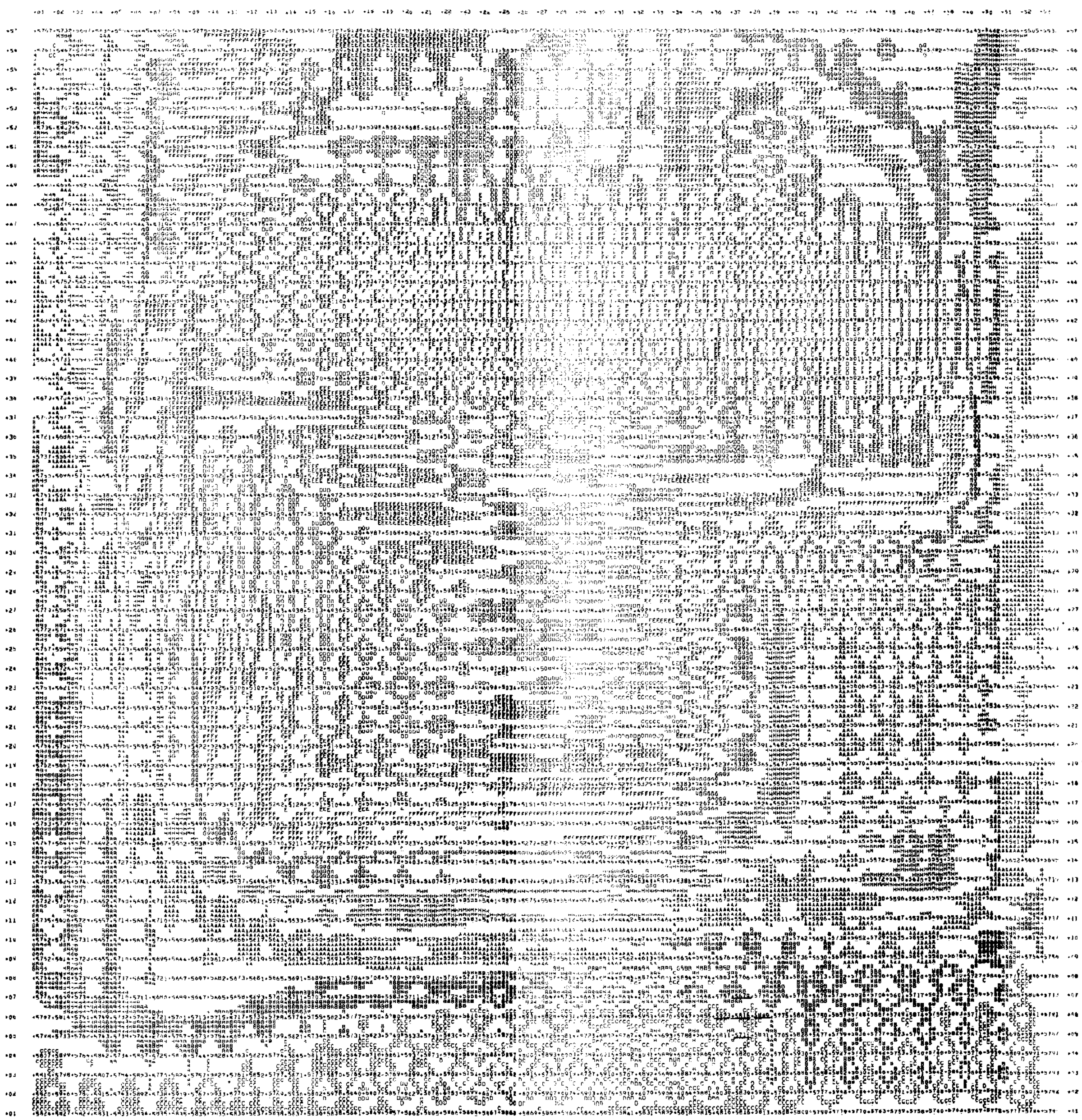

Figure 2.--Twelve-hour 500-mb height forecast from a limited-area primitive equation barotropic model using unbalanced initial data and constant boundary conditions.

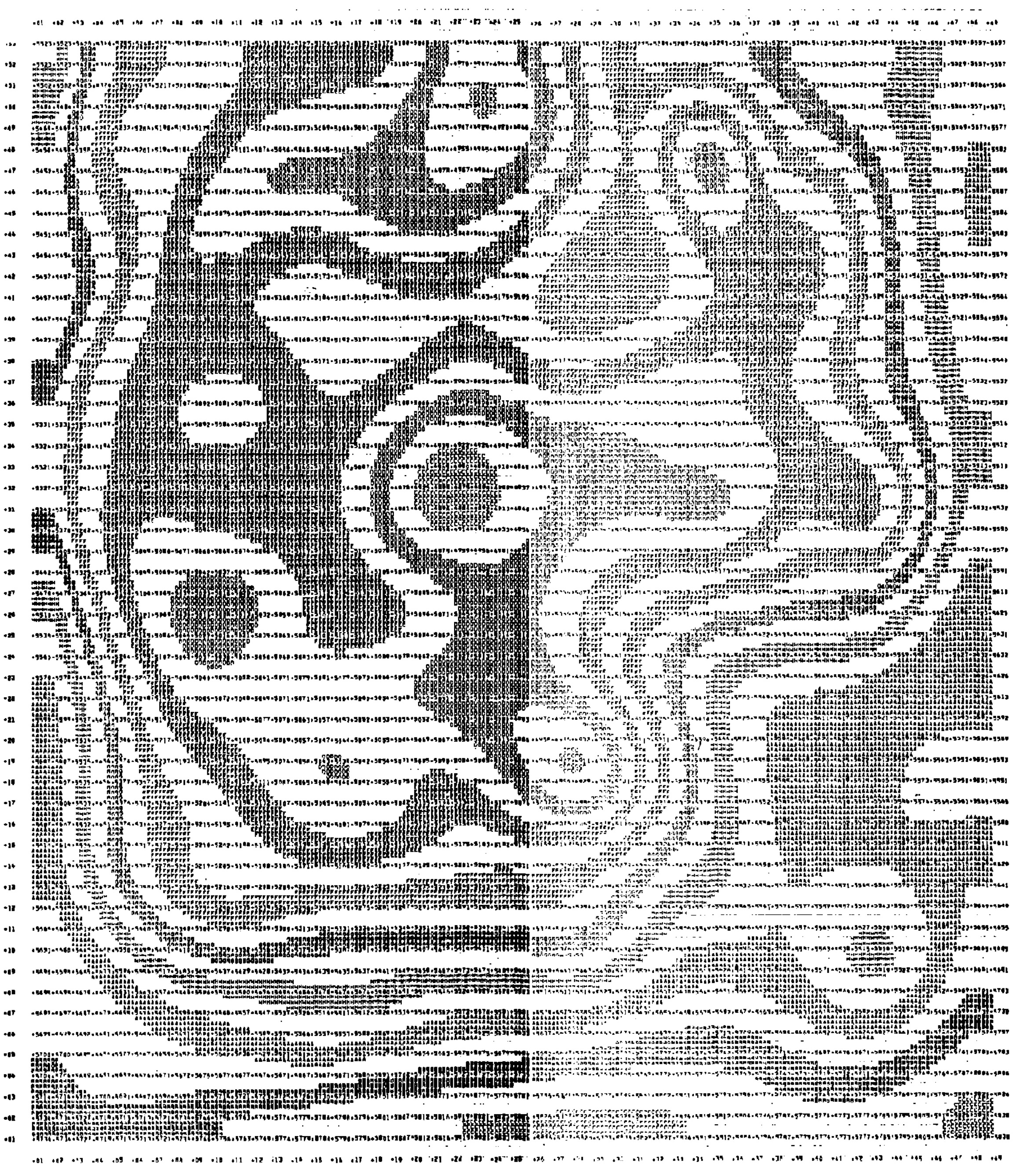

Figure 3.--Twelve-hour 500-mb forecast as in Figure 2, after passing a low-pass filter (Gerrity and McPherson, 1969) over the field prior to output.

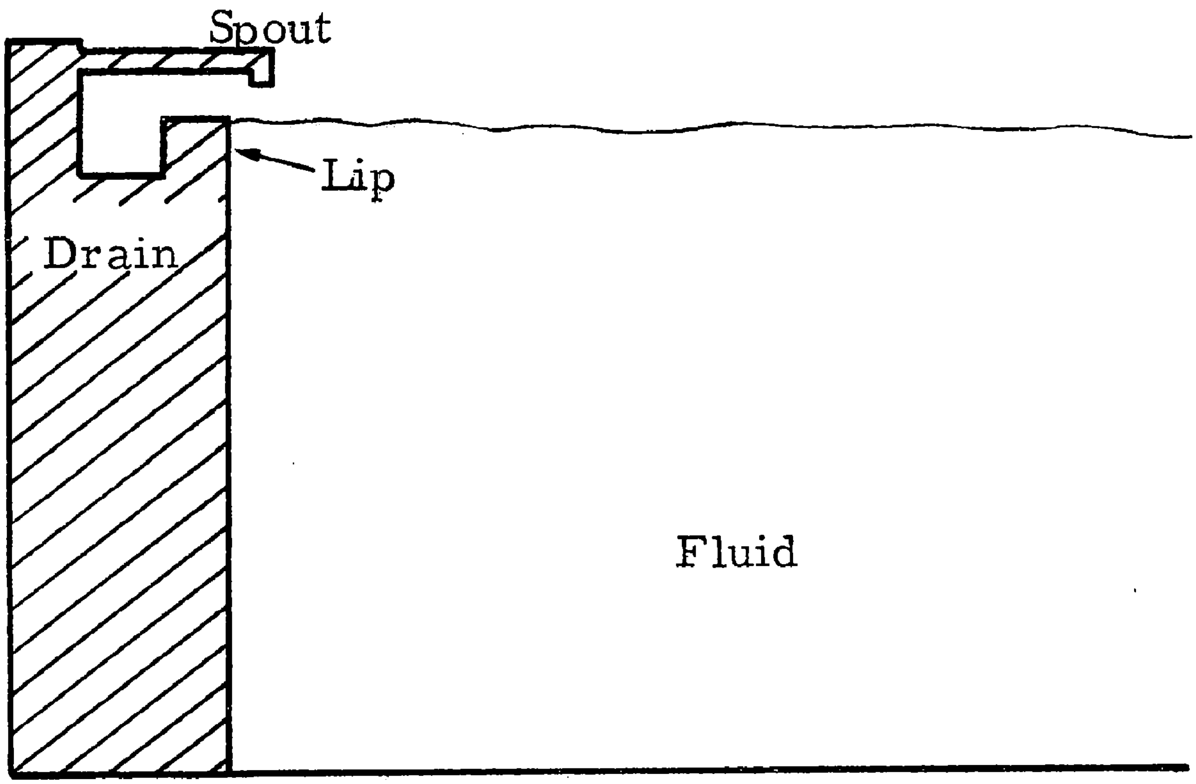

Figure 4.--Schematic illustrating the *swimming pool* physical analog of the mathematical one-dimensional simulation of the noise problem.

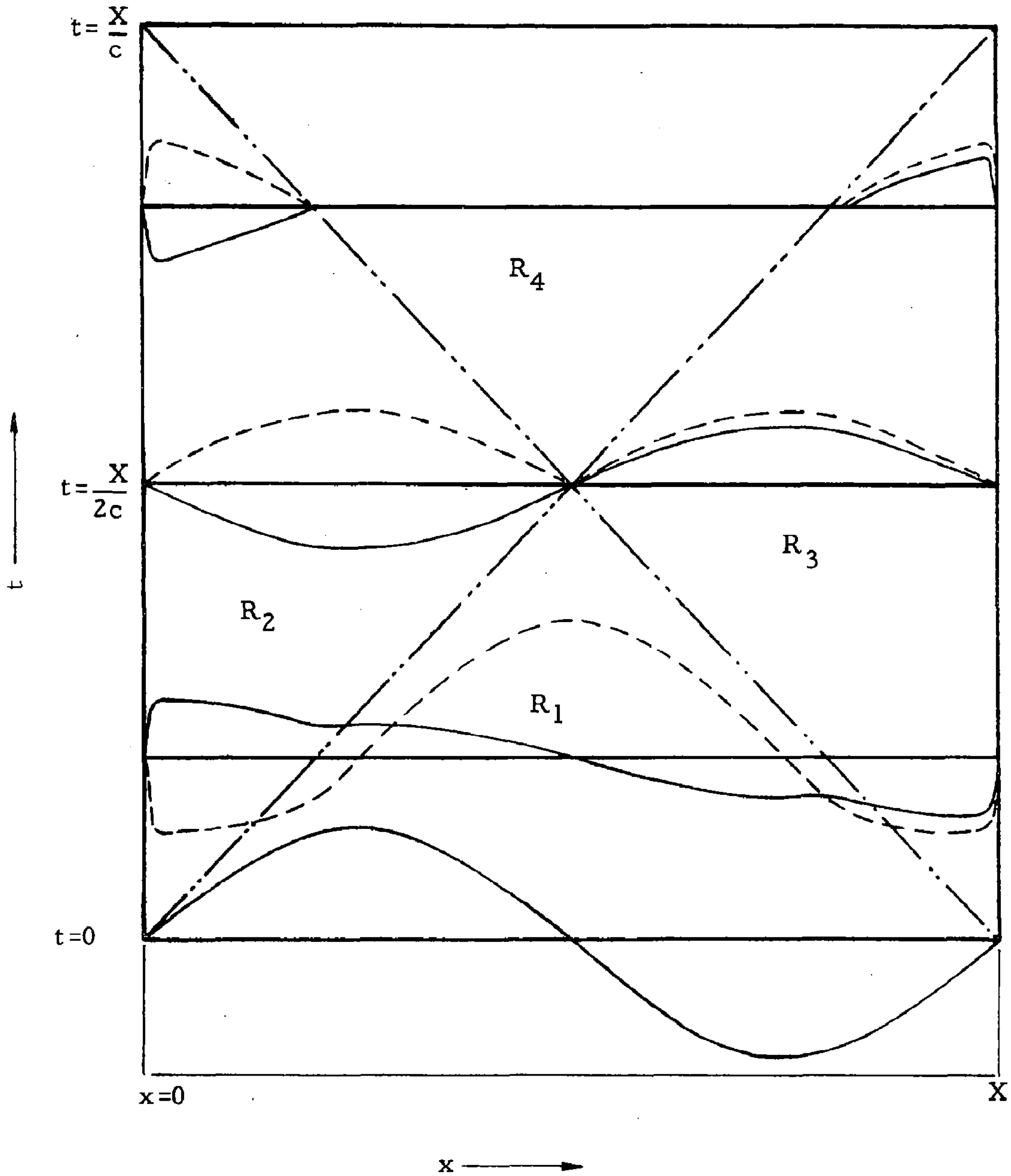

Figure 5.--Analytic solutions of the one-dimensional, constant boundaries, linear gravity wave problem, as obtained by the method of characteristics. The solid curves represent the velocity solution and the dashed curves represent the solution for the height of the free surface.

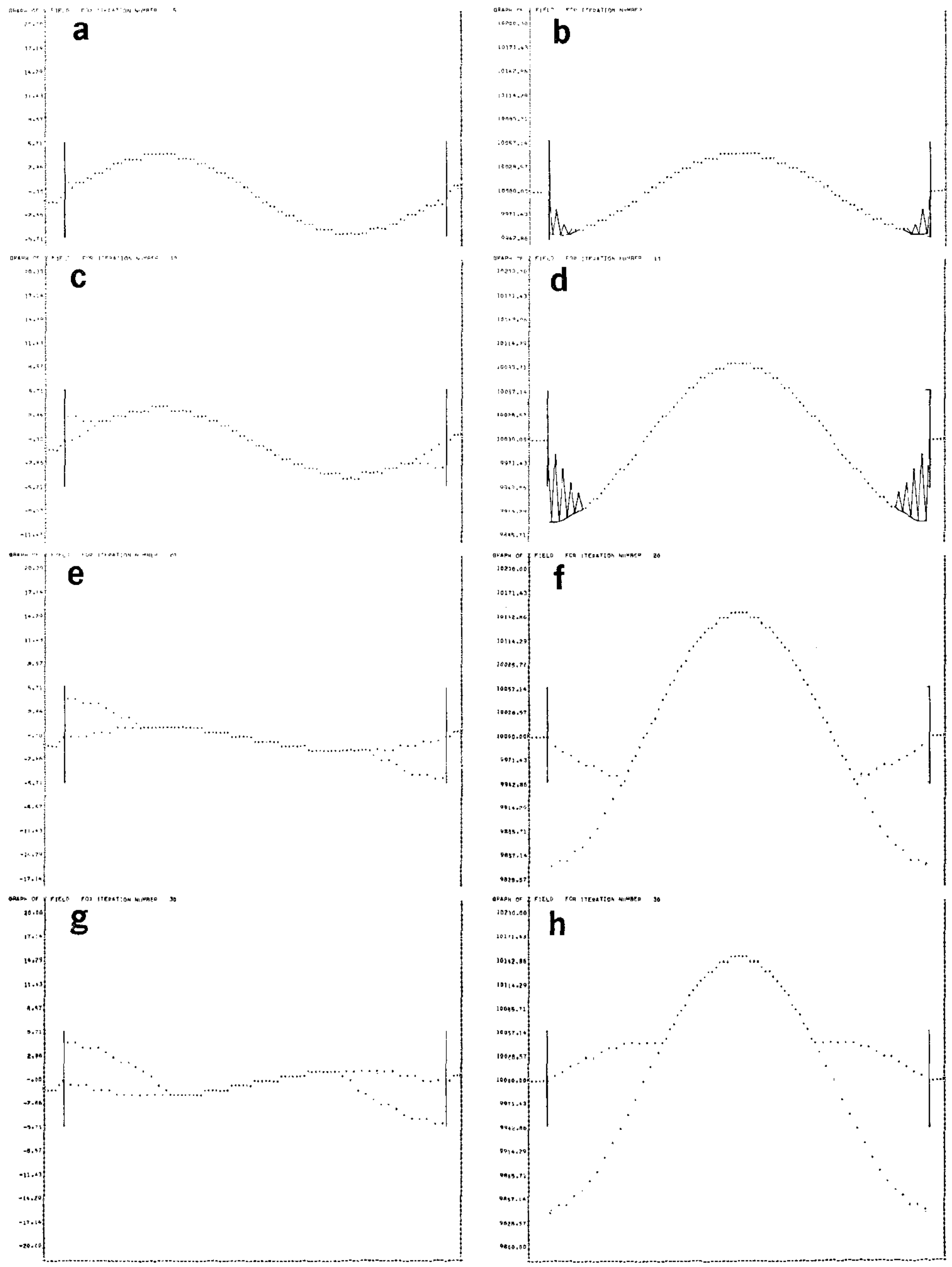

Figure 6.--Numerical solutions of the one-dimensional, constant-boundaries, linear gravity wave problem using centered differences in space and time. Solutions at each grid point are indicated by dots. The boundaries of the region of integration are denoted by short, solid vertical lines, $(\sqrt{gH})\Delta t/\Delta x = 1$.

a, b: Velocity and height, respectively, at time step 5;
c, d: Velocity and height, respectively, at time step 10;
e, f: Velocity and height, respectively, at time step 20;
g, h: Velocity and height, respectively, at time step 30.

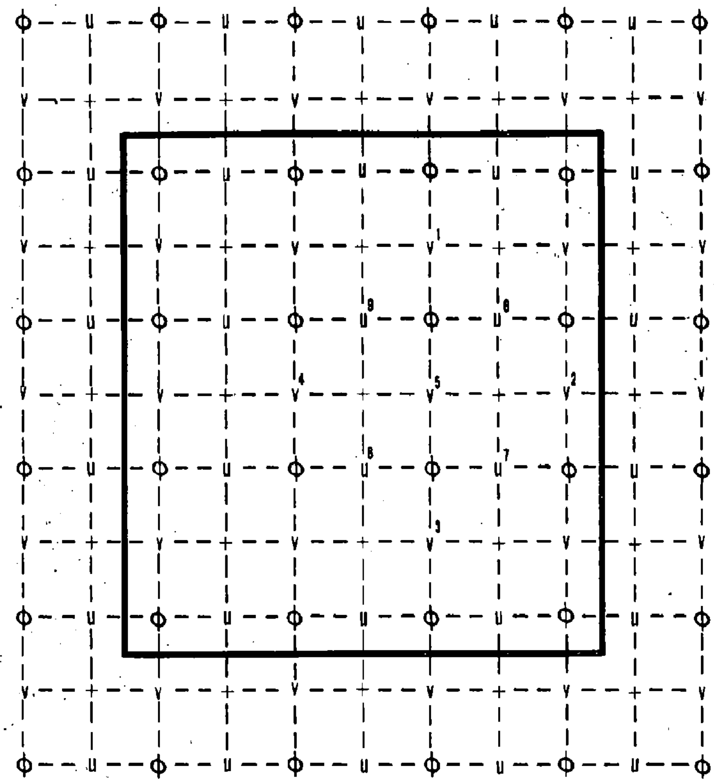

Figure 7.--Arrangement of the dependent variables on the decoupled grid. At adjacent time levels, the wind components exchange locations, and the geopotentials are located at the crosses. The intersections of the dashed lines indicate the grid points that would be used in conventional, explicit, regular-grid integrations. The heavy line distinguishes the lateral boundary zone from the domain of integration.

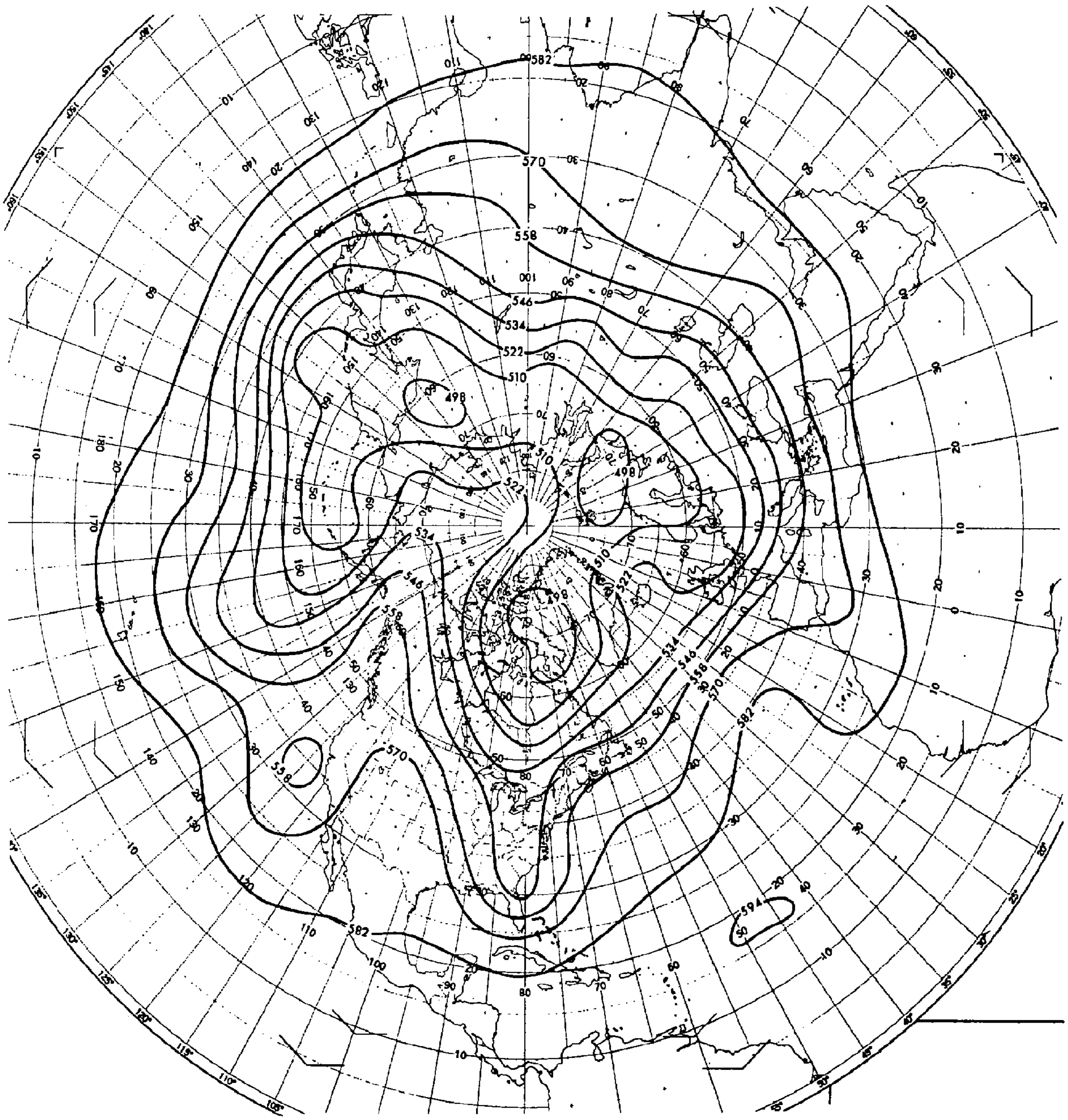

Figure 8a.--Analyzed configuration of the 500-mb height field at 1200 GMT February 10, 1969. Contours are labeled in decameters.

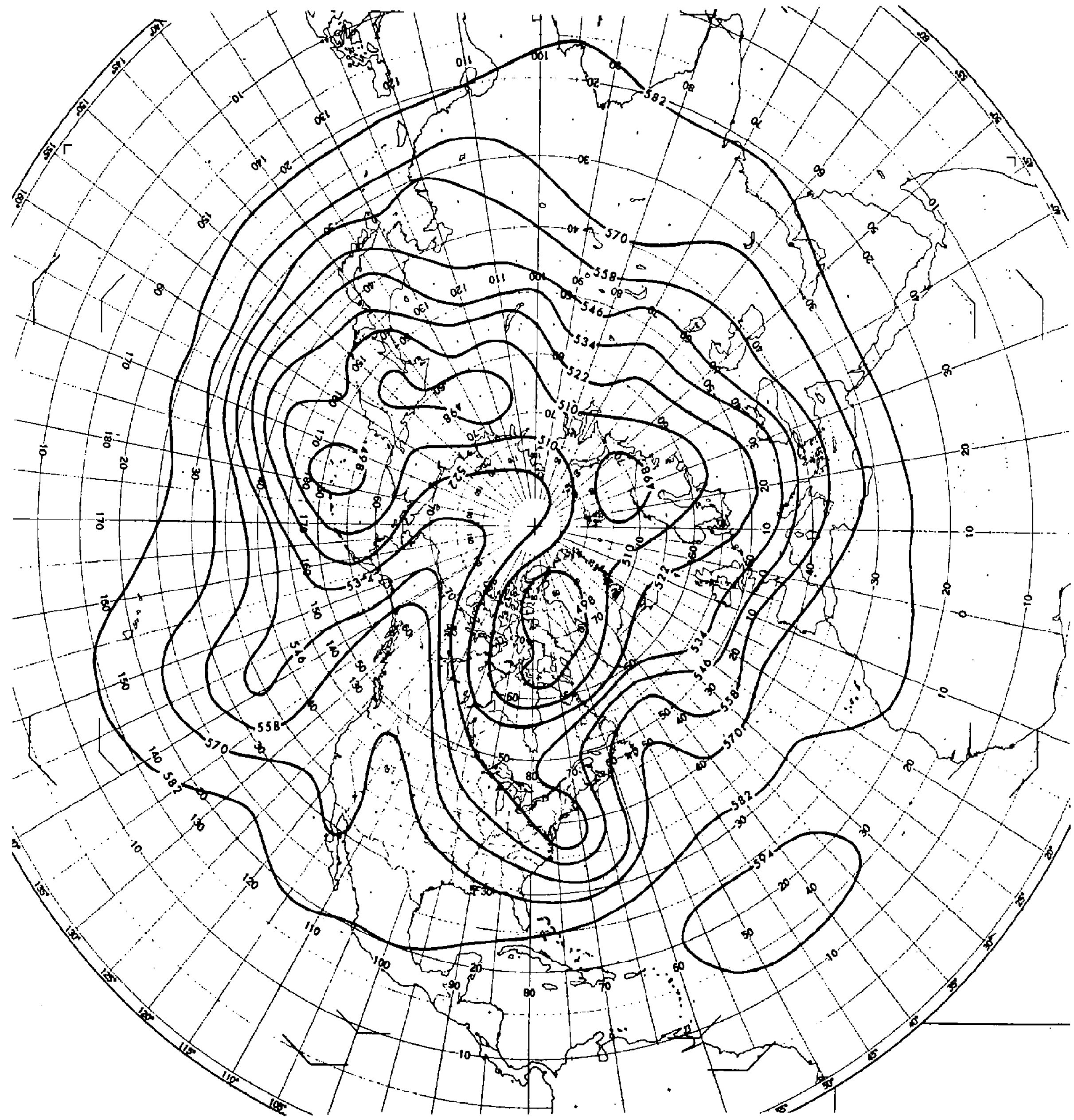

Figure 8b.--Analyzed configuration of the 500-mb height field at 1200 GMT February 11, 1969. Contours are labeled in decameters.

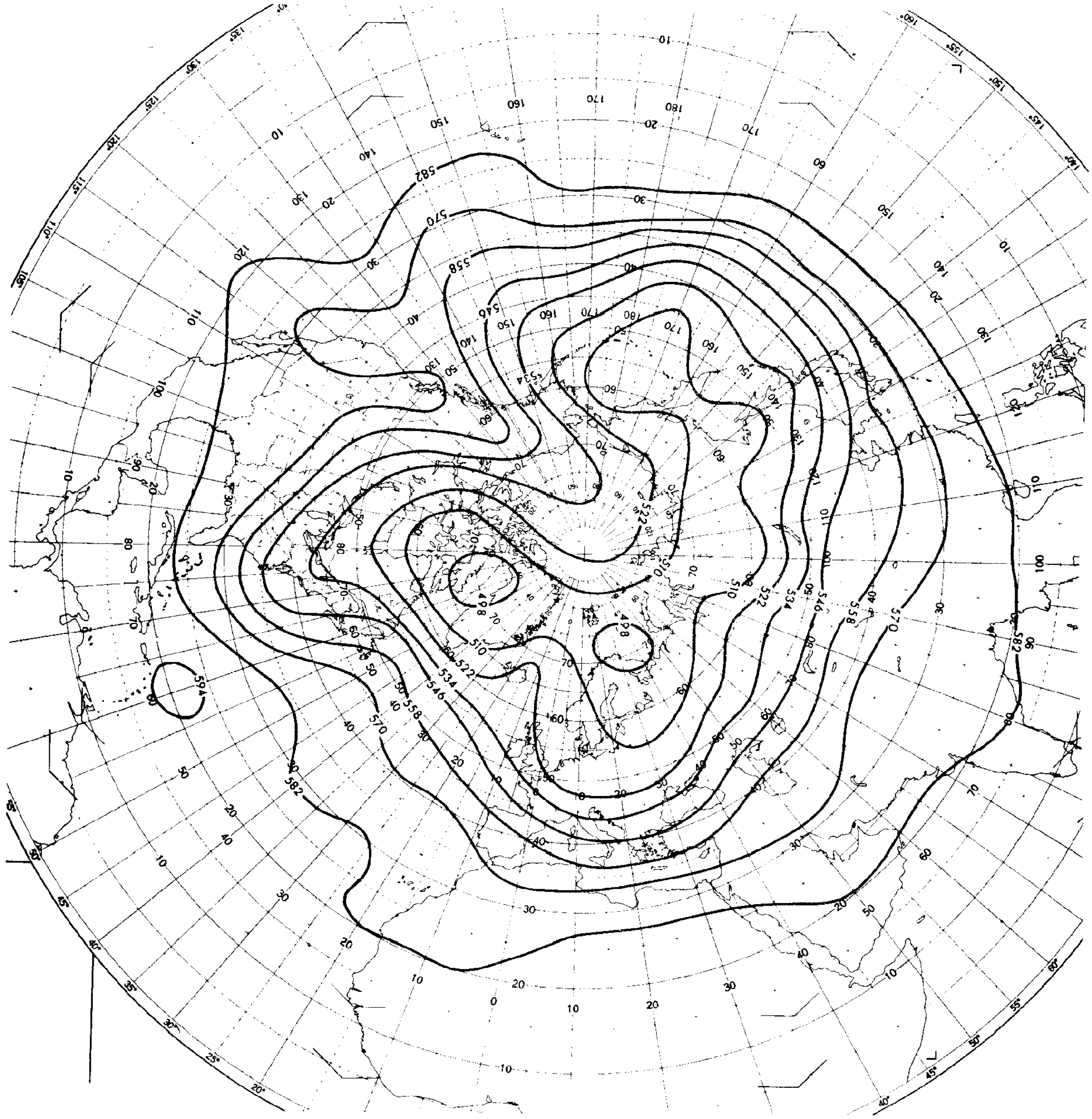

Figure 8c.--24-hour 500-mb height forecast based on initial data of 1200 GMT February 10, 1969, verifying at 1200 GMT February 11, 1969. Contours are labeled in decameters.

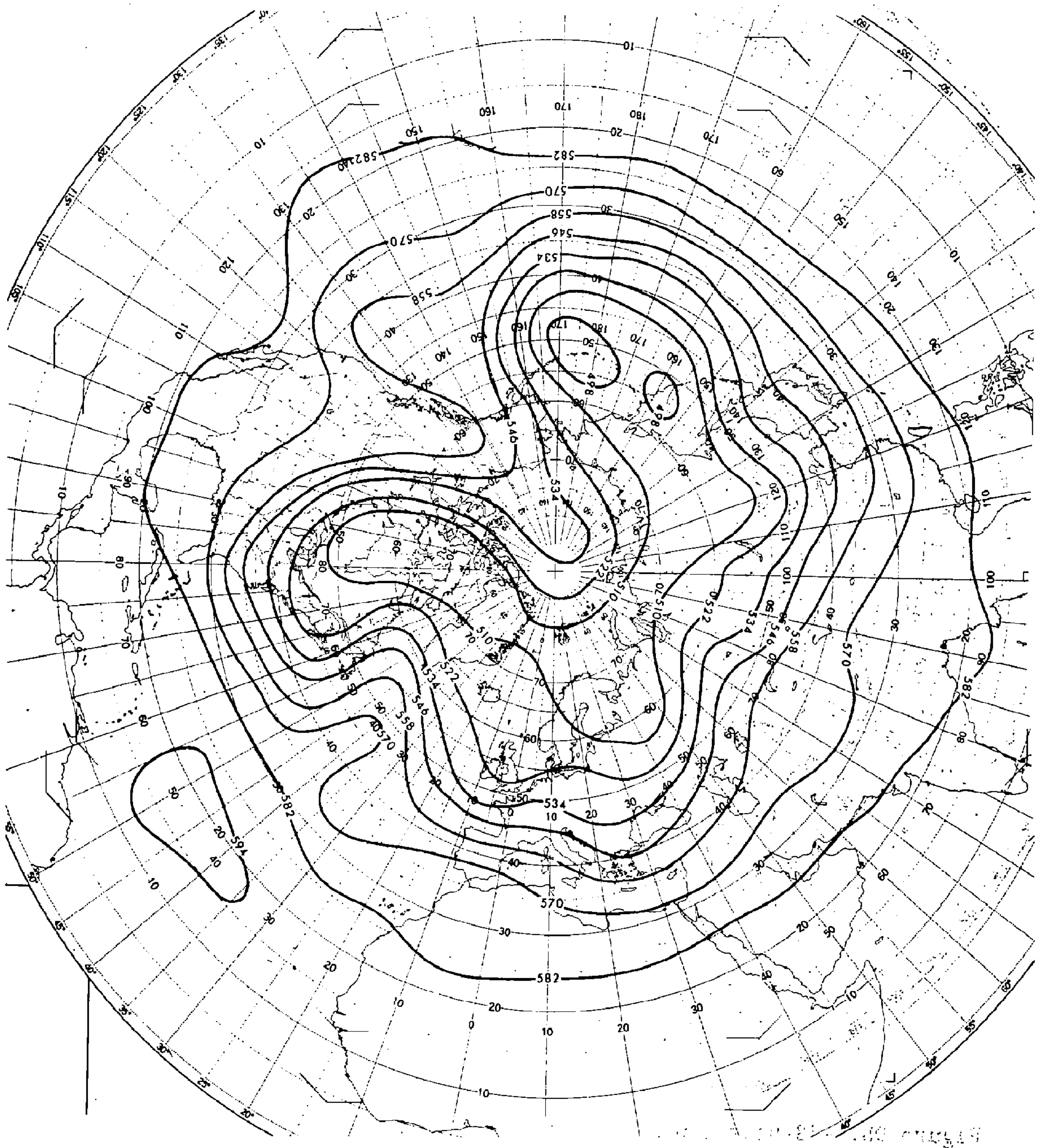

Figure 9a.--Analyzed configuration of the 500-mb height field at 1200 GMT February 12, 1969. Contours are labeled in decameters.

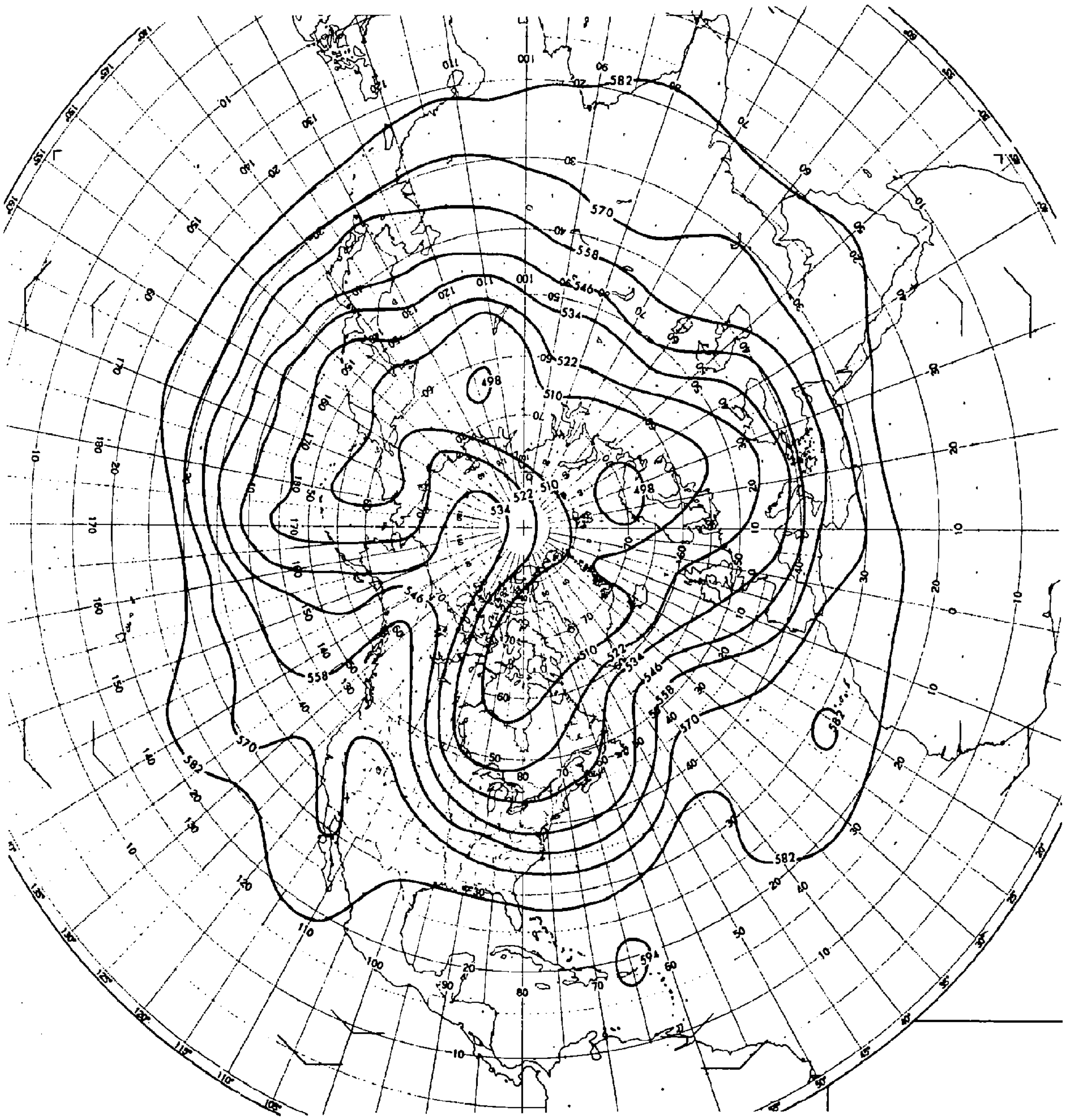

Figure 9b.--48-hour 500-mb height forecast based on initial data of 1200 GMT February 10, 1969, verifying at 1200 GMT February 12, 1969. Contours are labeled in decameters.

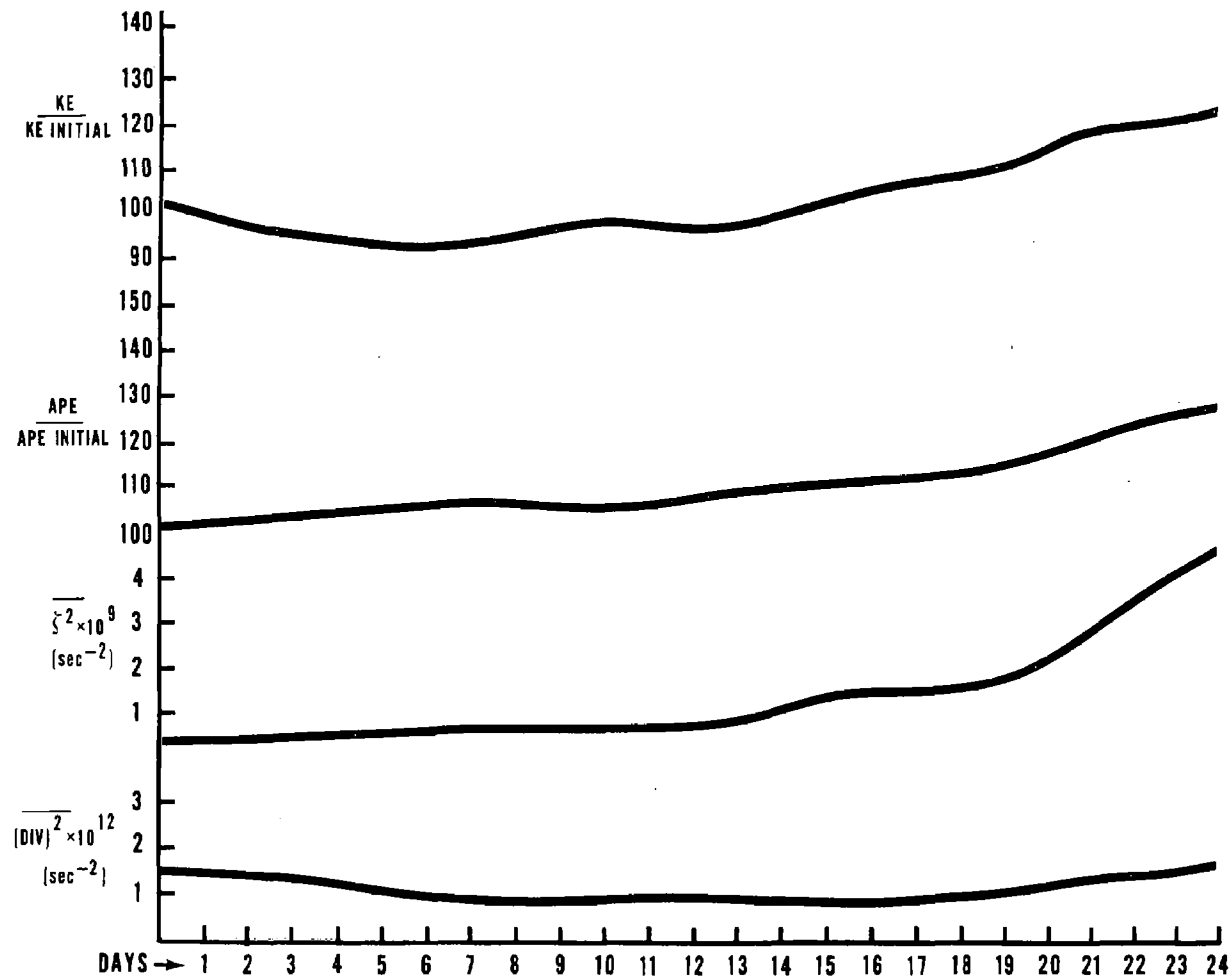

Figure 10.--Kinetic energy, available potential energy, mean-square vorticity, and mean-square divergence, plotted as a function of time for an extended integration based on initial data of 1200 GMT February 10, 1969. The energies have been normalized by their initial values.