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ABSTRACT

The predictability of the logistic map is investigated for the joint impact of initial-condition (IC) and model uncertainty (bias + random variability) as well as simulation variability. To this end, Monte Carlo simulations are carried out where IC bias is varied in a wide range of 10^{-15}–10^{-3}, and, similarly, model bias is introduced in comparable range. It is found that while the predictability limit of the logistic map can be continuously extended by reducing IC bias, the introduction of the model bias imposes an upper limit to the predictability limit beyond which further reductions in IC bias do not lead to an extension in the predictability limit, effectively restricting the feasible joint space spanned by the IC-model biases. It is further observed that imposing a lower limit to the allowed variability among ensemble solutions (so as to prevent the ensemble variability from collapse) results in a similar constraint in the joint IC-model-bias space; but this correspondence breaks down when the imposed variability limit is too high ($\gtrsim$0.7 for the logistic map). Finally, although increasing the IC random variability in an ensemble is found to consistently extend the allowed predictability limit of the logistic map, the same is not observed for model parameter random variability. In contrast, while low levels of model parameter variability have no impact on the allowed predictability limit, there appears to be a threshold at which an abrupt transition occurs toward a distinctly lower predictability limit.

When dynamical systems exhibit chaotic behavior, a key outcome is that their predictability becomes limited. One common way to measure the predictability limit is to initialize two simulations from slightly different states and observe whether and when the two solutions diverge beyond a predefined threshold. In this study, the logistic map is used to explore how various factors can impact the predictability limit of a simple chaotic system. Although under certain conditions the logistic map is chaotic and, thus, exhibits limited predictability, it is generally accepted that this predictability limit can be extended by simply reducing the initial difference (bias) between two simulations. However, it is shown here that when the model is slightly wrong (biased) in the way its parameter is prescribed, a radical change happens, and the predictability limit becomes constrained: The predictability limit can be extended only up to a certain limit, beyond which further decreases in initial bias lead to no additional reduction in the predictability limit. The consequences of this observation are investigated in various further numerical experiments.

I. INTRODUCTION

Predictability is one of the key features of chaotic systems: Even though they are deterministic, chaotic systems exhibit unstable dynamics that result in error growth over time, ultimately overwhelming the predictive capability of any simulation that attempts to reproduce their dynamic behavior. Traditionally, limits to predictability in chaotic systems are investigated for imperfections in both the initial conditions (ICs) of a system and the model that is used to predict it.[1]

Dependence on ICs in a perfect-model scenario can result in various types of predictability. Intrinsic predictability is defined as the asymptotic behavior valid for an infinitesimally small IC uncertainty and is typically measured by the Lyapunov exponents of a system.[2,3] However, in real-world applications of prediction, ICs can vary significantly both spatially and in time, resulting in varying magnitudes of practical predictability. In these situations, describing error growth by the leading Lyapunov exponents becomes unreliable, especially for the later stages of error growth.[4,5] It is this practical range of predictability that the present study is concerned about.

Meanwhile, the fact that the numerical models developed to represent real-world dynamical systems are not perfect imposes an entirely different class of limitations on predictability.[6] Even in the hypothetical scenario of perfect ICs, model limitations, such as incomplete dynamical representation, need for spatial, temporal, and/or spectral discretization, as well as imperfect knowledge of model parameters, are all known to result in chaotic departures from the true state.[7,8] However, the complexities of addressing a multitude of sources of model uncertainty even in low-dimensional nonlinear systems render it exceedingly prohibitive to systematically address their impact on predictability. The present study aims to address this complication for the combined impacts of IC and model uncertainty for the simple one-dimensional logistic map.

The logistic map[9] has been the subject of numerous studies focused on the chaotic properties of nonlinear systems. Through variations in a single parameter, it exhibits long-term behavior that ranges from stably convergent (single-valued and periodic) to chaotic that is easily visualized through bifurcation diagrams and attractor plots. In addition to its practical applications in various types of population growth studies, it is commonly used to explore the general properties of nonlinear chaos. For recent applications and comprehensive reviews, see Refs. 10–15.

In the present study, the predictability of the logistic map is investigated through an ensemble approach. Ensembles are Monte Carlo simulations, where representing the probabilistic nature of the underlying nonlinear system is otherwise computationally prohibitive, such as in complex numerical weather prediction (NWP) models.[16] Being approximations to the underlying true probability density function (PDF) of the model state, ensembles introduce a further degree of complexity to account for to fully investigate predictability in nonlinear chaotic systems. It is important to also note that the requirement to maintain sufficient ensemble variability becomes an additional important consideration. This is a common issue encountered in ensemble-based state-only and joint state-and-parameter estimation applications.[17–20]

This article presents a broad view of the combined impact of IC uncertainty, model uncertainty, and ensemble variability on the practical predictability of the one-dimensional logistic map. To appeal to a broad readership, a holistic approach is followed, and mathematical intricacies are avoided, with the goal and hope of inspiring further in-depth technical studies on the subject matter.

II. THE LOGISTIC MAP

The logistic map is one of the most frequently employed recurrence models to study the chaotic behavior of nonlinear systems. Its equation is given as follows:

$$x_{i+1} = rx_i(1 - x_i), \tag{1}$$

where the predicted variable $x_{\mathrm{i}+1}$ only depends on its value at the previous iteration (i) and the parameter r. The values of x are bounded within [0, 1] for the r values [0, 4], which also controls the behavior of the logistic map, as measured by the asymptotic values of x_n for a large n. For $r \in [0, 1]$, x converges to 0. For $r \in [1, 3]$, x converges to $(r - 1)/r$. Between $r \in [3, \sim 3.56995]$, a behavior typically referred to as period-doubling cascade[21] occurs, where a convergent x becomes periodic with the number of periods n doubling at a rate of

$$\delta = \lim_{n\to\infty} \frac{x_{n-1} - x_{n-2}}{x_n - x_{n-1}} \approx 4.6692. \tag{2}$$

Here, δ is known as the Feigenbaum constant[22] and x_n represents the value of (asymptotic) x, where period doubling for n occurs (i.e., where the number of periods becomes 2^n). Then finally, for most values within $r \in [\sim 3.56995, 4]$, the chaotic regime emerges, although there are still narrow windows of r where stability reappears. A well known example is the stable period-3 cycle[23] that begins at $r = 1 + \sqrt{8}$. The rich asymptotic behavior of the logistic map can be summarized by its bifurcation diagram as shown in Fig. 1.

III. ON THE RELATIONSHIP OF THE LOGISTIC MAP AND LINEAR RESPONSE THEORY

An important point needs to be made concerning the linear response theory (LRT) prior to the discussion of results. LRT is a commonly utilized mathematical tool in the analysis of a certain class of dynamical systems that, for small perturbations, allows them to be expressed as Taylor expansions for rigorous mathematical analysis. However, it is a well known fact that some simple dynamical systems such as the logistic map do not obey LRT,[24,25] and the discrepancy between this behavior and the seemingly successful application of LRT in higher-dimensional dynamical systems has been extensively studied (see, for example, the discussions in Refs. 26–28). Although this viewpoint is generally valid for most high-dimensional and/or stochastic dynamic systems, there still are real-world phenomena, especially at smaller atmospheric scales that involve a cloud-water phase transition and convective initiation that directly impact simulations of convective weather systems, where the validity of LRT is not well established. Therefore, in the present study, the lack of applicability of LRT to the logistic map is considered of secondary importance, especially considering the computational advantages that allow a large number of simulations to be performed spanning a wide range of parameter values considered.

IV. UNCERTAINTY, BIAS, AND RANDOM VARIABILITY

Before a detailed discussion of the study results, a clear distinction needs to be made between the terms uncertainty, bias, and

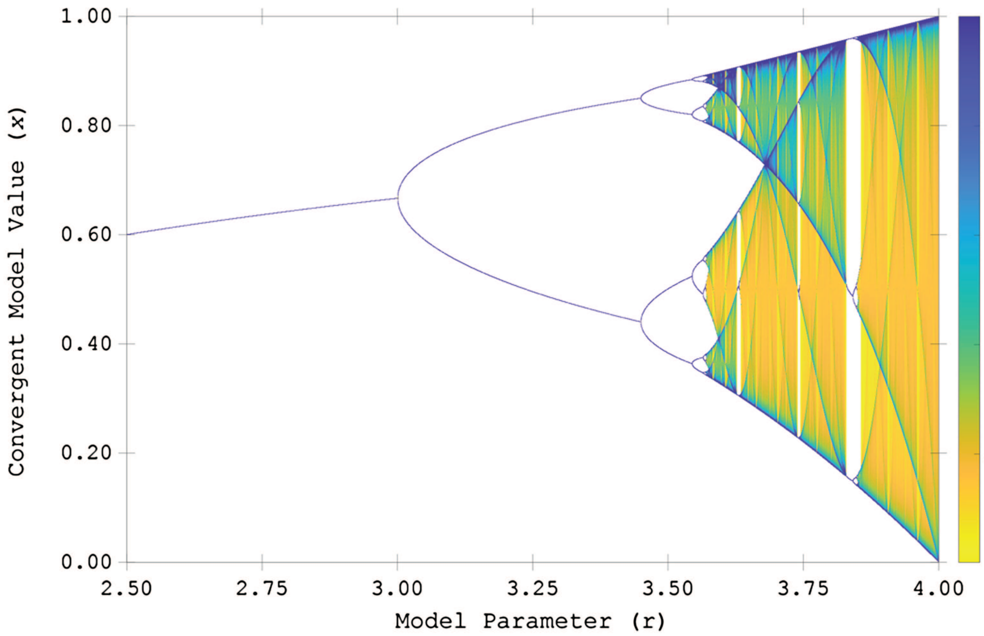

FIG. 1. The bifurcation diagram of the logistic map for $r \in [2.5, 4]$. Convergent model values for the given r are expressed as probability (%) to revisit a particular x range of width 0.01 and rescaled to improve discernibility.

variability that will be frequently used. Specifically, total uncertainty is defined as the sum of bias and random variability. This study utilizes a Monte Carlo approach,[29] where slightly different simulations are obtained by slightly varying the initial conditions and model properties. These slight variations, along with a predefined bias, represent the total uncertainty that is introduced to a particular simulation. For example, the total initial-condition uncertainty Δx_0 is represented as the sum of a prescribed bias $\overline{\Delta x_0}$ and randomly drawn deviations $\Delta x'$ according to a prescribed variability $\sigma_{\Delta x_0}$, i.e., $\Delta x_0 = \overline{\Delta x_0} + \Delta x'$ with $\Delta x' \sim N(0, \sigma_{\Delta x_0})$. Similarly, if model uncertainty is also present, then it is represented by the sum of a prescribed model parameter bias and variability as $\Delta r = \overline{\Delta r} + r'$ with $r' \sim N(0, \sigma_{\Delta r})$.

V. PREDICTABILITY

The goal of the present study is to investigate the practical predictability of the logistic map, given a small difference between two simulations. Since in the chaotic regime for r, the long-term solutions generally are not convergent, the definition of predictability here aims to capture a threshold at which such divergence exceeds a predefined ratio of the total signal. Also, because the logistic map solutions are bound within the range [0, 1], such a threshold can be specified as a ratio of this range. In the present study, this ratio is arbitrarily defined as 10%, corresponding to an absolute allowed difference between two solutions to be $\Delta_{\text{thresh}} = 0.1$. Beyond this threshold, it is deemed that predictability is lost.

A graphical demonstration of error growth is provided in Fig. 2, for which 50 Monte Carlo simulations are generated for $r = 3.7$, where total IC uncertainty Δx_0 is obtained as the sum of bias $\overline{\Delta x_0} \in \{10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}\}$ and random variability with $\sigma_{\Delta x_0} = 0.1 \times \overline{\Delta x_0}$. It can be observed that the rate of error growth is exponential for simulations initialized with a wide range of IC uncertainty but slows down as error saturation is approached. It should be noted here that much smaller IC uncertainty values than what is shown are not simulated to avoid issues with numerical precision impacting the analyses of error.[30,31]

For each level of IC bias, it is then possible to determine the corresponding predictability limit, as shown in Fig. 3, as the iteration number when the predictability threshold is first exceeded. Figure 3 (black lines) clearly demonstrates that the predictability limit of the logistic map can be steadily extended by reducing the magnitude of the IC bias, although at varying rates depending on the model parameter value itself (but saturating to climatic variability at high bias, not shown). It is also worth noting that the predictability limit depicted here is an average obtained from simulations initialized with various initial conditions within the range [0, 1], which in reality is, of course, also a function of the specific initial condition imposed.

VI. MODEL BIAS

In practical applications, the degree to which a particular model is correctly constructed is critically important to obtain accurate simulations. In the logistic map, the single parameter "r" controls the behavior of the model, which needs to be specified empirically in practical applications. For example, in population dynamics models, from where the use of the logistic equation originates, population

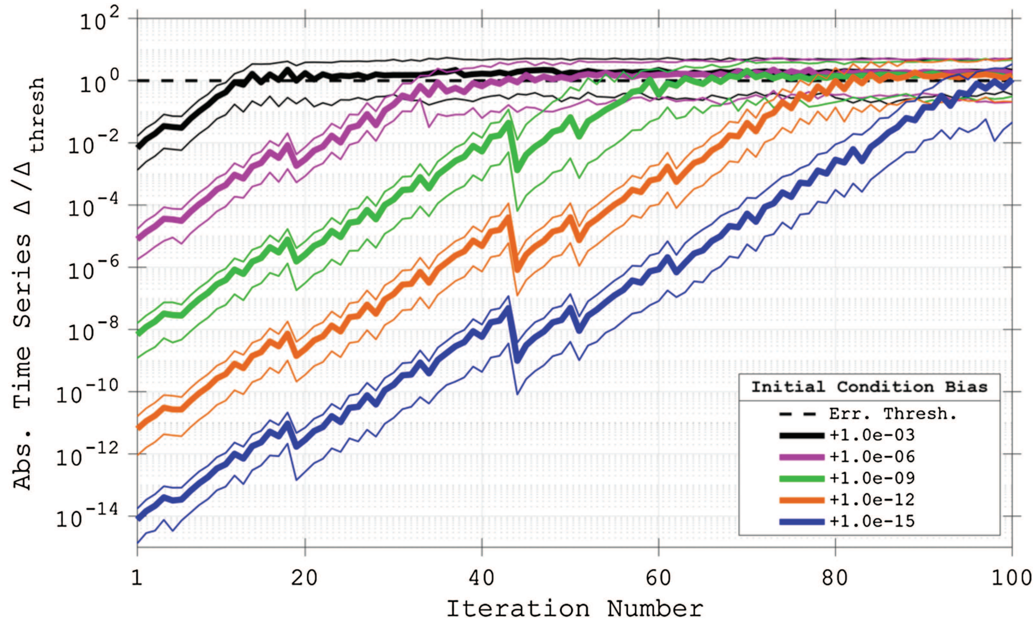

FIG. 2. Time evolution of a median absolute error (thick lines) for various IC bias values (colors), with $r = 3.7$. Error is relative to the assumed threshold of 0.1, which results in a nominal threshold (10^0, dashed black line). The 10th and 90th percentiles are shown as thin lines.

growth is typically described by the empirical parameters "relative growth rate coefficient" and "carrying capacity of the population," which both contribute to the parameter r,[32,33] and cannot be assumed to be known perfectly.

The more complex behavior of predictability in the presence of an additional model bias is illustrated in Fig. 3 (magenta and green lines), where two simulations now also differ as the model parameter bias $\overline{\Delta r}$ is varied. Several interesting facts are apparent in this

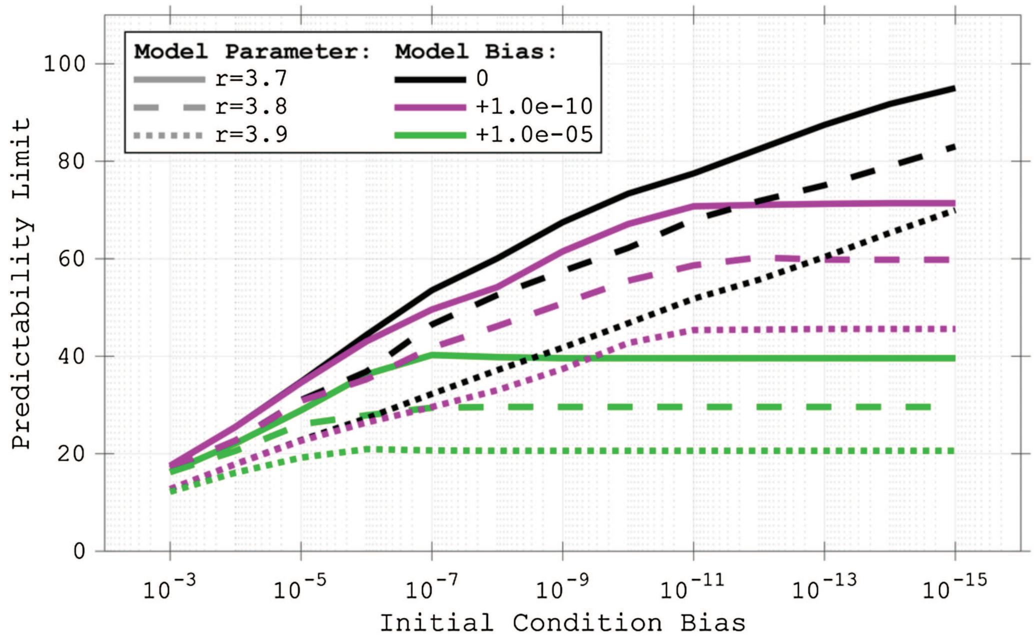

FIG. 3. Predictability limit of the logistic map as a function of IC bias, calculated for different model parameter values (line style) and model parameter bias values (line color).

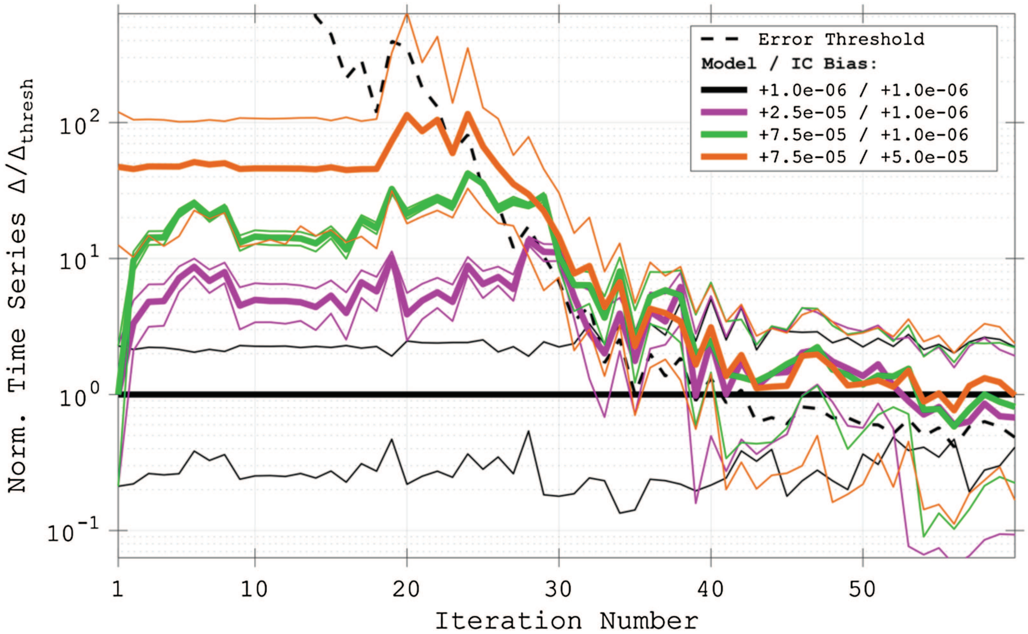

FIG. 4. Time evolution of the normalized median absolute error (thick lines) for various model and IC bias values. All error values are relative to the error for the combination $\overline{\Delta r} = 10^{-6}$, $\overline{\Delta x_0} = 10^{-6}$ (thick black line). The resulting normalized error threshold (0.1) is shown as the dashed black line. The 10th and 90th percentiles are shown as thin lines around the respective median errors.

scenario. First, model bias generally leads to lower predictability values at any given model parameter specification. This suggests a generally reduced "return on investment" to improve a model's predictability limit by the pure effort to lower the IC uncertainty. In other words, one gains a smaller increment in the predictability limit by the same amount of incremental reduction in IC uncertainty.

But even more importantly, in the presence of a model bias, the predictability limit also becomes abruptly flat when certain IC uncertainty levels are reached. Thus, when a model bias is present, one has only a limited window of opportunity to improve the predictability limit by lowering IC uncertainty. At that limit, effort needs to be shifted to improving one's model itself before any further gains in predictability can be achieved.

VII. MODEL SOLUTION VARIABILITY

In Fig. 2, it is already demonstrated that in the absence of a model bias, the emerging model solution variability, as measured by the difference between the respective 10th and 90th percentiles, remains stable until error saturation occurs. However, the picture changes remarkably when a model bias is introduced. The combined impact of model and IC biases on the resulting solutions is demonstrated in Fig. 4. In the base simulation (thick black line), both the IC and the model biases are set to 10^{-6}. Two more simulations are then carried out at the same IC bias but using a progressively higher model bias. The final simulation then increases the IC bias to 5×10^{-5}.

The impact of the higher model bias on the variability among the simulations is striking. It can be observed that at a constant IC bias, an increased model bias results in lower simulation variability. This suggests that as model bias increases, it becomes the dominant factor in restricting the variability among perturbed simulations. For IC bias to contribute again to maintaining simulation variability, it needs to be also further increased (i.e., the difference between green and orange lines). The result is that model and IC biases cannot be independently controlled if maintaining sufficient model solution variability is a requirement.

One good example for the need to maintain variability can be given from some NWP applications. NWP is an initial/boundary value problem where in any given cycle, the first step, known as data assimilation, comprises obtaining optimal ICs using the most recent observations to maximize the quality of the subsequent forecast. The sensitive nature of weather forecasts on initial conditions was demonstrated as early as in Lorenz's own contributions to chaos theory.[34]

Typically, atmospheric data assimilation combines information from two independent sources: The model state from a previous forecast cycle and the observed state from the atmospheric measurements not yet accounted for in the present forecast cycle. This necessitates the knowledge of uncertainty in both sources.[35] A common approach to estimating model uncertainty is the use of Monte Carlo forecasts, also known as ensembles.[36] However, maintaining sufficient variability in such ensembles is critical, especially when the prediction system is continuously cycled for long periods of time and ensembles tend to converge toward a particular region of the PDF. This leads to the systematic application of more weight to the model state, thereby disregarding the information content from new observations. This issue is known as filter divergence in NWP

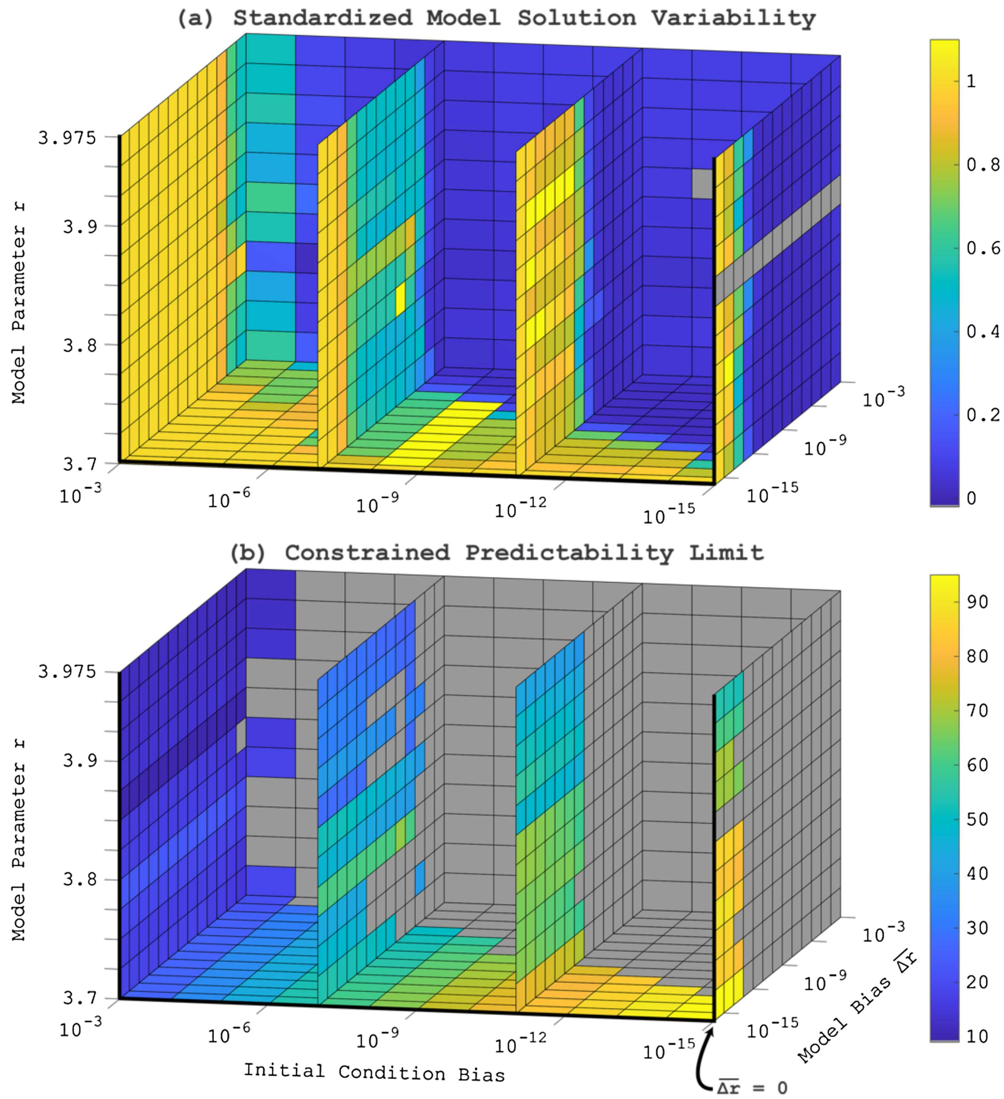

FIG. 5. (a) Standardized model solution variability and (b) predictability limit as constrained where standardized variability is greater than 0.50, as a function of IC and model biases as well as reference model parameter value. The gray areas in panel (b) indicate where the predictability limit is not allowed for the given standardized variability lower limit.

applications,[37–40] and to avoid it, special care is taken to maintain a sufficiently high ensemble variability.

VIII. IMPACT OF MODEL SOLUTION VARIABILITY ON PREDICTABILITY LIMIT

As seen in the examples shown in Figs. 2 and 4, the parameter space spanned by IC and model uncertainties can lead to a wide range of model solutions with variations in both the median and the variability of the ensemble solutions. A two-step standardization approach is adopted here to bring these solutions to a common reference frame that accounts for these variations. First, a nonparametric form of normalization is applied as follows:

$$\mathrm{V}_i = \frac{P^i_{90} - P^i_{10}}{P^i_{50}}. \tag{3}$$

Here, P^i_N represents the Nth percentile of the distribution valid at the ith iteration, and the difference between the 90th and the 10th percentiles is normalized by the median (50th percentile) of the corresponding distribution. Normalized variability V_i are further averaged over corresponding error growth periods to filter out temporal variability.

Despite normalization, variability V tends to increase more than three-fold with increasing IC bias (not shown). Therefore, an

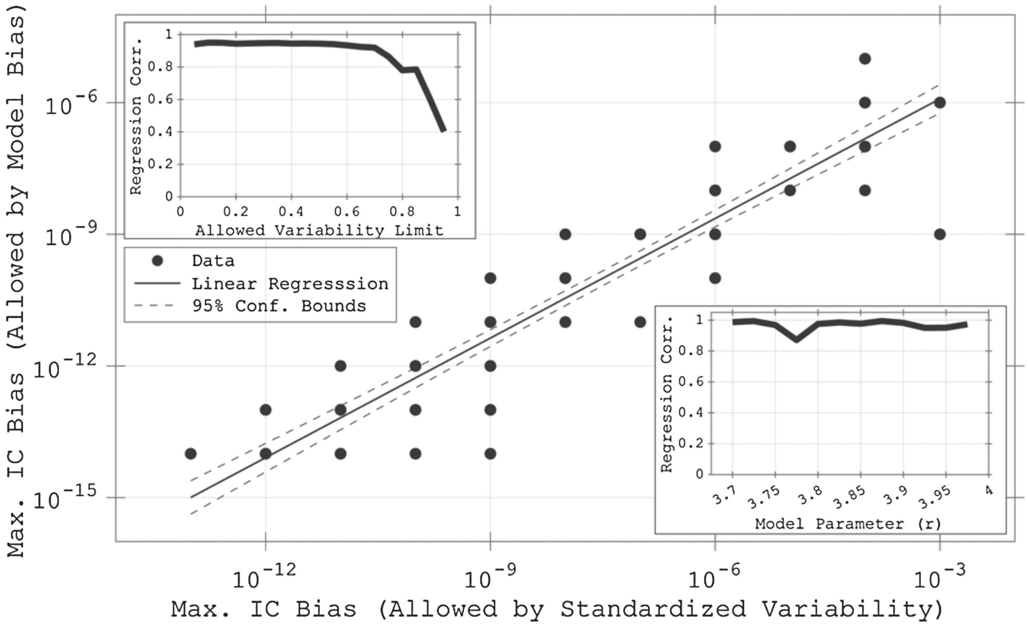

FIG. 6. Maximum allowed IC bias, as estimated from model bias vs from minimum allowed standardized variability (>0.50), using estimates from all model parameter (r) values sampled. Upper-left inset: Correlation coefficient R as a function of the maximum allowed standardized variability. Lower-right inset: Correlation coefficient R as a function of the model parameter.

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additional step of standardization is introduced by dividing normalized variability by the corresponding normalized variability at zero model bias, as $\hat{V} = V/V_0$, where V is normalized variability, V_0 is the corresponding normalized variability at zero model bias, and $\hat{V}$ is the resulting standardized variability. This is shown in detail in Fig. 5(a), where it can be observed that, while there is a tendency for the variability to increase with increasing IC bias, a sufficiently large model bias leads to the collapse of variability that occurs along a diagonal line in the IC-model-bias space. To prevent such collapse, Monte Carlo simulations can be restricted to a minimum level of standardized variability that would then lead to the constrained predictability limit shown in Fig. 5(b), for the example when standardized variability is greater than 0.50, i.e., when standardized variability is allowed to be reduced at most to 50% of its original value. Under this restriction, the theoretical maximum predictability limit is achieved only at combinations of very small IC and model bias values. A wider range of model bias values is allowed only if IC bias is also increased accordingly but at the cost of reduced predictability.

IX. PREDICTABILITY LIMIT UPPER BOUND: EQUIVALENCE BETWEEN MODEL BIAS AND STANDARDIZED VARIABILITY

It is established so far that the predictability limit of the logistic map can be bounded both by the existence of model bias (Fig. 3) and limitations imposed by a minimum standard variability among Monte Carlo simulations (Fig. 5). Specifically in Fig. 3, for a given level of model bias, the predictability limit can be increased only up to a certain value of IC bias, at which point growth in the predictability limit ceases. Similarly in Fig. 5, at an assumed level of model bias, the predictability limit can be increased with a decreased IC bias but only up to a certain limit that is determined by the allowed minimum standardized variability. Furthermore, it was also shown in Fig. 4 that increasing model bias directly results in reduced solution variability as was indicated by the decrease in the difference between the 10th and the 90th percentiles. In Fig. 6, this anecdotal relationship is generalized for a wide range of parameter values.

The main scatterplot in Fig. 6 indeed confirms that there is a clear linear relationship ($R^2 = 0.891$, $F(104) = 836$, $p = 5.94 \times 10^{-51}$) between the restricting IC bias that constrains predictability limit empirically (through minimum allowed standardized variability, x axis) and theoretically (through model bias, y axis). While this regression relationship is obtained over all sampled model parameter values, the lower-right panel also confirms that it remains strong when tested separately against individual model parameter values. Furthermore, the upper-left panel shows the regression coefficient when the predictability limit is constrained by a range of minimum allowed standardized variability values (0.05–0.95). Indeed, the strength of the linear relationship between empirical and theoretical predictability limits remains high for a wide range of constraints but breaks down at ∼0.7 (70% of allowed variability), suggesting that there is a trade-off between how strict one can be to maintain ensemble variability and how representative the resulting attainable range of predictability limit values are.

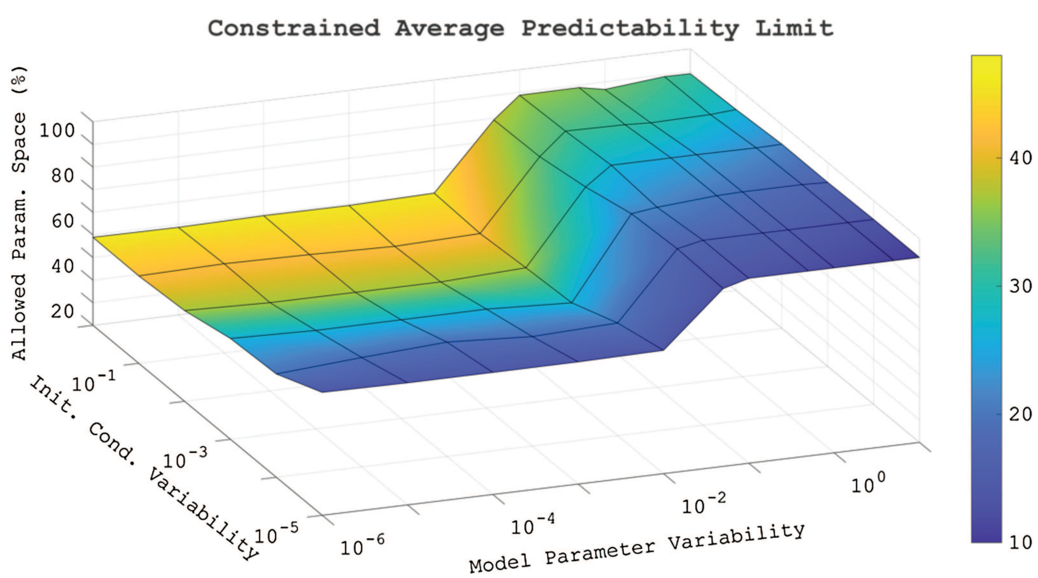

FIG. 7. Constrained average predictability limit of the logistic map (color shading) as a function of IC and model random variability (horizontal axes) and allowed parameter space (vertical axis), where standardized variability is greater than 0.5. Predictability limit is averaged over all IC bias, model bias, and model parameter values sampled.

X. IMPACT OF MODEL SIMULATION VARIABILITY ON PREDICTABILITY LIMIT

So far, the focus has been on how the predictability limit is impacted by the restrictions imposed by model solution variability. However, obviously, the variability in these Monte Carlo solutions is possible only because of the inherent variability in how the simulations are generated. Recall that simulations are so far randomized only in how a specific IC value x_0 is drawn from the assumed Gaussian distribution in IC variability with $\sigma_{\Delta x_0} = 0.1 \times \overline{\Delta x_0}$, as explained in Sec. III. Here, a more rigorous analysis is presented for the impact of generalized variability in both the ICs and the model parameter.

In Fig. 7, the dual impact of IC and model parameter random variability on allowed parameter space and average predictability limit is shown. Allowed parameter space is a measure of how model solution variability restricts the predictability limit to certain combinations of IC and model biases [i.e., Fig. 5(b)]. It is defined as the ratio of the number of parameter combinations allowed to the total number of parameter combinations. The average predictability limit is obtained likewise. This is carried out for all combinations of IC and model parameter random variability (i.e., $\sigma_{\Delta x_0}$ and $\sigma_{\Delta r}$, respectively) along the horizontal axes.

From Fig. 7, one can observe that the increase in IC variability consistently leads to improved average predictability regardless of model parameter variability. This means that an ensemble needs to be constructed with sufficient (random) variability in ICs to maximize the allowed predictability limit of the logistic map simulations. Meanwhile, introducing random variability to the simulations' model parameter value does not appear to result in any improvements in the allowed predictability limit. Highest values of predictability limit occur below 10^{-2}, at which point a quick transition occurs from a lower to a higher allowed percentage of parameter space. However, this is also accompanied by generally smaller values of the average predictability limit, indicating a trade-off between how much of the parameter space one would like to have at their disposal vs the sacrifice in the average predictability limit.

XI. SUMMARY

The present study revisits the chaotic regime of the logistic map with an emphasis on how several factors impact its predictability. The logistic map has been extensively studied for its long-term behavior, both for its critical dependence on the initial-condition (IC) and for model uncertainty. To generalize this existing body of work, the analysis of the impact of the combined Monte Carlo-based uncertainties in ICs and model specification is expanded by imposing limits on how much long-term model solutions are allowed to vary. The study is then generalized by accounting for how the Montel Carlo simulations are generated where both ICs and the model parameter are varied jointly. The findings can be summarized as follows:

- In the absence of model uncertainty, the predictability limit of the logistic map can be extended continuously by reducing IC bias. In this scenario, Monte Carlo simulations also exhibit stable variability among long-term solutions.
- When a model bias is introduced, the predictability limit becomes progressively lower. Furthermore, a fundamental change occurs where the predictability limit becomes bounded and cannot be extended beyond a specific level of IC bias. To improve predictability beyond this upper bound also requires lowering model bias.

- A standardization approach is introduced so that the variability among Monte Carlo simulations becomes comparable among experiments initialized with different IC and model biases. Using standardized variability as a minimum constraint results in significant restrictions on how much of the IC-model-bias space is attainable.
- A statistical equivalence is demonstrated for how one can arrive at an estimate of the maximum predictability limit through model bias and simulation variability limitations, respectively. However, this relationship holds only when the limitation on minimum variability is not too strict. Otherwise, the allowed combinations of IC and model biases become too restrictive compared to those implied by the theoretical predictability limit.
- Finally, in terms of random variability, it is shown that specifying sufficient variability in the ICs is critical to extend the predictability limit of the resulting simulations. Meanwhile, accounting for the ensemble variability of the model parameter results in no discernible impact. In contrast, beyond a threshold, a quick transition occurs toward a larger percentage of the allowed space of IC and model uncertainties, but at the cost of a reduced average predictability limit.

XII. DISCUSSION

It is well known that in some dynamical systems, predictability is constrained and cannot be extended indefinitely by reducing the uncertainty of ICs. The best example is that of the atmosphere, where Lorenz's seminal work and others that followed demonstrated that scale interactions and error energy cascade from turbulent to synoptic and larger scales result in strict limitations to the predictability of the atmosphere.[41–45] There are, of course, additional sources of error that limit predictability, such as those that arise from insufficient knowledge of the dynamical system, simplifications that are necessary to simulate them numerically, and existence of scales that cannot be resolved explicitly, given computational limitations.[46] The impact of such imperfections on predictability has generally been addressed in the linear approximation limit and the resulting Lyapunov exponents that are valid only for infinitesimally small perturbations.[47]

It is shown here that when IC and model uncertainties coexist, there can be profound impacts on the predictability of a nonlinear dynamical system. Even in the one-dimensional logistic map whose predictability limit can be extended with a reduced IC bias, model bias results in an upper bound of predictability, beyond which a further decrease in IC bias leads to no further improvement. This is significant because a restricted predictability limit is typically associated with scale interactions and upscale growth of error in a dynamical system. It is shown here that such behavior is not exclusive to models with interacting scales.

Another noteworthy outcome is regarding the high correlation found between the theoretical and the empirical predictability limit upper bounds obtained through model bias and simulation variability considerations, respectively. While this is expected because model bias leads to a smaller variance among Monte Carlo solutions, it is noteworthy that this relationship is valid only up to a certain degree of restriction in simulation variability. There is, thus, an apparent trade-off between the need to maintain variability among simulations and to maximize the allowed predictability limit of the underlying model, especially when the restriction on ensemble variability is high. This is a very important additional constraint to account for in ensemble design.

Finally, this study reveals that introducing model parameter variability in an ensemble yields no measurable impact on the predictability limit of the logistic map, whereas IC variability clearly results in extended predictability. In fact, it is shown that extensive parameter variability can lower the average predictability that can be attained. While it is plausible that sensitivity to a single parameter that controls a model's behavior may be too high to counteract the potential benefits of accounting for its variability in an ensemble, that such sensitivity does not exist in IC variability is somewhat surprising and worthy of future investigation. Nevertheless, because the logistic map is a chaotic system that does not obey the linear response theory (LRT), the extension of the validity of findings here to higher-order dynamical systems that obey the LRT is an open question and will be addressed in future studies.

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AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts of interest to disclose.

Author Contributions

The author has contributed entirely to all phases of this study, including conceptualization, model development and simulations, statistical analyses, graphical visualizations, and writing.

Altug Aksoy: Conceptualization (equal); Data curation (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Project administration (equal); Resources (equal); Software (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the author upon reasonable request.

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