Robust Observer-Based Control of Nonlinear Dynamical Systems with State Constraints

by

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Abstract

In this paper, we design a variable structure observer-based control system that guarantees asymptotic convergence of the plant's trajectory to the equilibrium point despite matched and unmatched uncertainties in the plant dynamics. Our control laws are functions of the estimated plant state and the proposed framework allows employing any estimator or observer, such as the Walcott and Żak observer, as long as the estimated state converges asymptotically to the plant state. *Barrier Lyapunov functions* guarantee that the closed-loop system's trajectory verifies the state constraints. This study is the first of its kind, since recently variable structure control architectures have been adapted to account for constraints on the state space or allow output-feedback, but observer-based variable structure control in the presence of state constraints has not been attempted before. A numerical simulation involving the roll dynamics of an unstable aircraft, whose aerodynamic coefficients are unknown, illustrates our theoretical framework.

Key Words: Sliding mode control, barrier Lyapunov function, state constraints, observer-based feedback control, finite-time stability, nonlinear robust control.

Running Title: Robust Feedback Stabilization of Constrained Systems

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I. Introduction

In most cases of practical interest, it is difficult to accurately model dynamical systems and estimate the parameters characterizing such models. Additional complexity in the problem of controlling nonlinear systems is given by the fact that it may not be possible to directly measure all the components of the system's state. Lastly, it is often needed to guarantee that the system's trajectory is confined to a constraint set at all times. In this paper, we present a novel nonlinear observer-based control architecture, which guarantees that the system's trajectory converges asymptotically to an equilibrium point and is constrained to a simply connected open set.

Notorious robust control techniques for nonlinear dynamical systems are adaptive control [1], [2] and sliding mode control [3], [4], [5, Ch. 7], [6, Ch. 14], [7]. The sliding mode control architecture, which was first devised in 1960s by Emel'yanov and Barbashin [4], consists in steering in finite-time the system's trajectory to a *sliding manifold*, which is designed so that if the system's trajectory reaches this manifold, then the system state asymptotically converges to zero. The control law that drives the system trajectory to the sliding manifold involves the signum function, and hence is discontinuous. It is well known that solutions of ordinary differential equations with discontinuous right-hand sides may not exist or may not be unique [8, Ch. 2], [9]. Furthermore, in most cases of practical interest discontinuous control inputs induce an undesired effect known as *chattering*, which consists in high-frequency oscillations of the system's state about the sliding manifold [6, Ch. 14], [10]. In spite of the theoretical and practical challenges concerning sliding mode control, this technique has drawn considerable interest in aerospace [11], chemical [12], electrical [13], marine [14], and mechanical engineering [15] for its ease of implementation [16] and ability to compensate for disturbances and uncertainties [17]; for further details, see [18], [19] and the numerous references therein.

The problem of designing sliding mode control laws, which account for constraints in the state space, received relatively less attention [20] and has been addressed for first-order sliding mode [21], second-order sliding mode [22], third-order sliding mode with box constraints [23], and within the context of model predictive control [24]. The classical sliding mode architecture has also been modified to allow observer-based and output-feedback control [25]–[28]. It is worthwhile to recall also both [29], where an output-feedback sliding mode control architecture is proposed for uncertain stochastic systems, and [30], where sensor faults are accounted for. However, to the authors' best knowledge, an observer-based sliding mode control architecture in the presence of state constraints has not been proposed before. Alternative approaches to the problem of designing control laws that guarantee some bounds on the tracking error involve the adaptive control framework [31], where single-input-single-output dynamical systems are considered, and the backstepping framework [32], where triangular systems are considered.

In this paper, we design robust, observer-based feedback control laws for nonlinear time-varying dynamical systems affected by matched and unmatched uncertainties and subject to state constraints on the closed-loop system's trajectory. A unique feature of this work is that the feedback control laws presented are functions of the estimated plant state, which is reconstructed by a dynamic observer using the information provided by the measured output. To meet our design goal, firstly we prove sufficient conditions for the closed-loop systems trajectory to converge to the sliding manifold in finite-time and asymptotically to the equilibrium point along the sliding manifold in the presence of constraints on the plant state. Successively, the effectiveness of our observer-based feedback controls is shown by proving that if a state-feedback control guarantees convergence of the plant trajectory and the estimated state verifies the constraints on the plant state and asymptotically converges to the actual plant state, then the feedback control law obtained by accounting for the *estimated* state guarantees convergence of the closed-loop system to the equilibrium point. *Barrier Lyapunov functions* [33], [34] are employed to certify that the constraints on the plant state are verified and it is worthwhile

to recall the recent publications [35], where time-varying barrier Lyapunov functions are used within the context of backstepping control, and [36], [37], where barrier Lyapunov functions are used within the context of adaptive control.

Although the framework provided in this paper does not depend on a *specific* plant observer, we show how a popular nonlinear robust observer, that is, the Walcott and Żak observer [38]–[44], can be used within our observer-based feedback control framework. Alternative observers that can be employed within our framework are presented, for instance, in [45]–[49].

This paper is organized in two parts. Firstly, we prove sufficient conditions for strong uniform finite-time stability and uniform asymptotic stability of time-varying dynamical systems in the presence of state constraints. Successively, we prove sufficient conditions for asymptotic and finite-time convergence of dynamical systems, whose feedback controls account for the system's estimated state. A numerical example illustrates the applicability of our robust nonlinear observer-based feedback control architecture. Specifically, we consider the problem of stabilizing the roll dynamics of a delta-wing aircraft, whose aerodynamic coefficients are unknown. The proofs of our main results are provided in the Appendix.

II. Notation, Definitions, and Mathematical Preliminaries

In this section, we establish notation, definitions, and review some preliminary results. Let $\mathbb{R}_+$ denote the set of positive real numbers, $\overline{\mathbb{R}}_+$ denote the set of nonnegative real numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denote the set of $n \times m$ real matrices, and $\mathcal{B}_\varepsilon(x)$ denote the *open ball centered* at x with *radius* ε. We write $\| \cdot \|$ both for the Euclidean vector norm and the corresponding equi-induced matrix norm, $\| \cdot \|_\infty$ both for the infinity vector norm and the corresponding equi-induced matrix norm, I_n or I for the $n \times n$ identity matrix, $0_{n \times m}$ or 0 for the zero $n \times m$ matrix, and A^{T} for the transpose of the matrix A.

Consider the nonlinear dynamical system given by

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0, \qquad t \in \mathcal{I}_{t_0, x_0}, \tag{1}$$

where, for every $t \in \mathcal{I}_{t_0, x_0}$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{I}_{t_0, x_0} \subseteq [t_0, \infty)$ is the maximal interval of existence of a solution $x(t)$ of (1), $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, and $f : \mathcal{I}_{t_0, x_0} \times \mathcal{D} \to \mathbb{R}^n$ is such that, for every $(t, x) \in \mathcal{I}_{t_0, x_0} \times \mathcal{D}$, $f(t, 0) = 0$ and $f(\cdot, \cdot)$ is continuous in t and x. A continuously differentiable function $x : \mathcal{I}_{t_0, x_0} \to \mathcal{D}$ is said to be the *solution* of (1) on the interval $\mathcal{I}_{t_0, x_0} \subset \mathbb{R}$ if $x(\cdot)$ satisfies (1) for all $t \in \mathcal{I}_{t_0, x_0}$. As shown in [50], it follows from Peano's theorem [8, Th. 2.24] that the joint continuity of $f(\cdot, \cdot)$ implies that, for every $x \in \mathcal{D}$, there exists $\tau_0 < t_0 < \tau_1$ and a solution $x(\cdot)$ of (1) defined on the open interval (τ_0, τ_1) such that $x(t_0) = x$. A solution $t \mapsto x(t)$ is said to be *right maximally* defined if x cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal solutions to (1) exist on $[t_0, \infty)$, and hence, we assume that (1) is *forward complete*.

We assume that (1) possesses unique solutions in forward time for all initial conditions except possibly the origin in the following sense. For every $x \in \mathcal{D} \backslash \{0\}$ there exists $\tau_x > t_0$ such that, if $y_1 : [t_0, \tau_1) \to \mathcal{D}$ and $y_2 : [t_0, \tau_2) \to \mathcal{D}$ are two solutions of (1) with $y_1(t_0) = y_2(t_0) = x$, then $\tau_x \leq \min\{\tau_1, \tau_2\}$ and $y_1(t) = y_2(t)$ for all $t \in [t_0, \tau_x)$. Without loss of generality, we assume that for each x, τ_x is chosen to be the largest such number in $[t_0, \infty)$. In this case, we denote by the continuously differentiable map $s^{t_0, x_0}(\cdot) \triangleq s(\cdot, t_0, x_0)$ the *trajectory* or the unique *solution curve* of (1) on $\mathcal{I}_{t_0, x_0}$ satisfying $s(0, t_0, x_0) = x_0$. Sufficient conditions for forward uniqueness in the absence of Lipschitz continuity can be found in [51] [9, Sec. 10], [52], and [53, Sec. 1].

The following definition introduces the notion of finite-time stability for time-varying nonlinear dynamical systems, which plays a key role in this paper. This statement is a generalization of the definition of finite-time stability for nonautonomous systems provided in [50], as the notion

of Lyapunov stability is provided in terms of open sets, and not open balls; the importance of defining finite-time stability in terms of open sets will become clear while analyzing the stability properties of constrained dynamical systems. For the statement of this definition, recall that if $s^{t_0,x_0}(t) \to 0$ as $t \to T$, where $s^{t_0,x_0}(\cdot)$ denotes the solution of (1), then for every $\hat{\varepsilon} > 0$, there exists $\hat{T}(\hat{\varepsilon}, t_0, x_0) \in (t_0, T)$, such that if $t \in (\hat{T}(\hat{\varepsilon}, t_0, x_0), T)$, then $\|s^{t_0,x_0}(t)\| < \hat{\varepsilon}$. Alternatively, if $s^{t_0,x_0}(t) \to 0$ as $t \to T$ *uniformly* in t_0, then for every $\hat{\varepsilon} > 0$, there exists $\hat{T}(\hat{\varepsilon}, x_0) \in (t_0, T)$, such that if $t \in (\hat{T}(\hat{\varepsilon}, x_0), T)$, then $\|s^{t_0,x_0}(t)\| < \hat{\varepsilon}$.

Definition 2.1. The nonlinear dynamical system (1) is *finite-time stable* if there exist an open neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of the origin and a function $T : [0, \infty) \times \mathcal{D}_0 \setminus \{0\} \to (t_0, \infty)$, called the *settling-time function*, such that the following statements hold:

i) *Finite-time convergence.* For every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0 \setminus \{0\}$, $s^{t_0,x_0}(t)$ is defined on $[t_0, T(t_0, x_0))$, $s^{t_0,x_0}(t) \in \mathcal{D}_0 \setminus \{0\}$ for all $t \in [t_0, T(t_0, x_0))$, and $s^{t_0,x_0}(t) \to 0$ as $t \to T(t_0, x_0)$.

ii) *Lyapunov stability.* For every $t_0 \in [0, \infty)$ and every open set $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_0$ containing $x = 0$, there exists an open set $\mathcal{N}_\delta \subseteq \mathcal{D}_0$ containing $x = 0$, such that for every $x_0 \in \mathcal{N}_\delta \setminus \{0\}$, $s^{t_0,x_0}(t) \in \mathcal{N}_\varepsilon$ for all $t \in [t_0, T(t_0, x_0))$.

The nonlinear dynamical system (1) is *uniformly finite-time stable* if (1) is finite-time stable and the following statement holds:

iii) *Uniform Lyapunov stability.* For every open set $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_0$ containing $x = 0$, there exists an open set $\mathcal{N}_\delta \subseteq \mathcal{D}_0$ containing $x = 0$, such that for every $x_0 \in \mathcal{N}_\delta \setminus \{0\}$, $s^{t_0,x_0}(t) \in \mathcal{N}_\varepsilon$ for all $t \in [t_0, T(t_0, x_0))$ and for all $t_0 \in [0, \infty)$.

The nonlinear dynamical system (1) is *strongly uniformly finite-time stable* if (1) is uniformly finite-time stable and the following statement holds:

iv) *Uniform finite-time convergence.* For every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0 \setminus \{0\}$, $s^{t_0,x_0}(t)$ is defined on $[t_0, T(t_0, x_0))$, $s^{t_0,x_0}(t) \in \mathcal{D}_0 \setminus \{0\}$ for all $t \in [t_0, T(t_0, x_0))$, and $s^{t_0,x_0}(t) \to 0$ as $t \to T(t_0, x_0)$ uniformly in t_0 for all $t_0 \in [0, \infty)$.

The nonlinear dynamical system (1) is *globally finite-time stable* (respectively, *globally uniformly finite-time stable* or *globally strongly uniformly finite-time stable*) if it is finite-time stable (respectively, uniformly finite-time stable or strongly uniformly finite-time stable) with $\mathcal{D}_0 = \mathbb{R}^n$.

The following result proves that if (1) is finite-time stable, then its solution exists, is unique, and is defined for all $t \in [t_0, \infty)$.

Proposition 2.1 [50], [54]. Consider the nonlinear dynamical system $\mathcal{G}$ given by (1). Assume $\mathcal{G}$ is finite-time stable and let $\mathcal{D}_0 \subseteq \mathcal{D}$ and $T : [0, \infty) \times \mathcal{D}_0 \setminus \{0\} \to (t_0, \infty)$ be defined as in Definition 2.1. Then, for every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$, there exists a unique solution $s(t, t_0, x_0)$, $t \geq t_0$, to (1) such that $s(t, t_0, x_0) \in \mathcal{D}_0$, $t \in [t_0, T(t_0, x_0))$, and such that $s(t, t_0, x_0) = 0$, $t \geq T(t_0, x_0)$, where $T(t_0, 0) \triangleq t_0$.

It follows from Proposition 2.1 that if the zero solution $x(t) \equiv 0$ to (1) is finite-time stable, then the solutions of (1) define a continuous *global semiflow* on $\mathcal{D}_0$; that is, $s : [t_0, \infty) \times [0, \infty) \times \mathcal{D}_0 \to \mathcal{D}_0$ is jointly continuous and satisfies the consistency property $s(t_0, t_0, x) = x$ and the semigroup property $s(t, \tau, s(\tau, t_0, x)) = s(t, t_0, x)$ for every $x \in \mathcal{D}_0$ and $t \geq \tau \geq t_0$. In addition, it follows from Proposition 2.1 that we can extend $T(t_0, \cdot)$ to all of $\mathcal{D}_0$ by defining $T(t_0, 0) \triangleq t_0$, for all $t_0 \in [0, \infty)$. Now, by uniqueness of solutions it follows that $s(T(t_0, x) + t, t_0, x) = 0$, $t \in [0, \infty)$, and hence, it is easy to see from Definition 2.1 that

$$T(t_0, x) = \inf\{t \in [t_0, \infty) : s(t, t_0, x) = 0\}, \qquad (t_0, x) \in [0, \infty) \times \mathcal{D}_0. \qquad (2)$$

Lastly, it follows from Definition 2.1 and Proposition 2.1 that if the zero solution $x(t) \equiv 0$ to (1) is finite-time stable, then it is asymptotically stable, and hence, finite-time stability is a stronger

condition than asymptotic stability.

In this paper, we consider controlled, nonlinear, time-varying dynamical systems of the form

$$\dot{x}(t) = F(t, x(t), u(t)), \qquad x(t_0) = x_0, \qquad t \geq t_0, \tag{3}$$

$$y(t) = H(x(t), u(t)), \tag{4}$$

where $x(t) \in \mathcal{D}$, $t \geq t_0$, denotes the *plant state*, $u(t) \in U \subseteq \mathbb{R}^m$ denotes the *control input*, $y(t) \in \mathbb{R}^l$ denotes the *measured output*, $0 \in U$, $F : [t_0, \infty) \times \mathcal{D} \times U \rightarrow \mathbb{R}^n$ is such that $F(t, 0, 0) = 0$, $t \geq t_0$, $F(\cdot, x, u)$ is continuous in t, for all $(x, u) \in \mathcal{D} \times U$, and $F(t, \cdot, \cdot)$ is jointly continuous in x and u uniformly in t, for all $t \in [t_0, \infty)$, and $H : \mathcal{D} \times U \rightarrow \mathbb{R}^l$ is continuous on $\mathcal{D} \times U$ and such that $H(0, 0) = 0$. A piecewise continuous function $\phi : \mathcal{D} \rightarrow U$ such that $\|\phi(\cdot)\|$ is continuous and $\phi(0) = 0$ is called a *control law*, and if $u(t) = \phi(x(t))$, $t \geq t_0$, where $\phi(\cdot)$ is a control law and $x(\cdot)$ denotes the unique solution of the *closed-loop system*

$$\dot{x}(t) = F(t, x(t), \phi(x(t))), \qquad x(t_0) = x_0, \qquad t \geq t_0, \tag{5}$$

then $u(\cdot)$ is a *state-feedback control law*.

To address the problem of designing feedback control laws that do not rely on the prefect knowledge of the state vector, we consider dynamical systems in the form

$$\dot{x}(t) = F(t, x(t), \phi(x(t) + e(t))), \qquad x(t_0) = x_0, \qquad t \geq t_0, \tag{6}$$

where $\phi(\cdot)$ is a control law and $e : [t_0, \infty) \rightarrow \mathbb{R}^n$ is continuous and such that $e(t) \rightarrow 0$ as $t \rightarrow \infty$. The vector function $e(\cdot)$ denotes the *estimation error*, that is, the difference between the solution $x(\cdot)$ of (6) and the state of a nonlinear observer, which is designed to estimate $x(\cdot)$, based on the measured output $y(\cdot)$. If $u(t) = \phi(x(t) + e(t))$, $t \geq t_0$, where $\phi(\cdot)$ is a control law, $e(t) \rightarrow 0$ as $t \rightarrow \infty$ uniformly in t_0, and $x(\cdot)$ denotes the unique solution of (6), then $u(\cdot)$ is an *observer-based feedback control law*.

III. Sufficient Conditions for Finite-Time and Uniform Asymptotic Stability in the Presence of State Constraints

In this section, we provide sufficient conditions for strong uniform finite-time stability and uniform asymptotic stability [8, Def. 4.2] of the time-varying nonlinear dynamical system (1), which guarantee that the solution $x(t)$, $t \geq t_0$, of (1) is contained in the simply connected *constraint set*

$$\mathcal{C} \triangleq \{x \in \mathcal{D} : h(x) \geq 0\}, \tag{7}$$

where $h : \mathcal{D} \rightarrow \mathbb{R}$ is continuously differentiable and $h(0) > 0$. For the statement of these results, we denote the *interior* of $\mathcal{C}$ by $\mathring{\mathcal{C}}$, that is, $\mathring{\mathcal{C}} \triangleq \{x \in \mathcal{D} : h(x) > 0\}$; in this paper, we assume that the constraint set is a proper subset of the plant state space, that is, $\mathring{\mathcal{C}} \subset \mathbb{R}^n$.

Theorem 3.1. Consider the nonlinear dynamical system (1) with $x_0 \in \mathring{\mathcal{C}}$. If there exist a continuously differentiable function $V : [t_0, \infty) \times \mathring{\mathcal{C}} \rightarrow \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and real numbers $\theta \in (0, 1)$ and $k > 0$, such that

$$\alpha(\|x\|) \leq \frac{V(t, x)}{h(x)} \leq \beta(\|x\|), \qquad (t, x) \in [t_0, \infty) \times \mathring{\mathcal{C}}, \tag{8}$$

$$\frac{1}{h(x)} \frac{\partial V(t, x)}{\partial t} + \left[\frac{1}{h(x)} \frac{\partial V(t, x)}{\partial x} - \frac{V(t, x)}{h^2(x)} \frac{\partial h(x)}{\partial x} \right] f(t, x) \leq -k \left(\frac{V(t, x)}{h(x)} \right)^\theta, \tag{9}$$

then (1) is strongly uniformly finite-time stable, and $x(t) \in \mathring{\mathcal{C}}$ for all $t \geq t_0$. Moreover, there exist a neighborhood $\mathcal{D}_0 \subseteq \mathring{\mathcal{C}}$ of the origin and a settling-time function $T : [0, \infty) \times \mathcal{D}_0 \to [t_0, \infty)$, such that

$$T(t_0, x_0) \leq \frac{1}{k(1-\theta)} \left[\frac{V(t_0, x_0)}{h(x_0)} \right]^{1-\theta}, \qquad (t_0, x_0) \in [0, \infty) \times \mathcal{D}_0, \qquad (10)$$

and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathcal{D}_0$.

It follows from the proof of Theorem 3.1, which is reported in the Appendix, that if the continuously differentiable function $\frac{V(t,x)}{h(x)}$ satisfies (8) and (9), then it is a *barrier Lyapunov function*, that is, $\frac{V(t,x)}{h(x)}$ is positive-definite, $\frac{V(t, x(t))}{h(x(t))}$ is finite for every $t \geq t_0$ along the trajectory of (1), and $\frac{V(t,x)}{h(x)} \to \infty$ as $\text{dist}(x, \partial\mathcal{C}) \to 0$, where $\partial\mathcal{C} \triangleq \{x \in \mathcal{D} : h(x) = 0\}$ denotes the *boundary* of $\mathcal{C}$ and $\text{dist}(\cdot, \cdot)$ denotes the *distance function* between subsets of $\mathbb{R}^n$ [55, p. 16]; the notion of barrier Lyapunov function is usually defined for time-invariant dynamical systems [33], [34], whereas the dynamical system (1) is time-varying.

A weaker form of Theorem 3.1 could have been proven as a direct consequence of Theorem 3.2 of [50]. Specifically, if (8) and (9) are satisfied, then it follows from Theorem 3.2 of [50] that for every $\varepsilon > 0$ and $t_0 \in [0, \infty)$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\mathcal{B}_\delta(0) \subseteq \mathring{\mathcal{C}}$, and if $x_0 \in \mathcal{B}_\delta(0)$, then $x(\cdot)$ converges to the equilibrium point $x = 0$ in finite-time and $x(t) \in \mathcal{B}_\varepsilon(0) \subseteq \mathring{\mathcal{C}}$, $t \geq t_0$. This is a considerably weaker result, since strong uniform finite-time stability of (1) is guaranteed only if the initial condition x_0 lays in an open ball, which is *entirely* contained in $\mathring{\mathcal{C}}$. Theorem 3.1, instead, proves that if (8) and (9) are satisfied, then for every $\varepsilon > 0$ and $t_0 \in [0, \infty)$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $x_0 \in \mathcal{B}_\delta(0) \cap \mathring{\mathcal{C}}$, then $x(\cdot)$ converges to $x = 0$ in finite-time, and $x(t) \in \mathcal{B}_\varepsilon(0) \cap \mathring{\mathcal{C}}$, $t \geq t_0$.

It is worth to remind that convex sets are diffeomorphic to $\mathbb{R}^n$ [56, p. 60]. Therefore, if $\mathring{\mathcal{C}}$ is convex, then there exists a continuously differentiable function $\mathcal{T} : \mathring{\mathcal{C}} \to \mathbb{R}^n$, whose inverse exists and is continuously differentiable, and one can apply the results on global strong uniform stability of time-varying dynamical systems [50] to prove sufficient conditions for the solutions of (1) to be finite-time stable and contained in $\mathring{\mathcal{C}}$ for any initial condition. However, there does not exist any systematic approach to devise such $\mathcal{T}(\cdot)$.

The next result provides sufficient conditions for the time-varying dynamical system (1) to be uniformly asymptotically stable and such that $x(t) \in \mathring{\mathcal{C}}$, for all $t \geq t_0$. The proof of this theorem is virtually identically to the proof of Theorem 3.1, and therefore is omitted for brevity.

Theorem 3.2. Consider the nonlinear time-varying dynamical system (1) with $x_0 \in \mathring{\mathcal{C}}$. If there exist a continuously differentiable function $V : [t_0, \infty) \times \mathring{\mathcal{C}} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\lambda(\cdot)$ such that

$$\alpha(\|x\|) \leq \frac{V(t,x)}{h(x)} \leq \beta(\|x\|), \qquad (t, x) \in [t_0, \infty) \times \mathring{\mathcal{C}}, \qquad (11)$$

$$\frac{1}{h(x)} \frac{\partial V(t,x)}{\partial t} + \left[\frac{1}{h(x)} \frac{\partial V(t,x)}{\partial x} - \frac{V(t,x)}{h^2(x)} \frac{\partial h(x)}{\partial x} \right] f(t,x) \leq -\lambda(\|x\|), \qquad (12)$$

then (1) is uniformly asymptotically stable, and $x(t) \in \mathring{\mathcal{C}}$ for all $t \geq t_0$.

It is worth to recall that the authors in [33] and [57] provide systematic approaches to find a barrier Lyapunov function and a feedback control law, such that a closed-loop system is

asymptotically stable and the system's trajectory is contained in some constraint set. These results are achieved in [33] by mean of an iterative algorithm and in [57] by extending Artstein necessary and sufficient conditions for the existence of control Lyapunov functions [58].

IV. Observer-Based Feedback Control and Trajectory Convergence

In this section, we provide sufficient conditions for the solution of the nonlinear dynamical system (6) to converge to the equilibrium point either asymptotically or in finite-time. Specifically, the next result proves that if $e : [t_0, \infty) \to \mathbb{R}^n$ is such that $\lim_{t \to \infty} e(t) = 0$ and the state-feedback control law $u(t) = \phi(x(t))$, $t \geq t_0$, guarantees strong uniform finite-time stability of (5), then the observer-based feedback control law $u(t) = \phi(x(t) + e(t))$ guarantees finite-time convergence of the solution of (6) to $x = 0$.

Theorem 4.1. Consider the nonlinear dynamical system (6), assume that $e(\cdot)$ is continuous on $[t_0, \infty)$ and $e(t) \to 0$ as $t \to \infty$ uniformly in t_0, and suppose there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, class $\mathcal{K}$ function $\alpha(\cdot)$ and $\beta(\cdot)$, and real numbers $\theta \in (0, 1)$ and $k > 1$, such that

$$\alpha(\|x\|) \leq V(t, x) \leq \beta(\|x\|), \qquad x \in \mathcal{D}, \tag{13}$$

$$\frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} F(t, x, \phi(x)) \leq -k \left(V(t, x) \right)^\theta. \tag{14}$$

Then, there exists a compact set $\mathcal{M} \subset \mathcal{D}$, such that $0 \in \mathcal{M}$ and if $x_0 \in \mathcal{M}$, then $x(t) \to 0$ as $t \to t_1$ uniformly in t_0, for some finite-time $t_1 \geq t_0$.

The proof of Theorem 4.1 is reported in the Appendix. In this paper, we address the problem of designing nonlinear robust observers and observer-based feedback control laws, which guarantee uniform finite-time convergence to the origin of nonlinear dynamical systems. It follows from Theorem 3.2 of [50] that if (13) and (14) are satisfied, then the closed-loop system (5) is strongly uniformly finite-time stable. Hence, Theorem 4.1 proves that if the state-feedback control law $u = \phi(x)$ guarantees strong uniform finite-time stability of the closed-loop system and the estimation error $e(\cdot)$ is such that $\lim_{t \to \infty} e(t) = 0$, then the observer-based feedback control law $u = \phi(x + e)$ guarantees uniform finite-time convergence of the system trajectory to the equilibrium point. Remarkably, it follows from the proof of Theorem 4.1 that larger values of k and smaller values of θ allow for larger sets $\mathcal{M}$.

The next result proves that if $\lim_{t \to \infty} e(t) = 0$ and the state-feedback control law $u(t) = \phi(x(t))$ guarantees uniform asymptotic stability of (5), then the observer-based feedback control law $u(t) = \phi(x(t) + e(t))$ guarantees uniform asymptotic convergence of the solution of (6) to $x = 0$. The proof of this theorem is omitted, since it is substantially identical to the proof of Theorem 4.1.

Theorem 4.2. Consider the nonlinear dynamical system (6), assume that $e(t) \to 0$ as $t \to \infty$ uniformly in t_0, and suppose there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, $\lambda(\cdot)$ such that (13) is satisfied and

$$\frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} F(t, x, \phi(x)) \leq -\lambda(\|x\|), \qquad (t, x) \in [t_0, \infty) \times \mathcal{D}. \tag{15}$$

Then, there exists a compact set $\mathcal{M} \subset \mathcal{D}$, such that $0 \in \mathcal{M}$ and if $x_0 \in \mathcal{M}$, then $x(t) \to 0$ as $t \to \infty$ uniformly in t_0.

It is worth to remark that neither Theorem 4.1 nor Theorem 4.2 infer Lyapunov stability of the equilibrium point $x = 0$. These results merely state finite-time or asymptotic convergence of the closed-loop system's trajectory to the equilibrium point.

V. A Robust Observer-Based Feedback Controller for Nonlinear Dynamical Systems

In this section, we apply the results of Sections III and IV to design a robust observer-based feedback control law, which guarantees that the closed-loop trajectory is constrained to a simply connected set and asymptotically converges to the equilibrium point. This result is inspired by the sliding mode control architecture, as our controller steers the system trajectory to a sliding manifold in finite-time. However, in spite of the terminal sliding mode framework, our control architecture is observer-based and guarantees that the closed-loop system trajectory is constrained to a simply connected set.

We consider time-varying nonlinear dynamical systems of the form

$$\dot{x}(t) = f(x(t)) + B(x(t))\left[G(x(t))E(x(t))u(t) + \delta_1(t, x(t), u(t))\right] + \delta_2(x(t)),$$
$$x(t_0) = x_0, \qquad t \geq t_0, \quad (16)$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \geq t_0$, $u(t) \in U \subseteq \mathbb{R}^m$, $f : \mathcal{D} \to \mathbb{R}^n$, $B : \mathcal{D} \to \mathbb{R}^{n \times m}$, $G : \mathcal{D} \to \mathbb{R}^{m \times m}$, and $E : \mathcal{D} \to \mathbb{R}^{m \times m}$ are continuous in x, $f(0) = 0$, $E(x)$ is invertible for all $x \in \mathcal{D}$, $G(\cdot)$ is an unknown positive-definite diagonal matrix, that is, $G(x) \geq g_0 I_m$ for some $g_0 > 0$, $\delta_1 : [t_0, \infty) \times \mathcal{D} \times U \to \mathbb{R}^m$ is such that $\delta_1(t, 0, 0) = 0$, $t \geq t_0$, $\delta_1(\cdot, x, u)$ is continuous in t, for all $(x, u) \in \mathcal{D} \times U$, and $\delta_1(t, \cdot, \cdot)$ is jointly continuous in x and u uniformly in t, for all $t \in [t_0, \infty)$, and $\delta_2 : \mathcal{D} \to \mathbb{R}^n$ is continuous in x and such that $\delta_2(0) = 0$. The terms $\delta_1(\cdot, \cdot, \cdot)$ and $\delta_2(\cdot)$ are *unknown* and capture the matched and unmatched uncertainties, respectively. Assuming that the elements of $G(\cdot)$ are unknown allows accounting for failures in the control mechanism. The dynamical system (16) is in the same form as the dynamical model used in classic sliding mode design; for details, see [6, pp. 569-570].

For the statement of the following assumption, it is worth to recall that if $\mathcal{T} : \mathcal{D} \to \mathbb{R}^n$ is continuously differentiable, invertible, and $\mathcal{T}^{-1}(\cdot)$ is continuously differentiable, then $\mathcal{T}(\cdot)$ is a *diffeomorphism* [6, p. 508].

Assumption 5.1. Consider the nonlinear dynamical system (16) and the constraint set (7). There exists a diffeomorphism $\mathcal{T} : \mathring{\mathcal{C}} \to \mathbb{R}^n$, such that $\mathcal{T}(0) = 0$ and

$$\frac{\partial \mathcal{T}(x)}{\partial x} B(x) = \begin{bmatrix} 0_{(n-m) \times m} \\ I_m \end{bmatrix}, \qquad x \in \mathring{\mathcal{C}}. \quad (17)$$

Let $\mathcal{T}(\cdot)$ be a diffeomorphism, such that Assumption 5.1 is satisfied and $\mathcal{T}(x) = [\eta^{\mathrm{T}}, \xi^{\mathrm{T}}]^{\mathrm{T}}$, where $x \in \mathring{\mathcal{C}}$, $\eta \in \mathbb{R}^{n-m}$, and $\xi \in \mathbb{R}^m$. Then (16) is equivalent to

$$\dot{\eta}(t) = f_\eta(\eta(t), \xi(t)) + \delta_\eta(\eta(t), \xi(t)), \qquad \eta(t_0) = [I_{n-m}, 0_{(n-m) \times m}]\mathcal{T}(x_0), \qquad t \geq t_0, \quad (18)$$
$$\dot{\xi}(t) = f_\xi(\eta(t), \xi(t)) + G(\eta(t), \xi(t))E(\eta(t), \xi(t))u(t) + \delta_\xi(t, \eta(t), \xi(t), u(t)),$$
$$\xi(t_0) = [0_{m \times (n-m)}, I_m]\mathcal{T}(x_0). \quad (19)$$

Assumption 5.1, which is fundamental to design also classical sliding mode controls [6, p. 564], postulates the separation of the matched uncertainties from the unmatched uncertainties. This assumption is usually verified by mechanical systems, where $f_\eta(\cdot, \cdot)$ captures the undisturbed kinematic equations and $f_\xi(\cdot, \cdot)$ captures the undisturbed, uncontrolled dynamic equations.

Define the simply connected sets

$$\mathcal{C}_\eta \triangleq \{\eta \in \mathbb{R}^{n-m} : h_\eta(\eta) \geq 0\}, \quad (20)$$
$$\mathcal{C}_\xi \triangleq \{\xi \in \mathbb{R}^m : h_\xi(\xi) \geq 0\}, \quad (21)$$

where $h_\eta : \mathbb{R}^{n-m} \to \mathbb{R}$ and $h_\xi : \mathbb{R}^m \to \mathbb{R}$ are continuously differentiable, $h_\eta(0) > 0$, $h_\xi(0) > 0$, and $\mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \subseteq \mathcal{T}(\mathring{\mathcal{C}})$; note that if $(\eta, \xi) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi$, then $\mathcal{T}^{-1}(\eta, \xi) \in \mathring{\mathcal{C}}$, since $\mathcal{T}(\cdot)$ is continuous and surjective. The next lemma provides sufficient conditions for the existence of a state-feedback control law $\phi : \mathbb{R}^{n-m} \to \mathbb{R}^m$, such that $\phi(\mathring{\mathcal{C}}_\eta) \subseteq \mathring{\mathcal{C}}_\xi$ and the closed-loop system

$$\dot{\eta}(t) = f_\eta(\eta(t), \phi(\eta(t))) + \delta_\eta(\eta(t), \phi(\eta(t))), \quad \eta(t_0) = [I_{n-m}, 0_{(n-m)\times m}]\mathcal{T}(x_0), \quad t \geq t_0, \quad (22)$$

is asymptotically stable and $\eta(t) \in \mathring{\mathcal{C}}_\eta$, $t \geq t_0$. The existence of the state-feedback control law $\phi(\cdot)$ is fundamental to define a sliding manifold; for details, see [6, p. 564].

Lemma 5.1. Consider the nonlinear time-invariant dynamical system (18) with $\eta(t_0) \in \mathring{\mathcal{C}}_\eta$. If there exist a continuously differentiable function $V : \mathring{\mathcal{C}}_\eta \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\lambda(\cdot)$, and a state-feedback control law $\phi : \mathbb{R}^{n-m} \to \mathring{\mathcal{C}}_\xi$, such that $\phi(\mathring{\mathcal{C}}_\eta) \subseteq \mathring{\mathcal{C}}_\xi$ and

$$\alpha(\|\eta\|) \leq \frac{V(\eta)}{h_\eta(\eta)} \leq \beta(\|\eta\|), \qquad \eta \in \mathring{\mathcal{C}}_\eta, \tag{23}$$

$$\left[\frac{1}{h_\eta(\eta)} \frac{\partial V(\eta)}{\partial \eta} - \frac{V(\eta)}{h_\eta^2(\eta)} \frac{\partial h_\eta(\eta)}{\partial \eta} \right] [f_\eta(\eta, \phi(\eta)) + \delta_\eta(\eta, \phi(\eta))] < -\lambda(\|\eta\|), \tag{24}$$

then the closed-loop system (22) is asymptotically stable, and $\eta(t) \in \mathring{\mathcal{C}}_\eta$ for all $t \geq t_0$.

Proof: The result is a direct consequence of Theorem 3.2 applied to the nonlinear time-invariant dynamical system (18). ∎

The next theorem, which is the main result of this section, provides a robust observer-based feedback control law, such that $x(t)$, $t \geq t_0$, lays in the interior of the constraint set $\mathcal{C}$ given by (7) and $x(t) \to 0$ as $t \to \infty$, in spite of the uncertainties captured by $G(\eta, \xi, u)$, $\delta_\xi(\eta, \xi, u)$, and $\delta_\eta(\eta, \xi)$. For the statement of this result, let

$$\mathcal{C}_z \triangleq \{z \in \mathbb{R}^m : h_z(z) \geq 0\} \tag{25}$$

be simply connected, $h_z : \mathbb{R}^m \to \mathbb{R}$ be continuously differentiable, $h_z(0) > 0$, and $\mathring{\mathcal{C}}_z \subseteq \{z \in \mathbb{R}^m : z = \xi - \phi(\eta), (\eta, \xi) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi\}$. Furthermore, let x_i denote the ith component of $x \in \mathbb{R}^n$, let the invertible matrix function $\hat{G} : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{m \times m}$ denote an *estimate* of $G(\cdot)$, and define

$$\psi(\eta, \xi, w) \triangleq -\lambda(\eta, \xi) + E^{-1}(\eta, \xi)w, \qquad (t, \eta, \xi, w) \in [t_0, \infty) \times \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \times \mathbb{R}^m, \tag{26}$$

$$\lambda(\eta, \xi) \triangleq E^{-1}(\eta, \xi)\hat{G}^{-1}(\eta, \xi) \left[f_\xi(\eta, \xi) - \frac{\partial \phi(\eta)}{\partial \eta} f_\eta(\eta, \xi) \right], \tag{27}$$

$$\Delta(t, \eta, \xi, w) \triangleq \delta(t, \eta, \xi, \psi(\eta, \xi, w)) + \left[I - G(\eta, \xi)\hat{G}^{-1}(\eta, \xi) \right] \left[f_\xi(\eta, \xi) - \frac{\partial \phi(\eta)}{\partial \eta} f_\eta(\eta, \xi) \right], \tag{28}$$

$$\delta(t, \eta, \xi, w) \triangleq \delta_\xi(t, \eta, \xi, w) - \frac{\partial \phi(\eta)}{\partial \eta} \delta_\eta(\eta, \xi), \tag{29}$$

where $\phi(\cdot)$ satisfies Lemma 5.1. Furthermore, let $e : [t_0, \infty) \to \mathcal{B}_{\delta_e}(0)$ be continuous and such that

$$[(\eta(t) + e_\eta(t))^{\mathrm{T}}, (\xi(t) + e_\xi(t))^{\mathrm{T}}]^{\mathrm{T}} \triangleq \mathcal{T}(x(t) + e(t)), \qquad t \geq t_0, \tag{30}$$

and $e(t) \to 0$ as $t \to \infty$ uniformly in t_0, where δ_e is sufficiently small, $x(\cdot)$ denotes the unique solution of (16) with $u = \psi(\eta + e_\eta, \xi + e_\xi, \gamma(\eta + e_\eta, \xi + e_\xi))$, $\gamma(\cdot, \cdot)$ is such that

$$\gamma_i(\eta, \xi) = -\text{sign}(z_i(\eta, \xi))\text{sign}\left(\frac{2}{h_z(z(\eta, \xi))} - \frac{z_i(\eta, \xi)}{h_z^2(z(\eta, \xi))}\frac{\partial h_z(z)}{\partial z_i}\right)\beta_i(\eta, \xi),$$
$$(\eta, \xi) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi, \qquad i = 1, \ldots, m, \quad (31)$$

$z_i(\cdot, \cdot)$ denotes the ith component of $z(\eta, \xi) \triangleq \xi - \phi(\eta)$, $i = 1, \ldots, m$, $k_i \in [0, 1)$, $\text{sign}(\cdot)$ denotes the signum function, $\beta_i : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \overline{\mathbb{R}}_+$ is continuous in its arguments and such that

$$\beta_i(\eta, \xi) \geq c + \frac{\rho_i(\eta, \xi)}{1 - k_i}, \tag{32}$$

$c > 0$, $\rho_i : \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \to \overline{\mathbb{R}}_+$ is continuous in its arguments, and

$$\left|\left[G^{-1}(\eta, \xi)\Delta(t, \eta, \xi, \psi(\eta, \xi, w))\right]_i\right| \leq \rho_i(\eta, \xi) + k_i|w_i|,$$
$$(t, \eta, \xi, w) \in [t_0, \infty) \times \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \times \mathbb{R}^m. \tag{33}$$

Theorem 5.1. Consider the nonlinear dynamical system (16) and the constraint sets (7), (20), and (25). Suppose Assumption 5.1 is verified, the conditions of Lemma 5.1 are satisfied, and there exist $k_i \in [0, 1)$, $i = 1, \ldots, m$, and continuous functions $\rho_i : \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \to \overline{\mathbb{R}}_+$, such that (33) is verified. Then, there exist compact sets $\mathcal{M}_\eta \subset \mathring{\mathcal{C}}_\eta$ and $\mathcal{M}_z \subset \mathring{\mathcal{C}}_z$, such that $0 \in \mathcal{M}_\eta$ and $0 \in \mathcal{M}_z$. Furthermore, if $(\eta(t_0), \xi(t_0) - \phi(\eta(t_0)) \in \mathcal{M}_\eta \times \mathcal{M}_z$, then the solution $x(\cdot)$ of (16), with

$$u = \psi(\eta + e_\eta, \xi + e_\xi, \gamma(\eta + e_\eta, \xi + e_\xi)), \qquad (\eta, \xi, e_\eta, e_\xi) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \times \mathcal{B}_{\delta_{e_\eta}}(0) \times \mathcal{B}_{\delta_{e_\xi}}(0), \tag{34}$$

where $\mathcal{B}_{\delta_{e_\eta}}(0) \times \mathcal{B}_{\delta_{e_\xi}}(0) \subseteq \mathcal{T}(\mathcal{B}_{\delta_e}(0))$, is such that $x(t) \in \mathring{\mathcal{C}}$, $t \geq t_0$, and $x(t) \to 0$ as $t \to \infty$ uniformly in t_0.

Theorem 5.1, whose proof is reported in the Appendix, provides sufficient conditions to guarantee that the trajectory $x(t)$, $t \geq t_0$, of (16) with feedback control law (34) is contained in the interior of the constraint set (7), and asymptotically convergences to the equilibrium point $x = 0$. The classical sliding mode architecture guarantees asymptotic convergence for uncertain nonlinear plants in the same form as (16), and accounts for uncertainties, such that

$$\left\|G^{-1}(\eta, \xi)\Delta(t, \eta, \xi, \psi(\eta, \xi, w))\right\|_\infty \leq \rho(\eta, \xi) + k\|w\|_\infty,$$
$$(t, \eta, \xi, w) \in [t_0, \infty) \times \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^m; \tag{35}$$

note that (35) implies (33). The classical sliding mode control does not account for constraints on the system's trajectory, which is assumed to be perfectly known.

The control law (34) is a function of the design parameters $c > 0$ and $k_i \in [0, 1)$, $i = 1, \ldots, m$. The parameter c is arbitrary and should be chosen as small as possible to minimize the amplitude of chattering in proximity of the sliding manifold. The parameters k_i, $i = 1, \ldots, m$, depend on the modeling assumptions on the plant and should be chosen as small as possible to reduce chattering. However, smaller values of c and k_i imply slower convergence to the sliding manifold.

Remark 5.1. If

$$\text{sign}\left(\frac{2}{h_z(z(\eta, \xi))} - \frac{z_i(\eta, \xi)}{h_z^2(z(\eta, \xi))}\frac{\partial h_z(z)}{\partial z_i}\right) > 0, \qquad i = 1, \ldots, m, \tag{36}$$

for all $(\eta, \xi) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi$, then (31) reduces to

$$\gamma_i(\eta, \xi) = -\text{sign}(z_i(\eta, \xi))\beta_i(\eta, \xi). \tag{37}$$

This condition is verified, for instance, if $h_z(z) < e^{2\omega} z_i^2$, for all $z \in \mathbb{R}^m$ and $i = 1, \ldots, m$, and

$$\mathring{\mathcal{C}}_z \subset \left\{ z \in \mathbb{R}^m : \|z\| \le e^{-\omega} \right\}, \tag{38}$$

where $\omega \in \mathbb{R}$.

In order to guarantee that the solution $x(\cdot)$ of (16) with observer-based feedback control (34) is such that $x(t) \in \mathring{\mathcal{C}}$, $t \ge t_0$, Theorem 5.1 requires the estimation error $e(\cdot)$ to be such that $e(t) \in \mathcal{B}_{\delta_e}(0)$, $t \ge t_0$, where $\delta_e > 0$ is sufficiently small. Specifically, one needs to design a robust nonlinear observer, such that $(\eta(t) + e_\eta(t)) \in \mathring{\mathcal{C}}_\eta$, $t \ge t_0$, and $(\xi(t) + e_\xi(t)) \in \mathring{\mathcal{C}}_\xi$. This observer design problem is addressed in the next section.

VI. The Walcott and Żak Observer

In this section, we show how a popular nonlinear robust observer, that is, the Walcott and Żak observer [38]–[44], can be employed within the proposed observer-based feedback control framework. Specifically, consider the uncertain nonlinear dynamical system

$$\dot{x}(t) = Ax(t) + B\big[u(t) + \delta_1(t, x(t), u(t))\big], \qquad x(t_0) = x_0, \qquad t \ge t_0, \tag{39}$$
$$y(t) = Cx(t), \tag{40}$$

where $x(t) \in \mathbb{R}^n$, $t \ge 0$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $\delta_1 : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is *unknown*, $\delta_1(\cdot, \cdot, \cdot)$ is jointly continuous in t, x, and u, $\delta_1(t, \cdot, \cdot)$ is locally jointly Lipschitz continuous in x and u uniformly in t, for all t in compact subsets of $[t_0, \infty)$, and $C \in \mathbb{R}^{l \times n}$ has full row rank. The plant model (39) is sufficiently broad to model numerous problems of practical interest, since several system identification techniques can be applied to compute linear models that provide satisfactory first-order approximations of nonlinear dynamical systems [59], [60]. To simplify the sensors' calibration process, it is common practice to consider some linear function of the measured state vector as the system output; the rank condition on the matrix C implies that there is no redundancy amongst the measurements captured by the output $y(\cdot)$.

Consider the *observer*

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B\big[u(t) - v(t)\big] - K\big[C\hat{x}(t) - y(t)\big], \qquad \hat{x}(t_0) = \hat{x}_0, \qquad t \ge t_0, \tag{41}$$

where the *observer gain* $K \in \mathbb{R}^{n \times l}$ is such that $A_e \triangleq A + KC$ is Hurwitz, P is the symmetric, positive-definite solution of the Lyapunov equation

$$0 = A_e^{\mathrm{T}} P + PA_e + Q, \tag{42}$$

$Q \in \mathbb{R}^{n \times n}$ is symmetric and positive-definite, and $F \in \mathbb{R}^{m \times l}$ is such that

$$PB = C^{\mathrm{T}} F^{\mathrm{T}}. \tag{43}$$

The next result provides a feedback control law for the virtual control $v : [t_0, \infty) \to \mathbb{R}^m$ so that the state $\hat{x}(\cdot)$ of (41) converges to the plant state $x(t)$ of (39) exponentially in time. For the statement of this result, let $e(t) = x(t) - \hat{x}(t)$, $t \ge t_0$, denote the *estimation error*, and note that (39)–(41) imply that

$$\dot{e}(t) = A_e e(t) + B\big[v(t) + \delta_1(t, x(t), u(t))\big], \qquad e(t_0) = x_0 - \hat{x}_0, \qquad t \ge t_0, \tag{44}$$

where $x(t)$ verifies (39).

Theorem 6.1 [38]. Consider the uncertain nonlinear dynamical system (39) and (40), the observer (41), and the estimation error dynamics (44). If there exist a continuous function $\rho : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \to \overline{\mathbb{R}}_+$ such that

$$\|\delta_1(t, x, u)\| \leq \rho(t, \hat{x}, y, u), \qquad (t, x, \hat{x}, y, u) \in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m, \qquad (45)$$

then the solution $e(t)$, $t \geq 0$, of (44) with feedback control law

$$v = \gamma(t, \hat{x}, y, u), \qquad (t, \hat{x}, y, u) \in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m, \qquad (46)$$

is such that $e(t) \in \mathcal{B}_{\delta_e}(0)$, $t \geq t_0$, and $e(t) \to 0$ as $t \to \infty$ uniformly in t_0 and exponentially in time, where $\gamma_i(t, \hat{x}, y, u) = -\text{sign}(z_i(\hat{x}, y))\beta(t, \hat{x}, y, u)$, $z_i(\hat{x}, y)$ denotes the ith component of $z(\hat{x}, y) \triangleq F[y - C\hat{x}]$, $i = 1, \ldots, m$, $\beta(t, \hat{x}, y, u) \geq c_e + \rho(t, \hat{x}, y, u)$, $c_e > 0$, and $\|x_0 - \hat{x}_0\| < \delta_e$.

Remark 6.1. Theorem 6.1 guarantees that the norm of the estimation error is a decreasing function of time and exponentially converges to zero [38]. Hence, if the estimation error $e(t_0) = x_0 - \hat{x}_0$ is sufficiently small, that is, $(\eta(t_0) + e_\eta(t_0)) \in \mathring{\mathcal{C}}_\eta$ and $(\xi(t_0) + e_\xi(t_0)) \in \mathring{\mathcal{C}}_\xi$, where $\mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \subseteq \mathcal{T}(\mathcal{B}_{\delta_e}(0))$, then the Walcott and Żak observer given by (41) and (46) is suitable estimator for the observer-based feedback control framework developed herein. Similarly, the observers presented in [39]–[44], which are variations of the Walcott and Żak observer, can be employed within the proposed framework.

Although the estimation error $e(\cdot)$ can be neither measured directly nor computed integrating (44), $Ce(t) = y(t) - C\hat{x}(t)$, $t \geq t_0$, is well-defined, since the measured output $y(t)$ is known and $\hat{x}(t)$ is computed integrating (41) with the same control input $u(\cdot)$ as in (39) and the virtual control input $v(\cdot)$ given by (46). Thus, (41) with feedback control law (46) is well-defined. The gain matrix $K \in \mathbb{R}^{l \times n}$ in (41) plays an important role in the Walcott and Żak observer. Specifically, since $A_e = A + KC$, it follows from (44) and (46) that K directly affects the rate of exponential convergence of the estimation error along the sliding manifold. Moreover, the matrices Q and K provide $n(n + 1)/2 + ml$ design variables so that P and F verify both (42) and (43).

VII. Roll Stabilization of a Delta-Wing Aircraft

In this section, we apply the results developed in Sections V and VI and provide an observer-based feedback robust nonlinear control that stabilizes the roll dynamics of a delta-wing aircraft; this aircraft configuration is unstable [61, pp. 285]. Specifically, our controller guarantees that the aircraft roll angle and roll rate are constrained to given intervals at all times and the system's state asymptotically converges to the equilibrium condition.

The roll dynamics of a delta-wing aircraft is captured by [61, pp. 285-290]

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\theta_1 & -\theta_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \theta_6 \end{bmatrix} \left[u(t) + \tilde{\delta}_1(x_1, x_2) \right], \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad t \geq 0, \quad (47)$$

$$y(t) = \hat{\theta}_6^{-1} x_1(t) + \hat{\theta}_6^{-1} x_2(t), \qquad (48)$$

where $x_1(t)$ and $x_2(t) \in \mathbb{R}$ denote the aircraft roll angle and roll rate, respectively, $u(t) \in \mathbb{R}$ denotes the voltage applied to the aileron motor to induce a deflection of the control surface [62, Ex. 3.4], $\Phi \in \mathbb{R}$, $\tilde{\delta}_1(x_1, x_2) = \theta_6^{-1}[-\theta_3|x_1|x_2 + \theta_4|x_2|x_2 + \theta_5 x_1^3]$, the parameters $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$, $i = 1, \ldots, 6$, are unknown, $\overline{\theta}_j, \underline{\theta}_j > 0$, $j = 1, 3, \ldots 6$, is known, $\overline{\theta}_2, \underline{\theta}_2 > 1$ is known, and
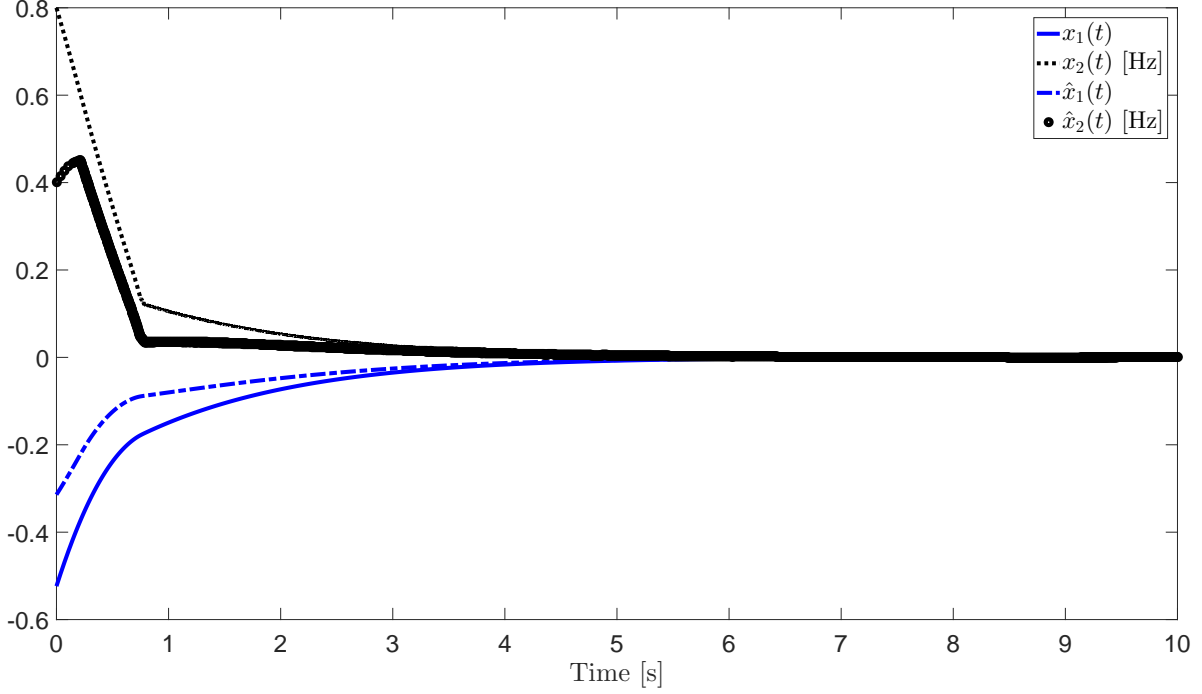
Figure VII.1. Closed-loop aircraft roll dynamics and estimated roll dynamics versus time.

$\hat{\theta}_6 \in \left[\frac{1}{2}\overline{\theta}_6, \overline{\theta}_6\right]$ is known. Our goal is to apply Theorems 5.1 and 6.1 and design an observer-based feedback control law, such that $x(t) \to 0$ as $t \to \infty$, where $x = [x_1, x_2]^{\mathrm{T}}$, and $x(t) \in \overset{\circ}{\mathcal{C}}$, $t \geq 0$, where

$$\mathcal{C} = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} : |x_1| \leq \overline{x}_1, |x_2| \leq \overline{x}_2\}$$

and $\overline{x}_1, \overline{x}_2 > 0$.

Let $\hat{\theta}_i \in \left[\underline{\theta}_i, \overline{\theta}_i\right]$, $i = 1, 2, 6$, denote an estimate of θ_i. The nonlinear dynamical system (47) and (48) is equivalent to

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\hat{\theta}_1 & -\hat{\theta}_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{\theta}_6 \end{bmatrix} [u(t) + \delta_1(x_1(t), x_2(t), u(t))],$$

$$[x_1(0), x_2(0)]^{\mathrm{T}} = [x_{10}, x_{20}]^{\mathrm{T}}, \qquad t \geq 0, \quad (49)$$

$$y(t) = \hat{\theta}_6^{-1} x_1(t) + \hat{\theta}_6^{-1} x_2(t), \tag{50}$$

where

$$\delta_1(x_1, x_2, u) = \hat{\theta}_6^{-1} \left[\theta_6 \tilde{\delta}_1(x_1, x_2) + (\hat{\theta}_1 - \theta_1)x_1 + (\hat{\theta}_2 - \theta_2)x_2 + (\theta_6 - \hat{\theta}_6)u \right],$$

and (49) and (50) is in the same form as (16) with $n = 2$, $m = 1$, $x = [x_1, x_2]^{\mathrm{T}}$, $t_0 = 0$, $f(x) = \begin{bmatrix} 0 & 1 \\ -\hat{\theta}_1 & -\hat{\theta}_2 \end{bmatrix} x$, $B(x) = \left[0, \hat{\theta}_6\right]^{\mathrm{T}}$, $G(x) = E(x) = 1$, and $\delta_2(x) = 0$. In this case, Assumption 5.1 is verified by $\mathcal{T}(x) = \hat{\theta}_6^{-1}x$, $x \in \mathbb{R}^2$, $\eta = \hat{\theta}_6^{-1}x_1$, and $\xi = \hat{\theta}_6^{-1}x_2$. Moreover, $f_\eta(\eta, \xi) = \xi$, $f_\xi(\eta, \xi) = -\hat{\theta}_1\eta - \hat{\theta}_2\xi$, $\delta_\eta(\eta, \xi) = 0$, $\delta_\xi(t, \eta, \xi, u) = \delta_1(\hat{\theta}_6\eta, \hat{\theta}_6\xi, u)$,

$$\mathcal{C}_\eta = \left\{ \eta \in \mathbb{R} : |\eta| \leq \hat{\theta}_6^{-1}\overline{x}_1 \right\}, \tag{51}$$
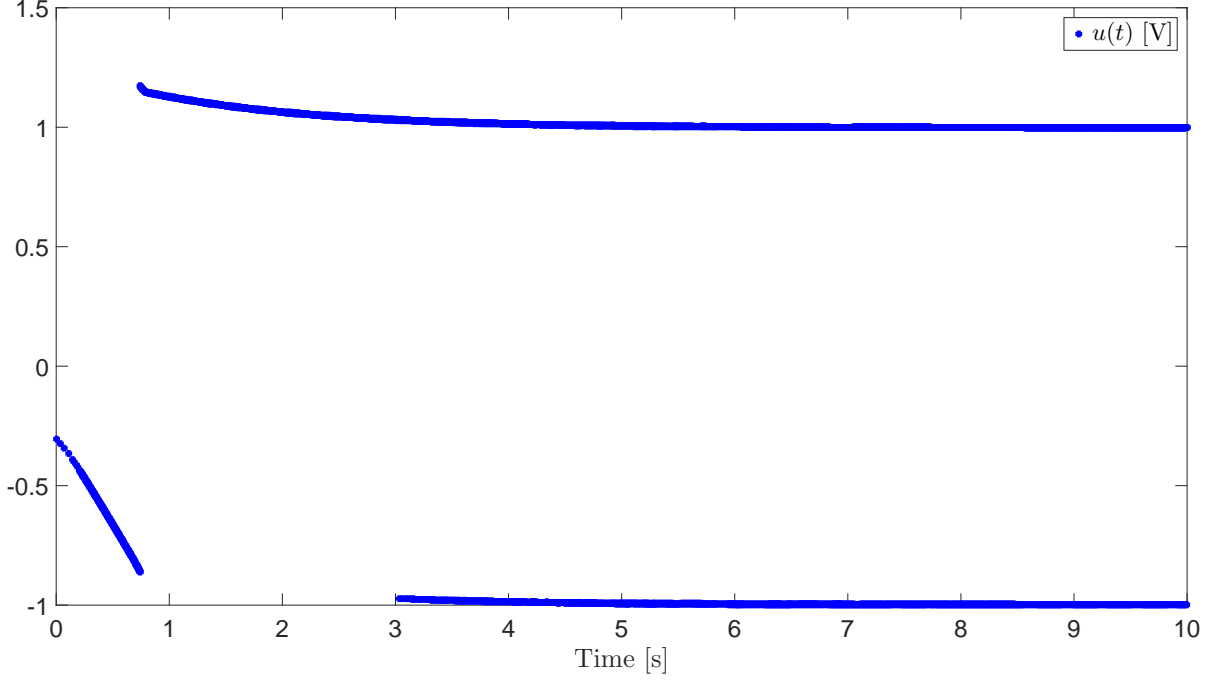
Figure VII.2. Output-feedback control versus time.

$$\mathcal{C}_\xi = \left\{ \xi \in \mathbb{R} : |\xi| \leq \hat{\theta}_6^{-1} \overline{x}_2 \right\}, \tag{52}$$

$$\mathcal{C}_z = \left\{ z \in \mathbb{R} : |z| \leq \hat{\theta}_6^{-1} \left(\overline{x}_2 + q \overline{x}_1 \right) \right\}, \tag{53}$$

$h_\eta(\eta) = \left(\hat{\theta}_6^{-1} \overline{x}_1 \right)^2 - \eta^2$, $h_\xi(\xi) = \left(\hat{\theta}_6^{-1} \overline{x}_2 \right)^2 - \xi^2$, and $h_z(z) = \hat{\theta}_6^{-2} \left(\overline{x}_2 + q \overline{x}_1 \right)^2 - z^2$.

Next, consider the continuously differentiable function

$$V(\eta) = \eta^2, \qquad \eta \in \mathring{\mathcal{C}}_\eta, \tag{54}$$

and the state-feedback control law $\phi(\eta) = -q\eta$, where $q \in \left(0, \overline{x}_2 \overline{x}_1^{-1} \right)$, and note that $\dfrac{V(\eta)}{h_\eta(\eta)}$ is

positive-definite for all $\eta \in \mathring{\mathcal{C}}_\eta$, (23) is verified with $\alpha(\|\eta\|) = \beta(\|\eta\|) = \dfrac{V(\eta)}{\left| (\hat{\theta}_6^{-1} \overline{x}_1)^2 - \eta^2 \right|}$, and

$\phi(\mathring{\mathcal{C}}_\eta) \subset \mathring{\mathcal{C}}_\xi$. Furthermore, it holds that

$$\left[\frac{1}{h_\eta(\eta)} \frac{\partial V(\eta)}{\partial \eta} - \frac{V(\eta)}{h_\eta^2(\eta)} \frac{\partial h_\eta(\eta)}{\partial \eta} \right] [f_\eta(\eta, \phi(\eta)) + \delta_\eta(\eta, \phi(\eta))] = -2q \frac{h_\eta(\eta) + \eta^2}{h_\eta^2(\eta)} \eta^2$$

$$= -2q \frac{\left(\hat{\theta}_6^{-1} \overline{x}_1 \right)^2 \eta^2}{\left(\hat{\theta}_6^{-1} \overline{x}_1 \right)^2 - \eta^2}, \quad \eta \in \mathring{\mathcal{C}}_\eta, \tag{55}$$

which implies that (24) is verified. Hence, the conditions of Lemma 5.1 are satisfied and the closed-loop system

$$\dot{\eta}(t) = -q\eta(t), \qquad \eta(0) = \hat{\theta}_6^{-1} x_{10}, \qquad t \geq 0, \tag{56}$$

is asymptotically stable and $\eta(t) \in \mathring{\mathcal{C}}_\eta$, $t \geq 0$. Finally, in this case it holds that

$$\left[G^{-1}(\eta, \xi) \Delta(t, \eta, \xi, \psi(\eta, \xi, w)) \right] = \delta_\xi(t, \eta, \xi, \psi(\eta, \xi, w)), \qquad [t_0, \infty) \times \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi \times \mathbb{R}^m,$$

where

$$\psi(\eta, \xi, w) = \hat{\theta}_1 \eta + \left(\hat{\theta}_2 - q\hat{\theta}_1 \right) \xi + w, \tag{57}$$

and (33) is satisfied with

$$\rho_1(\eta, \xi) = \overline{\theta}_3 \hat{\theta}_6 |\eta||\xi| + \overline{\theta}_4 \hat{\theta}_6 \xi^2 + \overline{\theta}_5 \hat{\theta}_6^2 |\eta|^3 + \left(\hat{\theta}_1 - \underline{\theta}_1 \right) |\eta| + \left(\hat{\theta}_2 - \underline{\theta}_2 \right) |\xi|$$

$$+ \hat{\theta}_6^{-1} \left(\overline{\theta}_6 - \hat{\theta}_6 \right) \left[\hat{\theta}_1 |\eta| + \left(\hat{\theta}_2 - q\hat{\theta}_1 \right) |\xi| \right], \qquad (\eta, \xi) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi, \tag{58}$$

$$k_1 = \hat{\theta}_6^{-1} \left(\overline{\theta}_6 - \hat{\theta}_6 \right). \tag{59}$$

In this case, the conditions of Theorem 5.1 are verified and it follows from Remark 5.1 and (31) that

$$\gamma_1(\eta, \xi) = -\text{sign}(\xi + q\eta)\beta_1(\eta, \xi), \qquad (\eta, \xi) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi, \tag{60}$$

where $\beta_1(\eta, \xi) \geq c + (1 - k_1)^{-1}\rho_1(\eta, \xi)$ and $c > 0$.

Let $\mathcal{M}_\eta$ be a compact set, such that $0 \in \mathcal{M}_\eta \subset (-\hat{\theta}_6^{-1}\overline{x}_1, \hat{\theta}_6^{-1}\overline{x}_1) = \mathring{\mathcal{C}}_\eta$ and let $\mathcal{M}_z$ be a compact set, such that $0 \in \mathcal{M}_z \subset \left(-\hat{\theta}_6^{-1}\overline{x}_2 - q\hat{\theta}_6^{-1}\overline{x}_1, \hat{\theta}_6^{-1}\overline{x}_2 + q\hat{\theta}_6^{-1}\overline{x}_1 \right) = \mathring{\mathcal{C}}_z$. It follows from Theorem 5.1 that if $\left(\hat{\theta}_6^{-1}x_{10}, \hat{\theta}_6^{-1}(x_{20} + qx_{10}) \right) \in \mathcal{M}_\eta \times \mathcal{M}_z$ and the estimated state $(\eta(t) + e_\eta(t), \xi(t) + e_\xi(t)) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi$, $t \geq 0$, then the solution $x(t) = [x_1(t), x_2(t)]^{\text{T}}$, $t \geq 0$, of (47) with feedback control (34) is such that $x(t) \in \mathring{\mathcal{C}}$, $t \geq 0$, and $x(t) \to 0$ as $t \to \infty$.

Next, we apply the Walcott and Żak observer outlined in Theorem 6.1 to estimate the trajectory of (49). Note that (49) and (50) is in the same form as (39) and (40) with $A = \begin{bmatrix} 0 & 1 \\ -\hat{\theta}_1 & -\hat{\theta}_2 \end{bmatrix}$, $B = \left[0, \hat{\theta}_6 \right]^{\text{T}}$, $C = \hat{\theta}_6^{-1}[1, 1]$, and $\delta_1(t, x, u) = \delta_1(x, u)$, $(t, x, u) \in [0, \infty) \times \mathbb{R}^2 \times \mathbb{R}$, and consider the observer (41), where $x(t)$ satisfies (49). Defining the estimation error as $e(t) = x(t) - \hat{x}(t)$, $t \geq 0$, it follows from (49), (41), and (57) that

$$\dot{e}(t) = A_e e(t) + B \left[v(t) + \delta_1(\hat{\theta}_6 \eta(t), \hat{\theta}_6 \xi(t), \psi(\eta(t), \xi(t), \gamma_1(\eta(t), \xi(t)))) \right],$$
$$e(0) = x_0 - \hat{x}_0, \qquad t \geq 0, \tag{61}$$

where $A_e = A + KC$.

In this case, it holds that

$$|\delta_1(\hat{\theta}_6 e_1, \hat{\theta}_6 e_2, \psi(e_1, e_2, \gamma_1(e_1, e_2)))| \leq \rho_2(e_1, e_2), \qquad (e_1, e_2) \in \mathbb{R} \times \mathbb{R}, \tag{62}$$

where $e = \hat{\theta}_6[e_1, e_2]^{\text{T}}$ and $\rho_2(e_1, e_2) \triangleq \rho_1(e_1, e_2) + k_1|\psi(e_1, e_2, \gamma_1(e_1, e_2))|$, and $\rho_2(e_1, e_2) \leq \rho(\alpha)$, $(e_1, e_2, \alpha) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, where

$$\rho(\alpha) \triangleq \max_{\alpha \in \{e_1, e_2\}} \rho_2(\alpha, \alpha).$$

Since both $e_1(\cdot)$ and $e_2(\cdot)$ are continuous in their arguments, it follows from (57), (58), and (60) that both $\alpha(t) = \hat{\theta}_6^{-1} \|x(t) - \hat{x}(t)\|_\infty$, $t \geq 0$, and $\rho(\alpha(t))$ are continuous, where $\| \cdot \|_\infty$ denotes the infinity norm of its argument. Now, if there exist $K \in \mathbb{R}^2$, $F \in \mathbb{R}$, and a symmetric positive-definite matrix $Q \in \mathbb{R}^{2 \times 2}$ such that (42) and (43) are verified, then it follows from Theorem 6.1

that the estimation error dynamics (61) with feedback control law (46) is such that $\|x(\cdot) - \hat{x}(\cdot)\|$ is a monotonically decreasing function and converges to zero exponentially fast. In this problem, the feedback control law $v = \gamma(\hat{x}, y)$ can be computed as follows. It follows from (58) that $\rho(\cdot)$ is a monotonically increasing function and note that

$$\hat{\theta}_6^{-1} \max\{|x_1 - \hat{x}_1|, |x_2 - \hat{x}_2|\} \leq \hat{\theta}_6^{-1}|x_1 - \hat{x}_1 + x_2 - \hat{x}_2| + \hat{\theta}_6^{-1} \min\{|x_1 - \hat{x}_1|, |x_2 - \hat{x}_2|\}$$
$$= |y - C\hat{x}| + \hat{\theta}_6^{-1} \min\{|x_1 - \hat{x}_1|, |x_2 - \hat{x}_2|\}.$$

Thus, it follows from [6, p. 188] that

$$\rho(\alpha(t)) \leq \rho(2|y(t) - C\hat{x}(t)|) + \rho\left(2\hat{\theta}_6^{-1} \min\{|x_1(0) - \hat{x}_1(0)|, |x_2(0) - \hat{x}_2(0)|\}\right), \qquad (63)$$

for all $t \geq 0$, and (61) specializes to

$$\gamma(\hat{x}, y) = -\operatorname{sign}(y - C\hat{x})\left[\rho(2|y - C\hat{x}|) + c_e\right], \qquad (\hat{x}, y) \in \mathbb{R}^2 \times \mathbb{R}, \qquad (64)$$

where $c_e \geq \rho\left(2\hat{\theta}_6^{-1} \min\{|x_1(0) - \hat{x}_1(0)|, |x_2(0) - \hat{x}_2(0)|\}\right)$; although $x(0)$ may not be precisely known, it is reasonable to assume that an estimate of $x(0)$ is available. Alternative approaches to estimate both the aircraft roll angle and the roll rate consist in applying Kalman filters [49, Ch. 8] or extended Kalman filters [49, Ch. 13] to data measured by the gyroscopes installed on the vehicle [63].

Let $\theta_1 = 0.018$, $\theta_2 = 1.113$, $\theta_3 = 0.062$, $\theta_4 = 0.009$, $\theta_5 = 0.0021$, $\theta_6 = 0.790$, $\bar{\theta}_i = 0.1$, $i = 1, 3, 4, 5$, $\bar{\theta}_2 = 2.1$, $\bar{\theta}_6 = 1$, $\underline{\theta}_j = 10^{-3}$, $j = 1, 3, \ldots, 6$, $\underline{\theta}_2 = 1.001$, $\hat{\theta}_p = 0.03$, $p = 1, 3, 4, 5$, $\hat{\theta}_2 = 1.05$, $\hat{\theta}_6 = 0.9$, $\bar{x}_1 = 2\pi/9$, $\bar{x}_2 = 1\,\text{Hz}$, $q = \frac{1}{2}\bar{x}_2\bar{x}_1^{-1}$, $K = 0$, $F = 1$,

$$P = \begin{bmatrix} 2F\left(\hat{\theta}_1 + \hat{\theta}_2\right) & F \\ F & F \end{bmatrix}, Q = \begin{bmatrix} 2F\hat{\theta}_1 & 0 \\ 0 & 2F\left(\hat{\theta}_2 - 1\right) \end{bmatrix}, \text{ and } c = 1. \text{ Figure VII.1 shows the}$$

trajectory of the closed-loop system using the observer-based feedback control law $u = \psi(\eta + e_\eta, \xi + e_\xi, \gamma_1(\eta + e_\eta, \xi + e_\xi))$, which is shown in Figure VII.2; clearly, $x(t) \in \mathring{\mathcal{C}}$, $t \geq 0$, $x(t) \to 0$ as $t \to \infty$, and $\|x(0) - \hat{x}(0)\| \geq \|x(t) - \hat{x}(t)\|$, $t \geq t_0$, and $\|x(t) - \hat{x}(t)\| \to 0$ as $t \to \infty$, that is, the closed-loop system converges to $x \equiv 0$ and the estimated state converges to the system's state. Figure VII.2 clearly shows that the control input is affected by chattering for all $t \geq 3$.

VIII. Conclusion

This paper presents an observer-based feedback sliding mode architecture for time-varying nonlinear dynamical systems in the presence of uncertainties in the plant model and constraints on the system's trajectory. The proposed control law guarantees finite-time convergence of the closed-loop trajectory to the sliding manifold, while the state constraints are verified by *barrier Lyapunov functions*. Moreover, it is shown how the proposed framework can be integrated with existing nonlinear estimators, such as the Walcott and Żak observer. Our theoretical results are illustrated by a numerical example involving the roll dynamics of an unstable aircraft.

The numerical example clearly shows that the proposed control algorithm is affected by chattering. Future work directions involve the development of a chattering-free observer-based feedback control algorithm, based on the framework presented in this paper. To this goal, one possibility involves the introduction of a *boundary layer*, where the signum function is approximated by the saturation, the sigmoid, or the arctangent functions [64], [65]. Alternatively, higher-order sliding mode methods [66] and observer-based sliding mode controllers [67] will be considered to mitigate chattering.

Appendix – Proofs of the Main Results

In this section, we present the proofs of the main theoretical results developed in this paper.

Proof of Theorem 3.1

Firstly, we use a contradiction argument to prove that if $x_0 \in \mathring{\mathcal{C}}$, then $x(t) \in \mathring{\mathcal{C}}$ for all $t \geq t_0$. Specifically, suppose *ad absurdum* there exists $T^* \in (t_0, \infty) \cap \mathcal{I}_{t_0,x_0}$, such that $\lim_{t \to T^*} h(x(t)) = 0$ along the trajectory of (1), where $\mathcal{I}_{t_0,x_0}$ denotes the maximal interval of existence of the solution $x(\cdot)$ of (1). It follows from the continuity of $h(\cdot)$ on $\mathcal{D}$ and $x(\cdot)$ on $\mathcal{I}_{t_0,x_0}$ that $\lim_{t \to T^*} h(x(t)) = h\left(\lim_{t \to T^*} x(t)\right) = h\left(x(T^*)\right) = 0$, which implies that $x(T^*) \neq 0$, since $0 \in \mathcal{C}$ and $h(x) > 0$, $x \in \mathring{\mathcal{C}}$. Moreover, it follows from (8) that $V(t, x) = 0$ if and only if $x = 0$ and hence it follows from the continuity of $V(\cdot, x(\cdot))$ that $\lim_{t \to T^*} V(t, x(t)) = V(T^*, x(T^*)) \neq 0$. Therefore, $\lim_{t \to T^*} \dfrac{V(t, x(t))}{h(x(t))} = \infty$. Now, since

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{V(t, x(t))}{h(x(t))} = \frac{1}{h(x(t))} \frac{\partial V(t, x(t))}{\partial t} + \left[\frac{1}{h(x(t))} \frac{\partial V(t, x(t))}{\partial x} \right. $$
$$\left. - \frac{V(t, x(t))}{h^2(x(t))} \frac{\partial h(x(t))}{\partial x} \right] f(t, x(t)), \qquad t \geq t_0, \quad (65)$$

along the trajectory of (1), it follows from (9) that $\dfrac{V(t, x(t))}{h(x(t))}$, $t \geq t_0$, is a decreasing function of time, that is, if $x_0 \in \mathring{\mathcal{C}}$ and $x_0 \neq 0$, then

$$\frac{V(t, x(t))}{h(x(t))} < \frac{V(t_0, x_0)}{h(x_0)} \leq \beta(\|x_0\|) < \infty, \qquad t > t_0. \quad (66)$$

However, the fact that $\lim_{t \to T^*} \dfrac{V(t, x(t))}{h(x(t))} = \infty$ contradicts (66). Hence, if $x_0 \in \mathring{\mathcal{C}}$, then $x(t) \in \mathring{\mathcal{C}}$ for all $t \geq t_0$, that is, $\mathring{\mathcal{C}}$ is positively invariant with respect to (1).

Next, we prove uniform Lyapunov stability of (1). Let $\varepsilon > 0$, consider the open set $\mathcal{B}_\varepsilon(0) \cap \mathring{\mathcal{C}}$ containing $x = 0$, and let $\delta = \delta(\varepsilon) > 0$ be such that $\beta(\delta) = \alpha(\varepsilon)$. It follows from (8) that, for all $(t_0, x_0) \in [0, \infty) \times \mathcal{B}_\delta(0) \cap \mathring{\mathcal{C}}$,

$$\alpha(\|x(t)\|) \leq \frac{V(t, x(t))}{h(x(t))} \leq \frac{V(t_0, x_0)}{h(x_0)} < \beta(\delta) = \alpha(\varepsilon), \qquad t \geq t_0. \quad (67)$$

Thus, $x(t) \in \mathcal{B}_\varepsilon(0) \cap \mathring{\mathcal{C}}$, $t \geq t_0$, since $\mathring{\mathcal{C}}$ is positively invariant with respect to (1) for every $x_0 \in \mathcal{B}_\delta(0) \cap \mathring{\mathcal{C}}$, and uniform Lyapunov stability of (1) is proven.

Next, note that the solution to

$$\dot{v}(t) = -kv^\theta(t), \qquad v(t_0) = v_0 = \frac{V(t_0, x_0)}{h(x_0)}, \qquad t \geq t_0, \quad (68)$$

with $(t_0, x_0) \in [0, \infty) \times \mathring{\mathcal{C}}$, is given by

$$v(t) = \begin{cases} \left[v_0^{1-\theta} - k(1-\theta)t \right]^{\frac{1}{1-\theta}}, & t_0 \leq t < t_1, \qquad v_0 \neq 0, \\ 0, & t \geq t_1, \qquad v_0 \neq 0, \\ 0, & t \geq t_0, \qquad v_0 = 0, \end{cases} \quad (69)$$

where

$$t_1 = \frac{1}{k(1-\theta)} \left[\frac{V(t_0, x_0)}{h(x_0)} \right]^{1-\theta}. \tag{70}$$

Now, let $w : [0, \infty) \to \mathbb{R}$ be a continuously differentiable function such that

$$\dot{w}(t) \le -kv^\theta(t), \qquad w(t_0) = \frac{V(t_0, x_0)}{h(x_0)}, \qquad t \ge t_0, \tag{71}$$

where $v(t)$ is given by (69). Then, it follows from (68), (71), and the comparison lemma [8, p. 126] that

$$w(t) \le v(t), \qquad t \ge 0. \tag{72}$$

Thus, it follows from (9), (68), (69), (71), and (72), with $w(t) = \left[\frac{V(t, x(t))}{h(x(t))} \right]^{1-\theta}$, $t \ge t_0$, that

$$\alpha(\|x(t)\|) \le \frac{V(t, x(t))}{h(x(t))} \le v(t), \qquad t \ge t_0, \tag{73}$$

and hence, using (8), (69), and (73),

$$x(t) = 0, \qquad t \ge t_1, \tag{74}$$

where t_1 is given in (70).

Next, we prove that there exist a neighborhood $\mathcal{D}_0 \subseteq \overset{\circ}{\mathcal{C}}$ of the origin and a settling-time function $T : [0, \infty) \times \mathcal{D}_0 \to (t_0, \infty)$ such that (10) is satisfied. Since $s(t_0, t_0, x_0) = x_0$ and $s(\cdot, \cdot, \cdot)$ is continuous, $\inf\{t \in (t_0, \infty) : s(t, t_0, x_0) = 0\} > t_0$, $x_0 \in \mathcal{B}_\delta(0) \cap \overset{\circ}{\mathcal{C}} \setminus \{0\}$. Furthermore, it follows from (74) that $\inf\{t \in (t_0, \infty) : s(t, t_0, x) = 0\} < \infty$, $x_0 \in \mathcal{B}_\delta(0) \cap \overset{\circ}{\mathcal{C}} \setminus \{0\}$. Now, defining $\mathcal{D}_0 \triangleq \mathcal{B}_\delta(0) \cap \overset{\circ}{\mathcal{C}}$ and $T : [0, \infty) \times \mathcal{D}_0 \to (t_0, \infty)$ by (69) and (70), (10) is immediate.

Next, we prove strong uniform finite-time stability of (1). For $t \ge T(t_0, x_0)$, uniform finite-time convergence of $x(t)$ to zero is immediate for all $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$. Alternatively, for every $t < T(t_0, x_0)$ and $\varepsilon > 0$, there exists $\delta = \alpha^{-1} \left(\frac{\varepsilon^{1-\theta}}{k(1-\theta)} \right)$ such that if $\|x(t)\| \le \alpha^{-1}(v(t)) < \varepsilon$, then $T(t_0, x_0) - t \le t_1 - t < \delta$, which proves uniform finite-time convergence of (1). Consequently, the time-varying nonlinear dynamical system (1) is strongly uniformly finite-time stable.

Lastly, it follows from the finite-time stability of (1) and Proposition 3.4 of [50] that $T(\cdot, \cdot)$ can be extended to $\overline{\mathbb{R}}_+$ and $T(t_0, 0) = 0$. Moreover, the right-hand side of (10) is jointly continuous at $(t_0, 0)$, $t_0 \in [0, \infty)$, and hence, by Proposition 3.4 of [50], it is jointly continuous on $[0, \infty) \times \mathcal{D}_0$. $\blacksquare$

Proof of Theorem 4.1

It follows from (13) that $x = 0$ is a local minimizer of $V(t, x)$, $(t, x) \in [t_0, \infty) \times \mathcal{D}$, and hence $\left[\frac{\partial V(t,x)}{\partial t}, \frac{\partial V(t,x)}{\partial x} \right]^{\mathrm{T}} = 0$, for all $t \in [t_0, \infty)$ and $x = 0$, since $V(\cdot, \cdot)$ is continuously differentiable on $[t_0, \infty) \times \mathcal{D}$. Furthermore, it follows from (14) and the continuity of $\left\| \frac{\partial V(t,x)}{\partial x} \right\|$ on $[t_0, \infty) \times \mathcal{D}$ that for every $\varepsilon > 0$, $k > 0$, and $\theta \in (0, 1)$ there exist a compact set $\mathcal{M} \subset \mathcal{D}$ and a continuous function $g : \mathcal{M} \to [0, k)$, such that $0 \in \mathcal{M}$ and

$$\varepsilon \left\| \frac{\partial V(t, x)}{\partial x} \right\| \le g(x)(V(t, x))^\theta, \qquad (t, x) \in [t_0, \infty) \times \mathcal{M}. \tag{75}$$

Next, it follows from (14) that the total derivative of $V(\cdot, \cdot)$ along the trajectory of (6) is given by

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} F(t,x,\phi(x+e)) = \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} F(t,x,\phi(x))$$
$$+ \frac{\partial V(t,x)}{\partial x} [F(t,x,\phi(x+e)) - F(t,x,\phi(x))]$$
$$\leq -k\,(V(t,x))^\theta + \frac{\partial V(t,x)}{\partial x} [F(t,x,\phi(x+e))$$
$$-F(t,x,\phi(x))]\,, \quad (76)$$

for all $(t,x,e) \in [t_0,\infty) \times \mathcal{D} \times \mathbb{R}^n$. Since $F(t,\cdot,\cdot)$ is jointly continuous in x and u uniformly in t, for all $t \in [t_0,\infty)$, and $\|\phi(\cdot)\|$ is continuous on $\mathcal{D}$, for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|e\| < \delta$, then $\|F(t,x,\phi(x+e)) - F(t,x,\phi(x))\| < \varepsilon$, for all $(t,x,e) \in [t_0,\infty) \times \mathcal{M} \times \mathcal{B}_\delta(0)$. Therefore,

$$\frac{\partial V(x)}{\partial x} F(t,x,\phi(x+e)) \leq -k\,(V(x))^\theta + \varepsilon \left\| \frac{\partial V(t,x)}{\partial x} \right\|, \quad (t,x,e) \in [t_0,\infty) \times \mathcal{M} \times \mathcal{B}_\delta(0),$$
$$(77)$$

and it follows from the continuity of $g(\cdot)$ on $\mathcal{M}$, (75), and Theorem 2.13 of [8] that

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} F(t,x,\phi(x+e)) \leq (\bar{g}-k)\,(V(t,x))^\theta, \quad (t,x,e) \in [t_0,\infty) \times \mathcal{M} \times \mathcal{B}_\delta(0),$$
$$(78)$$

where $\bar{g} \triangleq \max_{x \in \mathcal{M}} g(x)$ and $\bar{g} < k$.

Since $e(t) \to 0$ as $t \to \infty$ uniformly in t_0, for every $\varepsilon > 0$ there exists $T(\varepsilon) > t_0$, such that $\|e(t)\| < \delta(\varepsilon)$ for all $t > T(\varepsilon)$. Therefore, for every initial condition $x_0 \in \mathcal{M}$, it holds that

$$\frac{\partial V(t,x(t))}{\partial t} + \frac{\partial V(t,x(t))}{\partial x} F(t,x(t),\phi(x(t)+e(t))) \leq (\bar{g}-k)\,(V(t,x(t)))^\theta, \quad t > T(\varepsilon), \quad (79)$$

along the trajectory of (6). Uniform finite-time convergence of $x(\cdot)$ to zero now follows from (13) and (79) by proceeding as in the proof of Theorem 3.1. ∎

Proof of Theorem 5.1

It follows from Assumption 5.1 that there exists a diffeomorphism $\mathcal{T}(x) = [\eta^{\mathrm{T}}, \xi^{\mathrm{T}}]^{\mathrm{T}}$ such that $\mathcal{T}(0) = 0$, (17) is satisfied, and (16) is equivalent to (18) and (19). The nonlinear dynamical system (18) with $\xi = \phi(\eta)$ is equivalent to (22) and it follows from Lemma 5.1 that the equilibrium point $\eta(t) \equiv 0$, $t \geq t_0$, of (22) is asymptotically stable and $\eta(t) \in \mathring{\mathcal{C}}_\eta$, $t \geq t_0$.

Consider the nonlinear dynamical system

$$\dot{\eta}(t) = f_\eta(\eta(t), \phi(\eta(t)+e_\eta(t))) + \delta_\eta(\eta(t), \phi(\eta(t)+e_\eta(t))),$$
$$\eta(t_0) = [I_{n-m}, 0_{(n-m)\times m}]\mathcal{T}(x_0), \quad t \geq t_0, \quad (80)$$

and recall that by assumption, $e_\eta(t) \in \mathcal{B}_{\delta_{e_\eta}}(0)$, $t \geq t_0$, and $\lim_{t\to\infty} e_\eta(t) = 0$ uniformly in t_0. Thus, it follows from (23), (24), and Theorem 4.2 that there exists a compact set $\mathcal{M}_\eta \subset \mathring{\mathcal{C}}_\eta$, such that $0 \in \mathcal{M}_\eta$ and if $\eta(t_0) \in \mathcal{M}_\eta$, then $\eta(t) \to 0$ as $t \to \infty$ uniformly in t_0. Furthermore, proceeding as in the proofs of Theorems 4.1 and 3.1, one can show that $\eta(t) \in \mathcal{M}_\eta \subset \mathring{\mathcal{C}}_\eta$, $t \geq t_0$. Thus, the observer-based feedback control $\xi = \phi(\eta+e_\eta)$ guarantees that the solution of

the dynamical system (18) asymptotically converges to $\eta(t) \equiv 0$ and lays in the constraint set $\mathring{\mathcal{C}}_\eta$ at all times.

Let $[\eta^{\mathrm{T}}(t), \xi^{\mathrm{T}}(t)]^{\mathrm{T}}$, $t \geq t_0$, denote the solution of (18) and (19) with observer-based feedback control law (34). Next, we prove that $(\eta(t), \xi(t)) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi$, $t \geq t_0$, $\|\xi(t) - \phi(\eta(t))\| \to 0$ as $t \to t_1$, for some finite-time $t_1 \geq t_0$, and $\|\xi(t) - \phi(\eta(t))\|$ is sufficiently small, that is, $\|\xi(t) - \phi(\eta(t))\| < \|\phi(\eta(t) + e_\eta(t)) - \phi(\eta(t))\|$. Specifically, since $\|\phi(\cdot)\|$ is continuous on $\mathring{\mathcal{C}}_\eta$ and $e_\eta(t) \in \mathcal{B}_{\delta_{e_\eta}}(0)$, $t \geq t_0$, in the following we prove that the solution $[\eta^{\mathrm{T}}(t), \xi^{\mathrm{T}}(t)]^{\mathrm{T}}$, $t \geq t_0$, of (18) and (19) is such that $z(t) = (\xi(t) - \phi(\eta(t))) \in \mathcal{N}_z$, $t \geq t_0$, where $\mathcal{N}_z \subseteq \mathring{\mathcal{C}}_z$ is an open connected set such that $0 \in \mathcal{N}_z$, and $z(t) \to 0$ as $t \to t_1$ uniformly in t_0.

It follows from (18) and (19) that

$$
\dot{z}(t) = f_\xi(\eta(t), \xi(t)) - \frac{\partial \phi(\eta(t))}{\partial \eta} f_\eta(\eta(t), \xi(t)) + G(\eta(t), \xi(t)) E(\eta(t), \xi(t)) u(t)
$$
$$
+ \delta(t, \eta(t), \xi(t), u(t)), \qquad z(t_0) = \xi(t_0) - \phi(\eta(t_0)), \qquad t \geq t_0, \tag{81}
$$

and (81) with $u = \psi(\eta + e_\eta, \xi + e_\xi, w)$, where $\psi(\cdot, \cdot, \cdot)$ is given by (26), is equivalent to

$$
\dot{z}(t) = G(\eta(t), \xi(t)) w(t) + \Delta(t, \eta(t), \xi(t), w(t)) + \Gamma(t, \eta(t), \xi(t), w(t), e_\eta(t), e_\xi(t))
$$
$$
- \Gamma(t, \eta(t), \xi(t), w(t), 0, 0), \qquad z(t_0) = \xi(t_0) - \phi(\eta(t_0)), \qquad t \geq t_0, \tag{82}
$$

where $\Delta(\cdot, \cdot, \cdot, \cdot)$ is given by (28) and

$$
\Gamma(t, \eta, \xi, w, e_\eta, e_\xi) \triangleq \delta(t, \eta, \xi, \psi(\eta + e_\eta, \xi + e_\xi, w)) - G(\eta, \xi) E(\eta, \xi) \lambda(\eta + e_\eta, \xi + e_\xi)
$$
$$
+ G(\eta, \xi) E(\eta, \xi) E^{-1}(\eta + e_\eta, \xi + e_\xi) w,
$$
$$
(t, \eta, \xi, w, e_\eta, e_\xi) \in [t_0, \infty) \times \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathcal{B}_{\delta_{e_\eta}}(0) \times \mathcal{B}_{\delta_{e_\xi}}(0). \tag{83}
$$

Now, consider the nonlinear dynamical system (82) with $w = \gamma(\eta, \xi)$ and $e_\eta = e_\xi = 0$, let

$$
V(z) = \frac{\|z\|^2}{h_z(z)}, \qquad z \in \mathcal{N}_z, \tag{84}
$$

and define

$$
\delta_z \triangleq \min_{i=1,\ldots,m} \min_{z \in \hat{\mathcal{M}}_z} (1 - k_i) \left| \frac{2}{\sqrt{h_z(z)}} - \frac{z_i}{\sqrt{h_z^3(z)}} \frac{\partial h_z(z)}{\partial z_i} \right|, \tag{85}
$$

where $\hat{\mathcal{M}}_z \subset \mathcal{N}_z$ is a compact set, whose smallest open cover is $\mathcal{N}_z$ [55, pp. 618-619]; the existence of $\delta_z > 0$ is guaranteed by Theorem 2.13 of [8], since $\left| \dfrac{2}{\sqrt{h_z(z)}} - \dfrac{z_i}{\sqrt{h_z^3(z)}} \dfrac{\partial h_z(z)}{\partial z_i} \right|$ is continuous on $\hat{\mathcal{M}}_z$.

Since $\sum_{i=1}^m z_i^{2\theta} \geq \left(\sum_{i=1}^m z_i^2\right)^\theta$, $\theta \in (0, 1)$, it follows from (84), (82), (33), and (31) that

$$
\dot{V}(z) = \sum_{i=1}^m \left[\frac{2}{h_z(z)} - \frac{z_i}{h_z^2(z)} \frac{\partial h_z(z)}{\partial z_i} \right] z_i [g_i(\eta, \xi) \gamma_i(\eta, \xi) + \Delta_i(t, \eta, \xi, \gamma(\eta, \xi))]
$$
$$
\leq \sum_{i=1}^m g_i(\eta, \xi) \left[\left(\frac{2}{h_z(z)} - \frac{z_i}{h_z^2(z)} \frac{\partial h_z(z)}{\partial z_i} \right) z_i \gamma_i(\eta, \xi) \right.
$$
$$
\left. + \left| \frac{2}{h_z(z)} - \frac{z_i}{h_z^2(z)} \frac{\partial h_z(z)}{\partial z_i} \right| |z_i| (\rho_i(\eta, \xi) + k_i |\gamma_i(\eta, \xi)|) \right]
$$

$$\leq \sum_{i=1}^{m} \left| \frac{2}{h_z(z)} - \frac{z_i}{h_z^2(z)} \frac{\partial h_z(z)}{\partial z_i} \right| g_i(\eta, \xi)|z_i| \left[\rho_i(\eta, \xi) + (k_i - 1) \beta_i(\eta, \xi) \right]$$

$$\leq -cg_0 \delta_z \sqrt{V(z)}, \tag{86}$$

where $g_i(\cdot, \cdot)$ denotes the element on the ith row and ith column of $G(\cdot, \cdot)$ and g_0 is such that $G(\eta, \xi) \geq g_0 I_m > 0$, $(\eta, \xi) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$. Hence, (13) is satisfied with $\alpha(\|z\|) = \beta(\|z\|) = \|z\|^2|h_z(z)|^{-1}$ and (14) is satisfied with $k = cg_0 \delta_z$ and $\theta = \frac{1}{2}$. Since $e_\eta(t) \to 0$ and $e_\xi(t) \to 0$ as $t \to \infty$, and $V(\cdot)$ is a class $\mathcal{K}$ function, it follows from (84), (86), and Theorem 4.1 that there exists a compact set $\mathcal{M}_z \subseteq \hat{\mathcal{M}}_z \subset \mathcal{N}_z \subseteq \mathring{\mathcal{C}}_z$, such that $0 \in \mathcal{M}_z$ and if $z(t_0) \in \mathcal{M}_z$, then the solution $z(\cdot)$ of (82) is such that $z(t) \to 0$ as $t \to t_1$ uniformly in t_0, for some finite-time $t_1 > t_0$. Furthermore, proceeding as in the proofs of Theorems 4.1 and 3.1, one can prove that $z(t) \in \mathcal{M}_z \subset \mathcal{N}_z$, $t \geq t_0$. The result now follows recalling that the nonlinear dynamical system (16) is equivalent to (18) and (19) or, alternatively, (18) and (81), and if $(\eta(t), z(t)) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_z$, then $(\eta(t), \xi(t)) \in \mathring{\mathcal{C}}_\eta \times \mathring{\mathcal{C}}_\xi$ and hence, $x(t) \in \mathring{\mathcal{C}}$. ∎

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