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A STUDY OF NON-LINEAR COMPUTATIONAL INSTABILITY FOR A TWO-DIMENSIONAL MODEL

Paul D. Polger

National Meteorological Center

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ABSTRACT

Experiments were conducted to test methods of achieving stability for three finite-difference analogues to a two-dimensional system of forecast equations. The equations described barotropic motion in an inviscid, incompressible, hydrostatic fluid with a free surface. The primary integration procedure was the leapfrog method. The measure of stability of the model was defined as a function of the available energy.

No significant suppression of instability was achieved solely through manipulation of the boundary conditions of the finite-difference formulations. Increased stability was achieved for an invariant form of the forecast equations by applying initial data derived from an initialization procedure consistent with the formulation of the forecast system.

A significant increase in stability was obtained by applying a space-time averaging operator to the advective coefficient of the non-linear terms of an advective form of the primitive equations. The result lends experimental support to extending a theory of non-linear computational instability in one dimension to two dimensions.

Introduction

The study of computational instability arising from the numerical solution of non-linear differential systems related to meteorological problems has been the concern of researchers at the National Meteorological Center in recent years. In a summary of earlier investigations, Shuman (1962) presented the results of stability experiments for several finite-difference analogues to the primitive equation barotropic forecast. The study showed the importance of choosing the proper finite-difference scheme in order to achieve stability for short-range forecasting. Experiments in maintaining stability for extended integrations were reported by Shuman and Vanderman (1966). The stability of the forecast model was found to be highly sensitive to the boundary conditions as well as to the finite-difference analogues. A later experiment by Shuman and Stackpole (1969) used a system of equations corresponding to the NMC operational model, which has been discussed elsewhere by Shuman and Hovermale (1968). They showed that a spurious divergence-on-divergence interaction in the vorticity equation gave rise to instability. Corrective measures involved modifying the forecast equations to a new difference form. This gave further support to the necessity of choosing the proper finite-difference system.

Buck (1967) used spectral methods to examine the equations governing a one-dimensional model. The importance of the cascade of energy in time and the concept of aliasing were demonstrated to be a possible cause of instability. Gerrity and McPherson (1970) presented an analysis of the difference system for a fine-mesh barotropic model which they had developed (Gerrity and McPherson, 1969). They investigated noise components, defined as unwanted gravitational oscillations, which obscured the model's forecasts. The behavior of the noise components was identified and the effect of the boundaries was made evident.

The culmination of these studies in instability was provided by Robert, Shuman, and Gerrity (1970). They developed a general theory to explain the stability of space-averaged difference forms, certain integrations of the pure gravity wave, and the observed association of instability with high frequencies in space and time. The Robert, Shuman, and Gerrity (RSG) theory was used to demonstrate a simple technique for achievement of stability for a non-linear set. This involved applying a space-time averaging operator to the advective coefficient of the non-linear terms of the forecast equations, which were one-dimensional and given in an advective form. The results were promising enough to warrant experimental extension of the "RSG" theory to a fully two-dimensional system.

The purpose of this paper is to present an investigation of experimental studies of the "RSG" theory in two-dimensions. Additional experiments applying techniques to suppress instability are given. In general, these techniques involved the modification of the initial conditions and boundary conditions of the forecast equations. For the study, we chose a set of equations used by Grammeltvedt (1969) in his study of finite-difference schemes for the primitive equations of a barotropic fluid.

The Model

Grammeltvedt (1969) conducted a survey of several finite-difference schemes using a free-surface barotropic model describing an inviscid, incompressible, homogeneous, hydrostatic fluid in a rotating channel. The fluid flow is periodic in the east-west direction. The northern and southern boundaries are rigid walls where the normal velocity components vanish.

The Eulerian form of the model is given by the following equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + \frac{\partial \phi}{\partial x} = 0 \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + \frac{\partial \phi}{\partial y} = 0 \tag{2}$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial (\phi u)}{\partial x} + \frac{\partial (\phi v)}{\partial y} = 0 \tag{3}$$

The variables u and v are the velocities in the x and y directions, and correspond respectively to east-west and north-south flow. The geopotential is given by $\phi = gh$, where h is the height of the free surface of the fluid and g is the acceleration of gravity. The coriolis parameter f varies in the y-direction due to the β-plane assumption.

The initial height field, shown in Figure 1, chosen for our study was defined by Grammeltvedt as follows:

$$h(x,y) = H_o + H_1 \tanh \frac{9(y-y_o)}{2D} + H_2 \operatorname{sech}^2 \frac{9(y-y_o)}{D} \left[.8 \sin\left(\frac{2\pi x}{L}\right) + .5 \sin\left(\frac{12\pi x}{L}\right)\right] \tag{4}$$

where the channel length is given by L, the channel width by D, and the middle latitude by $y_o = \frac{D}{2}$. The values used in the calculations were as follows:

H_o = 2000 m	L = 6000 km
H_1 = - 220 m	D = 4000 km
H_2 = 133 m	$f_o = 10^{-4}$ sec^{-1}
g = 10 msec^{-2}	$\beta = 1.5 \times 10^{-11}$ m^{-1} sec^{-1}

The initial winds, figures 2 and 3, were computed geostrophically so that:

$$u = -\frac{g}{f}\overline{h_y}^y$$

$$v = \frac{g}{f}\overline{h_x}^x \tag{5}$$

where,

$$f = f_o + \beta(y - y_o).$$

The notation is given by equation (6). The data from these calculations were used in three schemes, which correspond to Grammeltvedt's schemes B, E, and H. These particular schemes were chosen because they were representative of the different finite difference forms treated by Grammeltvedt. The notations used to define the equations [Shuman (1962)] have the following form:

$$\alpha_x = \frac{1}{\Delta x}\left[\alpha(x_i + \frac{\Delta x}{2}) - \alpha(x_i - \frac{\Delta x}{2})\right]$$

and (6)

$$\overline{\alpha}^x = \frac{1}{2}\left[\alpha(x_i + \frac{\Delta x}{2}) + \alpha(x_i - \frac{\Delta x}{2})\right]$$

Scheme B, a *momentum* form, was used by Grimmer and Shaw (1967) for a spherical free surface model whose finite-difference equations are:

$$\overline{(\phi u)_t}^t + (\overline{\phi u}^x\,\overline{u}^x)_x + (\overline{\phi v}^y\,\overline{u}^y)_y - f\phi v + \phi\overline{\phi_x}^x = 0 \tag{7}$$

$$\overline{(\phi v)_t}^t + (\overline{\phi v}^x\,\overline{v}^x)_x + (\overline{\phi v}^y\,\overline{v}^y)_y + f\phi u + \phi\overline{\phi_y}^y = 0 \tag{8}$$

$$\overline{\phi_t}^t + \overline{(\phi u)_x}^x + \overline{(\phi v)_y}^y = 0 \tag{9}$$

Scheme E is a *total energy conservative scheme*, in invariant form, whose finite-difference equations are:

$$\overline{u_t}^t + \overline{\left(\frac{u^2 + v^2}{2} + \phi\right)_x}^x - v\left(f + \overline{v_x}^x - \overline{u_y}^y\right) = 0 \tag{10}$$

$$\overline{v_t}^t + \overline{\left(\frac{u^2 + v^2}{2} + \phi\right)_y}^y + u\left(f + \overline{v_x}^x - \overline{u_y}^y\right) = 0 \tag{11}$$

$$\overline{\phi_t}^t + \overline{(\phi u)_x}^x + \overline{(\phi v)_y}^y = 0 \tag{12}$$

Scheme H, given by Shuman and Vanderman (1966), has been used for numerous experiments at NMC and was once the form used for the operational model. The finite-difference formulation is:

$$\overline{u}^t_t + \overline{\left(\overline{u}^{xy}\, \overline{u}^y_x + \overline{v}^{xy}\, \overline{u}^x_y + \overline{\phi}^y_x - \overline{f}^{xy}\, \overline{v}^{xy} \right)}^{xy} = 0 \qquad (13)$$

$$\overline{v}^t_t + \overline{\left(\overline{u}^{xy}\, \overline{v}^y_x + \overline{v}^{xy}\, \overline{v}^x_y + \overline{\phi}^x_y + \overline{f}^{xy}\, \overline{u}^{xy} \right)}^{xy} = 0 \qquad (14)$$

$$\overline{\phi}^t_t + \overline{\left[\left(\overline{u}^y\, \overline{\phi}^y \right)_x + \left(\overline{v}^x\, \overline{\phi}^x \right)_y \right]}^{xy} = 0 \qquad (15)$$

The calculations were carried out on a 31 x 23 evenly spaced grid. The leapfrog method, with a ten-minute time step, was used for the time integration of each of the three schemes. The starting procedure was a single, forward time step. This explicit method has the advantage of not damping or amplifying the solution for the linearized set of equations given by equations (1), (2), and (3), provided the linear stability criterion is not violated. The stability criterion used by Grammeltvedt was a function of the available energy. Instability was defined to occur when the available energy became 10 percent greater than its initial value. The available energy, AE, was calculated over the region of integration as the summation over all i, j grid points in that area:

$$AE = \frac{1}{2g} \sum_{ij} \{ \phi_{ij} \, (u^2_{ij} + v^2_{ij}) + (\phi_{ij} - gH_o)^2 \} \qquad (16)$$

The results of our calculations, which became the reference cases for subsequent tests, are shown in Table 1 in comparison to Grammeltvedt's results. The discrepancies that appear remained unsolved after numerous attempts at replication of the three schemes.

Boundary Conditions Variation Experiments

In the introduction, it was noted that several authors had shown that minor adjustments to the boundary conditions could result in significant changes in behavior of a given model. Thus, the first phase of experimentation was to determine if increased stability could be obtained by simple alteration of the boundary conditions. Any changes introduced apply to the northern and southern boundaries since periodicity has been specified in the east-west direction. The periodicity was achieved by equating the variables at the first and thirty-first columns of gridpoints.

	Scheme B	Scheme E	Scheme H
Reference Case	13	7.5	29
Grammeltvedt's Results	18	11	28
Constant Boundary Case	13.5	6.5	15
Constant Boundary with Diffusive Operator Case	14.5	9	32
Reference Case with Appropriate Initialization Procedure	---	13	---
Constant Boundary with Diffusive Operator Case with Appropriate Initialization Procedure	---	14.5	---
RSG Technique with Constant Boundary and Diffusive Operator	---	< 3	46
Reference Case with Leapfrog-Trapezoidal Integration Procedure	---	---	34
Constant Boundary with Diffusive Operator Case and Leapfrog-Trapezoidal Procedure	---	---	15

Table 1 - The running time in days for Schemes B, E, and H under various initial conditions, boundary conditions, and integration techniques.

The boundary conditions used in the reference cases have the walls at gridpoints comprising the second and twenty-second rows. For schemes B and E, the boundary conditions are:

$$v = 0 \qquad \overline{\phi}^y_y = 0 \qquad \overline{u}^y_y = 0$$

and (17)

$$\overline{v}^y_y = \frac{2v_{N-1}}{\Delta y} \qquad \overline{v}^y_y = \frac{2v_{S+1}}{\Delta y}$$

where N and S are the northern and southern boundaries respectively. The boundary conditions for scheme H were:

$$v = 0 \qquad \overline{\phi}^y_y = 0 \qquad \overline{u}^y_y = 0$$

$$v_{N\pm1} = 0 \qquad v_{S\pm1} = 0 \tag{18}$$

Shuman and Vanderman (1966) tested two sets of boundary conditions, given by (19) and (20), for a primitive equation free surface barotropic model designed for a tropical belt. The forecast began "blowing up" at 13 days with the boundary conditions given by (19), which reflect pure gravity waves at the wall while neglecting the inertial effects. The boundary conditions are:

$$u_y = 0 \qquad \overline{v}^y = 0 \qquad \phi_y = 0 \tag{19}$$

where the wall lies between gridpoints, halfway between the outer and first interior rows. The forecast was stable to 100 days when the second set of boundary conditions were applied. The boundary conditions are:

$$\overline{u}^{ty}_t + \overline{m}^y \; \overline{\left(\overline{u}^{xy} \; \overline{u}^y_x + g\overline{h}^y_x \right)}^x = 0 \tag{20}$$

$$\overline{v}^y = 0$$

$$\overline{f}^y \; \overline{u}^y + \overline{m}^y \; gh_y = 0$$

Their result indicated the sensitivity of the forecast to the boundary conditions. A similar test was conducted for our model. The boundary conditions given by equation (19) were applied to schemes B, E, and H. There was no appreciable change in the stability of schemes B or E with these boundary conditions. However, scheme H became unstable after 9 days, which was 20 days sooner than the reference case. In an attempt to stabilize scheme H, the boundary conditions given by equation (20) were applied. The forecast became unstable after 7 days. This result is contrary to the experience of Shuman and Vanderman. The reason for the discrepancy appears to be associated with the initial conditions.

Previous experience [Gerrity and McPherson (1969, 1970)] indicated that increased stability might be achieved using "constant boundaries." In this case, the initial value of the velocity components and the geopotential were maintained throughout the integrations on the outer row of gridpoints. The velocity component normal to the wall vanished at the walls. When this boundary condition was applied to scheme E, instability occurred at 6.5 days, which was a day less than the reference case. The result for scheme B was that the model ran half a day longer than the reference case before the stability criterion was violated. Scheme H with constant boundary conditions became unstable after 15 days, which was 14 days earlier than in the reference case.

It was apparent that additional adjustments would be necessary in order to obtain a more stable formulation of each scheme. A close examination of scheme B and scheme E provided the insight for the next experiment. Both of these schemes suffered from complete obscuration of the fields by short wave-length noise after only a few days of integration. This "noise" appeared to contribute to the degeneration of the stability of the model for schemes B and E. A detailed discussion of noise of this type and its effect has been given by Gerrity and McPherson (1970). One method they employed to suppress the two-grid interval waves was to introduce a "diffusive" operator at the first row of gridpoints on the region encompassing the interior of the boundary. The equations then become:

$$u^{n+1} = \tilde{u}^n - \Delta t \cdot F^n_u \tag{21}$$

$$v^{n+1} = \tilde{v}^n - \Delta t \cdot F^n_v \tag{22}$$

$$\phi^{n+1} = \tilde{\phi}^n - \Delta t \cdot F^n_\phi \tag{23}$$

where the tilde indicates the average of the immediately adjacent points and F^n_u, F^n_v, F^n_ϕ are the tendencies of u, v, and ϕ respectively for time levels n. This is the first step of the Lax-Wendroff method (Richtmeyer, 1962). Applying this device with constant boundary conditions, scheme B ran 14.5 days, scheme E ran 9 days, and scheme H ran 32 days. Thus, in all three cases, there was improvement in stability over the reference cases, and in schemes B and E the two-grid interval waves were eliminated in the north-south direction. A clear picture of the meteorological features was obtained by applying a smoother (Gerrity and McPherson, 1969) to the height field at the time instability occurred. Because of the nature of the smoother, the three outermost rows are not smoothed. Figures 4-6 show the unfiltered height field, the height field with the diffusive operator applied, and the smoothed height field for scheme B after approximately 13 days. It is interesting to note that scheme H gained the most in terms of running time, even though it was the least susceptible to the noise we sought to suppress.

The alteration of the boundary conditions did not result in extended stability for any of the three schemes. In several experiments instability occurred earlier than the time established in the reference cases. The results indicate that "correct" specification of the boundary conditions is a necessary but not sufficient condition for extended stability.

Initial Conditions Variation Experiments

The experiments conducted prior to this point all used the initial conditions given in Figures 1-3. Grammeltvedt experimented with non-divergent initial conditions, but found no improvement in stability. Following a procedure used by Kwizak and Robert (1970), we attempted to achieve a better initial balance by effectively inverting the balance equation using a particular finite-difference scheme consistent with the forecast equations. In this case, scheme E was chosen. The first step in the initialization was to obtain a set of non-divergent winds by calculating the vorticity,

$$\zeta = \overline{v}^{x}_{x} - \overline{u}^{y}_{y} \tag{24}$$

and solving for the stream function from

$$\nabla^2 \psi = \zeta \tag{25}$$

which allows us to calculate non-divergent winds,

$$\begin{aligned} u &= -\overline{\psi}^{x}_{x} \\ v &= \overline{\psi}^{y}_{y} \end{aligned} \tag{26}$$

Differencing equation (10) with respect to x, and equation (11) with respect to y, and adding the two equations, forms the divergence tendency:

$$\left(\overline{\overline{u}^{x}_{x} + \overline{v}^{y}_{y}}^{t} \right)_{t} + \left(\overline{\phi}^{xx}_{xx} + \overline{\phi}^{yy}_{yy} \right) + \left(\overline{K}^{xx}_{xx} + \overline{K}^{yy}_{yy} \right) + \left(\overline{u\eta}\right)^{y}_{y} - \left(\overline{v\eta}\right)^{x}_{x} = 0 \tag{27}$$

where $K = \frac{1}{2} (u^2 + v^2)$

and $\eta = f + \overline{v}^{x}_{x} - \overline{u}^{y}_{y}$.

Setting $\phi = \phi_0 + \phi^1$ where ϕ_0 is the initial value of the geopotential and ϕ^1 is an adjustment needed to balance the winds, and setting the tendency of the divergence equal to zero, we obtain:

$$\overline{\phi^1}^{xx}_{xx} + \overline{\phi^1}^{yy}_{yy} = - \left(\overline{\phi}^{xx}_{o_{xx}} + \overline{\phi}^{yy}_{o_{yy}} \right) - \left(\overline{K}^{xx}_{xx} + \overline{K}^{yy}_{yy} \right) - \overline{(u\eta)}^{y}_{y} + \overline{(v\eta)}^{x}_{x} \tag{28}$$

or

$$\nabla^2 \phi^1 = F \tag{29}$$

where F is a forcing function formed by the terms on the right hand side of equation (28). Solving for ϕ^1 by relaxation, subject to $\phi^1 = 0$ at the boundary, the final value of the geopotential is obtained by adding ϕ^1 to ϕ_o.

The initialized data were tested on two cases of scheme E. The first corresponded to the reference case and it ran 13 days, which was an improvement of 5.5 days. The second case had the constant boundary condition with the diffusive operator. This model ran 14.5 days with the initialized data, also showing an increase of 5.5 days over the same case without the initialization. The mean divergence was calculated for all the cases. For the two cases of scheme E with the non-divergent initial data, the order of magnitude of the mean divergence never exceeded 10^{-8}. The order of magnitude of the mean divergence for the non-initialized cases was generally 10^{-7}. These results from scheme E indicate that a more stable forecast can be obtained by a small reduction in the growth of divergence with time through proper initialization.

A similar initialization procedure could be developed for scheme H. However, scheme B does not lend itself to an initialization procedure of this type because all the terms of equations (7) and (8) are non-linear.

Integration Procedure Variation Experiments

An experiment aimed toward stabilization was the application of the leapfrog-trapezoidal method for the numerical integration. Grammeltvedt applied this method to his scheme F, which is a quadratic advective scheme. He ran this scheme to 40 days without incurring instability. The leapfrog-trapezoidal method is a two-step procedure given by Kurihara (1965) as:

$$\begin{aligned} F^{\lambda} &= F^{n-1} - 2 \cdot \Delta t \cdot T^{n} \\ F^{n+1} &= F^{n} - \Delta t \cdot (T^{n} + T^{\lambda}) \end{aligned} \tag{30}$$

where T^{λ} is the tendency computed from F^{λ}. He notes that a major drawback to the system is that it doubles the calculating time. An advantage is the suppression of the computational mode but not the physical mode. The leapfrog-trapezoidal method was applied to scheme H with the conditions of the reference case, and with constant boundary conditions and the diffusive operator. The reference case with the leapfrog-trapezoidal method ran 34 days, which compares to 29 days for the same case without the leapfrog-trapezoidal procedure. The case with the constant boundaries and diffusive operator only ran 15 days, which was the same as prior to applying the leapfrog-trapezoidal method. These results suggest that the suppression of the temporal computational mode is not sufficient to achieve relative stability.

RSG Technique

It was apparent that the instability occurring in these experiments could be suppressed to only a moderate degree by the techniques employed at the boundaries and on the initial conditions. To test the application of the "RSG" theory to two dimensions, schemes E and H with constant boundaries and the diffusive operator were chosen. Both these cases represented relatively well-behaved runs under similar conditions. Again, scheme B was not used since the *momentum* form of the forecast equations does not lend itself to the "RSG" technique.

The time-space averaging operator of the "RSG" technique which is applied to the advective coefficient of the non-linear terms has been called the *star* operator. It is defined as $(\ )^* = \frac{1}{2}[(\ )_{n-1} + (\ )_n]$, which simply averages the present and immediate past time steps for a given function.

Previous experiments (Gerrity and McPherson, 1970) were conducted to test an invariant form of the equations with the star operator. For that study, the equations from scheme H were formulated in the invariant form:

$$\overline{u}^t_t - \overline{\zeta^* \, \overline{v}^{xy}}^{xy} + \overline{\overline{K}^y_x + \overline{\phi}^y_x}^{xy} = 0 \qquad (31)$$

$$\overline{v}^t_t + \overline{\zeta^* \, \overline{u}^{xy}}^{xy} + \overline{\overline{K}^x_y + \overline{\phi}^x_y}^{xy} \qquad (32)$$

where

$$\zeta = \overline{v}^y_x - \overline{u}^x_y + \overline{f}^{xy} \qquad (33)$$

and

$$K = \frac{1}{2}(u^2 + v^2) \qquad (34)$$

Applying the basic data used for our study, the invariant scheme without the star operator ran 16.5 days before becoming unstable. When the star was introduced into the equations, there was no instability after 23.7 days according to the criterion set by Grammeltvedt. However, the forecast exhibited a noisy behavior as shown in figure 7. The two experiments were repeated with data expected to produce a relatively well-behaved solution. In this case, the data corresponded to Grammeltvedt's initial condition 1 which is just one wave in the zonal flow. The application of the star operator resulted in a more rapid growth of noise than without the star.

One of the experiments run in our study was scheme E with the star operator. Scheme E is also in invariant form. This run became unstable after only a few days. Theoretically, the application of the star operator has been proven effective in extending stability only for the equations in an advective form. The instabilities which occurred in the foregoing experiments with the star operator could be associated with the invariant form of the equations.

The final experiment with the "RSG" theory was to apply the star operator to scheme H, which is an advective form of the forecast equations. The star was implemented in the momentum equations of scheme H in the following manner:

$$\overline{u}^{t}_{t} + \overline{\left(\overline{u^*}^{xy}\, \overline{u}^{y} + \overline{v^*}^{xy}\, \overline{u}^{x}_{y} + \overline{\phi}^{y}_{x} - \overline{f}^{xy}\, \overline{v}^{xy} \right)}^{xy} = 0 \qquad (35)$$

$$\overline{v}^{t}_{t} + \overline{\left(\overline{u^*}^{xy}\, \overline{v}^{y}_{x} + \overline{v^*}^{xy}\, \overline{v}^{x}_{y} + \overline{\phi}^{x}_{y} + \overline{f}^{xy}\, \overline{u}^{xy} \right)}^{xy} = 0 \qquad (36)$$

The result was a run of 46 days before instability was reached. This by itself is a significant contribution, but equally important is the fact that the height field was relatively well-behaved when the instability criterion was violated. The height field for this run (figure 8) can be compared to the height field for the invariant form of scheme H (figure 7). The available energy as a function of time for scheme H with the star operator is shown in figure 9 in comparison to that of the reference case and the leapfrog-trapezoidal case.

Summary of Results

Experiments were conducted to test methods of achieving stability for three finite-difference analogues to a system of forecast equations. The equations described barotropic motion in an inviscid, incompressible, hydrostatic fluid with a free surface. The primary integration procedure was the leapfrog method. The measure of stability of the model was defined as a function of the available energy.

The three finite-difference analogues chosen are referred to as schemes B, E, and H. They represent, respectively, the forecast equations in a *momentum* form, an invariant form, and an advective form. Each scheme was run using initial conditions and boundary conditions given by Grammeltvedt (1969). The number of days each of these three cases ran became the reference case for subsequent tests to increase the stability of each scheme.

The greatest increase in stability through alteration of the boundary conditions was obtained by applying *constant* boundary conditions with a diffusive operator. The diffusive operator is the first step of the Lax-Wendroff method (Richtmeyer, 1962) applied at points adjacent to the boundary. The reference cases for schemes B, E, and H ran 13 days, 7.5 days, and 29 days respectively. The application of the constant boundary conditions with the diffusive operator extended these runs, in order, to 14.5 days, 9 days, and 32 days. In other experiments, decreases in stability resulted from minor alterations in the boundary conditions specified in the reference cases.

A more stable forecast was achieved using an improved set of initial conditions. The initial geostrophic balance was replaced by an initial balance obtained by effectively inverting the balance equation for a finite-difference scheme consistent with the forecast equations. In this case, scheme E was chosen. Applying the initialized data and constant boundary conditions with the diffusive operator, scheme E ran 14.5 days. This compares to 9 days for the same case without the initialization procedure and 7.5 days for the reference case. Examination of the results showed that the non-divergent initial conditions reduced the growth of the mean divergence with time. The increased stability can be attributed to this decrease in the mean divergence.

Grammeltvedt achieved extended stability for an advective form of the equations by applying the leapfrog-trapezoidal method for the numerical integration. After duplicating his result, the technique was applied to scheme H, which is also an advective form of the forecast equations. When the boundary and initial conditions were the same as in the reference case, scheme H ran 34 days with the leapfrog-trapezoidal integration procedure. The case with constant boundary conditions and the diffusive operator ran only 15 days. These runs compare to 29 days and 32 days, respectively, for the leapfrog method. The suppression of the temporal computational mode was not sufficient to achieve extended stability for scheme H. Indeed, the leapfrog-trapezoidal technique in conjunction with the constant boundary conditions with the diffusive operator proved to be detrimental to the stability of scheme H.

An important aspect of the study was to test, on a two-dimensional model, a theory developed by Robert, Shuman, and Gerrity (1970) to suppress non-linear computational instability. From the "RSG" theory, a technique was developed which achieved relative stability for an advective form of the primitive equations in one dimension. The "RSG" technique applies a time-space averaging operator to the advective coefficient of the non-linear terms of the forecast equations. The technique was applied to scheme H with constant boundaries and the diffusive operator. The result was a run of 46 days, which compares to 32 days for the same case without the "RSG" technique. The height field was well-behaved for both cases at the time instability was reached. The result lends experimental support to extending the theory of non-linear computation instability, presented by Robert, Shuman, and Gerrity, from one dimension to two dimensions.

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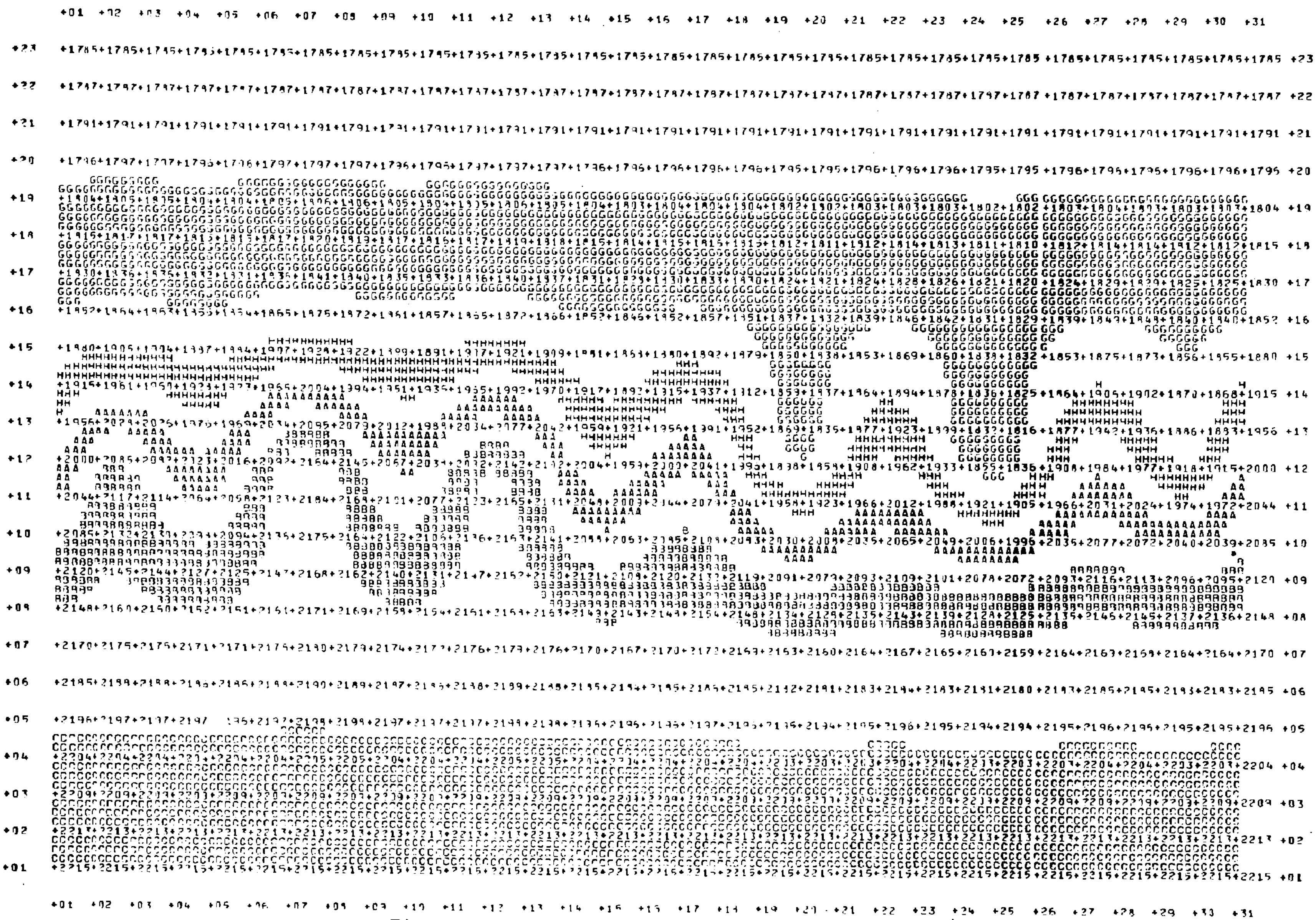

Figure 1.--Initial height field in meters.

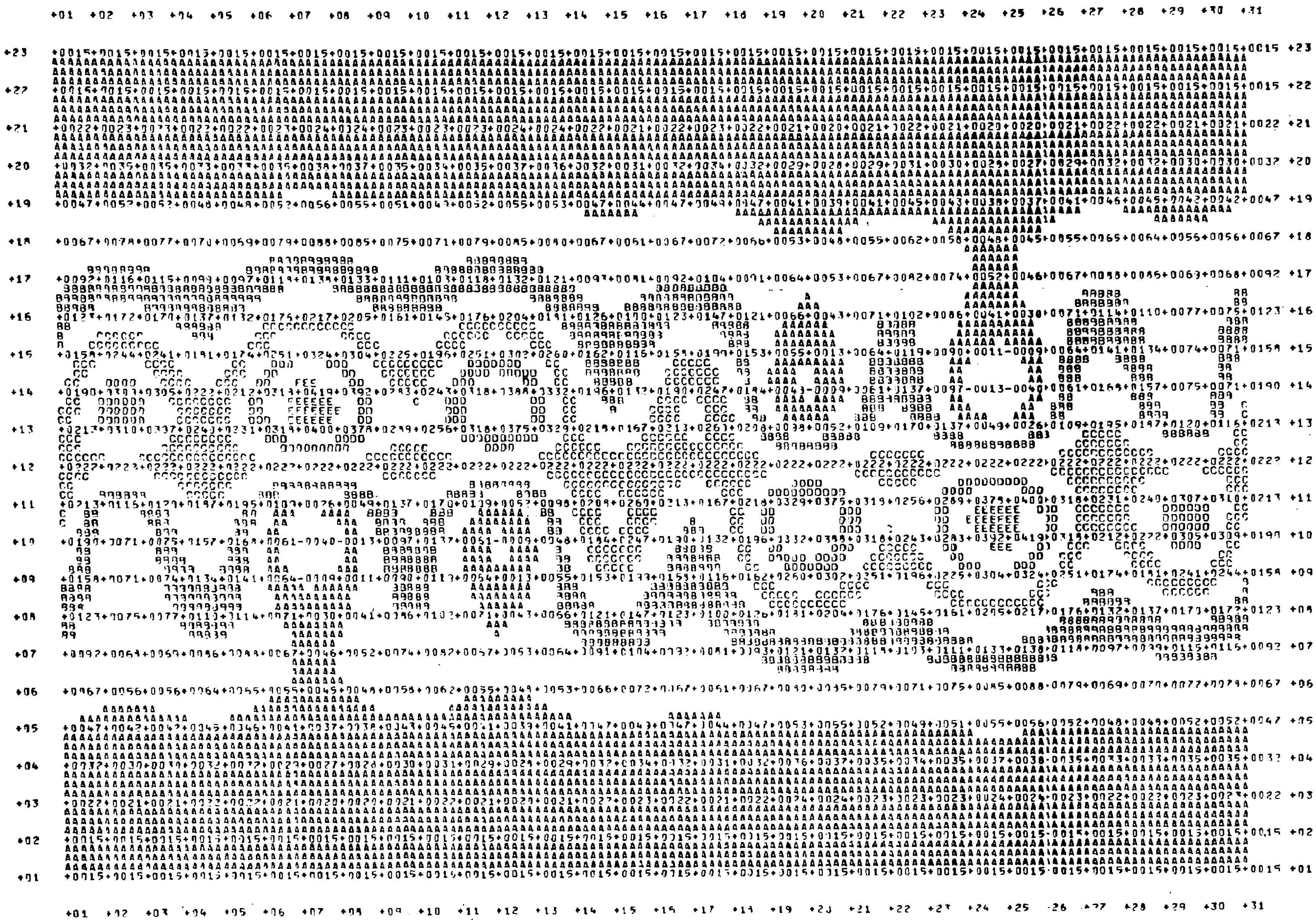

Figure 2.--Initial U component of the wind field in m sec^{-1} x 10^{-1}.

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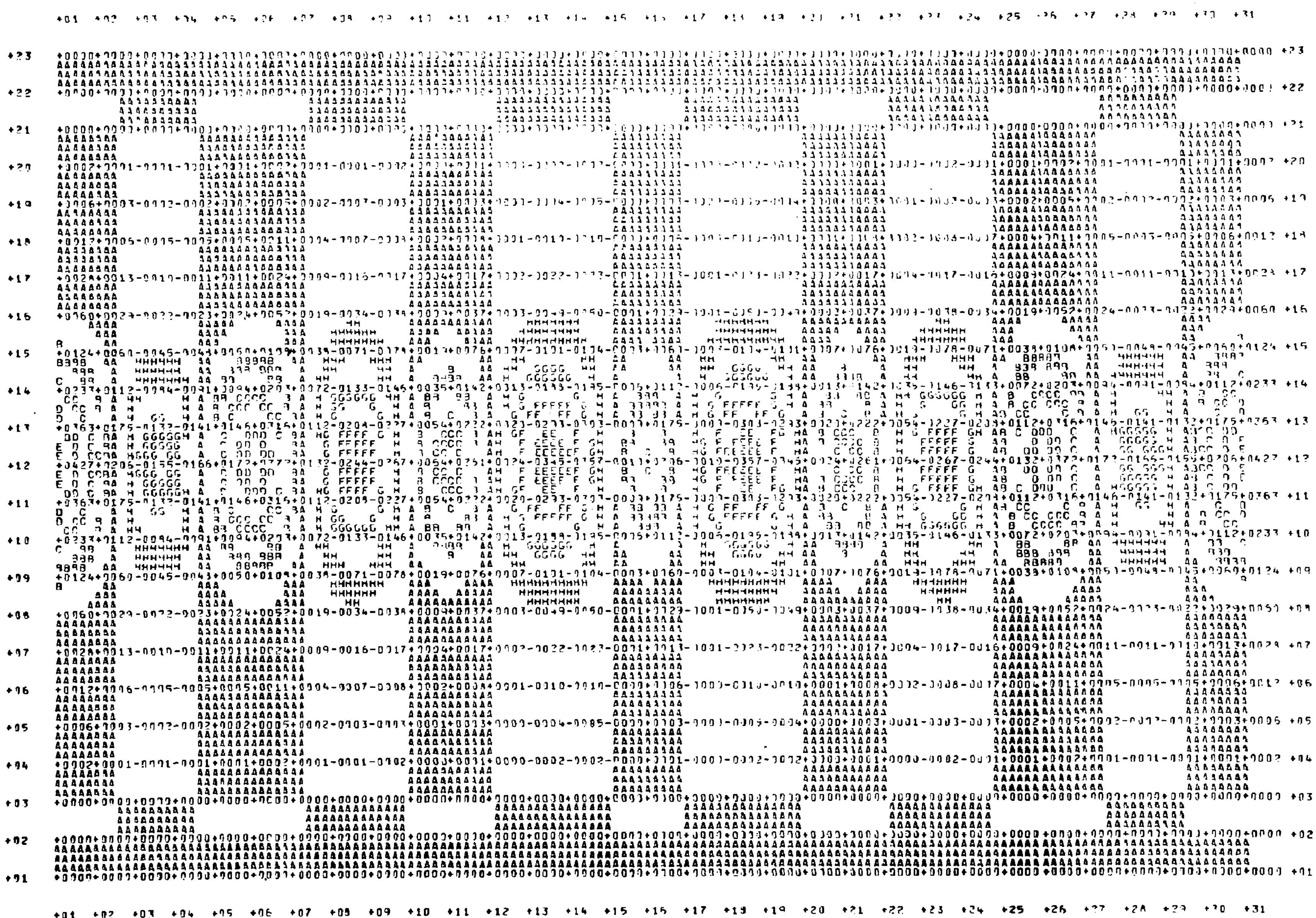

Figure 3.--Initial V component of the wind field in m $sec^{-1} \times 10^{-1}$.

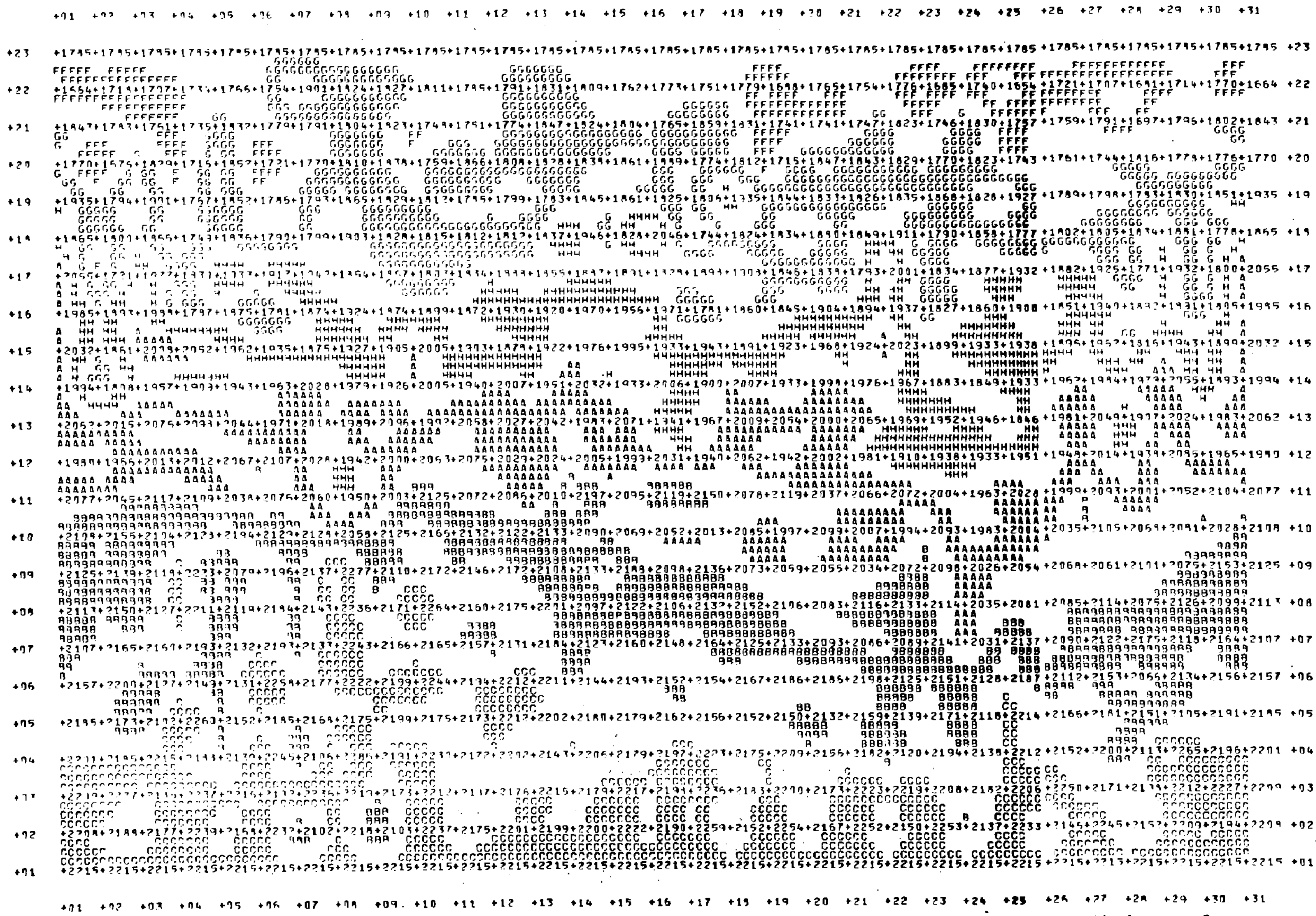

Figure 4.--Height field in meters obtained from Scheme B with constant boundary conditions after approximately 13 days.

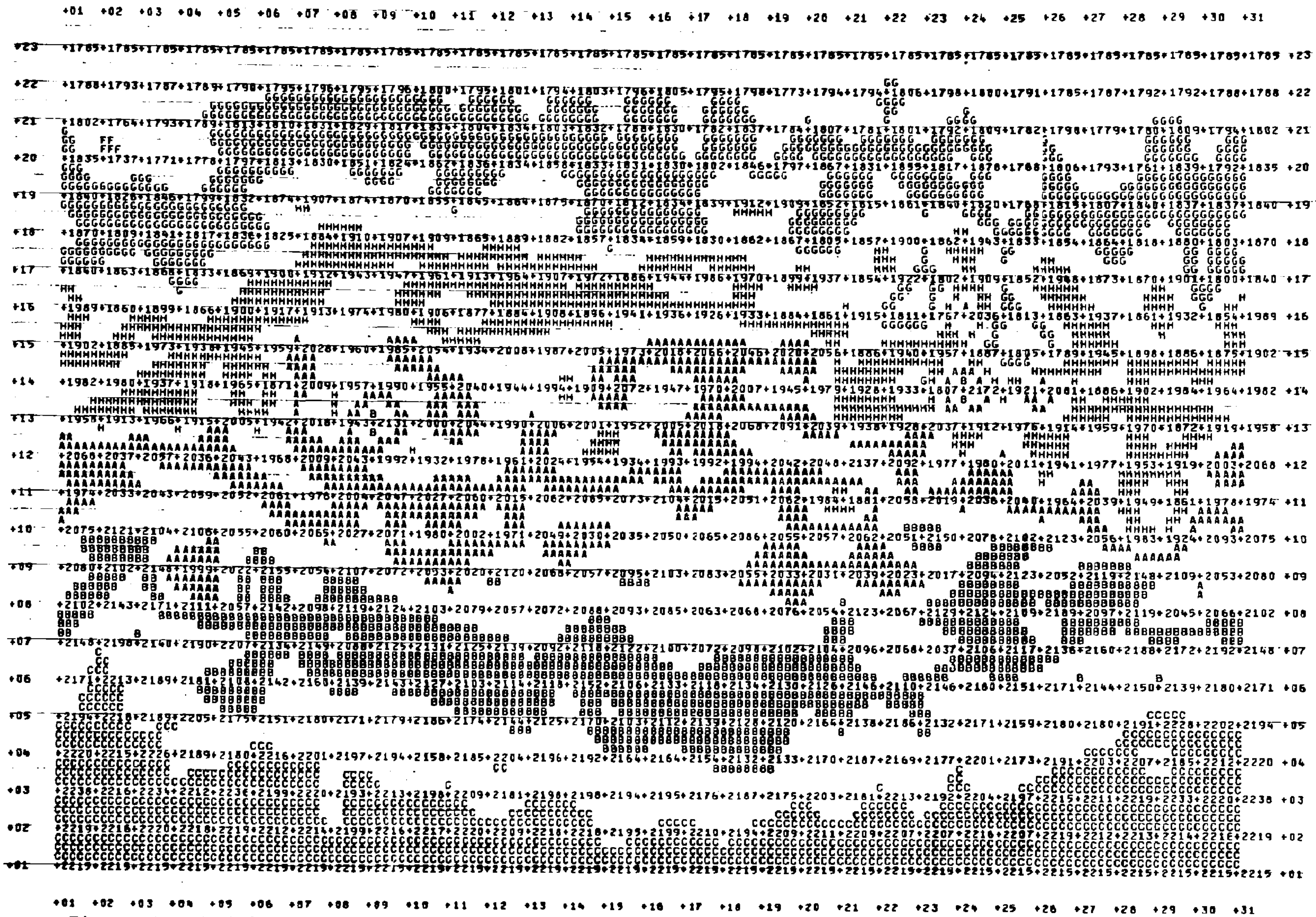

Figure 5.--Height field in meters obtained from Scheme B with constant boundary conditions and diffusive operator after approximately 13 days.

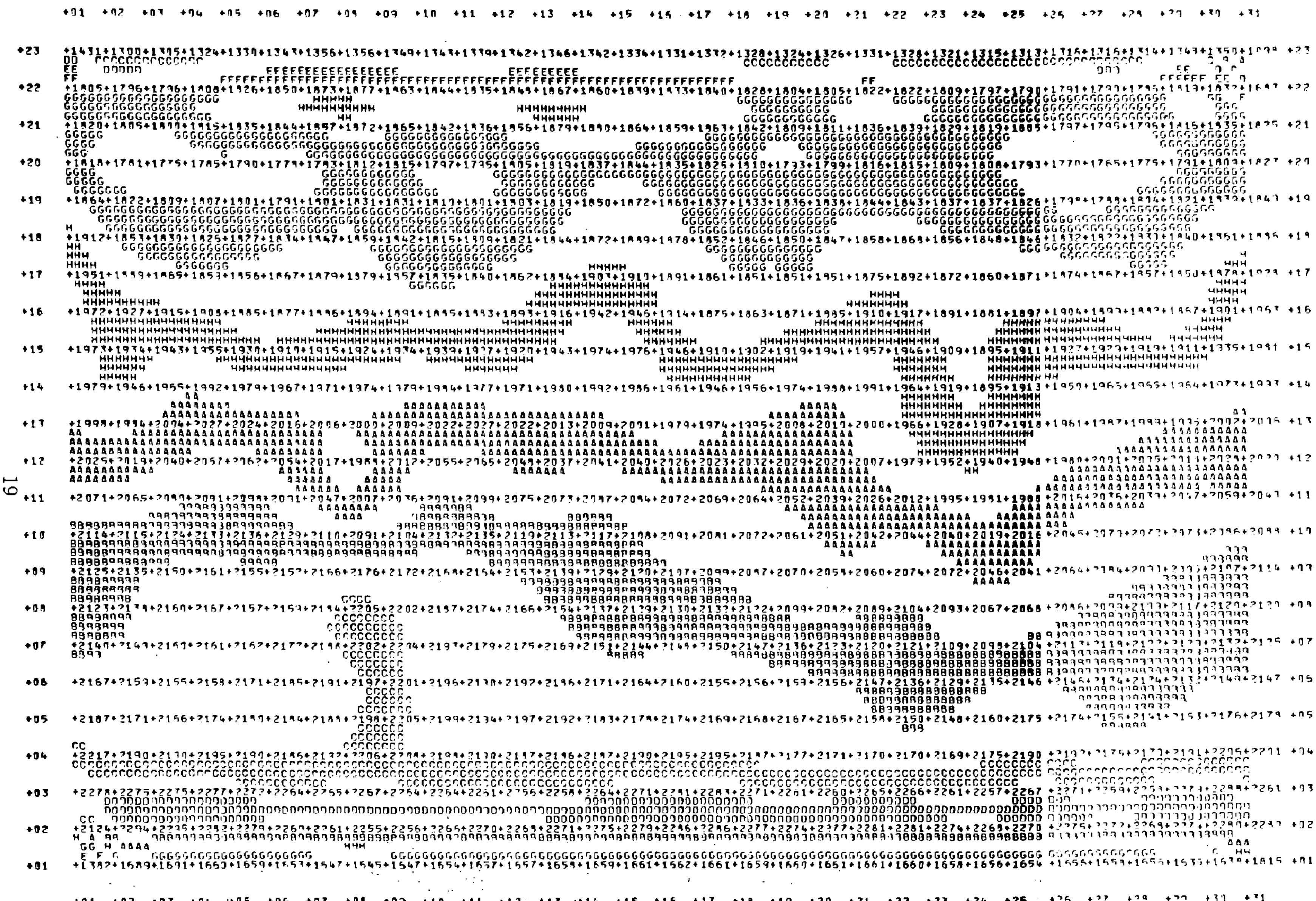

Figure 6.--Height field from Figure 4 after application of smoothing filter.

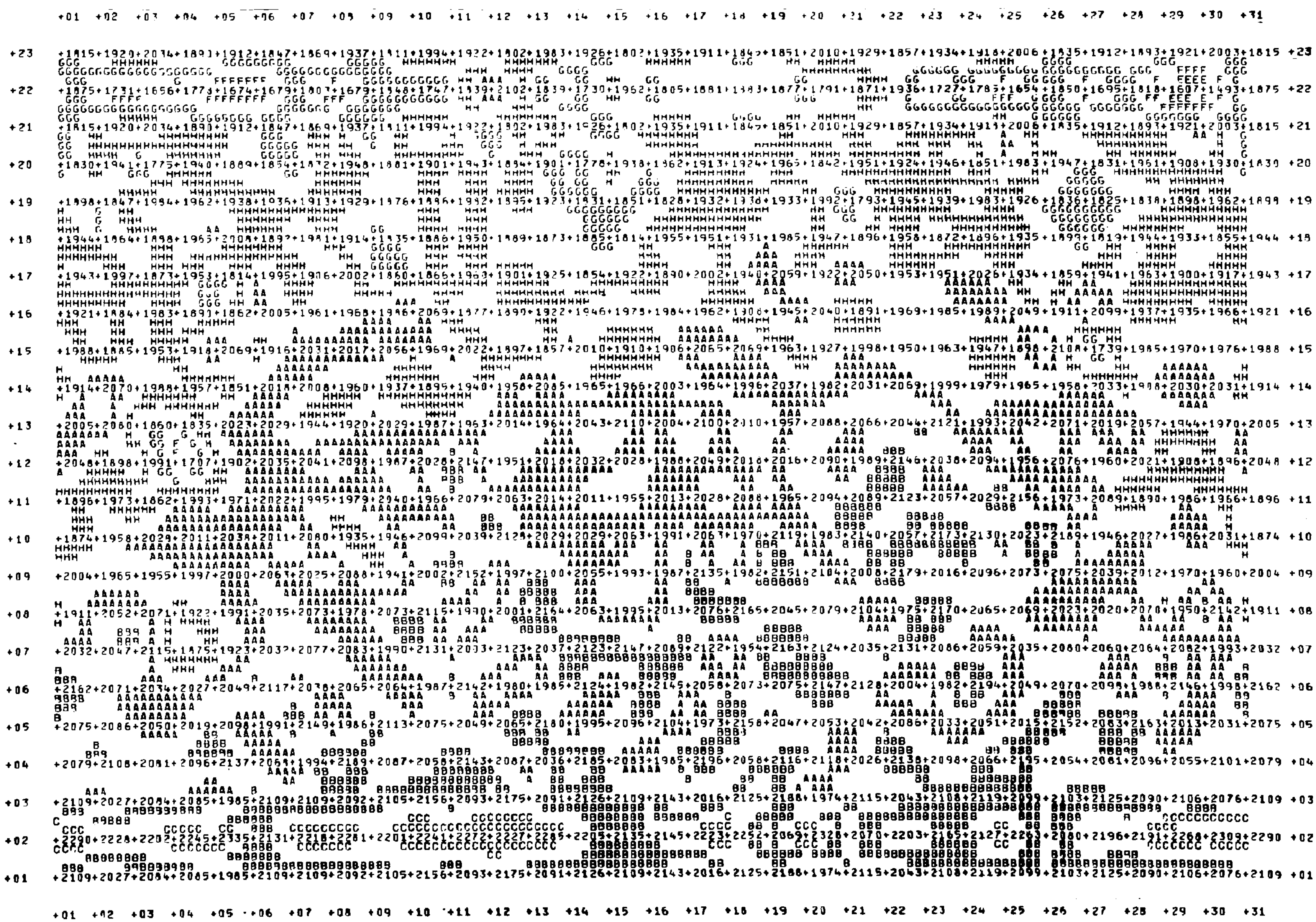

Figure 7. Height field in meters obtained from Scheme H in invariant form with averaging operator applied. Time is approximately 24 days.

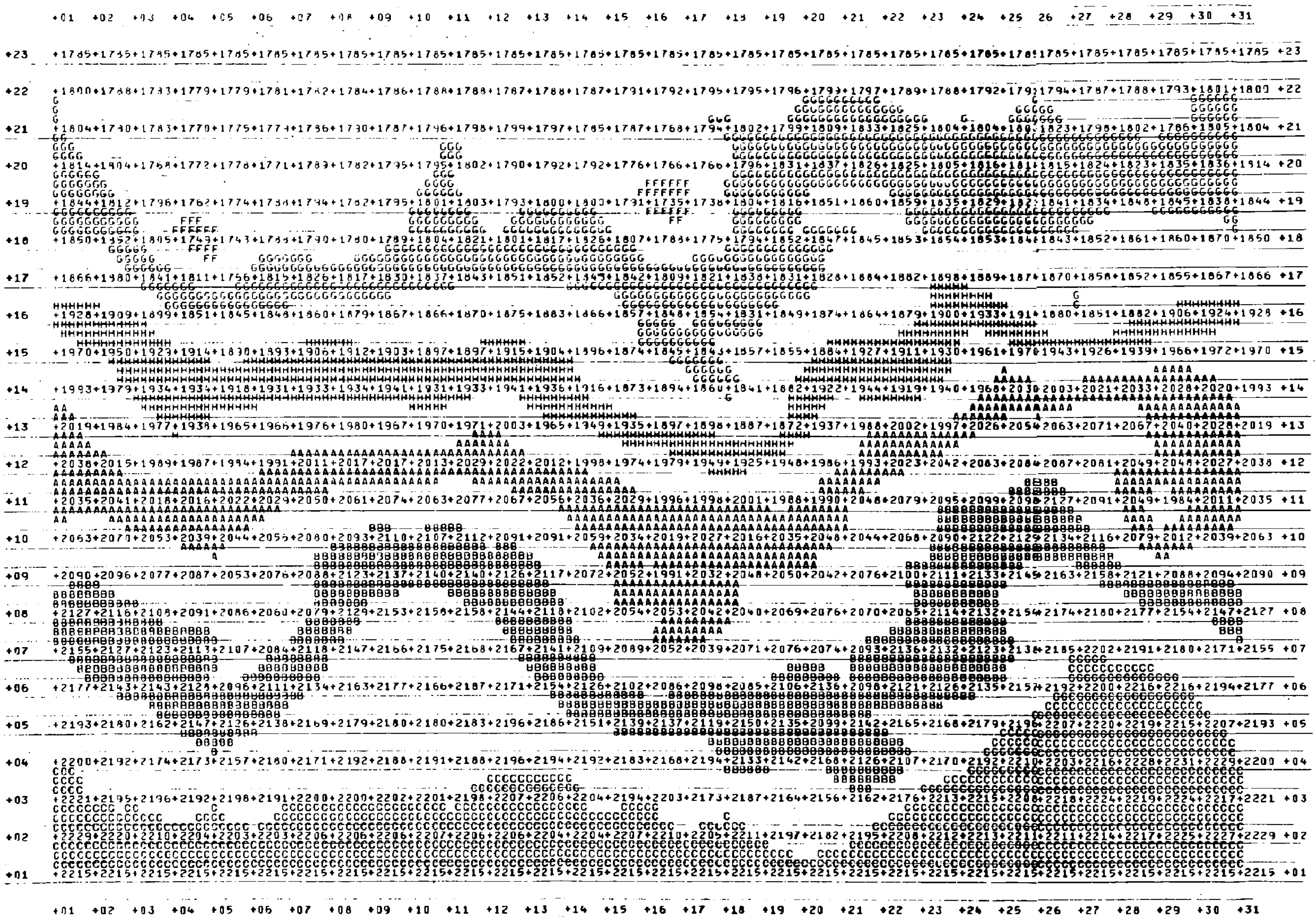

Figure 8.--Height field in meters obtained from Scheme H with averaging operator applied. Time is approximately 24 days.

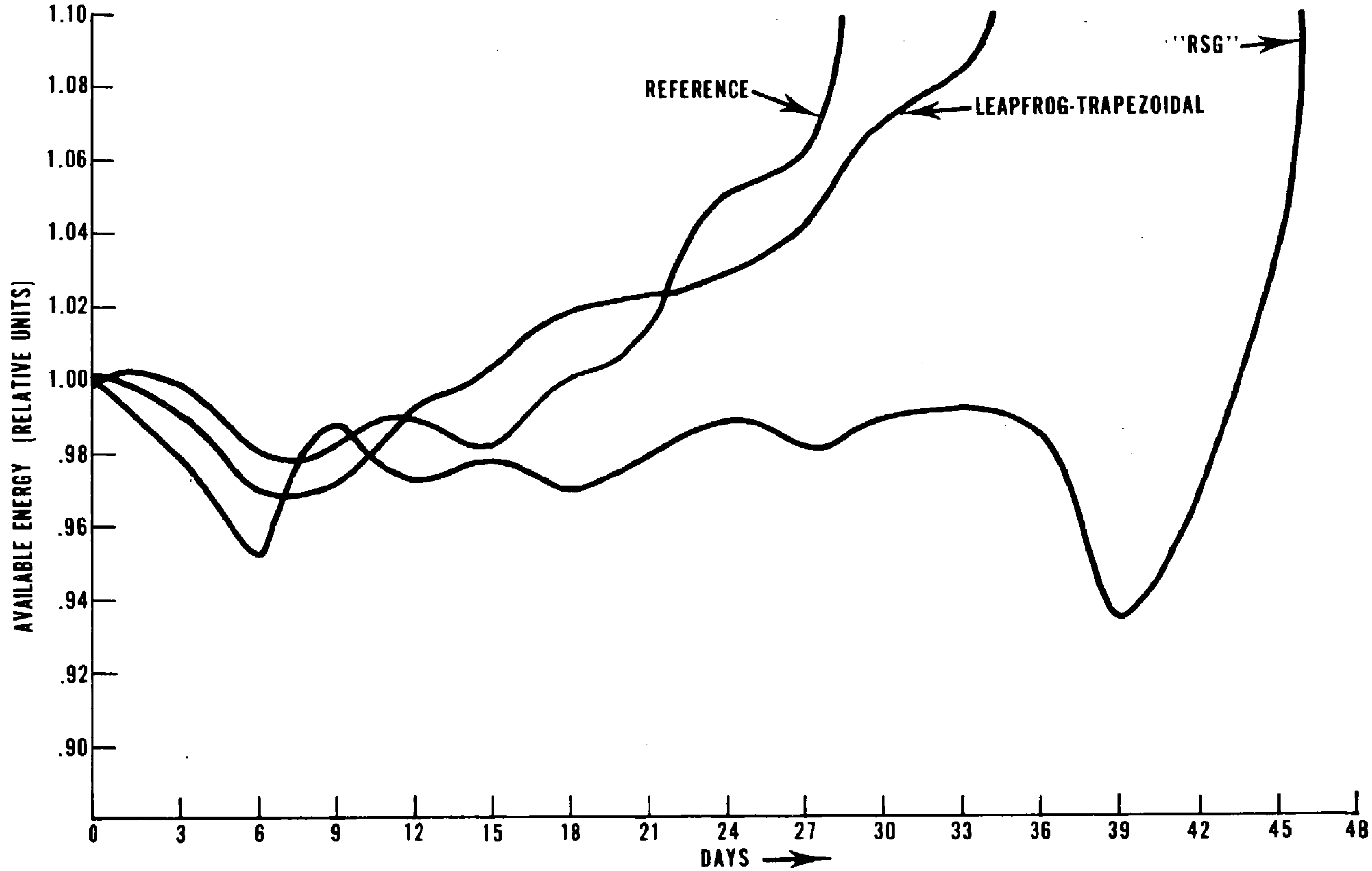

Figure 9.--The available energy as a function of time for three cases of Scheme H; the reference case, leapfrog trapezoidal integration method, "RSG" technique.