

1 On the Relationship between Conditional (CAR) and Simultaneous
2 (SAR) Autoregressive Models

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Abstract

7 We clarify relationships between conditional (CAR) and simultaneous (SAR) autoregressive mod-
8 els. We review the literature on this topic and find that it is mostly incomplete. Our main result is
9 that a SAR model can be written as a unique CAR model, and while a CAR model can be written
10 as a SAR model, it is not unique. In fact, we show how any multivariate Gaussian distribution
11 on a finite set of points with a positive-definite covariance matrix can be written as either a CAR
12 or a SAR model. We illustrate how to obtain any number of SAR covariance matrices from a
13 single CAR covariance matrix by using Givens rotation matrices on a simulated example. We also
14 discuss sparseness in the original CAR construction, and for the resulting SAR weights matrix.
15 For a real example, we use crime data in 49 neighborhoods from Columbus, Ohio, and show that
16 a geostatistical model optimizes the likelihood much better than typical first-order CAR models.
17 We then use the implied weights from the geostatistical model to estimate CAR model parameters
18 that provides the best overall optimization.

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20 KEY WORDS: lattice models; areal models, spatial statistics, covariance matrix

21

22 1 Introduction

23 Cressie (1993, p. 8) divides statistical models for data collected at spatial locations into two
24 broad classes: 1) geostatistical models with continuous spatial support, and 2) lattice models, also
25 called areal models (Banerjee et al., 2004), where data occur on a (possibly irregular) grid, or
26 lattice, with a countable set of nodes or locations. The two most common lattice models are the
27 conditional autoregressive (CAR) and simultaneous autoregressive (SAR) models, both notable
28 for sparseness of their precision matrices. These autoregressive models are ubiquitous in many
29 fields, including disease mapping (e.g., Clayton and Kaldor, 1987; Cressie and Chan, 1989; Lawson,
30 2013), agriculture (Cullis and Gleeson, 1991; Besag and Higdon, 1999), econometrics (Anselin,
31 1988; LeSage and Pace, 2009), ecology (Lichstein et al., 2002; Kissling and Carl, 2008), and image
32 analysis (Besag, 1986; Li, 2009). CAR models form the basis for Gaussian Markov random fields
33 (Rue and Held, 2005) and the popular integrated nested Laplace approximation methods (INLA,
34 Rue et al., 2009), and SAR models are popular in geographic information systems (GIS) with the
35 GeoDa software (Anselin et al., 2006). Hence, both CAR and SAR models serve as the basis for
36 countless scientific conclusions. Because these are the two most common classes of models for lattice
37 data, it is natural to compare and contrast them. There has been sporadic interest in studying
38 the relationships between CAR and SAR models (e.g., Wall, 2004), and how one model might or
39 might not be expressed in terms of the other (Haining, 1990; Cressie, 1993; Martin, 1987; Waller
40 and Gotway, 2004), but there is little clarity in the existing literature on the relationships between
41 these two classes of autoregressive models.

42 Historically, CAR and SAR models were obtained constructively, which naturally led to re-
43 sults on conditions of the constructions that yielded positive-definite covariance matrices. However,
44 our goal is the opposite. We investigate how to obtain the properties of CAR and SAR models from
45 a positive definite covariance matrix. We aim to clarify, and add to, the existing literature on the
46 relationships between CAR and SAR covariance matrices. Cressie and Wikle (2011, p. 185) show
47 how to obtain a CAR covariance matrix from a geostatistical covariance matrix, and, by extension,
48 from any valid covariance matrix. We add to this by showing that any positive-definite covariance

49 matrix for a multivariate Gaussian distribution on a finite set of points can be written as either a
50 CAR or a SAR covariance matrix, and hence any valid SAR covariance matrix can be expressed as
51 a valid CAR covariance matrix, and vice versa. This result shows that on a finite dimensional space,
52 both SAR and CAR models are completely general models for spatial covariance, able to capture
53 any positive-definite covariance. While CAR and SAR models are among the most commonly-used
54 spatial statistical models, this correspondence between them, and the generality of both models,
55 has not been fully described before now. These results also shed light on some previous literature.
56 CAR and SAR models are often developed with sparseness in mind, where sparseness is the notion
57 that the precision matrix has many zeros, allowing for the use of compact computer storage and fast
58 computing algorithms for sparse matrices. Our results do not necessarily lead to sparse precision
59 matrices for the SAR or CAR specifications, which is a desirable property for these models, so we
60 spend some time investigating this with examples and discussion.

61 This paper is organized as follows: In Section 2, we review SAR and CAR models and lay
62 out necessary conditions for these models. In Section 3, we provide theorems that show how to
63 obtain SAR and CAR covariance matrices from any positive definite covariance matrix, which also
64 establishes the relationship between CAR and SAR covariance matrices. In Section 4, we provide
65 examples of obtaining SAR covariance matrices from a CAR covariance matrix on fabricated data,
66 and a real example for obtaining a CAR covariance matrix from a geostatistical covariance matrix.
67 Finally, in Section 5, we conclude with a detailed discussion of the incomplete results of previous
68 literature.

69 **2 Review of SAR and CAR models**

70 In what follows, we denote matrices with bold capital letters, and their i th row and j th column
71 with small case letters with subscripts i, j ; for example, the i, j th element of \mathbf{C} is $c_{i,j}$. Vectors of
72 fixed values are denoted as lower case bold letters while vectors (or matrices) of random variables
73 are bold, capital, and italic; let $\mathbf{Z} \equiv (Z_1, Z_2, \dots, Z_n)^T$ be a vector of n random variables at the
74 nodes of a graph (or junctions of a lattice). The edges in the graph, or connections in the lattice,

75 define neighbors, which are used to model spatial dependency. Broad reviews of SAR and CAR
 76 can be found in Besag (1974), Wall (2004), and Ver Hoef et al. (2018), and in many books (e.g.,
 77 Anselin, 1988; Haining, 1990; Cressie, 1993; Schabenberger and Gotway, 2005; Cressie and Wikle,
 78 2011; Banerjee et al., 2014).

79 2.1 SAR Models

80 Consider the SAR model with mean zero. An explicit autocorrelation structure is imposed,

$$\mathbf{Z} = \mathbf{B}\mathbf{Z} + \boldsymbol{\nu}, \quad (1)$$

81 where the $n \times n$ spatial weights matrix, \mathbf{B} , is relating \mathbf{Z} to itself, and $\boldsymbol{\nu} \sim N(\mathbf{0}, \boldsymbol{\Omega})$, where con-
 82 ventionally $\boldsymbol{\Omega}$ is diagonal with positive diagonal values. These models are generally attributed to
 83 Whittle (1954). Solving for \mathbf{Z} , note that conventionally sites do not depend on themselves so \mathbf{B}
 84 has zeros on the diagonal, and that $(\mathbf{I} - \mathbf{B})^{-1}$ must exist (Cressie, 1993; Waller and Gotway, 2004),
 85 where \mathbf{I} is the identity matrix. Then $\mathbf{Z} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_{\text{SAR}})$, where

$$\boldsymbol{\Sigma}_{\text{SAR}} = (\mathbf{I} - \mathbf{B})^{-1} \boldsymbol{\Omega} (\mathbf{I} - \mathbf{B}^T)^{-1}; \quad (2)$$

86 see, for example, Cressie (1993, p. 409). The spatial dependence in the SAR model is due to the
 87 matrix \mathbf{B} which causes the simultaneous autoregression of each random variable on its neighbors.
 88 Note that \mathbf{B} does not have to be symmetric because it does not appear directly in the inverse of
 89 the covariance matrix (i.e., precision matrix). The covariance matrix must be positive definite.
 90 For SAR models, it is enough that $(\mathbf{I} - \mathbf{B})$ is nonsingular (i.e., that $(\mathbf{I} - \mathbf{B})^{-1}$ exists), because the
 91 quadratic form, writing it as $(\mathbf{I} - \mathbf{B})^{-1} \boldsymbol{\Omega} [(\mathbf{I} - \mathbf{B})^{-1}]^T$, with $\boldsymbol{\Omega}$ containing positive diagonal values,
 92 ensures $\boldsymbol{\Sigma}_{\text{SAR}}$ will be positive definite.

93 In summary, the following conditions must be met for $\boldsymbol{\Sigma}_{\text{SAR}}$ in (2) to be a valid SAR
 94 covariance matrix:

95 **S1** $(\mathbf{I} - \mathbf{B})$ is nonsingular,

96 **S2** $\mathbf{\Omega}$ is diagonal with positive diagonal elements, and
 97 **S3** $b_{i,i} = 0, \forall i$.

98 2.2 CAR models

99 The term “conditional,” in the CAR model, is used because the distribution of each element of
 100 the random process is specified conditionally on the values at the neighboring nodes. Let Z_i be a
 101 random variable at the i th location, again assuming that the expectation of Z_i is zero for simplicity,
 102 and let z_j be its realized value. The CAR model is typically specified as

$$Z_i | \mathbf{z}_{-i} \sim N \left(\sum_{j=1}^n c_{i,j} z_j, m_{i,i} \right), \quad (3)$$

103 where \mathbf{z}_{-i} is the vector of all z_j where $j \neq i$, \mathbf{C} is the spatial weights matrix with $c_{i,j}$ as its i, j th
 104 element, $c_{i,i} = 0$, and \mathbf{M} is a diagonal matrix with positive diagonal elements $m_{i,i}$. Note that $m_{i,i}$
 105 may depend on the values in the i th row of \mathbf{C} . In this parameterization, the conditional mean of
 106 each Z_i is a weighted linear combination of values at neighboring nodes. The variance component,
 107 $m_{i,i}$, often varies with node i , and thus \mathbf{M} is generally heteroscedastic. In contrast to SAR models,
 108 it is not obvious that (3) leads to a full joint distribution for \mathbf{Z} . Besag (1974) used Brook’s lemma
 109 (Brook, 1964) and the Hammersley-Clifford theorem (Hammersley and Clifford, 1971; Clifford,
 110 1990) to show that, when $(\mathbf{I} - \mathbf{C})^{-1}\mathbf{M}$ is positive definite, $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{\Sigma}_{\text{CAR}})$, with

$$\mathbf{\Sigma}_{\text{CAR}} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{M}. \quad (4)$$

111 $\mathbf{\Sigma}_{\text{CAR}}$ must be symmetric, requiring

$$\frac{c_{i,j}}{m_{i,i}} = \frac{c_{j,i}}{m_{j,j}}, \quad \forall i, j. \quad (5)$$

112 Most authors describe CAR models as the construction (3), with the condition that $\mathbf{\Sigma}_{\text{CAR}}$ must be
 113 positive definite given the symmetry condition (5). However, more specific statements are possible
 114 on the necessary conditions for $(\mathbf{I} - \mathbf{C})$, making a comparable condition to S1 for SAR models.

115 We provide a proof, Proposition 1 in the Appendix, showing that if \mathbf{M} is diagonal with positive
 116 diagonal elements, along with (5) (forcing symmetry on Σ_{CAR}), then Σ_{CAR} is positive definite if
 117 and only if $(\mathbf{I} - \mathbf{C})$ has positive eigenvalues. Note that it might be easier to model symmetric
 118 $\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{1/2}$, as it establishes a link between \mathbf{C} , \mathbf{M} , and (5) directly (Cressie and Chan, 1989),
 119 and then $(\mathbf{I} - \mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{1/2})$ must have positive eigenvalues for Σ_{CAR} to be positive definite (which
 120 is used as part of the proof to Proposition 1).

121 In summary, the following conditions must be met for Σ_{CAR} in (4) to be a valid CAR
 122 covariance matrix:

- 123 **C1** $(\mathbf{I} - \mathbf{C})$ has positive eigenvalues,
- 124 **C2** \mathbf{M} is diagonal with positive diagonal elements,
- 125 **C3** $c_{i,i} = 0, \forall i$, and
- 126 **C4** $c_{i,j}/m_{i,i} = c_{j,i}/m_{j,j}, \forall i, j$.

127 2.3 Weights Matrices

128 In practice, $\mathbf{B} = \rho_s \mathbf{W}$ and $\mathbf{C} = \rho_c \mathbf{W}$ are usually used to construct valid SAR and CAR models,
 129 where \mathbf{W} is a weights matrix with $w_{i,j} \neq 0$ when locations i and j are neighbors, otherwise
 130 $w_{i,j} = 0$. Neighbors are typically pre-specified by the modeler. When i and j are neighbors, we
 131 often set $w_{i,j} = w_{j,i} = 1$ (so \mathbf{W} is symmetric), or use row-standardization so that $\sum_{j=1}^n w_{i,j} = 1$;
 132 that is, dividing each row in unstandardized \mathbf{W} by $w_{i,+} \equiv \sum_{j=1}^n w_{i,j}$ yields an asymmetric row-
 133 standardized matrix that we denote as \mathbf{W}_+ . For CAR models, define \mathbf{M}_+ as the diagonal matrix
 134 with $m_{i,i} = 1/w_{i,+}$, then (5) is satisfied. The row-standardized CAR model's covariance matrix
 135 can be written equivalently as

$$\Sigma_+ = \sigma^2(\mathbf{I} - \rho_c \mathbf{W}_+)^{-1} \mathbf{M}_+ = \sigma^2(\text{diag}(\mathbf{W}\mathbf{1}) - \rho_c \mathbf{W})^{-1}, \quad (6)$$

136 where $\mathbf{1}$ is a vector of all ones, σ^2 is an overall variance parameter, and $\text{diag}(\cdot)$ creates a diagonal
 137 matrix from a vector. A special case of the CAR model, called the intrinsic autoregressive model
 138 (IAR) (Besag and Kooperberg, 1995), occurs when $\rho_c = 1$, but the covariance matrix does not
 139 exist, so we do not consider it further.

140 There can be confusion on how ρ is constrained for SAR and CAR models, which we now
141 clarify. Suppose that \mathbf{N} is a square matrix with real eigenvalues, as would be the case if $\mathbf{N} = \mathbf{W}$
142 for symmetric \mathbf{W} . If \mathbf{W} is asymmetric with possibly complex eigenvalues, then for the CAR
143 covariance matrix (4), Cressie and Chan (1989) use $\mathbf{N} = \mathbf{M}^{-1/2}\mathbf{W}\mathbf{M}^{1/2}$, which is symmetric with
144 real eigenvalues due to CAR condition (C4). Note that Li et al. (2012) use $\mathbf{N} = \mathbf{\Omega}^{-1/2}\mathbf{W}\mathbf{\Omega}^{1/2}$ in
145 the SAR setting (2) and estimate $\mathbf{\Omega}$ from data, and deal directly with the possibility of complex
146 eigenvalues. In addition, if $\mathbf{\Omega}$ is unconstrained, then condition (C4) could be applied for SAR models
147 to $\mathbf{\Omega}$ such that $\mathbf{N} = \mathbf{\Omega}^{-1/2}\mathbf{W}\mathbf{\Omega}^{1/2}$ is symmetric (Wall, 2004). Let $\{\lambda_i\}$ be the set of eigenvalues of
148 \mathbf{N} , and let $\{\omega_i\}$ be the set of eigenvalues of $(\mathbf{I} - \rho\mathbf{N})$. Then, in the Appendix (Propositions 2, 3),
149 we show that $\omega_i = (1 - \rho\lambda_i)$. Li et al. (2007) note that when all $\lambda_i \neq 0$, $(\mathbf{I} - \rho\mathbf{N})$ will be nonsingular
150 for all $\rho \notin \{\lambda_i^{-1}\}$. However, notice that if $\lambda_i = 0$, then $\omega_i = 1$ for all ρ . In fact, if all $\lambda_i = 0$,
151 then all $\omega_i = 1$, and $(\mathbf{I} - \rho\mathbf{N})$ will be nonsingular, satisfying SAR model condition S1. In general,
152 then, $\rho \notin \{\lambda_i^{-1}\}$ whenever $\lambda_i \neq 0$ is necessary and sufficient for SAR model condition S1. If any
153 $\lambda_i \neq 0$, then at least two λ_i are nonzero because $\text{tr}(\mathbf{N}) = \sum_{i=1}^n \lambda_i = 0$. If at least two eigenvalues
154 are nonzero, then $\lambda_{[1]}$, the smallest eigenvalue of \mathbf{N} , must be less than zero, and $\lambda_{[N]}$, the largest
155 eigenvalue of \mathbf{N} , must be greater than zero. Then $1/\lambda_{[1]} < \rho < 1/\lambda_{[N]}$ ensures that $(\mathbf{I} - \rho\mathbf{N})$ has
156 positive eigenvalues (Appendix, Propositions 2, 3) and satisfies condition C1 for CAR models.

157 In practice, the restriction $1/\lambda_{[1]} < \rho < 1/\lambda_{[N]}$ is often used for both CAR and SAR
158 models. When considering \mathbf{W}_+ , the restriction becomes $1/\lambda_{[1]} < \rho < 1$ (Haining, 1990, p. 82),
159 where usually $1/\lambda_{[1]} < -1$. Wall (2004) shows irregularities for negative ρ values near the lower
160 bound for both SAR and CAR models, thus many modelers simply use $-1 < \rho < 1$. In fact,
161 in many cases, only positive autocorrelation is expected, so a further restriction is used where
162 $0 < \rho < 1$ (e.g., Li et al., 2007). For these constructions, $\mathbf{\Sigma}_{\text{SAR}}$ and $\mathbf{\Sigma}_{\text{CAR}}$ typically show more
163 positive marginal autocorrelation with increasing positive ρ values, and more negative marginal
164 autocorrelation with decreasing negative ρ values (Wall, 2004). There has been little research on
165 the behavior of autocorrelation outside of these limits for SAR models.

166 Weights in \mathbf{W} can be based on distance (Cressie and Chan, 1989) or may be modeled as
167 asymmetric for SAR models (Burden et al., 2015). Cressie et al. (2005) establish a link between \mathbf{W}

168 and \mathbf{M} that allows ρ_c to be constrained and interpreted as a partial correlation parameter when
 169 working with spatial rates rather than row-standardization, and this idea, of parameterizing so that
 170 ρ_c is a partial correlation, is generalized for numbers of neighbors in the ACAR model of Cressie
 171 and Wikle (2011, p. 188). A useful parameterization for CAR models was given by Pettitt et al.
 172 (2002); if $\gamma_{i,j}$ represents some function of distance between sites i and j (and often set to zero
 173 beyond a certain range), then \mathbf{C} in (4) is constructed as $c_{i,i} = 0$,

$$c_{i,j} = \frac{\phi\gamma_{i,j}}{1 + |\phi| \sum_{j \neq i} \gamma_{i,j}}, \quad (7)$$

174 for $i \neq j$, and \mathbf{M} in (4) is constructed as

$$m_{i,i} = \frac{1}{1 + |\phi| \sum_{j \neq i} \gamma_{i,j}}. \quad (8)$$

175 Here, ϕ is an unbounded parameter, obviating the need to find eigenvalues for \mathbf{C} .

176 Our goal is to develop relationships that allow a CAR covariance matrix, satisfying conditions
 177 C1 - C4, to be obtained from a SAR covariance matrix, satisfying conditions S1 - S3, and vice versa.
 178 We develop these in the next section, and, in the Discussion and Conclusions section, we contrast
 179 our results to the incomplete results of previous literature.

180 3 Relationships between CAR and SAR models

181 Assume a covariance matrix for a SAR model as given in (2) and a covariance matrix for a CAR
 182 model as given in (4). Cressie and Wikle (2011, p. 185-186) give the result that any Gaussian
 183 distribution on a finite set of points, $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{\Sigma})$, can be written with a covariance matrix
 184 parameterized either as a CAR model, $\mathbf{\Sigma} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{M}$. We show the additional result that the
 185 distribution of \mathbf{Z} can be written as a SAR model with covariance matrix, $\mathbf{\Sigma} = (\mathbf{I} - \mathbf{B})^{-1}\mathbf{\Omega}(\mathbf{I} - \mathbf{B}^T)^{-1}$.
 186 It is straightforward to generalize to the case where the mean is nonzero so, for simplicity of notation,
 187 we use the zero mean case. A corollary is that any CAR covariance matrix can be written as a
 188 SAR covariance matrix, and vice versa.

189 We now state the theorems that both SAR and CAR covariance matrices are sufficiently
190 general to represent any finite-dimensional positive-definite covariance matrix. We outline the
191 proofs, which is useful for discussion. Detailed proofs, showing that conditions S1-S3 are satisfied
192 for SAR models, and conditions C1-C4 are satisfied for CAR models, are given in the Appendix.

193 **Theorem 1.** *Any positive definite covariance matrix Σ can be expressed as the covariance matrix*
194 *of a SAR model $(\mathbf{I} - \mathbf{B})^{-1}\mathbf{\Omega}(\mathbf{I} - \mathbf{B}^T)^{-1}$, (2), for a (non-unique) pair of matrices \mathbf{B} and $\mathbf{\Omega}$.*

195 For a basic outline to the proof of Theorem 1, write $\Sigma^{-1} = \mathbf{L}\mathbf{L}^T$, where \mathbf{L} will be full
196 rank and suppose it has positive diagonal elements. Note that \mathbf{L} is *not* unique. For example, a
197 Cholesky decomposition (Harville, 1997, p.229) is different from a square-root matrix (Harville,
198 1997, p.543), yet either could be used to obtain \mathbf{L} , and each will have strictly positive diagonal
199 elements. Decompose \mathbf{L} into $\mathbf{L} = \mathbf{G} - \mathbf{P}$ where \mathbf{G} is diagonal and \mathbf{P} has zeros on the diagonal, so
200 $\mathbf{L}\mathbf{L}^T = (\mathbf{G} - \mathbf{P})(\mathbf{G}^T - \mathbf{P}^T)$. Then set $\mathbf{\Omega}^{-1} = \mathbf{G}\mathbf{G}$ and $\mathbf{B}^T = \mathbf{P}\mathbf{G}^{-1}$, so $\Sigma^{-1} = (\mathbf{I} - \mathbf{B}^T)\mathbf{\Omega}^{-1}(\mathbf{I} - \mathbf{B})$,
201 expressed in SAR form (2).

202 The result that follows was given by Cressie and Wikle (2011, p. 185-186), that any multi-
203 variate Gaussian distribution can be written as a CAR model. Their construction of the equivalent
204 CAR model corrects an earlier one given by Cressie (1993, p. 434) where the diagonal elements of
205 \mathbf{C} were not necessarily zero.

206 **Theorem 2.** *Any positive-definite covariance matrix Σ can be expressed as the covariance matrix*
207 *of a CAR model $(\mathbf{I} - \mathbf{C})^{-1}\mathbf{M}$, (4), for a unique pair of matrices \mathbf{C} and \mathbf{M} .*

208 For a basic outline to the proof of Theorem 2, let $\mathbf{Q} = \Sigma^{-1}$ and decompose it into $\mathbf{Q} =$
209 $\mathbf{D} - \mathbf{R}$, where \mathbf{D} is diagonal with elements $d_{i,i} = q_{i,i}$ (the diagonal elements of the precision
210 matrix \mathbf{Q}), and \mathbf{R} has zeros on the diagonal ($r_{i,i} = 0$) and off-diagonals equal to $r_{i,j} = -q_{i,j}$. Set
211 $\mathbf{C} = \mathbf{D}^{-1}\mathbf{R}$ and $\mathbf{M} = \mathbf{D}^{-1}$. Then $\Sigma^{-1} = \mathbf{D} - \mathbf{R} = \mathbf{D}(\mathbf{I} - \mathbf{D}^{-1}\mathbf{R}) = \mathbf{M}^{-1}(\mathbf{I} - \mathbf{C})$, with Σ expressed
212 in CAR form (4)

213 Having shown that any positive definite matrix Σ can be expressed as either the covariance
214 matrix of a CAR model or the covariance matrix of a SAR model, we have the following corollary.

215 **Corollary 1.** *Any SAR model can be written as a unique CAR model, and any CAR model can be*
216 *written as a non-unique SAR model.*

217 *Proof.* The proof follows directly by first noting that a SAR model yields a positive-definite covari-
 218 ance matrix, and applying Theorem 2, and then noting that a CAR model yields a positive-definite
 219 covariance matrix, and applying Theorem 1. \square

220 The following corollary gives more details on the non-unique nature of the SAR models.

221 **Corollary 2.** *Any positive-definite covariance matrix can be expressed as one of an infinite number*
 222 *of \mathbf{B} matrices that define the SAR covariance matrix in (2).*

223 *Proof.* Write $\Sigma^{-1} = \mathbf{L}\mathbf{L}^T$ as in Theorem 1. Let $\mathbf{A}_{h,s}(\theta)$ be a Givens rotation matrix (Golub and
 224 Van Loan, 2013), which is a sparse orthonormal matrix that rotates angle θ through the plane
 225 spanned by the h and s axes, where h indexes the row of \mathbf{A} , and s indexes the column. The
 226 elements of $\mathbf{A}_{h,s}(\theta)$ are as follows. For $i \notin \{h, s\}$, $a_{i,i} = 1$. Also $a_{h,h} = a_{s,s} = \cos(\theta)$, $a_{h,s} = \sin(\theta)$
 227 and $a_{s,h} = -\sin(\theta)$. All other entries of $\mathbf{A}_{h,s}(\theta)$ are equal to zero. That is, $\mathbf{A}_{h,s}(\theta)$ has form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cos(\theta) & 0 & \cdots & 0 & \sin(\theta) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -\sin(\theta) & 0 & \cdots & 0 & \cos(\theta) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

228 Notice that $\Sigma^{-1} = \mathbf{L}\mathbf{L}^T = \mathbf{L}(\mathbf{A}_{h,s}^T(\theta)\mathbf{A}_{h,s}(\theta))\mathbf{L}^T = \mathbf{L}_*\mathbf{L}_*^T$, where $\mathbf{L}_* = \mathbf{L}\mathbf{A}_{h,s}^T(\theta)$. A SAR covariance
 229 matrix can be developed as readily for \mathbf{L}_* as for \mathbf{L} in the proof of Theorem 1. Any of the infinite
 230 values of $\theta \in [0, 2\pi)$ will result in a unique $\mathbf{A}_{h,s}(\theta)$, leading to a different \mathbf{L}_* , and a different \mathbf{B}
 231 matrix in (A.1), but yielding the same positive-definite covariance matrix Σ . \square

232 3.1 Implications of Theorems and Corollaries

233 Note that for Corollary 2, additional \mathbf{B} matrices that define a fixed positive-definite covariance
 234 matrix in Corollary 2 could also be obtained by repeated Givens rotations. For example, let
 235 $\mathbf{L}_* = \mathbf{L}\mathbf{A}_{1,2}^T(\theta)\mathbf{A}_{3,4}^T(\eta)$ for angles θ and η . Then a new \mathbf{B} can be developed for this \mathbf{L}_* just as
 236 readily as those in the proof to Corollary 2. We use this idea extensively in the examples.

237 Theorem 1 helps clarify the use of $\mathbf{\Omega}$. Authors often write the SAR model covariance matrix
 238 as $(\mathbf{I}-\mathbf{B})^{-1}(\mathbf{I}-\mathbf{B}^T)^{-1}$, assuming that $\mathbf{\Omega} = \mathbf{I}$ in (2). In the proofs to Theorem 1 and Corollary 2, this
 239 requires finding \mathbf{L} with ones on the diagonal so that $\mathbf{G} = \mathbf{I}$. It is interesting to consider if one can
 240 always find such \mathbf{L} , which would justify the practice of using the simpler form, $(\mathbf{I}-\mathbf{B})^{-1}(\mathbf{I}-\mathbf{B}^T)^{-1}$,
 241 for SAR models. We now show that this formulation does not allow the use of Theorem 1. Consider
 242 the case where the dimensions of the matrices involved are 2×2 . Then a matrix \mathbf{L} that has ones
 243 on the diagonal, but is otherwise completely general, is

$$\mathbf{L} = \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix},$$

244 so

$$\mathbf{L}\mathbf{L}^T = \begin{pmatrix} 1+a^2 & a+b \\ a+b & 1+b^2 \end{pmatrix}.$$

245 Now the matrix,

$$\begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix},$$

246 is positive definite, but is not expressible as $\mathbf{L}\mathbf{L}^T$ if \mathbf{L} is restricted to having ones on the diagonal.
 247 We conclude that not every possible positive definite covariance matrix can be written in the form
 248 $(\mathbf{I}-\mathbf{B})^{-1}(\mathbf{I}-\mathbf{B}^T)^{-1}$, so $\mathbf{\Omega}$ is necessary in $(\mathbf{I}-\mathbf{B})^{-1}\mathbf{\Omega}(\mathbf{I}-\mathbf{B}^T)^{-1}$ for Theorem 1 to hold.

249 In Section 2.3, we discussed how most CAR and SAR models are constructed by constraining
 250 ρ in $\rho\mathbf{W}$. Consider Theorem 1, where \mathbf{L} is a lower-triangular Cholesky decomposition. Then \mathbf{P} has
 251 zero diagonals and is strictly lower triangular, and so $\mathbf{B}^T = \mathbf{P}\mathbf{G}^{-1}$ is strictly lower triangular. In
 252 this construction, all of the eigenvalues of \mathbf{B} are zero. Thus, for SAR models, there are unexplored
 253 classes of models that do not depend on the typical construction $\mathbf{B} = \rho\mathbf{W}$.

254 Most CAR and SAR models are developed such that \mathbf{C} and \mathbf{B} are sparse matrices, containing
 255 mostly zeros, but containing positive elements whose weights depend locally on neighbors. Although
 256 we demonstrated how to obtain a CAR covariance matrix from a SAR covariance matrix, and vice
 257 versa, there is no guarantee that using a sparse \mathbf{C} in a CAR model will yield a sparse \mathbf{B} in a

258 SAR model, or vice versa. Note, however, that Zimmerman and Nunez-Anton (2009, p. 244-245)
259 show how antedependence models, with conditional dependence in one dimension where data are
260 ordered by time, can be viewed as CAR models. These one-dimensional CAR models lead to a
261 sparse Cholesky decomposition (Zimmerman and Nunez-Anton, 2009, Theorem 2.3, p. 41), which
262 would lead to a sparse SAR model formulation. The antedependence models work in more than one
263 dimension, and Zimmerman and Nunez-Anton (2009, Theorem 2.3, p. 41) generalizes to obtaining
264 a sparse Cholesky decomposition from a CAR model. In fact, that is the goal of (Rue and Held,
265 2005, Section 2.4, p. 40), who give algorithms to concentrate nonzero values near the diagonal, and
266 show that if an outer (away from the diagonal) triangular part of the matrix \mathbf{C} in a CAR model
267 is all zeros, then that same outer triangular part of \mathbf{L} will also be zeros. We return to this idea in
268 the examples.

269 4 Examples

270 We provide two examples, one where we illustrate Theorem 1 primarily, and a second where we
271 use Theorem 2. In the first, we fabricated a simple neighborhood structure and created a positive
272 definite matrix by a CAR construction. Using Givens rotation matrices, we then obtained various
273 non-unique SAR covariance matrices from the CAR covariance matrix. We also explore sparseness
274 in \mathbf{B} for SAR models when they are obtained from sparse \mathbf{C} for CAR models.

275 For a second example, we used real data on neighborhood crimes in Columbus, Ohio. We
276 model the data with the two most common CAR models, using a first-order neighborhood model
277 where \mathbf{C} is both unstandardized and row-standardized. Then, from a positive-definite covariance
278 matrix obtained from a geostatistical model, we obtain the equivalent and unique CAR covariance
279 matrix. We use the weights obtained from the geostatistical covariance matrix to allow further
280 CAR modeling, finding a better likelihood optimization than both the unstandardized and row-
281 standardized first-order CAR models.

282 **4.1 Uniqueness and sparseness for SAR models**

283 Consider the graph in Figure 1a, which shows an example of neighbors for a CAR model. Using one
 284 to indicate a neighbor, and zeros elsewhere, the \mathbf{W} matrix was used to create the row-standardized
 285 \mathbf{W}_+ matrix in (6). Values of $\rho_c \mathbf{W}_+$, where $\rho_c = 0.9$, are shown graphically in Figure 1b. For the
 286 resulting covariance matrix, Σ_+ in (6), the Cholesky decomposition was used to create \mathbf{L} as in
 287 Theorem 1. Using (A.1) in Theorem 1, the weights matrix \mathbf{B} created from \mathbf{L} is shown in Figure 1c.
 288 Note from Figure 1b that, beyond indices separated by more than 5, all elements are zero. Those
 289 indices separated by 5 can be seen with a vertical orientation in Figure 1a. Consequently, the
 290 off-diagonal elements in the Cholesky decomposition shown in Figure 1c, with indices separated
 291 by more than 5, are all zero (in keeping with Theorem 2.3, Zimmerman and Nunez-Anton, 2009,
 292 p. 41), and many of those indices separated by less than 5 are non-zero “fill-in” values (Rue and
 293 Held, 2005, p. 44). One approach to obtain sparseness works by concentrating non-zero values in
 294 \mathbf{W} to be near the diagonal by re-indexing the data (Rue and Held, 2005, p. 47). Other approaches
 295 include tapering (Furrer et al., 2006) and thresholding (Bickel and Levina, 2008) for covariance
 296 matrices (for a broad treatment, see Pourahmadi, 2013).

297 For the same covariance matrix Σ_+ , we also used the spectral decomposition to create
 298 \mathbf{L} as in Theorem 1. The weights matrix \mathbf{B} created from this \mathbf{L} , using (A.1) in Theorem 1, is
 299 shown in Figure 1d. Note that the \mathbf{B} matrix in Figure 1d is less sparse than \mathbf{B} in Figure 1c,
 300 although they both yield exactly the same covariance matrix by the SAR construction (2), which
 301 we verified numerically. Figure 1c, because it is strictly upper triangular, also verifies our comments
 302 in Section 3.1; that there exists some \mathbf{B} whose eigenvalues are all zero.

303 In addition to re-indexing data with the Cholesky decomposition to obtain sparseness, we
 304 sought to transform the \mathbf{B} matrix in Figure 1d to a sparser form using the proof to Corollary 2 and
 305 the Givens rotations. For a nonzero vector $\mathbf{x} = (x_1, \dots, x_n)$, an index of sparseness (Hoyer, 2004)
 306 is

$$\text{sparseness}(\mathbf{x}) = \frac{\sqrt{n} - \frac{\sum_i |x_i|}{\sqrt{\sum_i x_i^2}}}{\sqrt{n} - 1},$$

307 which ranges from zero to one. Ignoring the dimensions of a matrix, we create the matrix function

$$f(\mathbf{B}) = \frac{\sum_{i,j} |b_{i,j}|}{\sqrt{\sum_{i,j} b_{i,j}^2}},$$

308 which is a measure of the fullness of a matrix. We propose an iterative algorithm to minimize
 309 $f(\mathbf{B})$ for orthonormal Givens rotations as explained in Corollary 2. Let $\mathbf{L}_{h,s}(\theta) = \mathbf{L}\mathbf{A}_{h,s}^T(\theta)$, where
 310 $\mathbf{L} = \mathbf{V}\mathbf{E}^{-1/2}\mathbf{V}^{-1}$ used the spectral decomposition of Σ_+ as in the proof of Theorem 1, and $\mathbf{A}_{h,s}(\theta)$
 311 is a Givens rotation matrix as in the proof of Corollary 2. Denote θ_k^* as the value of θ that minimizes
 312 $f(\mathbf{B})$ when \mathbf{B} is created by decomposing $\mathbf{L}\mathbf{A}_{h,s}^T(\theta)$ into \mathbf{P} and \mathbf{G} (as in (ii) in Theorem 1), while
 313 constraining θ to values satisfying $b_{i,j} \geq 0 \forall i, j$. Then $\mathbf{L}_{1,2}^{[1]} \equiv \mathbf{L}\mathbf{A}_{1,2}^T(\theta_1^*)$, where $k = 1$ is the first
 314 iteration. For the second iteration, let θ_2^* be the value that minimizes $f(\mathbf{B})$ for \mathbf{B} created from
 315 $\mathbf{L}_{1,2}^{[1]}\mathbf{A}_{1,3}^T(\theta)$, and hence for $k = 2$, $\mathbf{L}_{1,3}^{[2]} \equiv \mathbf{L}_{1,2}^{[1]}\mathbf{A}_{1,3}^T(\theta_2^*)$. We cycled through $h = 1, 2, \dots, 24$ and
 316 $s = (h + 1), \dots, 25$ for each iteration k in a coordinate descent minimization of $f(\mathbf{B})$. We cycled
 317 through all of h and s eight times for a total of $8(25)(25 - 1)/2 = 2400$ iterations. The value
 318 of $f(\mathbf{B})$ for each iteration is plotted in Figure 1e and the final \mathbf{B} matrix is given in Figure 1f.
 319 Although we did not achieve the sparsity of Figure 1c, we were able to increase sparseness from
 320 the starting matrix in Figure 1d. Note that the \mathbf{B} matrix depicted in Figure 1f yields exactly the
 321 same covariance matrix as the \mathbf{B} matrices shown in Figures 1c,d. There are undoubtedly better
 322 ways to minimize $f(\mathbf{B})$, such as simulated annealing (Kirkpatrick et al., 1983), and there may be
 323 alternative optimization criteria. We do not pursue these here. Our goal was to show that it is
 324 possible to explore many configurations of matrix weights in SAR models, which produce equivalent
 325 covariance matrices, by using orthonormal Givens rotations of the \mathbf{L} matrix.

326 4.2 Columbus Crime Data

327 The Columbus data are found in the `spdep` package (Bivand et al., 2013; Bivand and Piras, 2015)
 328 for R (R Core Team, 2016). Figure 2 shows 49 neighborhoods in Columbus, Ohio. We used
 329 residential burglaries and vehicle thefts per thousand households in the neighborhood (Anselin,
 330 1988, Table 12.1, p. 189) as the response variable. Spatial pattern among neighborhoods appeared

331 autocorrelated (Figure 2), with higher crime rates in the more central neighborhoods. When
 332 analyzing rate data, it is customary to account for population size (e.g., Clayton and Kaldor,
 333 1987), which affects the variance of the rates. However, for illustrative purposes, we used raw
 334 rates. A histogram of the data appeared approximately bell-shaped, thus we assumed a Gaussian
 335 distribution with a covariance matrix containing autocorrelation among locations.

336 First-order neighbors were also taken from the `spdep` package for R, and are shown by white
 337 lines in Figure 2. Using a one to indicate a neighbor, and zero otherwise, we denote the 49×49
 338 matrix of weights as \mathbf{W}_{un} , and the CAR precision matrix has $\mathbf{C} = \rho_{\text{un}}\mathbf{W}_{\text{un}}$ and $\mathbf{M} = \sigma_{\text{un}}^2\mathbf{I}$ in
 339 (4). Using the eigenvalues of \mathbf{W}_{un} , the bounds for ρ_{un} were $-0.335 < \rho_{\text{un}} < 0.167$. We added
 340 a constant independent diagonal component, $\delta_{\text{un}}^2\mathbf{I}$ (also called the nugget effect in geostatistics),
 341 so the covariance matrix was $\Sigma_{\text{un}} = \sigma_{\text{un}}^2(\mathbf{I} - \rho_{\text{un}}\mathbf{W}_{\text{un}})^{-1} + \delta_{\text{un}}^2\mathbf{I}$. Denote the crime rates as \mathbf{y} .
 342 We assumed a constant mean, so $\mathbf{y} \sim \mathcal{N}(\mathbf{1}\mu, \Sigma_{\text{un}})$, where $\mathbf{1}$ is a vector of all ones. Let $\mathcal{L}(\boldsymbol{\theta}_{\text{un}}|\mathbf{y})$
 343 be minus 2 times the restricted maximum likelihood function (REML, Patterson and Thompson,
 344 1971, 1974) for the crime data, where the set of covariance parameters is $\boldsymbol{\theta}_{\text{un}} = (\sigma_{\text{un}}^2, \rho_{\text{un}}, \delta_{\text{un}}^2)^T$.
 345 We optimized the REML likelihood and obtained $\mathcal{L}(\hat{\boldsymbol{\theta}}_{\text{un}}|\mathbf{y}) = 388.83$. Recall that CAR models are
 346 generally heteroscedastic (e.g., Wall, 2004). The marginal variances of the estimated model are
 347 shown in Figure 3a, and the marginal correlations are shown in Figure 4a.

348 We also optimized the likelihood using the row-standardized weights matrix, \mathbf{W}_+ in (6),
 349 which we denote \mathbf{W}_{rs} . In this case, the CAR precision matrix has $\mathbf{C} = \rho_{\text{rs}}\mathbf{W}_+$, $-1 < \rho_{\text{rs}} < 1$,
 350 and $\mathbf{M} = \sigma_{\text{rs}}^2\mathbf{M}_+$ in (4). Again we added a nugget effect, so $\Sigma_{\text{rs}} = \sigma_{\text{rs}}^2(\mathbf{I} - \rho_{\text{rs}}\mathbf{W}_+)^{-1}\mathbf{M}_+ +$
 351 $\delta_{\text{rs}}^2\mathbf{I}$. For the set of covariance parameters $\boldsymbol{\theta}_{\text{rs}} = (\sigma_{\text{rs}}^2, \rho_{\text{rs}}, \delta_{\text{rs}}^2)^T$, we obtained $\mathcal{L}(\hat{\boldsymbol{\theta}}_{\text{rs}}|\mathbf{y}) = 397.25$.
 352 This shows that the unstandardized weights matrix \mathbf{W}_{un} provides a substantially larger REML
 353 likelihood optimization than \mathbf{W}_{rs} . The marginal variances of the row-standardized model are
 354 shown in Figure 3b, and the marginal correlations are shown in Figure 4b. The difference between
 355 $\mathcal{L}(\hat{\boldsymbol{\theta}}_{\text{un}}|\mathbf{y})$ and $\mathcal{L}(\hat{\boldsymbol{\theta}}_{\text{rs}}|\mathbf{y})$ indicates that the weights matrix \mathbf{C} has a substantial effect for these data.

356 To show that a CAR covariance matrix can be developed from any covariance matrix (The-
 357 orem 2), next we fit a geostatistical model and derive the corresponding CAR covariance matrix.
 358 We optimized the likelihood with a geostatistical model using a spherical autocorrelation model.

359 Denote the geostatistical correlation matrix as \mathbf{S} , where

$$s_{i,j} = [1 - 1.5(e_{i,j}/\alpha) + 0.5(e_{i,j}/\alpha)^3]\mathcal{I}(d_{i,j} < \alpha),$$

360 and $\mathcal{I}(\cdot)$ is the indicator function, equal to one if its argument is true, otherwise it is zero, and $e_{i,j}$
 361 is Euclidean distance between the centroids of the i th and j th polygons in Figure 2. We included
 362 a nugget effect, so $\Sigma_{\text{sp}} = \sigma_{\text{sp}}^2 \mathbf{S} + \delta_{\text{sp}}^2 \mathbf{I}$. For the set of covariance parameters $\theta_{\text{sp}} = (\sigma_{\text{sp}}^2, \alpha, \delta_{\text{sp}}^2)^T$,
 363 we obtained $\mathcal{L}(\hat{\theta}_{\text{sp}}|\mathbf{y}) = 374.61$. The geostatistical model provides a substantially better optimized
 364 likelihood than either the unstandardized or row-standardized CAR model. The marginal variances
 365 of geostatistical models are equal (Figure 3c). The estimated range parameter, $\hat{\alpha}$, is shown by the
 366 lower bar in Figure 2. Responses at locations separated by a distance greater than that shown by
 367 the bar are estimated to have zero correlation (Figure 4c).

368 It appears that the geostatistical model provides a much better optimized likelihood than the
 369 two most commonly-used CAR models. Others have compared CAR to geostatistical models (e.g.,
 370 Banerjee et al., 2003; Hrafnkelsson and Cressie, 2003; Song et al., 2008), and Rue and Tjelmeland
 371 (2002) and Cressie and Verzele (2008) use a “closeness” criteria to approximate a geostatistical
 372 model with a CAR model, but where they enforce some sparsity in the CAR weight matrix. Here,
 373 we try another way to find a CAR model to compete with the geostatistical model, but where
 374 the weight matrix is not sparse. Using Theorem 2, as in Cressie and Wikle (2011, p. 185), we
 375 created \mathbf{C}_{cg} and \mathbf{M}_{cg} as in (A.2) in the Appendix from the positive definite covariance matrix
 376 of the geostatistical model, $\Sigma_{\text{sp}} = (\mathbf{I} - \mathbf{C}_{\text{cg}})^{-1}\mathbf{M}_{\text{cg}}$. Here, we have a CAR representation that
 377 is equivalent to the spherical geostatistical model. Now consider scaling \mathbf{C}_{cg} with ρ_{cg} , so $\Sigma_{\text{cg}} =$
 378 $\sigma_{\text{cg}}^2(\mathbf{I} - \rho_{\text{cg}}\mathbf{C}_{\text{cg}})^{-1}\mathbf{M}_{\text{cg}} + \delta_{\text{cg}}^2\mathbf{I}$, which we optimized for $\theta_{\text{cg}} = (\sigma_{\text{cg}}^2, \rho_{\text{cg}}, \delta_{\text{cg}}^2)^T$. For Σ_{cg} to be positive
 379 definite, $\sigma_{\text{cg}}^2 > 0$, $-1.104 < \rho_{\text{cg}} < 1.013$, and $\delta_{\text{cg}}^2 \geq 0$. Because $\theta_{\text{cg}} = (1, 1, 0)^T$ is in the parameter
 380 space, we can do no worse than the spherical geostatistical model. In fact, upon optimizing,
 381 we obtained $\mathcal{L}(\hat{\theta}_{\text{cg}}|\mathbf{y}) = 373.95$, where $\hat{\sigma}_{\text{cg}}^2 = 0.941$, $\hat{\rho}_{\text{cg}} = 1.01$, and $\hat{\delta}_{\text{cg}}^2 = 0$, a slightly better
 382 optimization than the spherical geostatistical model. The marginal variances for this geostatistical-
 383 assisted CAR model are shown in Figure 3d, and the marginal correlations are shown in Figure 4d.

384 Note the rather large changes from Figure 3c to Figure 3d, and from Figure 4c to Figure 4d, with
385 seemingly minor changes in $\hat{\sigma}_{cg}^2$, from 1 to 0.941, and in ρ_{cg} , from 1 to 1.01. Others have documented
386 rapid changes in CAR model behavior near the parameter boundaries, especially for ρ_{cg} (Besag and
387 Kooperberg, 1995; Wall, 2004). Note that for optimizing likelihoods, we transform ρ_{cg} with a logit
388 so that it is unbounded, and scaled so that the inverse logit is between -1.104 and 1.013. On the
389 logit-transformed scale, the rapid changes in ρ_{cg} near the bounds are no longer dramatic as they
390 get stretched toward $-\infty$ and ∞ .

391 5 Discussion and Conclusions

392 Some detailed comparisons of the mathematical relationships between CAR and SAR models have
393 been given in Besag (1974), Haining (1990, p. 89), and Cressie (1993, p. 408). Haining (1990, p.
394 89) provided several results that we restate using notation from Sections 2.1 and 2.2, and show
395 that some are incorrect or incomplete.

396 In an attempt to create a CAR covariance matrix from a SAR covariance matrix, assume
397 that \mathbf{B} satisfies conditions S1-S3 and $\boldsymbol{\Omega} = \mathbf{I}$ in (2). Let $\mathbf{M} = \mathbf{I}$ and \mathbf{C} be symmetric in (4) [which
398 omits the important case (6)]. Then setting SAR and CAR covariances matrix equal to each other,
399

$$(\mathbf{I} - \mathbf{C})^{-1} = [(\mathbf{I} - \mathbf{B})(\mathbf{I} - \mathbf{B}^T)]^{-1} = (\mathbf{I} - \mathbf{B} - \mathbf{B}^T + \mathbf{B}\mathbf{B}^T)^{-1}, \quad (9)$$

400 and Haining (1990) claims that \mathbf{C} can be obtained from \mathbf{B} by setting

$$\mathbf{C} = \mathbf{B} + \mathbf{B}^T - \mathbf{B}\mathbf{B}^T, \quad (10)$$

401 which is repeated in texts by Waller and Gotway (2004, p. 372) and Schabenberger and Gotway
402 (2005, p. 339), and in the literature (e.g., Dormann et al., 2007). However, aside from the lack of
403 generality due to assumptions $\mathbf{M} = \mathbf{I}$, $\boldsymbol{\Omega} = \mathbf{I}$, and symmetric \mathbf{C} , we note that (10) is incomplete
404 and too limiting to be useful, as given in the following remark.

405 **Remark 1.** *Condition C3 in Section 2.2 is not satisfied for \mathbf{C} in (10) except when \mathbf{B} contains all*

406 zeros.

407 *Proof.* Because \mathbf{B} has zeros on the diagonal, $\mathbf{B} + \mathbf{B}^T$ will have zeros on the diagonal. Denote \mathbf{b}_i as
408 the i th row of \mathbf{B} . Then the i th diagonal element of $\mathbf{B}\mathbf{B}^T$ will be $\mathbf{b}_i\mathbf{b}_i^T$, which will be zero only if all
409 elements of \mathbf{b}_i are zero. Hence, $\mathbf{B} + \mathbf{B}^T - \mathbf{B}\mathbf{B}^T$ will have zeros on the diagonal only if \mathbf{B} contains
410 all zeros. \square

411 In an attempt to create a SAR covariance matrix from a CAR covariance matrix, assume
412 the same conditions as for (9), and that \mathbf{C} satisfies conditions C1-C4. Let $(\mathbf{I} - \mathbf{C}) = \mathbf{S}\mathbf{S}^T$, where
413 \mathbf{S} is a Cholesky decomposition. Haining (1990) suggested $\mathbf{S} = \mathbf{I} - \mathbf{B}$ and setting \mathbf{B} equal to $\mathbf{I} - \mathbf{S}$.
414 However, this is incomplete because condition S3 in Section 2.1 will be satisfied only if \mathbf{S} has all
415 ones on the diagonal, which is also extremely limiting.

416 For another approach to relate SAR and CAR covariance matrices, Haining (1990) described
417 the model $\mathbf{F}(\mathbf{Z} - \boldsymbol{\mu}) = \mathbf{H}\boldsymbol{\varepsilon}$, where $\text{var}(\boldsymbol{\varepsilon}) = \mathbf{V}$. Then $E((\mathbf{Z} - \boldsymbol{\mu})(\mathbf{Z} - \boldsymbol{\mu})^T) = \mathbf{F}^{-1}\mathbf{H}\mathbf{V}\mathbf{H}^T(\mathbf{F}^{-1})^T$.
418 Now let $\mathbf{F} = (\mathbf{I} - \mathbf{C})$, $\mathbf{H} = \mathbf{I}$, and $\mathbf{V} = (\mathbf{I} - \mathbf{C})$ (this appears to originate in Martin (1987)).
419 The constructed model is really a SAR model except that it violates condition S2 by allowing
420 $\mathbf{V} = (\mathbf{I} - \mathbf{C})$. Alternatively, this can be seen as an attempt to create a SAR model from a CAR
421 model by assuming an inverse CAR covariance matrix for the error structure of the SAR model,
422 which gains nothing. Because these arguments are unconvincing, and other authors argue that one
423 cannot go uniquely from a CAR to a SAR (e.g., Mardia, 1990), we can find no further citations
424 for the arguments of Haining (1990) on obtaining a SAR covariance matrix from a CAR covariance
425 matrix.

426 Besag (1974) provided a demonstration of how a SAR covariance matrix with first-order
427 neighbors in \mathbf{B} leads to a CAR covariance matrix with third-order neighbors in \mathbf{C} , which we
428 reproduce here. Assume a rectangular lattice, as in Figure 1a, but with all first order neighbors,
429 and assume it is on a torus (making the top row neighbors of the bottom row, and the left side
430 neighbors of the right side, so all sites have 4 neighbors. Let $Z_{i,j}$ be a random variable in the i th
431 row and j column of the lattice. Assume a SAR model (2) with non-symmetric \mathbf{B} , created from

$$Z_{i,j} = \beta_1 Z_{i-1,j} + \beta_1' Z_{i+1,j} + \beta_2 Z_{i,j-1} + \beta_2' Z_{i,j+1} + \varepsilon_{i,j}, \quad (11)$$

432 where assume that $\text{var}(\{\varepsilon_{i,j}\}) = \mathbf{\Omega} = \mathbf{I}$. Then Besag (1974) showed that the corresponding CAR
 433 model is

$$\begin{aligned}
 E(Z_{i,j}|\{z_{k,\ell} : (k, \ell) \neq (i, j)\}) &= (1 + \beta_1^2 + \beta_1'^2 + \beta_2^2 + \beta_2'^2)^{-1}\{(\beta_1 + \beta_1')(z_{i-1,j} + z_{i+1,j}) \\
 &+ (\beta_2 + \beta_2')(z_{i,j-1} + z_{i,j+1}) - (\beta_1\beta_2' + \beta_1'\beta_2)(z_{i-1,j-1} + z_{i-1,j+1}) \\
 &- (\beta_1\beta_2 + \beta_1'\beta_2')(z_{i-1,j+1} + z_{i+1,j-1}) - (\beta_1\beta_1')(z_{i-2,j} + z_{i+2,j}) - (\beta_2\beta_2')(z_{i,j-2} + z_{i,j+2})\}, \quad (12)
 \end{aligned}$$

434 which follows by creating the covariance matrix from the SAR weights (11) and applying Theorem 2.
 435 Cressie (1993, p. 409) gave a version of (12) where \mathbf{B} was symmetric, although his formula had
 436 terms in it with incorrect signs. Besag's result (12) is useful for its generality in either the symmetric
 437 or non-symmetric case. If all $\beta_1, \beta_1', \beta_2, \beta_2'$ are nonzero the first-order SAR (11) leads to third-order
 438 CAR weights (12). It appears that, generally, there will be no equivalent SAR covariance matrices
 439 for first and second-order CAR covariance matrices. However, consider setting $\beta_1' = \beta_2 = \beta_2' = 0$,
 440 in which case an asymmetric first-order SAR weighting leads to a first-order CAR matrix (although
 441 only for the row weightings). Moreover, our demonstration in Figure 1c shows that a sparse \mathbf{B}
 442 may be obtained from a sparse CAR model, although it is asymmetric and may not have the usual
 443 neighborhood interpretation.

444 There are other parameterizations for CAR models. In (6) and (7) we introduced parame-
 445 terizations for a CAR model that can more generally be written as

$$\mathbf{\Sigma}_{\text{CAR}} = (\mathbf{D} - \mathbf{C}^\#)^{-1},$$

446 where the diagonal elements of $\mathbf{C}^\#$ are zero, and \mathbf{D} is diagonal. A reviewer pointed out that it is
 447 also useful to parameterize a CAR model as $\mathbf{\Sigma}_{\text{CAR}}^{-1} = \mathbf{C}^*$, in which case the conditional specification
 448 in terms of elements of \mathbf{C}^* is

$$Z_i|\mathbf{z}_{-i} \sim N\left(-\sum_{j \neq i} \frac{c_{i,j}^*}{c_{i,i}^*} z_j, \frac{\sigma^2}{c_{i,i}^*}\right), \quad (13)$$

449 which can be compared to (3). Each parameterization has its virtues. The unscaled weights are
 450 given directly in (3) including a zero weight for Z_i . To build models, one could simply say that \mathbf{C}^*
 451 must be positive definite, but more specific conditions on the diagonals and off-diagonals, similar
 452 to conditions C1 - C4 in Section 2.2, are useful, as illustrated by the construction of weights in
 453 Section 2.3. On the other hand, we can go from a SAR covariance matrix to a CAR covariance
 454 matrix simply by using $\mathbf{C}^* = (\mathbf{I} - \mathbf{B})\mathbf{\Omega}^{-1}(\mathbf{I} - \mathbf{B}^T)$. It is also easy to see from the proof of Theorem 1
 455 that it is possible to obtain a non-unique SAR covariance matrix from \mathbf{C}^* , as we did for \mathbf{C} .

456 From Section 2.3, we showed that, for SAR models, pre-specified weights $\mathbf{B} = \rho\mathbf{W}$ are often
 457 scaled by ρ , and that ρ is often constrained by the eigenvalues of \mathbf{W} . However, we also discussed in
 458 Section 3.1 that weights can be chosen so that all eigenvalues are zero for SAR models. Figure 1c
 459 provides an example where all diagonal elements of \mathbf{B} are zero, and hence a SAR model where all
 460 eigenvalues of \mathbf{B} are zero. We have little information or guidance for developing models where all
 461 eigenvalues of \mathbf{B} are zero, and this provides an interesting topic for future research.

462 Wall (2004) provided a detailed comparison on properties of marginal correlation for various
 463 values of ρ when \mathbf{B} or \mathbf{C} are parameterized as $\rho_s\mathbf{W}$ and $\rho_c\mathbf{W}$, respectively, but did not develop
 464 mathematical relationships between CAR and SAR models. Lindgren et al. (2011) showed that
 465 approximations to point-referenced geostatistical models based on a finite element basis expansion
 466 can be expressed as CAR models. In his discussion of the same, Kent (2011) noted that, for a
 467 given geostatistical model of the Matern class, one could construct either a CAR or SAR model
 468 that would approximate the Matern model. This indicates a correspondence between CAR and
 469 SAR models when used as approximations to continuous-space processes, but does not address the
 470 relationship between CAR and SAR models on a native areal support.

471 Our literature review and discussion showed that there have been scattered efforts to es-
 472 tablish mathematical relationships between CAR and SAR models, and some of the reported re-
 473 lationships are incomplete on the conditions for those relationships. With Theorems 1 and 2 and
 474 Corollary 1, we demonstrated that any zero-mean Gaussian distribution on a finite set of points,
 475 $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{\Sigma})$, with positive-definite covariance matrix $\mathbf{\Sigma}$, can be written as either a CAR or a
 476 SAR model, with the important difference that a CAR model is uniquely determined from $\mathbf{\Sigma}$ but

477 a SAR model is not so uniquely determined. This equivalence between CAR and SAR models
478 can also have practical applications. In addition to our examples, the full conditional form of the
479 CAR model allows for easy and efficient Gibbs sampling (Banerjee et al., 2004, p. 163) and fully
480 conditional random effects (Banerjee et al., 2004, p. 86). However, spatial econometricians often
481 employ SAR models (LeSage and Pace, 2009), so easy conversion from SAR to CAR models may
482 offer computational advantages in hierarchical models and provide insight on the role of fully condi-
483 tional random effects. We expect future research will extend our findings on relationships between
484 CAR and SAR models and explore novel applications.

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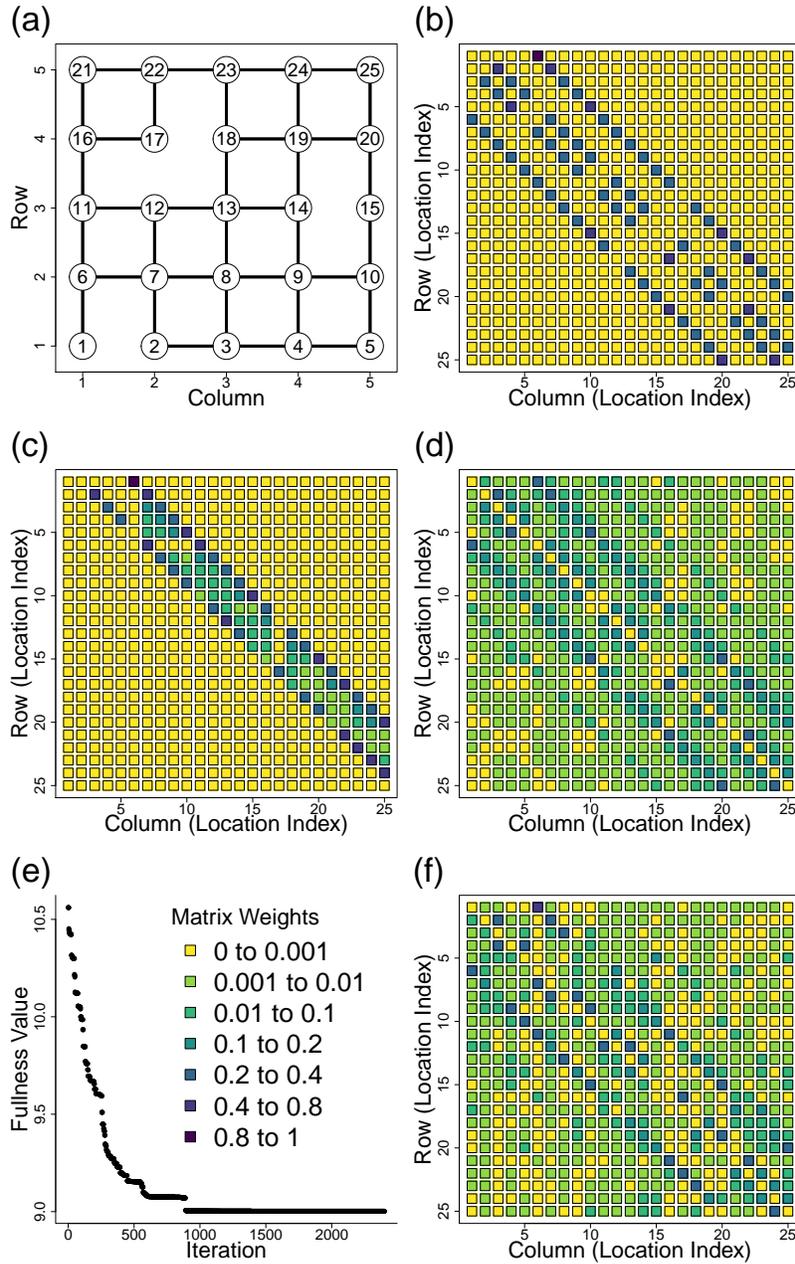


Figure 1: Sparseness in CAR and SAR models. (a) 5×5 grid of spatial locations, with lines connecting neighboring sites. The numbers in the circles are indices of the locations. (b) Graphical representation of weights in the $\rho\mathbf{W}_+$ matrix in the CAR model. The color legend is given below. (c) Graphical representation of weights in the \mathbf{B} matrix when using the Cholesky decomposition, and (d) when using spectral decomposition. (e) Fullness function during minimization when searching for sparseness. (f) Graphical representation of weights in the \mathbf{B} matrix at the termination of an algorithm to search for sparseness using Givens rotations on the spectral decomposition in (d).

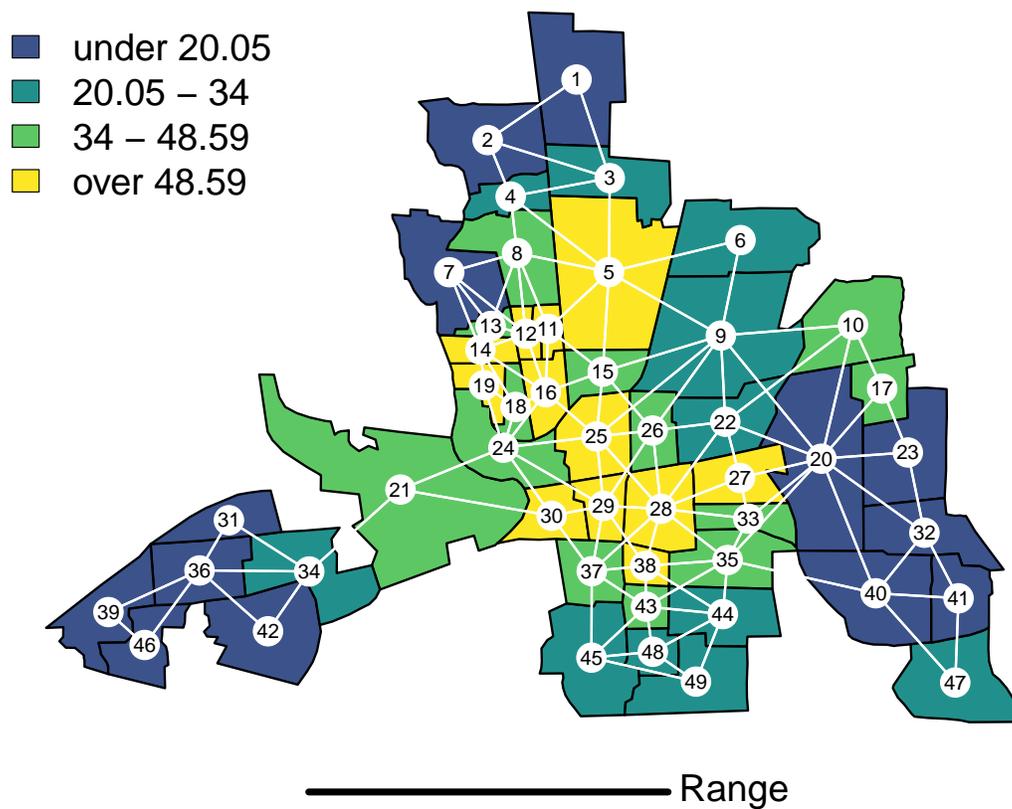


Figure 2: Columbus crime map, in rates per 1000 people. Numbers in each polygon are the indices for locations, and the white lines show first-order neighbors. The estimated range parameter from the spherical geostatistical model is shown at the bottom.

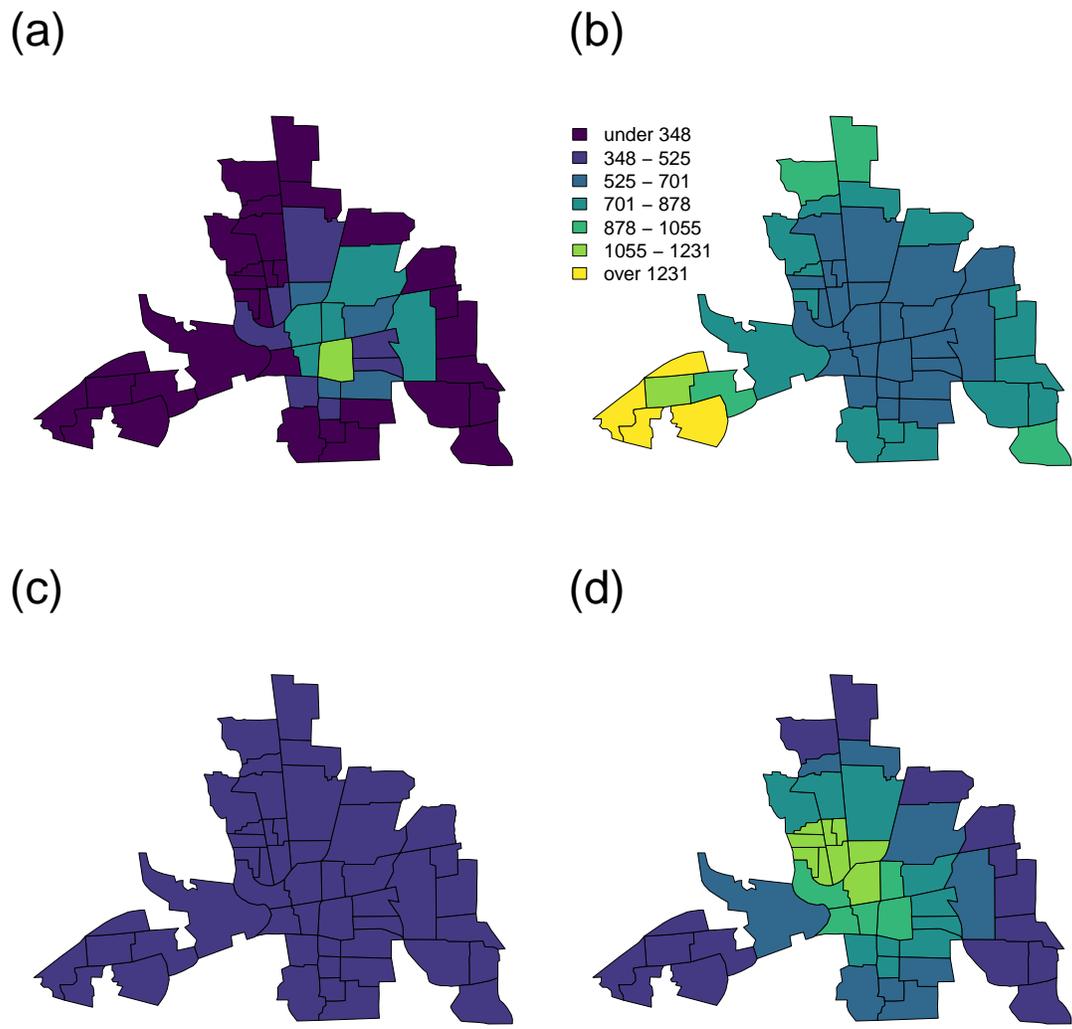


Figure 3: Marginal variances by location for Columbus crime data. (a) Unstandardized first-order CAR model, (b) Row-standardized first-order CAR model, (c) spherical geostatistical model, (d) CAR model using weights obtained from geostatistical model.

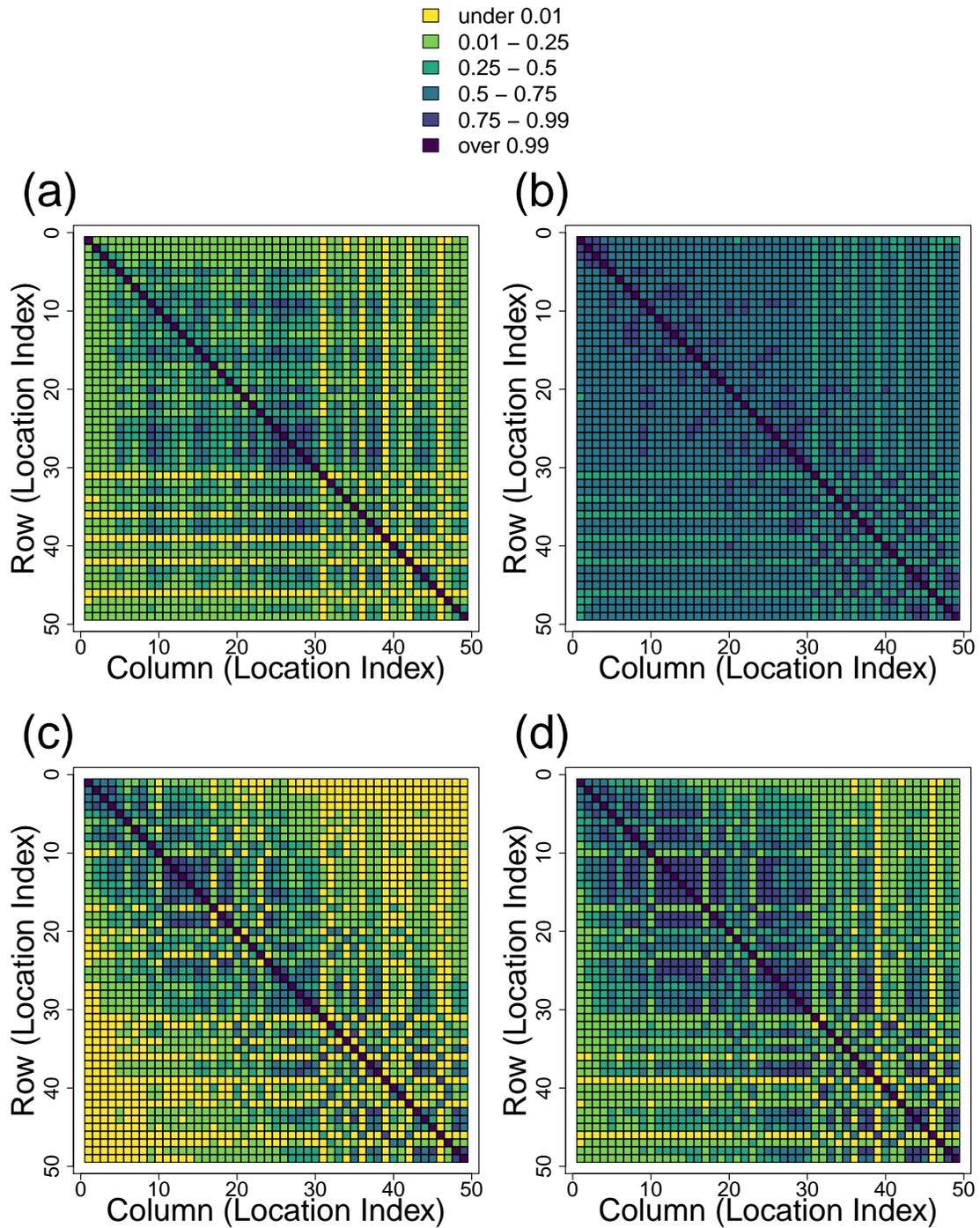


Figure 4: Marginal correlations for Columbus crime data, none of which were below zero. The location indices are given by the numbers in Figure 2. (a) Unstandardized first-order CAR model, (b) Row-standardized first-order CAR model, (c) spherical geostatistical model, (d) CAR model using weights obtained from geostatistical model.

618 **APPENDIX: Propositions and Proofs**

619 The following proposition is used to show condition C1 for CAR models.

620 **Proposition 1.** *For the CAR model covariance matrix, $\Sigma_{CAR} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{M}$ in (4), Σ_{CAR} is*
 621 *positive definite if and only if all eigenvalues of $(\mathbf{I} - \mathbf{C})$ are positive.*

622 *Proof.* Note that $\mathbf{I} - \mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{1/2}$ will be symmetric because of condition C4, and hence will have
 623 real eigenvalues. Write $\Sigma_{CAR}^{-1} = \mathbf{M}^{-1/2}(\mathbf{I} - \mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{1/2})\mathbf{M}^{-1/2}$. Then Σ_{CAR} and Σ_{CAR}^{-1} are
 624 positive definite if and only if $\mathbf{I} - \mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{1/2}$ is positive definite, i.e., if and only if all eigenvalues
 625 of $(\mathbf{I} - \mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{1/2})$ are positive. Now, $(\mathbf{I} - \mathbf{C}) = \mathbf{M}^{1/2}(\mathbf{I} - \mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{1/2})\mathbf{M}^{-1/2}$ has positive
 626 eigenvalues if and only if $(\mathbf{I} - \mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{1/2})$ has positive eigenvalues because they are similar
 627 matrices (Harville, 1997, p. 525). \square

628 Next, we show the conditions on ρ that ensure that $(\mathbf{I} - \rho\mathbf{W})$ has either nonzero eigenvalues,
 629 or positive eigenvalues.

630 **Proposition 2.** *Consider the $N \times N$ matrix $(\mathbf{I} - \rho\mathbf{W})$, where $w_{i,i} = 0$. Let $\{\lambda_i\}$ be the set of*
 631 *eigenvalues of \mathbf{W} , and suppose all eigenvalues are real. Then*

632 (i) $\mathbf{I} - \rho\mathbf{W}$ is nonsingular if and only if $\rho \notin \{\lambda_i^{-1}\}$ for all nonzero λ_i .

633 (ii) Assume at least two eigenvalues of \mathbf{W} are nonzero, and let $\lambda_{[1]}$ and $\lambda_{[N]}$ be the smallest and
 634 largest eigenvalues, respectively, of \mathbf{W} . Then all eigenvalues of $\mathbf{I} - \rho\mathbf{W}$ are positive if and
 635 only if $1/\lambda_{[1]} < \rho < 1/\lambda_{[N]}$.

636 *Proof.* Let λ_i be an eigenvalue of \mathbf{W} , with \mathbf{v}_i a corresponding eigenvector. Then $\mathbf{W}\mathbf{v}_i = \lambda_i\mathbf{v}_i$,
 637 implying that $\mathbf{v}_i - \rho\mathbf{W}\mathbf{v}_i = \mathbf{v}_i - \rho\lambda_i\mathbf{v}_i = (1 - \rho\lambda_i)\mathbf{v}_i$, i.e., $(\mathbf{I} - \rho\mathbf{W})\mathbf{v}_i = (1 - \rho\lambda_i)\mathbf{v}_i$. Thus, for
 638 every eigenvalue/eigenvector pair $(\lambda_i, \mathbf{v}_i)$ of \mathbf{W} , there is a corresponding eigenvalue/eigenvector
 639 pair (ω_i, \mathbf{v}_i) of $(\mathbf{I} - \rho\mathbf{W})$ where $\omega_i = 1 - \rho\lambda_i$. Observe that $\mathbf{I} - \rho\mathbf{W}$ is nonsingular if and only if
 640 all $\omega_i \neq 0$, i.e., if and only if $\rho\lambda_i \neq 1$ for all i , i.e., if and only if $\rho \neq 1/\lambda_i$ for all nonzero λ_i . This
 641 establishes part (i). Furthermore, all eigenvalues of $\mathbf{I} - \rho\mathbf{W}$ are positive if and only if $\rho\lambda_i < 1$ for
 642 all i , i.e., if and only if $\rho < 1/\lambda_i$ for all i such that $\lambda_i > 0$, $\rho > 1/\lambda_i$ for all i such that $\lambda_i < 0$,
 643 and $\rho \in (-\infty, \infty)$ for all i such that $\lambda_i = 0$. This last set of three conditions can be restated as
 644 $1/\lambda_{[1]} < \rho < 1/\lambda_{[N]}$, which establishes part (ii).

645 \square

646 For CAR models (4), Cressie and Chan (1989) consider the symmetric matrix $\mathbf{M}^{-1/2}\mathbf{W}\mathbf{M}^{1/2}$
 647 and (see Cressie, 1993, p. 559) shows bounds on ρ for $\mathbf{I} - \rho\mathbf{M}^{-1/2}\mathbf{W}\mathbf{M}^{1/2}$ so that all eigenvalues
 648 are positive. Here, we state the proposition without proof, as it proceeds in a similar fashion to
 649 Proposition 2.

650 **Proposition 3.** Consider the $N \times N$ matrix $(\mathbf{I} - \rho \mathbf{M}^{-1/2} \mathbf{W} \mathbf{M}^{1/2})$, where $w_{i,i} = 0$ and \mathbf{M} is
651 diagonal with positive values such that $\mathbf{M}^{-1/2} \mathbf{W} \mathbf{M}^{1/2}$ is symmetric with eigenvalues $\{\lambda_i\}$. Then

652 (i) $\mathbf{I} - \rho \mathbf{M}^{-1/2} \mathbf{W} \mathbf{M}^{1/2}$ is nonsingular if and only if $\rho \notin \{\lambda_i^{-1}\}$ for all nonzero λ_i .

653 (ii) Assume at least two eigenvalues of $\mathbf{M}^{-1/2} \mathbf{W} \mathbf{M}^{1/2}$ are nonzero, and let $\lambda_{[1]}$ and $\lambda_{[N]}$ be the
654 smallest and largest eigenvalues, respectively, of $\mathbf{M}^{-1/2} \mathbf{W} \mathbf{M}^{1/2}$. Then all eigenvalues of $\mathbf{I} -$
655 $\rho \mathbf{M}^{-1/2} \mathbf{W} \mathbf{M}^{1/2}$ are positive if and only if $1/\lambda_{[1]} < \rho < 1/\lambda_{[N]}$.

656 While this result was developed for CAR models, note that these bounds would also work
657 for a SAR covariance matrix (2) if $\mathbf{\Omega}$ had diagonal elements such that $\mathbf{\Omega}^{-1/2} \mathbf{W} \mathbf{\Omega}^{1/2}$ was symmetric
658 (Wall, 2004). Before proving Theorems 1 and 2, some preliminary results are useful.

659 **Proposition 4.** If \mathbf{D} is a diagonal matrix and \mathbf{Q} is a square matrix with zeros on the diagonal of
660 the same dimensions as \mathbf{D} , then both $\mathbf{D}\mathbf{Q}$ and $\mathbf{Q}\mathbf{D}$ have zeros on the diagonal.

661 *Proof.* We omit the proof because it is apparent from the algebra of matrix products. □

662 **Proposition 5.** Let \mathbf{A} , \mathbf{B} , and \mathbf{C} be square matrices. If $\mathbf{A} = \mathbf{B}\mathbf{C}$, and \mathbf{A} and \mathbf{C} have inverses,
663 then \mathbf{B} has an inverse.

664 *Proof.* Because \mathbf{C} has an inverse, $\mathbf{B} = \mathbf{A}\mathbf{C}^{-1}$, and because \mathbf{A} has an inverse, $\mathbf{B}^{-1} = \mathbf{C}\mathbf{A}^{-1}$. □

665 Finally, we show the proofs of Theorems 1 and 2.

666 **Theorem 1.** Any positive definite covariance matrix $\mathbf{\Sigma}$ can be expressed as the covariance matrix
667 of a SAR model $(\mathbf{I} - \mathbf{B})^{-1} \mathbf{\Omega} (\mathbf{I} - \mathbf{B}^T)^{-1}$, (2), for a (non-unique) pair of matrices \mathbf{B} and $\mathbf{\Omega}$.

668 *Proof.* We consider a constructive proof and show that the matrices \mathbf{B} and $\mathbf{\Omega}$ that we construct
669 satisfy conditions S1 - S3.

670 (i) Since $\mathbf{\Sigma}$ is positive definite, so is $\mathbf{\Sigma}^{-1}$ and we may write $\mathbf{\Sigma}^{-1} = \mathbf{L}\mathbf{L}^T$ where \mathbf{L} is full rank with
671 positive diagonal elements. Note that \mathbf{L} is *not* unique. A Cholesky decomposition (Harville,
672 1997, p.229) could be used, where \mathbf{L} is lower triangular, or a spectral (eigen) decomposition
673 could be used to obtain a square-root matrix (Harville, 1997, p.543), where $\mathbf{\Sigma}^{-1} = \mathbf{V}\mathbf{E}\mathbf{V}^T$,
674 with \mathbf{V} containing orthonormal eigenvectors and \mathbf{E} containing eigenvalues on the diagonal
675 and zeros elsewhere. Then $\mathbf{L} = \mathbf{V}\mathbf{E}^{1/2}\mathbf{V}^T$ is symmetric with positive diagonal elements,
676 where the diagonal matrix $\mathbf{E}^{1/2}$ contains the positive square roots of the eigenvalues in \mathbf{E} .

677 (ii) Decompose \mathbf{L} into $\mathbf{L} = \mathbf{G} - \mathbf{P}$ where \mathbf{G} is diagonal and \mathbf{P} has zeros on the diagonal. Then
678 $\mathbf{L}\mathbf{L}^T = (\mathbf{G} - \mathbf{P})(\mathbf{G}^T - \mathbf{P}^T)$ by construction.

679 (iii) Then set

$$\mathbf{\Omega}^{-1} = \mathbf{G}\mathbf{G} \quad \text{and} \quad \mathbf{B}^T = \mathbf{P}\mathbf{G}^{-1}. \quad (\text{A.1})$$

680 Note that because \mathbf{L} has positive diagonal elements, then $\ell_{i,i} > 0$, and because \mathbf{G} is diagonal
681 with $g_{i,i} = \ell_{i,i}$, \mathbf{G}^{-1} exists.

682 Then $\mathbf{\Sigma}^{-1} = (\mathbf{I} - \mathbf{B}^T)\mathbf{\Omega}^{-1}(\mathbf{I} - \mathbf{B})$, expressed in SAR form (2). The matrices \mathbf{B} and $\mathbf{\Omega}$ satisfy S1 -
683 S3, as follows.

684 (S1) Note that $\mathbf{P} = \mathbf{B}^T\mathbf{G}$, so $\mathbf{L} = \mathbf{G} - \mathbf{P} = (\mathbf{I} - \mathbf{B}^T)\mathbf{G}$ and $\mathbf{L}^T = \mathbf{G}(\mathbf{I} - \mathbf{B})$. Then, by Proposition
685 5, $(\mathbf{I} - \mathbf{B})^{-1}$ exists, and hence so does its transpose $(\mathbf{I} - \mathbf{B}^T)^{-1}$.

686 (S2) Because \mathbf{G} is diagonal, $\mathbf{\Omega}$ is diagonal with $\omega_{i,i} = g_{i,i}^2 > 0$.

687 (S3) By Proposition 4, $b_{i,i} = 0$ because $\mathbf{B}^T = \mathbf{P}\mathbf{G}^{-1}$. □

688 **Theorem 2.**

689 Any positive-definite covariance matrix $\mathbf{\Sigma}$ can be expressed as the covariance matrix of a
690 CAR model $(\mathbf{I} - \mathbf{C})^{-1}\mathbf{M}$, (4), for a unique pair of matrices \mathbf{C} and \mathbf{M} .

691 *Proof.* We add an explicit, constructive proof of the result given by Cressie and Wikle (2011, p.
692 185-186) by showing that matrices \mathbf{C} and \mathbf{M} are unique and satisfy conditions C1 - C4.

693 (i) Let $\mathbf{Q} = \mathbf{\Sigma}^{-1}$ and decompose it into $\mathbf{Q} = \mathbf{D} - \mathbf{R}$, where \mathbf{D} is diagonal with elements $d_{i,i} = q_{i,i}$
694 (the diagonal elements of the precision matrix \mathbf{Q}), and \mathbf{R} has zeros on the diagonal ($r_{i,i} = 0$)
695 and off-diagonals equal to $r_{i,j} = -q_{i,j}$.

696 (ii) Set

$$\mathbf{C} = \mathbf{D}^{-1}\mathbf{R} \quad \text{and} \quad \mathbf{M} = \mathbf{D}^{-1}. \quad (\text{A.2})$$

697 Then $\mathbf{\Sigma}^{-1} = \mathbf{D} - \mathbf{R} = \mathbf{D}(\mathbf{I} - \mathbf{D}^{-1}\mathbf{R}) = \mathbf{M}^{-1}(\mathbf{I} - \mathbf{C})$, which shows that $\mathbf{\Sigma}$ may be expressed in
698 CAR form (4), satisfying C1 - C4.

699 (C1) \mathbf{M} is strictly diagonal with positive values, so \mathbf{M} and \mathbf{M}^{-1} are positive definite. By hypothesis,
700 $\mathbf{\Sigma}$, and hence $\mathbf{\Sigma}^{-1}$, are positive definite. Then $\mathbf{\Sigma} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{M}$, so by Proposition 1, $(\mathbf{I} - \mathbf{C})^{-1}$
701 has positive eigenvalues and thus so does $\mathbf{I} - \mathbf{C}$.

702 (C2) $m_{i,i} = 1/q_{i,i}$, and because $\mathbf{Q} = \mathbf{\Sigma}^{-1}$ is positive definite, we have that $q_{i,i} > 0, i = 1, 2, \dots, n$.
703 Thus, each $m_{i,i} > 0$. By construction, $m_{i,j} = 0$ for $i \neq j$.

704 (C3) By Proposition 4, $c_{i,i} = 0$ because $\mathbf{C} = \mathbf{D}^{-1}\mathbf{R}$.

705 (C4) For $i \neq j$, we have that $c_{i,j} = d_{i,i}^{-1}r_{i,j}$. As $m_{i,i} = d_{i,i}^{-1}$, we have that

$$\frac{c_{i,j}}{m_{i,i}} = \frac{d_{i,i}^{-1}r_{i,j}}{d_{i,i}^{-1}} = r_{i,j} = -q_{i,j}.$$

706 Because $\mathbf{Q} = \mathbf{\Sigma}^{-1}$ is symmetric, $q_{i,j} = q_{j,i}$, hence $c_{i,j}/m_{i,i} = c_{j,i}/m_{j,j}$. The above proof shows
707 existence of a CAR representation of any covariance matrix $\mathbf{\Sigma}$. We now show uniqueness of

708 this CAR representation. Assume that there exists $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{M}}$, possibly different from \mathbf{C} and \mathbf{M}
709 in (A.2), that satisfy C1 - C4, and also satisfy $\boldsymbol{\Sigma} = \tilde{\mathbf{M}}^{-1}(\mathbf{I} - \tilde{\mathbf{C}})$. From Proposition 4, we have
710 that $\text{diag}(\mathbf{M}) = \text{diag}(\tilde{\mathbf{M}}) = \text{diag}(\boldsymbol{\Sigma}^{-1})$, and so $\mathbf{M} = \tilde{\mathbf{M}}$, since \mathbf{M} and $\tilde{\mathbf{M}}$ are diagonal matrices.
711 Furthermore, since $\mathbf{M}^{-1}(\mathbf{I} - \mathbf{C}) = \boldsymbol{\Sigma} = \tilde{\mathbf{M}}^{-1}(\mathbf{I} - \tilde{\mathbf{C}})$, we have that $\tilde{\mathbf{C}} = \mathbf{I} - \mathbf{M}\mathbf{M}^{-1}(\mathbf{I} - \mathbf{C})$,
712 but since $\mathbf{M} = \tilde{\mathbf{M}}$, it immediately follows that $\mathbf{C} = \tilde{\mathbf{C}}$, and we thus conclude that the CAR
713 representation of $\boldsymbol{\Sigma}$ is unique. □