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Final Report

THEORETICAL STUDIES OF MAGNETIC FIELD LINE RECONNECTION AND ITS APPLICATION TO SOLAR FLARES AND MAGNETOSPHERIC DYNAMICS

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I. WORK ACCOMPLISHED DURING THE CONTRACT PERIOD

The research supported by this contract has dealt essentially with the problem of magnetic field line reconnections. In particular, we addressed ourselves to the fundamental question of what determines the physical conditions and processes of the actual field line merging. Mathematically, the regions where this occurs can be described quite generally by the similarity solutions of the magnetohydrodynamic equations obtained by Yeh and Axford (1970), which permit us to discuss, quantitatively, magnetic merging in a collision-dominated fluid. The simplest and most tractable member of this family of solutions is the piecewise-uniform flow as described by Sonnerup (1970). In particular, these solutions can be used to represent convective hydromagnetic flows which are typically found in the region surrounding neutral points. Since in the vicinity of such neutral points magnetic diffusion must take place to accommodate the required flux transfer, we are faced immediately in this situation with the difficult and fundamental problem of how the boundary is determined between such a diffusive region and a convective region.

To study this question further, and to obtain specific solutions, we used the model of a stagnant resistive fluid in the middle of a piecewise-uniform flow of a dissipationless fluid. With this model we have been able to show that the matching of the physical parameters along such a boundary can be achieved in a self-consistent manner. It is clearly indicated by the solutions thus obtained that the location of the boundary,and therefore the shape and size of the diffusive region, are completely determined by the incident flow. This implies further that the physical conditions in the region of magnetic field line merging are governed by those distant boundary conditions which determine the fluid flows in the first place. These results are reported in detail in the paper:" A composite solution of field line reconnection ", which has been accepted for publication in the Journal of Plasma Physics.

In another phase of this research, we have studied an application of this process of field line reconnection to magnetospheric dynamics. The convective motion inside the magnetosphere is commonly thought of as being driven by the field line reconnection on the dayside of the magnetopause. The flux transfer in this process of reconnection between the geomagnetic field and the incident interplanetary field induces an electric field, and thus a voltage, across the dayside magnetopause. It is essential that one is able to calculate the magnitude of this induced voltage in order to determine its dependence on the magnitude and direction of the incident interplanetary field. For this purpose, we use a model in which the field line reconnection takes place along the separator of field line connectivity in the magnetic topology. The separator is a field line which, in general, separates different types of magnetic field lines (i.e. open, closed, etc.). In the magnetospheric case, the separator results from the interpermeation of the dipolar geomagnetic field with the uniform interplanetary field. With this model we are able to calculate the electric field induced at various segments of the separator. The integration of this induced electric field along the entire separator thus yields the total voltage between the dawn and dusk points. We find that this voltage increases with the incident field strength with a two-thirds power law, and diminishes as the incident field direction changes from southward to northward. In particular, this induced voltage has the characteristic that the dawn point is at a higher potential than the dusk point, whenever the incident magnetic field has a southward component or a small component perpendicular to the dawn-dusk plane. With a southward field of 5 gammas and a solar wind velocity of 300 km/sec, we obtain 353 kv as a typical value of the induced voltage. Our method of calculation of this voltage differs from that of Stern (1973) in making no assumption of equipotentials on individual field lines and from that of Gonzales and Mozer (1974) in making no assumption regarding the size and orientation of the separator. The results of this research will be published in a paper entitled:" Dayside Reconnection between a Dipolar Geomagnetic Field and a Uniform Interplanetary Field " in the Journal of Geophysical Research, Space Physics.

II. PUBLICATIONS

1. Tyan Yeh, " A composite solution of field line reconnection ", accepted for publication in J. Plasma Phys.

   Abstract: A solution of the MHD equations is presented which describes field line reconnection by stagnant diffusion in the middle of a piecewise-uniform convective hydromagnetic flow. This composite solution demonstrates that the diffusion region adjusts its shape and size in accommodation to the incident merging fluids.

2. Tyan Yeh, " Dayside Reconnection between a Dipolar Geomagnetic Field and a Uniform Interplanetary Field ", accepted for publication in J.Geophys. Res., Space Phys.

   Abstract: Field line reconnection on the dayside magnetopause is assumed to take place along the separator of field line connectivity in the magnetic topology resulting from the interpermeation of a dipolar geomagnetic field and a uniform interplanetary field. The induced voltage is calculated to show its dependence on the magnitude and direction of the incident magnetic field.

III. REFERENCES

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A composite solution of field line reconnection

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ABSTRACT

A solution of the MHD equations is presented which describes field line reconnection by stagnant diffusion in the middle of a piecewise-uniform convective hydromagnetic flow. This composite solution demonstrates that the diffusion region adjusts its shape and size in accommodation to the incident merging fluids.

1. Introduction

Among the similarity solutions for the steady motion of an incompressible, dissipationless, conducting fluid discussed by Yeh & Axford (1970), the piecewise-uniform flow obtained by Sonnerup (1970) is the simplest, self-consistent mathematical solution which describes the hydromagnetic flow around a neutral point. Its validity for representing certain physically realizable configurations of field line reconnection depends on, among other things, the resolution of two main difficulties. First, the significance of the discontinuity fronts which account for the bending of the streamlines and field lines in forming the desirable topology has to be clarified. Second, the feasibility of matching the convective flow to a diffusion region in the middle must be established. Following Petschek's (1964) explanation for the wave action in his model, the trailing discontinuity fronts have been thought of as slow shocks originating from the kinks in the newly reconnected field lines (cf. Yeh & Dryer (1973)). The leading discontinuity fronts have been regarded as slow expansion waves generated at distant "corners" or as the lumping of all MHD interactions in front of Petschek's waves (cf. Sonnerup (1973)). Such interpretations will become more certain only when the matching of the convective flow to the diffusion core is resolved.

In this paper we present a composite solution which consists of field line diffusion in a stagnant resistive fluid matched to field line convection in a surrounding perfectly conducting fluid. This is an exact solution covering the whole plane, with the appropriate physical laws satisfied everywhere. This composite solution shows that the piecewise-uniform flow can be matched self-consistently to a diffusion region and demonstrates that the diffusion region adjusts its shape and size in accommodation to the incident merging flows.

2. Piecewise-uniform hydromagnetic flow

We recapitulate the piecewise-uniform hydromagnetic flow in rectangular coordinates (x,y,z). Each streamline traverses two discontinuity fronts where the normal component of the velocity equals the Alfven speed associated with the normal component of the magnetic field. Relative to the incident streamlines, the orientation angles of the leading fronts, the separatrices, and the trailing fronts are given by

$$\tan\theta_1 = 1/A \quad , \tag{1}$$

$$\tan\theta_s = (1+\sqrt{2})/A \quad , \tag{2}$$

$$\tan\theta_2 = (1+\sqrt{2})^2/A \quad , \tag{3}$$

in which $A = (\mu\rho)^{\frac{1}{2}}u_1/B_1$ is the Alfvén number for the incident flows. ρ is the mass density, μ is the magnetic permeability, $u_1 = (\mu^{-1}\rho^{-1}E^2)^{\frac{1}{4}}A^{\frac{1}{2}}$ and $B_1 = (\mu\rho E^2)^{\frac{1}{4}}A^{-\frac{1}{2}}$. The electric field

$$\vec{E} = E\,\vec{e}_z \tag{4}$$

is uniform. The velocity, the magnetic field, and the pressure are given by

$$\vec{u}_1 = -\frac{x}{|x|}u_1\,\vec{e}_x \quad , \tag{5}$$

$$\vec{B}_1 = \frac{x}{|x|}B_1\,\vec{e}_y \tag{6}$$

in the inflow regions,

$$\vec{u}_2 = \frac{y}{|y|} u_1 \tan \theta_s \vec{e}_y \quad , \tag{7}$$

$$\vec{B}_2 = \frac{y}{|y|} B_1 \cot \theta_s \vec{e}_x \quad , \tag{8}$$

$$p_2 = p_1 + \frac{1}{2} \mu^{-1} B_1^2 (1 - \cot^2\theta_s) \tag{9}$$

in the outflow regions, and

$$\vec{u}_s = \frac{u_1}{\sqrt{2}\,(1 + \sqrt{2}) \cos \theta_s} \left(- \frac{x}{|x|} \cos \theta_s \vec{e}_x + \frac{y}{|y|} \sin \theta_s \vec{e}_y \right), \tag{10}$$

$$\vec{B}_s = \frac{(1 + \sqrt{2})\, B_1}{\sqrt{2} \sin \theta_s} \left(\frac{y}{|y|} \cos \theta_s \vec{e}_x + \frac{x}{|x|} \sin \theta_s \vec{e}_y \right) , \tag{11}$$

$$p_s = p_1 - \tfrac{1}{2} \mu^{-1} B_1^{\,2} \left(\tfrac{1}{2} + \sqrt{2} + \tfrac{1}{2} A^2 \right) \tag{12}$$

in the regions containing the separatrices. The concentrated vorticities and currents (per unit length)

$$\vec{\Omega}_1 = - \frac{u_1}{\sqrt{2} \cos \theta_1} \frac{x}{|x|} \frac{y}{|y|} \vec{e}_z \quad , \tag{13}$$

$$\vec{I}_1 = - \frac{\mu^{-1} B_1}{\sqrt{2} \sin \theta_1} \vec{e}_z \tag{14}$$

at the leading fronts and

$$\vec{\Omega}_2 = - \frac{u_1}{\sqrt{2}\,(1 + \sqrt{2}) \cos \theta_2} \frac{x}{|x|} \frac{y}{|y|} \vec{e}_z \quad , \tag{15}$$

$$\vec{I}_2 = \frac{(1 + \sqrt{2})\, \mu^{-1} B_1}{\sqrt{2} \sin \theta_2} \vec{e}_z \tag{16}$$

at the trailing fronts cause an overall deflection of 90° in the streamlines and field lines.

It is tempting to construct a solution covering the whole plane by extending the piecewise-uniform flow up to the neutral point. However, the convergence of all discontinuity fronts at the neutral point will form a discontinuity point which is not removable by resistivity. The reason is that, even with a diffusive flow in the vicinity of the neutral point which blends into the surrounding convective flow, it is still impossible to satisfy the requirement

$$\mu^{-1} \oint \vec{B} \cdot d\vec{\ell} = \eta^{-1}\left(\iint E \, dS + \oint \phi \, \vec{B} \times \vec{e}_z \cdot d\vec{\ell}\right) . \qquad (17)$$

In equation (17), the left side represents the total current enclosed by the integration contour according to Ampere's law. So does the right side by Ohm's law $\vec{E} + \vec{u} \times \vec{B} = \eta^{-1}\vec{J}$ (η is the resistivity), since

$$\iint \vec{u} \times \vec{B} \, dS = \iint \nabla \times (\phi \, \vec{B} \times \vec{e}_z) \, dS = \vec{e}_z \oint \phi \, \vec{B} \times \vec{e}_z \cdot d\vec{\ell} , \qquad (18)$$

ϕ being the stream function related to $\vec{u}$ by $\nabla\phi = \vec{u} \times \vec{e}_z$. To see the impossibility, we consider a contour lying wholly inside the convection region so that $\oint \vec{B}\cdot d\vec{\ell}$ and $\oint \phi\, \vec{B}\times\vec{e}_z\cdot d\vec{\ell}$ are solely determined by the convective flow while $\iint E\, dS$ is determined by the area enclosed, part of it being in the convection region and the remainder in the diffusion region. Thus, all three integrals in equation (17) can be evaluated without resort to the detail of the diffusive flow. As the contour of integration we may choose an octagon, with length 2ℓ along the y axis, whose sides align with steamlines in the regions containing the separatrices and with field lines elsewhere, yielding

$$\oint \vec{B}\cdot d\vec{\ell} = 4\sqrt{2}\,(1 - \cot^2\theta_s)\, B_1 \ell \quad , \tag{19}$$

$$\iint E\, dS = \frac{8}{1+\sqrt{2}} \cot\theta_s\, u_1\, B_1 \ell^2 \quad , \tag{20}$$

$$\oint \phi\, \vec{B}\times\vec{e}_z\cdot d\vec{\ell} = -\frac{8}{1+\sqrt{2}} \cot\theta_s\, u_1\, B_1 \ell^2 \quad . \tag{21}$$

In evaluating (21), we may choose $\phi = \int \vec{u}\times\vec{e}_z\cdot d\vec{\ell}$ integrated along the octagon from any point; the additive constant for ϕ makes no contribution to the integral since $\oint \vec{B}\times\vec{e}_z\cdot d\vec{\ell} = 0$ by the solenoidality of the magnetic field. Therefore, the right side of equation (17) is zero but the left side is not.

3. Stagnant diffusion

Now we consider a configuration in which the discontinuity fronts either emanate from or terminate at the vertices of a rhombic region which contains a stagnant resistive fluid (see figure 1). The rhombic region

$$|x| + |y| \cot \theta_s < \delta \tag{22}$$

has a width equal to

$$\delta = \frac{\eta}{\mu u_1}(2 + \sqrt{2})(1 - \cot^2\theta_s) \tag{23}$$

so that the requirement (17) with $\phi = 0$ is satisfied. In the stagnant fluid there is no motion, so the uniform electric field produces a uniform current density

$$\vec{J} = \frac{E}{\eta}\vec{e}_z \quad . \tag{24}$$

There the magnetic field $\vec{B} = \vec{e}_z \times \nabla\psi$ has a flux function ψ satisfying the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}\right)\psi = \frac{\mu E}{\eta} \tag{25}$$

and the boundary condition

$$\frac{x}{|x|}\frac{\partial\psi}{\partial x}\cos\theta_s - \frac{y}{|y|}\frac{\partial\psi}{\partial y}\sin\theta_s = |\vec{B}_s|\sin 2\theta_s \tag{26}$$

that the normal component is continuous. Therefore,

$$\psi = \frac{\mu E}{2\eta} \frac{x^2 \tan^2\theta_s - y^2}{\tan^2\theta_s - 1} , \qquad (27)$$

and hence

$$\vec{B} = \frac{\mu E}{\eta} \frac{y\,\vec{e}_x + x \tan^2\theta_s\,\vec{e}_y}{\tan^2\theta_s - 1} . \qquad (28)$$

There are surface vorticity and current

$$\vec{\Omega}_s = \frac{x}{|x|}\frac{y}{|y|}|\vec{u}_s|\,\vec{e}_z , \qquad (29)$$

$$\vec{I}_s = \mu^{-1}|\vec{B}_s|\left(1 - 2\frac{|x|}{\delta}\right)\vec{e}_z \qquad (30)$$

at the rhombic boundary. Integrating the equation $\nabla p = \vec{J} \times \vec{B}$, we obtain the pressure distribution

$$p = p_s + \frac{\mu E^2}{2\eta^2} \frac{y^2 - x^2 \tan^2\theta_s}{\tan^2\theta_s - 1} , \qquad (31)$$

which is constant on individual field lines. The pressure jump across the boundary has been chosen zero at the separatrices since the surface current is zero there.

The composite solution shows that, if the piecewise-uniform convective flow is a valid representation of certain physically realizable configurations, the merging fluids will form a diffusion region with its shape and size so adjusted to accommodate the surrounding convective flow. This means that there is no predetermined scale length and, therefore, field line reconnection is controlled more by distant boundary conditions than by regulatory constraints at the neutral point.

Acknowledgments

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FIGURE CAPTION

FIGURE 1. A stagnant diffusion region (dotted area) surrounded by a piecewise-uniform convective hydromagnetic flow. Solid lines represent magnetic field lines and dashed lines represent streamlines. Surface currents are shown by lines of dots (flowing out of the page) or lines of crosses (flowing into the page), with the magnitude depicted by the size of dot or cross.

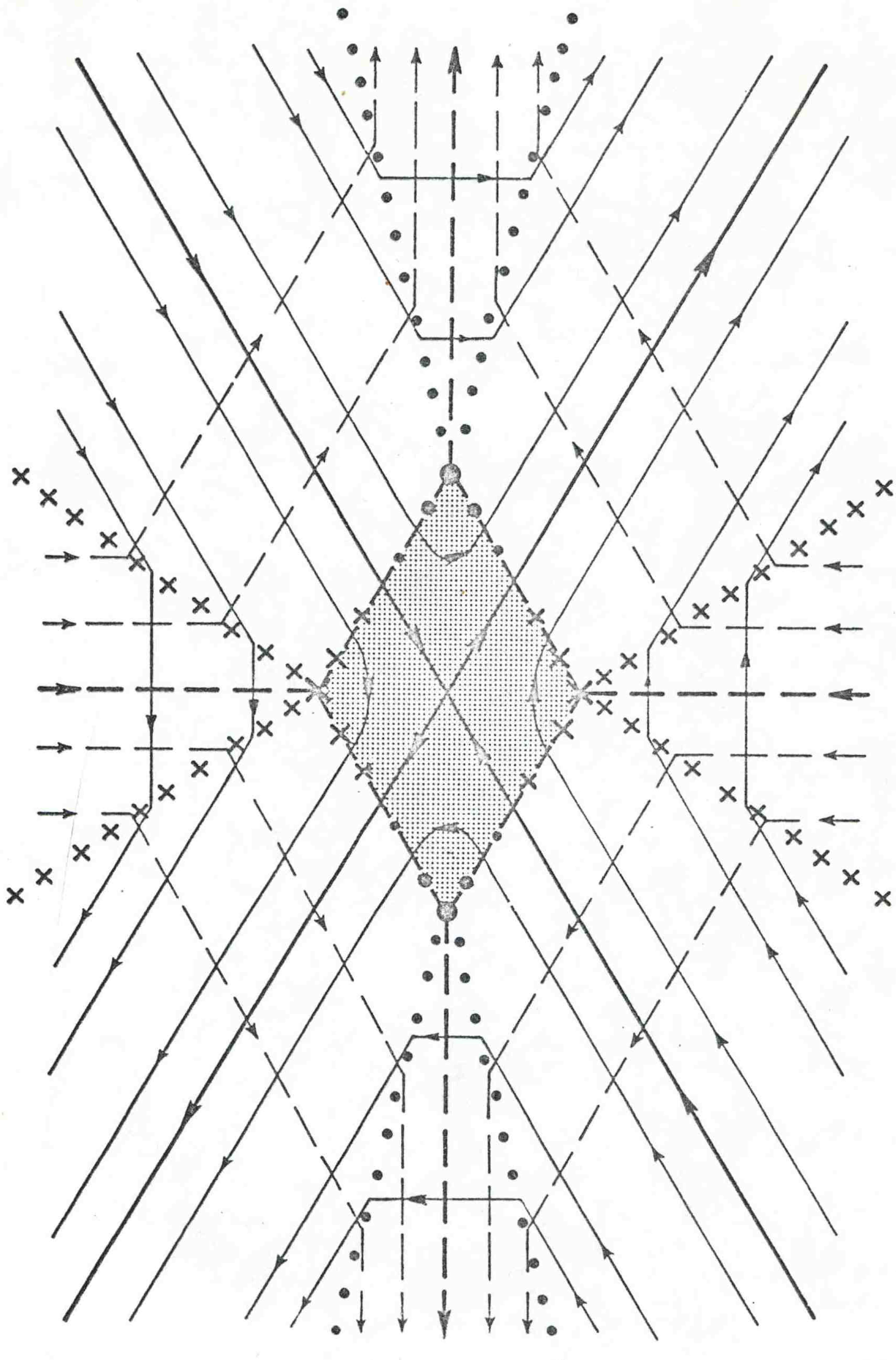

Figure 1

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Dayside Reconnection between a Dipolar Geomagnetic Field and a Uniform Interplanetary Field

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ABSTRACT Field line reconnection on the dayside magnetopause is assumed to take place along the separator of field line connectivity in the magnetic topology resulting from the interpermeation of a dipolar geomagnetic field and a uniform interplanetary field. The induced voltage is calculated to show its dependence on the magnitude and direction of the incident magnetic field.

I. INTRODUCTION

Since the suggestion by Dungey [1961], field line reconnection on the dayside magnetopause has been regarded as a basic driving mechanism for the convective motion inside the magnetosphere (e.g. Axford [1969]) and the flux erosion of the dayside magnetosphere (e.g. Holzer and Reid [1975]). The important feature of the field line reconnection between the geomagnetic field and the incident interplanetary field is the induction of an electric field by the transfer of magnetic fluxes along the separator (variously referred to as the X-line, neutral line, reconnection line, or separation line in the literature) of field line connectivity on the magnetopause. This induced electric field, which produces a voltage across the dayside magnetopause, may be viewed as derived from the incident electric field in the impinging solar wind (e.g. Stern [1973], Gonzales and Mozer [1974 Hence, field line reconnection acts as a means by which the incident electric field penetrates into the magnetosphere. This impartation of an electric field depends strongly on the direction of the incident magnetic field.

Without a consideration of the magnetic topology, an order-of-magnitude estimate for the induced voltage is the interplanetary electric field times the width of the magnetopause. Since the reconnection process diminishes when the interplanetary magnetic field turns its direction from southward to northward(for observational evidence, see the references cited in Gonzales and Mozer [1974]), the geometr of the separator is important in the calculation of the induced voltage. In this paper we shall, based on the magnetic topology resulting from the interpermeation of a dipolar magnetic field and a uniform magnetic field , elucidate some geometric features of field line reconnection between the geomagnetic field and the incident interplanetary field and calculate the induced voltage to show its dependence on the magnitude and direction of the incident magnetic field.

II. MAGNETIC TOPOLOGY

The magnetic topology in a reconnection configuration is characterized by a separator along which separatrices meet, each separatrix being a flux surface that separates field lines of different connectivity. Geometrically, magnetic field lines are directed smooth endless curves. An idealized exception is that they can emanate from or terminate at isolated neutral points, where the magnetic field vanishes and no direction of the magnetic field is defined. The separator consists of certain field lines which emanate from and terminate at the neutral points of the magnetic topology. In the interaction between the geomagnetic field and the incident interplanetary field, the magnetic topology possesses two neutral points and three kinds of field line connectivity. A terrestrial field line has its two ends connected to the earth, an open field line has only one end connected to the earth, and an interplanetary field line has no ends connected to the earth. The terrestrial field lines are enclosed by the separatrices which form a closed flux surface made up of field lines with one end connected to a neutral point and the other end connected to the earth. The interplanetary field lines are flanked by the separatrices which form an open flux surface made up of field lines with one end connected to a neutral point and the other end receding into interplanetary space. These features were outlined by Dungey [1963], and described in more detail by Stern [1973] and Cowley [1973], based on the magnetic topology resulting from the interpermeation of a dipolar geomagnetic field and a uniform interplanetary field. In particular, Cowley pointed out that the separator is circular when the incident magnetic field is perpendicular to the dipole axis. We shall show that this is generally true for an incident field in any direction other than northward.

We use spherical coordinates (r, θ, ϕ), with the origin at the center of the earth, the polar axis directed northward, and the azimuth measured eastward from the sunward azimuthal plane. The dipolar geomagnetic field is given by

$$\vec{B}_G = B_E\, r_E^3\, \frac{-2\cos\theta\, \vec{e}_r - \sin\theta\, \vec{e}_\theta}{r^3} \qquad (1)$$

where B_E is its strength at the equator and r_E is the earth's radius. The incident uniform interplanetary magnetic field can be represented by

$$\vec{B}_I = B_I\, (1, \theta_I, \phi_I) \qquad (2)$$

where B_I is its magnitude and the radial vector, specified by the polar angle θ_I and azimuthal angle ϕ_I, denotes its direction.

The interpermeation of the two fields may be obtained by their superposition. The interpermeated field

$$\vec{B} = \vec{B}_G + \vec{B}_I \tag{3}$$

has two neutral points, both being hyperbolic null points. The northern neutral point is located at

$$N_+ = (r_N, \theta_N, \phi_I) \tag{4}$$

in the $\phi = \phi_I$ half-plane and the southern neutral point is located at

$$N_- = (r_N, \pi - \theta_N, \phi_I + \pi) \tag{5}$$

in the $\phi = \phi_I + \pi$ half-plane. The radial distance is given by

$$r_N = \left(\frac{B_E}{B_I}\right)^{1/3} r_E \, R(\theta_I) \tag{6}$$

in which

$$R = \left[\frac{1}{2}\left(8 + \cos^2\theta_I\right)^{1/2} + \frac{1}{2}\cos\theta_I\right]^{1/3} \tag{7}$$

decreases from $2^{1/3}$ to 1 as θ_I varies from 0° to 180° and equals 1.12 when $\theta_I = 90^\circ$. The polar angle

$$\theta_N = \arctan \frac{4 \sin \theta_I}{\left(8 + \cos^2\theta_I\right)^{1/2} + 3\cos\theta_I} \tag{8}$$

increases from 0° to 90° as θ_I varies from 0° to 180° and equals 54.7° when $\theta_I = 90^\circ$.

The differential equation for the field lines is

$$\frac{B_I}{r_N}\frac{d\vec{r}}{d\tau} = \vec{B} \tag{9}$$

τ being a dimensionless variable which increases in the direction of the magnetic field. In component form, (9) can be written

$$\frac{1}{r_N}\frac{dr}{d\tau} = \cos\theta_I \cos\theta + \sin\theta_I \sin\theta \cos(\phi-\phi_I) - 2\frac{r_N^3}{R^3}\frac{\cos\theta}{r^3} \tag{10}$$

$$\frac{r}{r_N}\frac{d\theta}{d\tau} = -\cos\theta_I \sin\theta + \sin\theta_I \cos\theta \cos(\phi-\phi_I) - \frac{r_N^3}{R^3}\frac{\sin\theta}{r^3} \tag{11}$$

$$\frac{r}{r_N}\sin\theta \frac{d\phi}{d\tau} = -\sin\theta_I \sin(\phi-\phi_I) \tag{12}$$

Their linearized equations have three real-valued eigenvalues, whose magnitudes are

$$\lambda_1 = 3R^{-3}\frac{\left(4+5\cos^2\theta_N\right)^{1/2} + \cos\theta_N}{2} \tag{13}$$

$$\lambda_2 = 3R^{-3}\frac{\left(4+5\cos^2\theta_N\right)^{1/2} - \cos\theta_N}{2} \tag{14}$$

$$\lambda_3 = 3R^{-3}\cos\theta_N \tag{15}$$

The three corresponding eigenvectors are

$$\vec{\Lambda}_1 = (1,\ \theta_B,\ \phi_I) \tag{16}$$

$$\vec{\Lambda}_2 = (1,\ \theta_B + \frac{\pi}{2},\ \phi_I) \tag{17}$$

$$\vec{\Lambda}_3 = (1,\ \frac{\pi}{2},\ \phi_I + \frac{\pi}{2}) \tag{18}$$

in which the polar angle

$$\theta_B = \theta_N + \arctan \frac{2 \sin\theta_N}{(4 + 5 \cos^2\theta_N)^{1/2} + 3 \cos\theta_N} \tag{19}$$

increases from 0° to 135° as θ_I varies from 0° to 180° and equals 76.4° when $\theta_I = 90^\circ$. The first two eigenvectors have no azimuthal components and the third has only the azimuthal component. In the neighborhood of the northern neutral point the field lines are

$$\frac{1}{r_N}(\vec{r} - \vec{r}_{N_+}) = C_1\, \vec{\Lambda}_1 \exp(\lambda_1 \tau) + C_2\, \vec{\Lambda}_2 \exp(-\lambda_2 \tau) + C_3 \vec{\Lambda}_3 \exp(-\lambda_3 \tau) \tag{20}$$

along which the magnetic field changes as

$$\frac{1}{B_I}\, \vec{B} = C_1\, \lambda_1\, \vec{\Lambda}_1 \exp(\lambda_1 \tau) - C_2\, \lambda_2\, \vec{\Lambda}_2 \exp(-\lambda_2 \tau) - C_3\, \lambda_3 \vec{\Lambda}_3 \exp(-\lambda_3 \tau) \tag{21}$$

Similarly, in the neighborhood of the southern neutral point the field lines are

$$\frac{1}{r_N}(\vec{r} - \vec{r}_{N\_}) = C_1\vec{\Lambda}_1 \exp(-\lambda_1\tau) + C_2\vec{\Lambda}_2 \exp(\lambda_2\tau) + C_3\vec{\Lambda}_3 \exp(\lambda_3\tau) \quad (22)$$

along which the magnetic field changes as

$$\frac{1}{B_I}\vec{B} = -C_1\lambda_1\vec{\Lambda}_1 \exp(-\lambda_1\tau) + C_2\lambda_2\vec{\Lambda}_2 \exp(\lambda_2\tau) + C_3\lambda_3\vec{\Lambda}_3 \exp(\lambda_3\tau) \quad (23)$$

Each individual field line is specified by the values of the three constants C_1, C_2, and C_3. The three eigenvectors define three orthogonal directions (since the current is zero in the interpermeation model) along each of which two field lines leave (if the eigenvalue is positive) or enter (if the eigenvalue is negative) a neutral point in opposite directions. These field lines are sketched in Figure 1. Since the first eigenvalue is opposite in sign to the other two, each neutral point appears as a saddle point in both the $(\vec{\Lambda}_1, \vec{\Lambda}_2)$ plane and the $(\vec{\Lambda}_1, \vec{\Lambda}_3)$ plane, but as a nodal point in the $(\vec{\Lambda}_2, \vec{\Lambda}_3)$ plane. Furthermore, since $\lambda_1 > \lambda_2 > \lambda_3$, all field lines except those running along the eigenvectors approach a neutral point tangentially along the direction of $\vec{\Lambda}_3$ and osculatorily in the $(\vec{\Lambda}_2, \vec{\Lambda}_3)$ plane. In fact, in the azimuthal plane containing the two neutral points the magnetic field has no azimuthal component at all. There the planar field lines satisfy the differential equation

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{2\, r_N^3 \cos\theta - R^3\, r^3 \cos(\theta - \theta_I)}{r_N^3 \sin\theta + R^3\, r^3 \sin(\theta - \theta_I)} \quad (24)$$

with θ here interpreted as the polar angle for the entire plane, being positive in the $\phi = \phi_I$ half-plane and negative in the $\phi = \phi_I + \pi$ half-plane. However, two bundles of field lines, one on each side of the azimuthal plane, leave the southern neutral point azimuthally and another two bundles enter the northern neutral point.
In particular, there are two semi-circular field lines which run from the southern neutral point to the northern neutral point. They form a circle:

$$r = r_N \tag{25}$$

$$\tan\theta \cos(\phi-\phi_I) = \tan\theta_N \tag{26}$$

at the intersection of the sphere (25) and the slant plane (26) which passes through the origin and whose normal vector is

$$\vec{n} = (1,\ \theta_N+\frac{\pi}{2},\ \phi_I) \tag{27}$$

The two semi-circular field lines intersect the dawn-dusk plane, at the dusk point and the dawn point. The dusk point is located at

$$D_+ = (r_N,\ \theta_D,\ \frac{\pi}{2}) \tag{28}$$

and the dawn point at

$$D_- = (r_N,\ \pi-\theta_D,\ -\frac{\pi}{2}) \tag{29}$$

in which the polar angle

$$\theta_D = \arctan \frac{(8 + 9 \cot^2\theta_I)^{1/2} - 3 \cot\theta_I}{2 \sin\phi_I} \tag{30}$$

increases from $0°$ to $90°$ if $0° < \phi_I < 180°$ or decreases from $180°$ to $90°$ if $0° > \phi_I > -180°$ as θ_I varies from $0°$ to $180°$, but oscillates about $90°$ as ϕ_I varies from $-180°$ to $180°$. The portion of the circle from the dawn point to the dusk point on the sunward side of the dawn-dusk plane is the separator, whose projection in the dawn-dusk plane can be represented by

$$\vec{D} = 2\, r_N\, (1,\ \theta_D,\ \tfrac{\pi}{2}) \tag{31}$$

The whole magnetic configuration resulting from the interpermeation of a dipolar magnetic field and a uniform magnetic field is symmetric with respect to the origin, hence it possesses reflective symmetry with the azimuthal plane containing the two neutral points and skew symmetry with the slant plane containing the separator. Above the slant plane the upper bundles of the field lines which emanate from the southern neutral point as a nodal point will terminate at the north tip of the dipole, some of them after running inside the separator and near the northern neutral point as a saddle point. They, together with the field line running upward from the southern neutral point and the field line running inward from the northern neutral point, both toward the dipole, form the upper half of the doughnut-shaped closed separatrices (see Cowley [1973] for an isometric drawing). Likewise, below the slant plane the lower bundles of the field lines from the southern neutral point will recede into infinity, some of them after running outside the separator and near the northern neutral point. They, together with the field line running downward from the southern neutral point and the field line running outward from the northern neutral point, both to infinity, form the lower half of the tube-shaped open separatrices (see Cowley [1973] for an isometric drawing). Between the closed and the open separatrices, the region above the circular separator contains those open field lines running from infinity into the north tip of the dipole, and the region below contains those running from the south tip of the dipole to infinity. Since the magnetic field has no transverse components along the separator, each point of the separator appears as a transverse null point in the X-configuration formed by the local transverse components of the magnetic field in a plane perpendicular to the separator.

III. MAGNETIC MERGING

As far as the magnetic topology is concerned, a separator will provide the necessary requirement for field line reconnection. However, reconnection will take place only when the flux transfer at the separator is achieved by merging and diffusive motions of the field lines. Thus, reconnection is necessarily a magnetohydrodynamic process. In the absence of self-consistent treatments for such a dynamic interaction, we shall first approximate the magnetic configuration by that resulting from the magnetostatic interaction of a dipolar geomagnetic field and a uniform interplanetary field and then superpose the fluid motion in the region upstream of the separator to account for the merging motion. The only *a priori* justification for such a decoupling treatment is that the resultant interpermeation field gives a correct dayside topology of an open magnetosphere and that the geometry is presumably altered only slightly by the fluid motion if the motion is small. But caution must be exercised when the electric field is being considered. By Ohm's law ($\vec{E} = \vec{B} \times \vec{U} + \eta \vec{J}$), zero current in the interpermeated field means equipotentials on individual field lines. The entire neglect of the currents would imply a zero voltage across the two neutral points in the case of a southward incident field because the interpermeated magnetic field and hence the electric field are zero along the separator in this case. This fallacy means that field lines are not equipotential in the neighborhood of the separator due to the presence of strong currents there.

When the separator is rectilinear and the configuration has no variation along the separator, steady state field line reconnection is characterized by having a uniform electric field in the direction of the separator. This means that the electric field induced at the separator is equal to the electric field convected by the merging plasma, regardless the details of the diffusion in the neighborhood of the separator. In other words, as long as a steady state is maintained, the diffusion is adjusted to accommodate the incoming flows (e.g. Yeh and Axford [1970]). However, a curvilinear separator will couple the longitudinal flow to the transverse flow so that the electric field induced at the separator may differ considerably from that in the incident solar wind. This explains why geometrical considerations are important in the interaction between the geomagnetic field and the interplanetary field. Consider the case of a southward incident field, for which the separator is a semi-circle in the equatorial plane. In approaching the subsolar point of the magnetopause the solar wind fluid has to make a detour . Field line reconnection will be facilitated greatly by a detour in the meridional plane, but less by detours in the equatorial plane. A full impartation of the incident electric field at the subsolar point amounts to a total detour in the meridional plane. This is an upper bound for the induced electric field at the subsolar point. This upper bound will be lowered at other points of the separator due to an oblique angle between the directions of the incident electric field and the separator. The induced electric field diminishes to zero at the dawn point and the dusk point. The complication caused by the curvilinearity of the separator is circumvented when its curvature is small and the local variation in the direction of the separator is small (this being the case since $r_N/r_E \gg 1$), for then the local transverse hydromagnetic flow is approximately planar and the local induced electric field is equal to the transverse merging magnetic field times the flow velocity toward the local segment of the separator.

Let $\vec{e}_S$ denote the local unit vector along the separator traced from the dawn point to the dusk point. The transverse component of the geomagnetic field is

$$\vec{B}'_G = \vec{B}_G - (\vec{B}_G \cdot \vec{e}_S)\vec{e}_S \qquad (32)$$

and that of the incident interplanetary field is

$$\vec{B}'_I = \vec{B}_I - (\vec{B}_I \cdot \vec{e}_S)\vec{e}_S \qquad (33)$$

These two transverse components are oppositely directed and equal in magnitude, as is seen from

$$\vec{B}_G' + \vec{B}_I' = \vec{B}_G + \vec{B}_I - \left[(\vec{B}_G + \vec{B}_I)\cdot\vec{e}_S\right]\vec{e}_S = 0 \qquad (34)$$

noting that $\vec{e}_S$ is a unit vector parallel to $\vec{B}_G+\vec{B}_I$. The field lines of the interplanetary field approach the separator with the solar wind velocity

$$\vec{U}_I = U_I\,(1, \tfrac{\pi}{2}, \pi) \qquad (35)$$

whose transverse component is

$$\vec{U}_I' = \vec{U}_I - (\vec{U}_I\cdot\vec{e}_S)\vec{e}_S \qquad (36)$$

In response, the field lines of the geomagnetic field will approach the separator from the other side by magnetospheric convection. The passage of the magnetic fluxes induces an electric field

$$\vec{E}_S = \vec{B}_I' \times \vec{U}_I' = \vec{B}_I\times\vec{U}_I + (\vec{B}_I\cdot\vec{e}_S)\vec{U}_I\times\vec{e}_S - (\vec{U}_I\cdot\vec{e}_S)\vec{B}_I\times\vec{e}_S \qquad (37)$$

in the direction of the separator, which is only a part of the interplanetary electric field

$$\vec{E}_I = \vec{B}_I \times \vec{U}_I = B_I U_I (1 - \sin^2\theta_I\,\cos^2\phi_I)^{\frac{1}{2}}\left(1,\ \arctan\frac{|\cot\theta_I|}{\sin\phi_I},\ -\frac{\pi}{2}\frac{|\cot\theta_I|}{\cot\theta_I}\right) \qquad (38)$$

in the incident solar wind. Integration of the induced electric field, whose magnitude varies along the separator, from the dusk point to the dawn point along the separator gives the induced voltage

$$V = \int_{D_+}^{D_-} -\vec{E}_S \cdot d\vec{l} = 2\, r_E\, B_E^{1/3} B_I^{2/3}\, U_I\, S(\theta_I, \phi_I) \qquad (39)$$

which is the potential of the dawn point relative to the dusk point. Since $d\vec{l}$ is in the direction of $\vec{e}_S$, the integral yields

$$S = \frac{\left[(8+\cos^2\theta_I)^{\frac{1}{2}} + \cos\theta_I\right]^{\frac{1}{3}} \left[3\cos^2\theta_I - (8+\cos^2\theta_I)^{\frac{1}{2}}\cos\theta_I + 2\sin^2\theta_I \sin^2\phi_I\right]}{2^{\frac{5}{6}}\left[4 + 5\cos^2\theta_I - 3(8+\cos^2\theta_I)^{\frac{1}{2}}\cos\theta_I + 2\sin^2\theta_I \sin^2\phi_I\right]^{\frac{1}{2}}} \qquad (40)$$

The result can be written simply as

$$V = \vec{E}_I \cdot \vec{D} \qquad (41)$$

The simplification comes from the constancy of the interplanetary electric field, irrespective of the circularity of the separator. In Figure 2, contours of constant S are shown in (θ_I, ϕ_I) polar plane. The value of S decreases from 1 to 0 as θ_I varies from 180° to 0° if $\sin^2\phi_I \geq 2/3$ or to $\arctan((2-3\sin^2\phi_I)^{\frac{1}{2}}/\sin^2\phi_I)$ otherwise. In the latter ranges of ϕ_I, the negative value of S does not have a uniform limit when θ_I diminishes toward 0°. This indicates the inaccuracy of the modelling in that regime. When $\theta_I = 90^\circ$, S increases from 0 to 0.648 as ϕ_I varies from 0° to 90°.

IV. DISCUSSION

In our calculation of the dayside magnetopause voltage induced by field line reconnection between the geomagnetic field and the incident interplanetary field, we have made two simplifying assumptions. First, current-free interpermeation is used in determining the separator of the magnetic topology. Second, the local longitudinal variation of the configuration along the separator is neglected in considering the magnetic merging. The first assumption causes overestimates in the radius of the hemisphere-shaped dayside magnetopause and hence also in the induced voltage. The second assumption seems reasonable in view of the small curvature of the separator. It should be noted that although we use the current-free interpermeation of the two interacting fields in locating the separator, we do not invoke its peculiarity of zero current in the calculation of the induced electric field.

According to (6) and (39), typically r_N is 15 r_E and V is 532 kv for a southward field of 9.24γ and a solar wind velocity of 300 km/sec. The radius of the magnetopause decreases, but the induced voltage increases generally when the incident interplanetary field changes from northward to southward. Since the radius decreases with the incident field strength following a one-third power, the induced voltage increases with the incident field strength only as a two-third power. The induced voltage is such that the dawn point is at a higher potential than the dusk point whenever the incident field has a southward component or a small component perpendicular to the dawn-dusk plane.

Without considerations of magnetic topology, Gonzales and Mozer [1974] assumed that the separator is a semi-circle of an assigned radius with its midpoint located at the subsolar point of the magnetopause and its orientation determined by the two magnetic fields meeting there. Since they restricted their considerations to interplanetary fields without components perpendicular to the dawn-dusk plane and assumed that in the interaction region the geomagnetic field is approximately northward as at the subsolar point, namely $\vec{B}_I = B_I(1, \theta_I, \pm\frac{\pi}{2})$ and $\vec{B}_G = B_g(1,0,0)$, the assumed separator lies in a slant plane whose normal vector

$$\vec{n}' = \left(1,\ \frac{\pi}{2} + \arctan\frac{B_g/B_I - \cos\theta_I}{\sin\theta_I},\ \pm\frac{\pi}{2}\right)$$

is in the direction of $\vec{B}_I - \vec{B}_G$. The induced voltage is $2r'_N U_I \vec{B}_I \cdot \vec{n}'$ if $B_I > B_g \cos\theta_I$ and zero otherwise. The result can be expressed as

$$V' = \vec{E}_I \cdot \vec{D}'$$

in which

$$\vec{D}' = 2\, r'_N \left(1, \pm\arctan\frac{B_g/B_I - \cos\theta_I}{\sin\theta_I},\ \frac{\pi}{2}\right)$$

The requirement $B_I > B_g \cos\theta_I$ is to ensure that after subtracting the equal components parallel to the slant plane, the two magnetic fields have oppositely directed unequal components perpendicular to the slant plane. The difference between $\vec{D}$ and $\vec{D}'$ can be attributed to the fact that in Gonzales and Mozer's treatment the radius of the separator, r'_N, is assigned whereas in our treatment it is determined from the geometrical consideration of the interpermeated field.

A more simple-minded method for the calculation of the voltage was suggested by Stern [1973], without explicit resort to the separator. When equipotentials are assumed along individual field lines, each neutral point will be at the same potential as the field line which maps from it to infinity on the sunward side. Hence the potential difference between the two neutral points is equal to the interplanetary electric field times the distance between those two field lines at infinity. The result can be expressed as

$$V'' = \vec{E}_I \cdot \vec{D}''$$

For a southward interplanetary field, the field lines of the interpermeated magnetic field are given by

$$\left(\frac{1}{3} \frac{r^2}{r_N^2} + \frac{2}{3} \frac{r_N}{r} \right) \sin^2\theta = C$$

which is the integral of (24) when $\theta_I = 180^\circ$. The two field lines mapping to the neutral points, for which the constant C equals 1 are separated by a distance of $3^{\frac{1}{2}} r_N$. Hence

$$\vec{D}'' = 2\sqrt{3}\, r_N \left(1, \frac{\pi}{2}, \frac{\pi}{2}\right)$$

In general, the difference between V and V'' can be attributed to the voltage drop along field lines in the current-filled neighborhood of the neutral points, which is completely excluded when field lines are assumed equipotential. These currents must appear when the merging motion is superposed to the current-free interpermeation field.

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FIGURE CAPTIONS

Fig. 1 The separator of field line connectivity in the magnetic topology resulting from the interpermeation of a dipolar field and a uniform field. The two dots represent the two neutral points. The circle in a slant plane represents the separator, and the lines in the azimuthal plane containing the two neutral points represent the field lines mapped to the neutral points.

Fig. 2 Contour plot of $S(\theta_I,\phi_I)$, showing the dependence of the induced voltage on the direction of the incident interplanetary magnetic field.

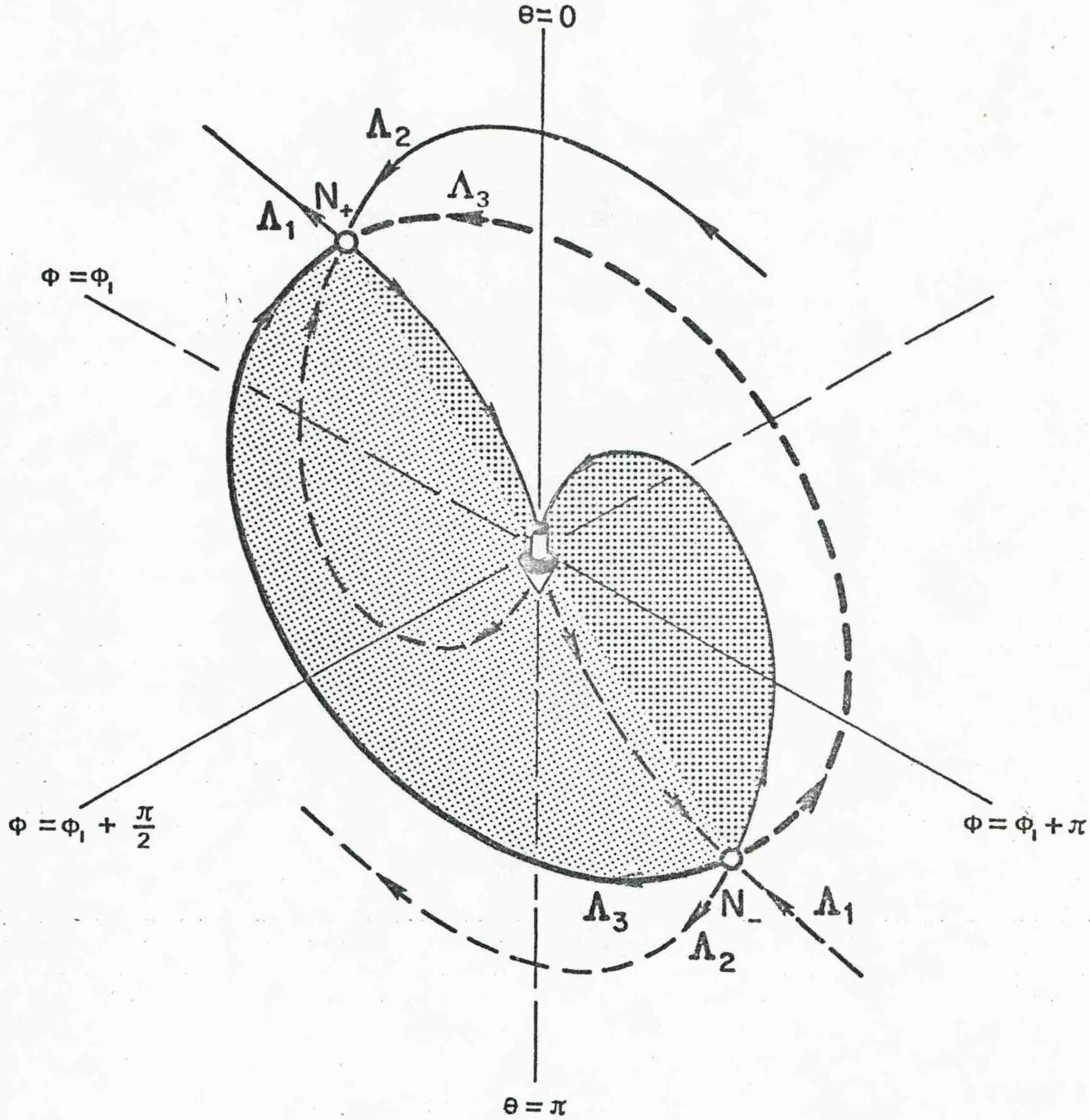

Fig. 1

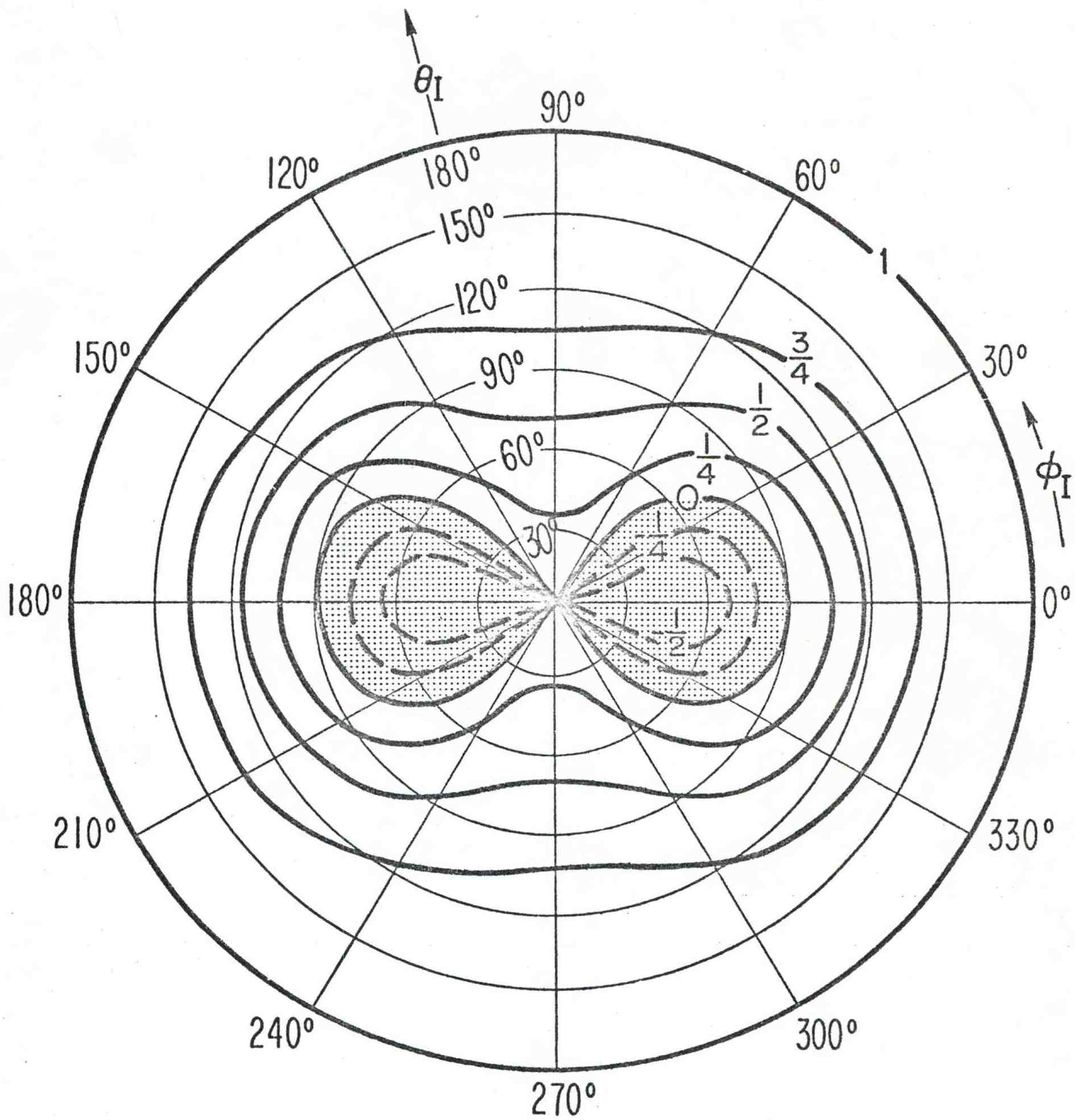
θI
90°
120°
180°
60°
150°
1
120°
150°
3/4
30°
90°
1/2
60°
1/4
φI
0
1/4
30°
180°
0°
1/2
210°
330°
240°
300°
270°

Fig. 2