Title

A mathematical proof comparing the statistical properties between two common approaches for parameterizing sex-composition likelihoods in fishery stock assessments

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Abstract

Two primary methods for parameterizing sex-specific age and length composition likelihoods in fishery stock assessments exist, which we refer to as the 'Joint and 'Split' approaches. When using the 'Joint' approach, sex-composition data are assumed to arise from a single statistical model that describes the probability of sampling across all ages and sexes in a given year. By contrast, the 'Split' approach assumes that sex-composition data arises from several statistical models: sex-specific models that describe the probability of sampling ages within each sex, and an additional model that describes the sex-ratio information from composition data. In this mathematical proof, we derive the statistical properties of both approaches under multinomial and Dirichlet-multinomial sampling and show that they produce equivalent model expectations. However, we illustrate that the 'Split' approach leads to smaller assumed variances when sampling follows a Dirichlet-multinomial distribution, because overdispersion acts independently within each sex rather than jointly across sexes. Given that both approaches yield equivalent model expectations, we generally recommend using the 'Joint' approach for parameterizing sex-composition likelihoods. The 'Joint' approach is simpler to implement, aligns with most fisheries sampling designs, and is able to jointly account for overdispersion and sampling correlations across sexes. However, we acknowledge that in some cases, the 'Split' approach may be more appropriate.

Keywords: sex-structure, compositional likelihoods, age composition, length composition, stock assessment models

Introduction

Sex-specific composition data (i.e., age and length) are commonly used to estimate sex-specific processes (e.g., natural mortality, selectivity) within fishery stock assessments (Maunder and Wong, 2011). Two primary methods for parameterizing sex-specific composition likelihoods exist. In the first approach, which we term 'Joint', compositions sum to 1 across all ages and sexes within a given year, and a single likelihood function is used to describe these data. In the second approach, which we term 'Split', composition data are divided into datasets that describe the observed age-or size-structure of each sex, along with a dataset that represents the sex-ratio of these data. Thus, in the 'Split' approach, sex-specific likelihood functions are parameterized to describe composition data, along with an additional likelihood component to describe sex-ratios (Francis, 2014). In our experience, sex-structured stock assessments for federal fisheries in the Alaska region are primarily parameterized utilizing the 'Joint' approach (> 90%; e.g., Shotwell et al., 2021; Spies et al., 2019). Additionally, several assessment applications exist outside of the Alaska region that employ the 'Split' approach (e.g., Rudd et al., 2021; Wang et al., 2007, 2005). In some instances, a variant of the 'Split' approach has also been utilized, where sex-specific likelihood functions are used for composition data, but does not include a likelihood component to describe sex-ratios (e.g., Cope et al., 2023; Goethel et al., 2023). Nonetheless, the general statistical properties of the 'Joint' and 'Split' approaches have yet to be mathematically described in the peer-reviewed literature (to our knowledge) and understanding these properties can provide insights into the benefits and drawbacks of each approach.

In the following sections, we describe the probability mass function (PMF), expectation, and variance of the 'Joint' approach using the multinomial and Dirichlet-multinomial distributions as examples. We then proceed to derive these same quantities for the 'Split' approach, which involves a hierarchical (compound) process initially conditioned on a binomial distribution, followed by a multinomial distribution or a Dirichlet-multinomial distribution. Finally, we compare the similarities and differences between the two approaches, both analytically and numerically.

Probability Mass Function, Expectation, and Variance of the 'Joint' Approach (Multinomial)

For a given year, sampling composition data following the 'Joint' approach are determined by probabilities across age or length (here we use the subscript *a* to denote the category for age, but note that *l* could be used interchangeably for length) categories and sex categories (*s,* where we derive the following for two sexes: female, F, and male, *M*), resulting in a vector $\boldsymbol{\pi}^{Joint}$:

$$\sum_{A}\sum_{S}\boldsymbol{\pi}^{Joint}=1 \qquad Eq.1$$

The vector $\boldsymbol{\pi}^{Joint}$ then governs the probability of observing a given age and sex category. In the subsequent sections, we use subscripts $\{(1,F),(2,F)\ldots(a,M)\}$ to abbreviate $\{(1,F),(2,F)\ldots(a,F),(1,M),(2,M)\ldots(a,M)\}$ for conciseness. The corresponding PMF is then defined as (ignoring normalization constant for brevity):

$$P\left(\boldsymbol{X}^{Joint}=\boldsymbol{x}\right)\propto\left(\pi_{1,F}^{Joint}\right)^{x_{1,F}}\left(\pi_{2,F}^{Joint}\right)^{x_{2,F}}\ldots\left(\pi_{a,M}^{Joint}\right)^{x_{a,M}} \qquad Eq.2$$

where $\boldsymbol{X}^{Joint}$ represents a random variable arising from a multinomial sampling process, $\boldsymbol{x}$ are the observed counts for age *a* and sex *s*, and n represents the total sample size (in the sampling context would represent the nominal sample size; in a stock assessment model context, this would represent the input sample size; see Hulson and Williams (2024) for standardized definitions of these terms). Therefore, $\boldsymbol{X}^{Joint}$ has an expectation of:

$$\mathbb{E}\left[\boldsymbol{X}^{Joint}\right]=n\boldsymbol{\pi}^{Joint} \qquad Eq.3$$

and variance of:

$$\mathbb{V}\left[\boldsymbol{X}^{Joint}\right]=n\boldsymbol{\pi}^{Joint}\left(1-\boldsymbol{\pi}^{Joint}\right) \qquad Eq.4$$

where both equations 3 and 4 are known properties of the multinomial distribution.

Probability Mass Function, Expectation, and Variance of the 'Split' Approach (Multinomial)

In the 'Split' approach, sampling composition data represents a hierarchical process. Sex-specific sample sizes (N_s) are random variables initially determined by a binomial distribution, governed by the sex-ratio (ϕ_s) observed by the sampling unit and the total sample size (n). In a scenario where two sexes are modeled, this results in the following PMF (using females as an example and ignoring normalization constant for brevity):

$$P\left(N_f = k\right) \propto \phi_F^k (1-\phi_F)^{n-k} \qquad Eq. 5$$

$$\phi_F = \sum_A \boldsymbol{\pi}_F^{Joint}$$

where k are sample sizes for females, and $n-k$ are sample sizes for males. Note that the term $1-\phi_F = \phi_M = \sum_A \boldsymbol{\pi}_M^{Joint}$.This sampling process results in the following expectation for N_f:

$$\mathbb{E}\left[N_f\right] = n\phi_F \qquad Eq. 6$$

with a variance of:

$$\mathbb{V}\left[N_f\right] = n\phi_F(1-\phi_F) \qquad Eq. 7$$

both of which are also known properties of the binomial distribution.

Following the determination of sex-specific sample sizes, sampling sex-specific composition data using the 'Split' approach normalizes probabilities within a given sex:

$$\boldsymbol{\pi}_F^{Split} = \frac{\boldsymbol{\pi}_F^{Joint}}{\sum_A \boldsymbol{\pi}_F^{Joint}}, \qquad \boldsymbol{\pi}_M^{Split} = \frac{\boldsymbol{\pi}_M^{Joint}}{\sum_A \boldsymbol{\pi}_M^{Joint}} \qquad Eq. 8$$

$$\sum_A \boldsymbol{\pi}_F^{Split} = 1\,, \sum_A \boldsymbol{\pi}_M^{Split} = 1$$

Sex-specific composition data are then assumed to arise from two sex-specific multinomial processes, conditioned on a binomial process:

$$P\left(\boldsymbol{X}_F^{Split} = \boldsymbol{x_F}, \boldsymbol{X}_M^{Split} = \boldsymbol{x_M} \middle| N_f = k\right) \propto \phi_F^k (1-\phi_F)^{n-k} \left(\pi_{1,F}^{Split}\right)^{x_{1,F}} \left(\pi_{2,F}^{Split}\right)^{x_{2,F}} \ldots \left(\pi_{a,M}^{Split}\right)^{x_{a,M}} Eq. 9$$

Given the hierarchical sampling process, we next invoke the law of iterated expectations to derive the expectation and variance arising from the 'Split' approach for $\boldsymbol{X}_F^{Split}$ and $\boldsymbol{X}_M^{Split}$. The iterated expectation for compositions for females is as follows:

$$\begin{aligned} \mathbb{E}\left[\boldsymbol{X}_F^{Split}\right] &= \mathbb{E}\left[\mathbb{E}\left(\boldsymbol{X}_F^{Split} \middle| N_F\right)\right] \qquad Eq. 10 \\ &= \mathbb{E}\left[N_F \boldsymbol{\pi}_F^{Split}\right] \\ &= \mathbb{E}[N_F]\boldsymbol{\pi}_F^{Split} \\ &= n\phi_F \boldsymbol{\pi}_F^{Split} \end{aligned}$$

Similarly, the expectation for compositions from males simply replaces the subscript *F* with *M* in equation 10. We next derive the variances in a similar manner, where the variance for compositions for females is:

$$
\begin{aligned}
\mathbb{V}\left[\boldsymbol{X}_F^{Split}\right] &= \mathbb{E}\left[\mathbb{V}\left(\boldsymbol{X}_F^{Split}\middle|N_F\right)\right] + \mathbb{V}\left[\mathbb{E}\left(\boldsymbol{X}_F^{Split}\middle|N_F\right)\right] \qquad Eq.11\\
&= \mathbb{E}\left[N_F\boldsymbol{\pi}_F^{Split}\left(1-\boldsymbol{\pi}_F^{Split}\right)\right] + \mathbb{V}\left[\boldsymbol{\pi}_F^{Split}N_F\right]\\
&= \boldsymbol{\pi}_F^{Split}\left(1-\boldsymbol{\pi}_F^{Split}\right)\mathbb{E}[N_F] + \left(\boldsymbol{\pi}_F^{Split}\right)^2\mathbb{V}[N_F]\\
&= \boldsymbol{\pi}_F^{Split}\left(1-\boldsymbol{\pi}_F^{Split}\right)n\phi_F + \left(\boldsymbol{\pi}_F^{Split}\right)^2 n\phi_F(1-\phi_F)
\end{aligned}
$$

Likewise, the variance for compositions for males simply replaces the subscript F with M in equation 11.

Comparison of Probability Mass Function, Expectation, and Variance (Multinomial)

Comparing the PMFs in equation 2 and 9, we find that the PMF of the 'Split' approach can be simplified to align with the PMF of the 'Joint' approach. Simplifying equation 9, substituting $\sum_A \boldsymbol{\pi}_F^{Joint}$ for ϕ_F, $\sum_A \boldsymbol{\pi}_M^{Joint}$ for $1-\phi_F$, and noting that $\sum_a \boldsymbol{x_F} = k$ and $\sum_a \boldsymbol{x_M} = n-k$ results in the following expression:

$$
\begin{aligned}
&P\left(\boldsymbol{X}_F^{Split} = \boldsymbol{x_F}, \boldsymbol{X}_M^{Split} = \boldsymbol{x_M}\middle|N_f = k\right) \qquad Eq.12\\
&\propto \phi_F^k(1-\phi_F)^{n-k}\left(\pi_{1,F}^{Split}\right)^{x_{1,F}}\left(\pi_{2,F}^{Split}\right)^{x_{2,F}} \dots \left(\pi_{a,M}^{Split}\right)^{x_{a,M}}\\
&\propto \left(\sum_A \boldsymbol{\pi}_F^{Joint}\right)^k \left(\sum_A \boldsymbol{\pi}_M^{Joint}\right)^{n-k} \left(\frac{\pi_{1,F}^{Joint}}{\sum_A \boldsymbol{\pi}_F^{Joint}}\right)^{x_{1,F}} \left(\frac{\pi_{2,F}^{Joint}}{\sum_A \boldsymbol{\pi}_F^{Joint}}\right)^{x_{2,F}} \dots \left(\frac{\pi_{a,M}^{Joint}}{\sum_A \boldsymbol{\pi}_M^{Joint}}\right)^{x_{a,M}}\\
&\propto \left(\sum_A \boldsymbol{\pi}_F^{Joint}\right)^k \left(\sum_A \boldsymbol{\pi}_M^{Joint}\right)^{n-k} \frac{\left(\pi_{1,F}^{Joint}\right)^{x_{1,F}}\left(\pi_{2,F}^{Joint}\right)^{x_{2,F}} \dots \left(\pi_{a,M}^{Joint}\right)^{x_{a,M}}}{\left(\sum_A \boldsymbol{\pi}_F^{Joint}\right)^k\left(\sum_A \boldsymbol{\pi}_M^{Joint}\right)^{n-k}}\\
&\propto \left(\pi_{1,F}^{Joint}\right)^{x_{1,F}}\left(\pi_{2,F}^{Joint}\right)^{x_{2,F}} \dots \left(\pi_{a,M}^{Joint}\right)^{x_{a,M}}\\
&\propto P\left(\boldsymbol{X}^{Joint} = \boldsymbol{x}\right)
\end{aligned}
$$

indicating that the PMF from both approaches are proportional. Given this, we can conclude that these approaches are equivalent under the assumption of multinomial sampling. Consequently, their expectations and variances are also equivalent, and we do not elaborate further on their equivalency in these aspects below. To further confirm these derivations, we conducted 100,000 simulations with $n = 100$ and $A = 15$. In these simulations, $\boldsymbol{\pi}^{Joint}$ and $\boldsymbol{\pi}_s^{Split}$ were randomly generated from $Uniform\sim(0,1)$ and normalized according to equation 1 and equations 5 and 8, respectively, while random variables $\boldsymbol{X}^{Joint}$ and $\boldsymbol{X}_s^{Split}$ were both drawn from multinomial distributions. As expected, the two approaches resulted in similar values for expected values and variances (approximate given the simulated nature; Fig. 1A and 1B). Additionally, a comparison of the empirical cumulative distribution function of multinomial samples generated indicated that both approaches resulted in the same distributional form (Fig. 1C) (refer to https://github.com/chengmatt/SexCompProof for a code example of simulations).

Probability Mass Function, Expectation, and Variance of the 'Joint' Approach (Dirichlet-multinomial)

Following the notation described in earlier sections, we next describe the PMF, expectation, and variance under Dirichlet-multinomial sampling when utilizing the 'Joint' approach, which is governed by $n,\ \theta$, and $\boldsymbol{\pi}^{Joint}$ (linear parameterization; Thorson et al., 2017). Here, $n\theta = \alpha_0^{Joint}$, which represents a parameter that controls the degree of overdispersion. The resulting PMF under the 'Joint' approach is given by the following (ignoring normalization constants):

$$P\left(\boldsymbol{X}^{Joint} = \boldsymbol{x}\right) \qquad Eq. 13$$

$$\propto \frac{\Gamma\left(\alpha_0^{Joint}\right)}{\Gamma\left(n + \alpha_0^{Joint}\right)} \frac{\Gamma\left(nx_{1,F} + \alpha_0^{Joint}\pi_{1,F}^{Joint}\right)\Gamma\left(nx_{2,F} + \alpha_0^{Joint}\pi_{2,F}^{Joint}\right) \dots \Gamma\left(nx_{a,M} + \alpha_0^{Joint}\pi_{a,M}^{Joint}\right)}{\Gamma\left(\alpha_0^{Joint}\pi_{1,F}^{Joint}\right)\Gamma\left(\alpha_0^{Joint}\pi_{2,F}^{Joint}\right) \dots \Gamma\left(\alpha_0^{Joint}\pi_{a,M}^{Joint}\right)}$$

where $\boldsymbol{X}^{Joint}$ here is a random variable arising from a Dirichlet-multinomial process, and $\boldsymbol{x}$ represent the associated observations. Random variable $\boldsymbol{X}^{Joint}$ then has an expectation of:

$$\mathbb{E}\left[\boldsymbol{X}^{Joint}\right] = n\boldsymbol{\pi}^{Joint} \qquad Eq. 14$$

and variance of:

$$\mathbb{V}\left[\boldsymbol{X}^{Joint}\right] = n\boldsymbol{\pi}^{Joint}\left(1 - \boldsymbol{\pi}^{Joint}\right)\left(\frac{n + \alpha_0^{Joint}}{1 + \alpha_0^{Joint}}\right) \qquad Eq. 15$$

where both equations 14 and 15 are properties of the Dirichlet-multinomial distribution.

Probability Mass Function, Expectation, and Variance of the 'Split' Approach (Dirichlet-multinomial)

When utilizing the 'Split' approach and assuming that composition samples arise from a Dirichlet-multinomial process, the same hierarchical process described in previous sections applies. Sex-specific sample sizes initially arise from a binomial distribution (equation 5) with expectation and variance of this binomial process following equations 6 and 7. The binomial process then results in k samples for females and $n - k$ samples for males. In the subsequent derivations, we assume that the parameter θ is consistent for both sexes to maintain comparability with previous sections. Therefore, the overdispersion for females is expressed as $k\theta = \alpha_{0F}^{Split}$ and for males as $(n-k)\theta = \alpha_{0M}^{Split}$. The probabilities of sampling composition data are then normalized within a given sex (equation 8), and composition data arise from sex-specific Dirichlet-multinomial processes with the following PMF:

$$P\left(\boldsymbol{X}_F^{Split} = \boldsymbol{x_F}, \boldsymbol{X}_M^{Split} = \boldsymbol{x_M} \middle| N_f = k\right) \qquad Eq. 16$$

$$\propto \phi_F^k(1 - \phi_F)^{n-k}$$

$$\frac{\Gamma\left(\alpha_{0F}^{Split}\right)\Gamma\left(\alpha_{0M}^{Split}\right)}{\Gamma\left(k + \alpha_{0F}^{Split}\right)\Gamma\left((n-k) + \alpha_{0M}^{Split}\right)} \frac{\Gamma\left(nx_{1,F} + \alpha_{0F}^{Split}\pi_{1,F}^{Split}\right)\Gamma\left(nx_{2,F} + \alpha_{0F}^{Split}\pi_{2,F}^{Split}\right) \dots \Gamma\left(nx_{a,M} + \alpha_{0M}^{Split}\pi_{a,M}^{Split}\right)}{\Gamma\left(\alpha_{0F}^{Split}\pi_{1,F}^{Split}\right)\Gamma\left(\alpha_{0F}^{Split}\pi_{2,F}^{Split}\right) \dots \Gamma\left(\alpha_{0M}^{Split}\pi_{a,M}^{Split}\right)}$$

Invoking the law of iterated expectations, the expected values for female composition samples using the 'Split' approach under the assumption of Dirichlet-multinomial sampling is:

$$\begin{aligned}\mathbb{E}\left[\boldsymbol{X}_F^{Split}\right] &= \mathbb{E}\left[\mathbb{E}\left(\boldsymbol{X}_F^{Split}\middle|N_F\right)\right] \\ &= \mathbb{E}\left[N_f\boldsymbol{\pi}_F^{Split}\right] \\ &= \mathbb{E}\left[N_f\right]\boldsymbol{\pi}_F^{Split} \\ &= n\phi_F\boldsymbol{\pi}_F^{Split}\end{aligned} \qquad Eq.17$$

Similarly, the expectation for composition samples from males replaces the subscript *F* with *M* in equation 17. Deriving the variances in a similar fashion results in the following expression:

$$\begin{aligned}\mathbb{V}\left[\boldsymbol{X}_F^{Split}\right] &= \mathbb{E}\left[\mathbb{V}\left(\boldsymbol{X}_F^{Split}\middle|N_F\right)\right] + \mathbb{V}\left[\mathbb{E}\left(\boldsymbol{X}_F^{Split}\middle|N_F\right)\right] \\ &= \mathbb{E}\left[N_F\boldsymbol{\pi}_F^{Split}\left(1-\boldsymbol{\pi}_F^{Split}\right)\left(\frac{N_F+\alpha_{0F}^{Split}}{1+\alpha_{0F}^{Split}}\right)\right] + \mathbb{V}\left[\boldsymbol{\pi}_F^{Split}N_F\right] \\ &= \boldsymbol{\pi}_F^{Split}\left(1-\boldsymbol{\pi}_F^{Split}\right)\mathbb{E}\left[N_F\left(\frac{N_F+\alpha_{0F}^{Split}}{1+\alpha_{0F}^{Split}}\right)\right] + \left(\boldsymbol{\pi}_F^{Split}\right)^2\mathbb{V}[N_F] \\ &= \boldsymbol{\pi}_F^{Split}\left(1-\boldsymbol{\pi}_F^{Split}\right)\left(\frac{1}{1+\alpha_{0F}^{Split}}\right)\mathbb{E}[N_F^2] + \alpha_{0F}^{Split}\mathbb{E}[N_F] + \left(\boldsymbol{\pi}_F^{Split}\right)^2\mathbb{V}[N_F] \\ &= \frac{\boldsymbol{\pi}_F^{Split}\left(1-\boldsymbol{\pi}_F^{Split}\right)n\phi_F(1-\phi_F)}{1+\alpha_{0F}^{Split}} + (n\phi_F)^2 + \alpha_{0F}^{Split}n\phi_F + \left(\boldsymbol{\pi}_F^{Split}\right)^2 n\phi_F(1-\phi_F)\end{aligned} \qquad Eq.18$$

In the same way, the variance for male compositions is determined by replacing the subscript *F* with *M* in equation 18.

Comparison of Probability Mass Function, Expectation, and Variance (Dirichlet-multinomial)

Comparing equations 13 and 16 (PMFs of Dirichlet-multinomial under the 'Joint' and 'Split' approaches), we find that equation 16 cannot be reduced into the form of equation 13 given the presence of sex-specific overdispersion parameters (i.e., α_{0F}^{Split} and α_{0M}^{Split}). Despite that, we find that the derived expected values are consistent between the two approaches, under Dirichlet-multinomial sampling. To show this, we substitute $\sum_A \boldsymbol{\pi}_F^{Joint}$ for ϕ_F and $\frac{\boldsymbol{\pi}_F^{Joint}}{\sum_A \boldsymbol{\pi}_F^{Joint}}$ for $\boldsymbol{\pi}_F^{Split}$ into equation 17:

$$\begin{aligned}\mathbb{E}\left[\boldsymbol{X}_F^{Split}\right] &= n\phi_F\boldsymbol{\pi}_F^{Split} \\ &= n\sum\nolimits_A \boldsymbol{\pi}_F^{Joint}\frac{\boldsymbol{\pi}_F^{Joint}}{\sum_A \boldsymbol{\pi}_F^{Joint}} \\ &= n\boldsymbol{\pi}_F^{Joint} \\ &= \mathbb{E}\left[\boldsymbol{X}_F^{Joint}\right]\end{aligned} \qquad Eq.19$$

While the expected values are identical, the variances between the two approaches differ (cf. equations 15 and 18). In particular, sampling via the 'Split' approach is expected to result in lower sampling variability because the overdispersion propagates independently within each sex, when sampling follows a Dirichlet-multinomial process. Note that the variance expression derived in equation 18 cannot be simplified to align with equation 15. To further verify the derivations for both approaches under a Dirichlet-multinomial sampling process, 10,000 simulations were conducted with $n = 100$, $A = 15$, $\theta = 1$. $\boldsymbol{\pi}^{Joint}$ and $\boldsymbol{\pi}_s^{Split}$ were generated from $Uniform \sim (0,1)$, normalized according to equation 1 and equations 5 and 8, respectively, and $\boldsymbol{X}^{Joint}$ and $\boldsymbol{X}_s^{Split}$ were drawn from Dirichlet-multinomial distributions. Consistent with the derivations described above, these simulations indicated that while both approaches yielded similar expected values, the 'Split' approach exhibited lower variability (Fig. 1D and 1E). Further comparison of the empirical cumulative distribution function of Dirichlet-multinomial samples revealed that the two approaches produced different distributional forms (Fig. 1F).

Discussion

We have mathematically demonstrated that the 'Joint' and 'Split' parameterizations of sex-composition likelihoods yield equivalent model expectations, irrespective of whether compositions arise from a multinomial or Dirichlet-multinomial sampling process. However, under Dirichlet-multinomial sampling, the 'Split' approach produces smaller variances. This occurs because the parameter θ here is assumed to be consistent between sexes, where overdispersion acts independently within each sex and across fewer bins. Conversely, the 'Joint' approach considers overdispersion jointly across sexes and a larger number of bins. Therefore, in extreme scenarios of clustered sampling, the 'Split' approach can produce realized samples of a limited number of ages, but assumes they are distributed more consistently across sexes, whereas the 'Joint' approach may generate samples that favor a particular sex. In theory, estimating sex-specific parameters for θ when utilizing the 'Split' approach, as opposed to assuming a consistent θ among sexes, should enable overdispersion to more closely resemble that of the 'Joint' approach. However, this process still does not yield the same distribution, given the introduction of new parameters, and is also a less parsimonious parameterization. Nonetheless, the conclusion that both 'Joint' and 'Split' approaches produce identical model expectations should hold regardless of the multivariate likelihood function used. Additionally, the reduced sampling variability observed when using the 'Split' approach is also expected to apply across various multivariate likelihoods capable of accommodating over-dispersed sampling processes (when over-dispersion parameters are not sex-specific), given that overdispersion acts independently within each sex and across fewer bins.

Findings from this proof should also generally hold true regardless of the number of sexes represented in compositional data. For instance, in the New Zealand Rock Lobster (*Jasus edwardsii*) stock assessment, where sexes are divided into three categories: immature females, mature females, and males (Rudd et al., 2021), the 'Split' approach is employed to analyze sex-composition data. In this particular case, rather than using a binomial likelihood, the stock assessment first utilizes a multinomial likelihood to first describe the sex ratio among these three categories, followed by three additional multinomial likelihoods to model composition data. Given that the multinomial distribution is a generalization of the binomial, this approach is equivalent to the 'Joint' parameterization under the assumption of multinomial sampling (i.e., equations 10 and 11 remains consistent for a given bin).

Overdispersion when sampling composition data is common in fisheries, because individuals are often clustered in space and time, and likely acts across sizes, ages, and sexes (Pennington and Volstad, 1994), typically reflecting the 'Joint' parameterization. Within the context of integrated stock assessment models, multiple data sources are combined into a single analysis, where model fits to particular data sources are influenced by the relative weighting applied or estimated (Maunder and Piner, 2015; Maunder and Punt, 2013). While both approaches outlined in this study lead to equivalent expectations, the method by which overdispersion is applied to sexes (i.e., 'Joint' or 'Split') in likelihoods that account for overdispersion may influence the relative weighting of data sources, potentially influencing model fits and resulting parameter estimates. In general, differences in variances between the two approaches were relatively minor (although these differences depend on the number of bins modeled and the specific values of θ) when compared to other sources of uncertainty in the stock assessment process, including survey abundance indices, natural mortality, and recruitment. Consequently, addressing disparities between these two approaches in an integrated assessment model may not be the highest priority. Nonetheless, additional research within the framework of an integrated model is necessary to ascertain the specific effects of this type of model misspecification. Considering that both approaches yield equivalent model expectations, we generally recommend the 'Joint' approach for parameterizing sex-composition likelihoods, due to its simplicity, alignment with fisheries sampling designs, along with its ability to account for overdispersion and correlated processes jointly across sexes. However, the 'Split' approach may also occur in certain fishery sampling schemes. For example, in cases where sexes can be visually distinguished (i.e., crustaceans), animals might first be separated by sex and then randomly sampled within each sex. Thus, practitioners should also carefully consider whether composition data arises from a 'Joint' or 'Split' approach.

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References

Cope, J.M., Hillier, L.K., Niles, C.B., Tsou, T.-S., Hinton, K.E., Caltabellotta, F.P., 2023. Status of Black Rockfish (Sebastes melanops) in 2023 in state and federal waters off Washington state.

Francis, R.I.C.C., 2014. Replacing the multinomial in stock assessment models: A first step. Fisheries Research 151, 70–84. https://doi.org/10.1016/j.fishres.2013.12.015

Goethel, D.R., Cheng, M.L.H., Echave, K.B., Marsh, C., Rodgveller, C.J., Shotwell, K., Siwicke, K., 2023. 3. Assessment of the Sablefish Stock in Alaska. North Pacific Fishery Management Council, Anchorage, AK.

Hulson, P.-J.F., Williams, B.C., 2024. Inclusion of ageing error and growth variability using a bootstrap estimation of age composition and conditional age-at-length input sample size for fisheries stock assessment models. Fisheries Research 270, 106894. https://doi.org/10.1016/j.fishres.2023.106894

Maunder, M.N., Piner, K.R., 2015. Contemporary fisheries stock assessment: many issues still remain. ICES Journal of Marine Science 72, 7–18. https://doi.org/10.1093/icesjms/fsu015

Maunder, M.N., Punt, A.E., 2013. A review of integrated analysis in fisheries stock assessment. Fisheries Research 142, 61–74. https://doi.org/10.1016/j.fishres.2012.07.025

Maunder, M.N., Wong, R.A., 2011. Approaches for estimating natural mortality: Application to summer flounder (Paralichthys dentatus) in the U.S. mid-Atlantic. Fisheries Research 111, 92–99. https://doi.org/10.1016/j.fishres.2011.06.016

Pennington, M., Volstad, J.H., 1994. Assessing the Effect of Intra-Haul Correlation and Variable Density on Estimates of Population Characteristics from Marine Surveys. Biometrics 50, 725. https://doi.org/10.2307/2532786

Rudd, M.B., Pons, M., Roberts, J., Webber, D.N., Starr, P.J., 2021. The 2020 stock assessment of red rock lobsters (Jasus edwardsii) in CRA 4. Fisheries New Zealand, Tini a Tangaroa, Wellington.

Shotwell, S.K., Spies, I., Ianelli, J.N., Aydin, K., Hanselman, D.H., Palsson, W., Siwicke, K., Sullivan, J., Yasumiishi, E., 2021. 7. Assessment of the Arrowtooth flounder stock in the Gulf of Alaska.

Spies, I., Wilderbuer, T.K., Nichol, D.G., Ianelli, J., 2019. Assessment of the Yellowfin Sole Stock in the Bering Sea and Aleutian Islands.

Thorson, J.T., Johnson, K.F., Methot, R.D., Taylor, I.G., 2017. Model-based estimates of effective sample size in stock assessment models using the Dirichlet-multinomial distribution. Fisheries Research 192, 84–93. https://doi.org/10.1016/j.fishres.2016.06.005

Wang, S.-P., Sun, C.-L., Punt, A.E., Yeh, S.-Z., 2007. Application of the sex-specific age-structured assessment method for swordfish, Xiphias gladius, in the North Pacific Ocean. Fisheries Research 84, 282–300. https://doi.org/10.1016/j.fishres.2006.11.027

Wang, S.-P., Sun, C.-L., Punt, A.E., Yeh, S.-Z., 2005. Evaluation of a sex-specific age-structured assessment method for the swordfish, Xiphias gladius, in the North Pacific Ocean. Fisheries Research 73, 79–97. https://doi.org/10.1016/j.fishres.2005.01.001

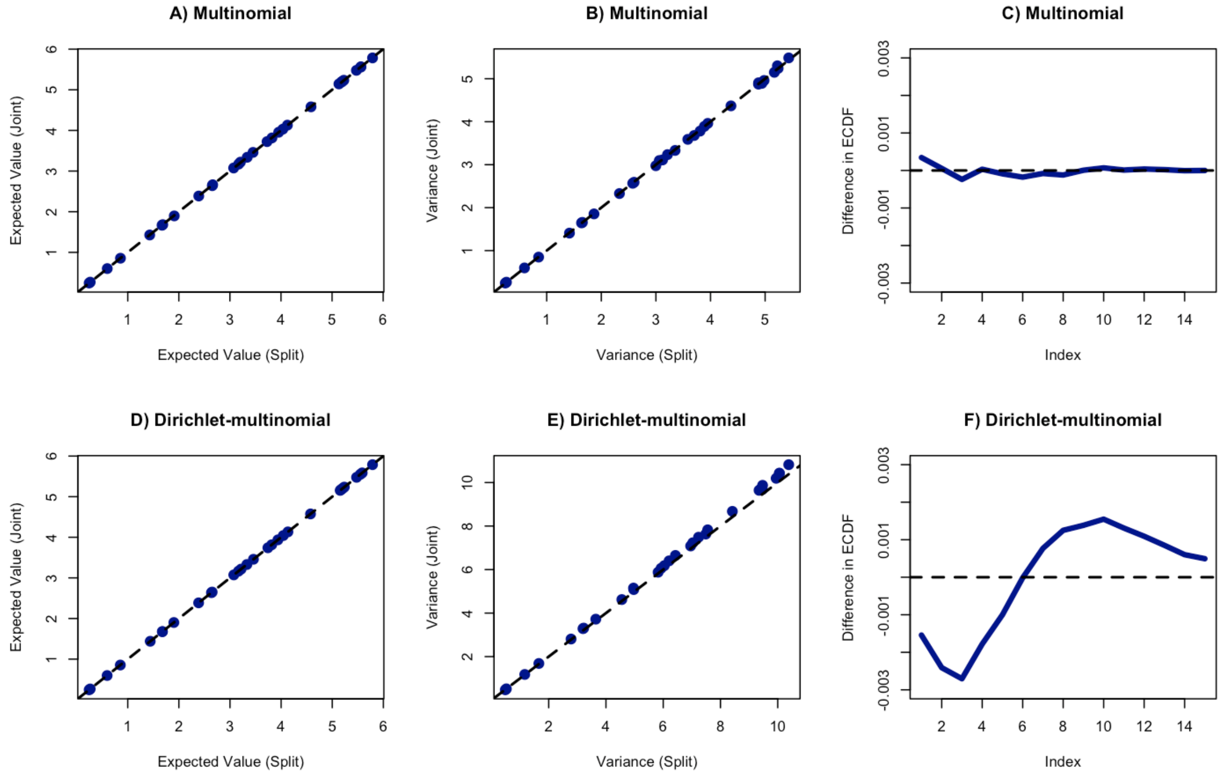

Figure 1. Comparison of expected values, variances, and empirical cumulative distribution functions of simulated composition data generated under the ‘Joint’ or ‘Split’ approaches, assuming either multinomial (top row) or Dirichlet-multinomial (bottom row) sampling. Panels in the first and second columns compare expected values and variances between the ‘Joint’ and ‘Split’ approaches, with the dashed sloped line representing a 1:1 relationship. Panels in the third column depict average differences in empirical cumulative distribution functions (‘Split’ minus ‘Joint’), with the dashed horizontal line indicating no difference.