Formulas for Parcel Velocity and Vorticity in a Rotating Cartesian Coordinate System

ROBERT DAVIES-JONES

NOAA/National Severe Storms Laboratory, Norman, Oklahoma

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ABSTRACT

Formulas in an Eulerian framework are presented for the absolute velocity and vorticity of individual parcels in inviscid isentropic flow. The analysis is performed in a rectangular Cartesian rotating coordinate system. The dependent variables are the Lagrangian coordinates, initial velocities, cumulative temperature, entropy, and a potential. The formulas are obtained in two different ways. The first method is based on finding a matrix integrating factor for the Euler equations of motion and a propagator for the vector vorticity equation. The second method is a variational one. Hamilton's principle of least action is used to minimize the fluid's absolute kinetic energy minus its internal energy and potential energy subject to the Lin constraints and constraints of mass and entropy conservation. In the first method, the friction and diabatic heating terms in the governing equations are carried along in integrands so that the generalized formulas lead to Eckart's circulation theorem. Using them to derive other circulation theorems, the helicity-conservation theorem, and Cauchy's formula for the barotropic vorticity checks the formulas further.

The formulas are suitable for generating diagnostic fields of barotropic and baroclinic vorticity in models if some simple auxiliary equations are added to the model and integrated stably forward in time alongside the model equations.

1. Introduction

Understanding of atmospheric vortices, such as tornadoes [see reviews by Davies-Jones et al. (2001) and Davies-Jones (2015)], lee vortices (Smolarkiewicz and Rotunno 1989; Davies-Jones 2000), and larger-scale cyclones (Lackmann 2011, 101–102) often involves determining the mechanisms by which air parcels obtain large vorticities. One approach to investigating tornadogenesis is to use a ''bare-bones computer model'' that forms a tornado (Davies-Jones 2008). The results are easy to interpret, but a loss of realism naturally comes with the simplifying assumptions. More realistic three-dimensional models of supercell storms produce tornadoes in favorable environments, but the origins of these simulated vortices are difficult to decipher, because these models are complex and use Eulerian coordinates, whereas the laws governing vorticity are Lagrangian in nature (Salmon 1988, p. 226). The modeler usually resorts to computing how circulation evolves around a material circuit drawn around the near-ground vorticity maximum and traced back to an arbitrary ''initial time'' using computed backward parcel trajectories (Rotunno and Klemp 1985; Davies-Jones and Brooks 1993; Adlerman et al. 1999; Markowski et al. 2012). This method allows the analyst to determine the barotropic (i.e., the initial) circulation and the change in circulation around the circuit, which is the baroclinic circulation plus circulation generated by frictional torque (when significant). Computational restraints generally limit the analysis to only one circuit. A diagnostic method using millions of forward trajectories has been developed recently (Dahl et al. 2014), but the resulting evaluations of barotropic and nonbarotropic vorticity (the latter computed as the residual vorticity) are along individual trajectories, which are spaced irregularly in Eulerian coordinates. The barotropic and baroclinic vorticity cannot be presented easily as fields by this method.

This paper develops formulas for the velocity and vorticity of an individual parcel in a general flow with friction and diabatic heating that can be used to generate the baroclinic- and barotropic vorticity fields if auxiliary

Corresponding author address: Dr. Robert Davies-Jones, Doggetts Farm, New Street, Stradbroke, Eye, Suffolk IP21 5JG, United Kingdom.
E-mail: bobdj1066@yahoo.com

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equations are added to the model and integrated forward in time alongside the model equations. The formulas are relatively simple if the flow is isentropic and inviscid. We use a rectangular Cartesian coordinate system that is rotating with Earth rather than the inertial systems used in previous work and assume that all fields are continuous and differentiable. The Lagrangian coordinates and initial velocities appear in the formulas as dependent variables. We find a matrix integrating factor for integration of the Euler equations of motion and a propagator for integration of the vector vorticity equation. We then show how the formulas can be obtained variationally from Hamilton's principle of least action. We check the formulas by showing that the barotropic part of the vorticity is equivalent to Cauchy's formula (Dutton 1976, p. 385) and by using them to derive Ertel's potential vorticity theorem, the helicity-conservation theorem, and the circulation theorems.

2. Previous work

The quest for these formulas starts with the fact that a velocity field (or any other vector field) $\mathbf{u}$ and its curl ζ can be represented locally by

$$\mathbf{u} = -\nabla\phi + \psi\nabla\chi \quad \text{and} \tag{1}$$

$$\boldsymbol{\zeta} \equiv \nabla\times\mathbf{u} = \nabla\psi\times\nabla\chi, \tag{2}$$

(Truesdell 1954) where ϕ, ψ, and χ are called Clebsch potentials (Lamb 1932; Serrin 1959) or Monge potentials (Truesdell 1954; Aris 1962). These solutions predict that flows in which all the vortex tubes are closed have no helicity $\mathcal{H}$ (Bretherton 1970; Salmon 1988, p. 241), because

$$\mathcal{H} \equiv \iiint_{\Xi} \mathbf{u}\cdot\zeta\, d\Xi = -\iiint_{\Xi} \nabla\phi\cdot\zeta\, d\Xi = -\iiint_{\Xi} \nabla\cdot(\phi\zeta)\, d\Xi = -\oiint_{\Sigma} \phi\zeta\cdot\mathbf{n}\, d\Sigma = 0. \tag{3}$$

(Here, Ξ is the fluid volume, Σ is its bounding surface, and $\mathbf{n}$ is the outward unit normal to the surface.) Thus, they fail to represent all flows. For instance, they exclude the Beltrami flows in closed domains (e.g., Davies-Jones 2008). Consequently, the representation (1) is generally local rather than global. We can make the formulas global by adding more terms such that

$$\mathbf{u} = -\nabla\phi + \sum_{i=1}^{N}\psi_i\nabla\chi_i \quad \text{and} \tag{4}$$

$$\nabla\times\mathbf{u} = \sum_{i=1}^{N}\nabla\psi_i\times\nabla\chi_i. \tag{5}$$

According to Salmon (1988, 236–237), two is the minimum value of N for representing either a general homentropic (uniform entropy) or a general isentropic flow.

For further elaboration, we need the following definitions. The Eulerian coordinates are (x, y, z, t), where t is the current time and $\mathbf{x} \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector in terms of eastward, northward, and upward unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. The corresponding Lagrangian coordinates are (X, Y, Z, τ), where τ is the symbol for time in the Lagrangian system, and $\mathbf{X} \equiv X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ is the initial position vector of a parcel at the initial time τ_0. The coordinates are measured from the origin of a tangent plane, which is rotating with the earth. The trajectory of a parcel is denoted by

$$\tilde{\mathbf{x}}(\tilde{\tau}) \quad \text{for} \quad \tau_0 \le \tilde{\tau} \le \tau \quad \text{where} \quad \tilde{\mathbf{x}}(\tau_0) = \mathbf{X}, \quad \tilde{\mathbf{x}}(\tau) = \mathbf{x}, \tag{6}$$

where the tilde denotes quantities at time $\tilde{\tau}$. The current and initial velocities of a parcel are $\mathbf{u} \equiv (u, v, w)$ and $\mathbf{U} \equiv (U, V, W)$, respectively. The gradient operator in Lagrangian coordinates is

$$\hat{\nabla} \equiv \mathbf{i}\frac{\partial}{\partial X} + \mathbf{j}\frac{\partial}{\partial Y} + \mathbf{k}\frac{\partial}{\partial Z}, \tag{7}$$

and it is

$$\tilde{\nabla} \equiv \mathbf{i}\frac{\partial}{\partial \tilde{x}} + \mathbf{j}\frac{\partial}{\partial \tilde{y}} + \mathbf{k}\frac{\partial}{\partial \tilde{z}} \tag{8}$$

at an intermediate time $\tilde{\tau}$ in $\tilde{\mathbf{x}}$ space. The material derivative in the rotating frame is D/Dt in the Eulerian framework and $\partial/\partial\tau$ in the Lagrangian one. Thus, $\partial\mathbf{x}/\partial\tau = \mathbf{u} = u\nabla x + v\nabla y + w\nabla z$, $\partial\mathbf{X}/\partial\tau = \partial\mathbf{U}/\partial\tau = 0$, and $\mathbf{U} = U\hat{\nabla}X + V\hat{\nabla}Y + W\hat{\nabla}Z$. The potential energy of a parcel is gz, where g is the gravitational acceleration. The thermodynamic properties of a parcel are its entropy S; its specific volume α; its density $\rho = 1/\alpha$; its pressure p; its enthalpy c_pT, where T is its temperature and c_p is the specific heat at constant pressure; and its internal energy $E \equiv c_vT$, where c_v is the specific heat at constant volume (Salmon 1988, p. 47). By the ideal gas law, $p\alpha = RT$, where $R = c_p - c_v$ is the gas constant for air. The internal energy is related to α and S by

$$\ln E(\alpha, S) = S/c_v - (R/c_v)\ln\alpha + \text{constant}. \tag{9}$$

Consequently,

$$\left(\frac{\partial E}{\partial S}\right)_\alpha = T, \quad \left(\frac{\partial E}{\partial \alpha}\right)_S = -p, \quad -\alpha\left(\frac{\partial E}{\partial \alpha}\right)_S + E = c_pT. \tag{10}$$

Ertel's potential vorticity is $\alpha\boldsymbol{\omega}\cdot\nabla S$. It is conserved following a parcel in inviscid isentropic flow.

In inertial reference frames, Serrin (1959), Dutton (1976), Mobbs (1981), Epifanio and Durran (2002), and Davies-Jones (2000, 2006, hereafter DJ06) found integrals of the vector vorticity equation for inviscid isentropic flow and decomposed the vorticity $\boldsymbol{\omega}$ into a barotropic part $\boldsymbol{\omega}_{\mathrm{BT}}$ and a baroclinic part $\boldsymbol{\omega}_{\mathrm{BC}}$. Serrin generalized Weber's transformed equations of motion to isentropic flow by adding an additional term. Transformation back to Eulerian coordinates results in the $N = 4$ formula for parcel velocity:

$$\mathbf{u} = -\nabla\Psi + U\nabla X + V\nabla Y + W\nabla Z + \Lambda\nabla S \qquad (11)$$

(Mobbs 1981), where Λ is the parcel's cumulative temperature or the integral over time of its temperature T from the initial time τ_0 to the current time τ, Ψ is defined by $D\Psi/Dt \equiv c_pT + gz - \mathbf{u}\cdot\mathbf{u}/2$, $\Psi = 0$ initially. By definition, $\Lambda = 0$ initially, and $D\Lambda/Dt = T$. The corresponding vorticity formula is

$$\nabla\times\mathbf{u} = \nabla U\times\nabla X + \nabla V\times\nabla Y + \nabla W\times\nabla Z + \nabla\Lambda\times\nabla S, \qquad (12)$$

where the last term is the baroclinic vorticity, and the remaining terms on the right are the barotropic vorticity. Alternatively, the barotropic vorticity is expressed by Cauchy's formula, which dates back to 1815 (Dutton 1976, p. 385). In general, there is a frictional part to vorticity. For nonisentropic flows with friction, Epifanio and Durran (2002) and DJ06 derived more complicated formulas involving propagators, also known as state-transition matrices.

Using variational analysis in the Eulerian framework with constraints of entropy and mass conservation but without the Lin constraints, Sasaki (2014) obtained formulas of the types of equations (1) and (2) for inviscid isentropic flow: namely,

$$\mathbf{u} = -\nabla\phi - S\nabla\Lambda, \quad \text{and} \qquad (13)$$

$$\boldsymbol{\omega} = \nabla\Lambda\times\nabla S \qquad (14)$$

where ϕ and Λ (α and β in Sasaki's notation) are Lagrange multipliers for mass and entropy conservation, and S is entropy. The rotational velocity is not unique because it contains an arbitrary irrotational part that can be exchanged with the potential term. For example, an equally valid decomposition of the wind is

$$\mathbf{u} = -\nabla(\phi + \Lambda S) + \Lambda\nabla S. \qquad (15)$$

Since $N = 1$, these formulas must apply only to a subset of solutions of the perfect-fluid equations (Salmon 1988, p. 234). The vorticity in (14) is, in fact, the baroclinic vorticity in isentropic flow $\boldsymbol{\omega}_{\mathrm{BC}}$ (Dutton 1976). The formulas exclude homentropic flows with barotropic (i.e., initial) vorticity, all flows with nonzero potential vorticity, and some flows with nonzero helicity (see above). The barotropic vorticity is missing because it depends only on the initial and current states of the flow, which is when the variations vanish, not on the flow configurations at intermediate times, which is when variations are allowed.

The Lin constraints are

$$\frac{D\mathbf{a}}{Dt} = \mathbf{0}, \qquad (16)$$

where $\mathbf{a} \equiv (a, b, c)$ is a unique parcel label or identifier. Justification for the Lin constraints is provided in section 7. Classically, the initial position vector (X, Y, Z) is used to identify each parcel, but other conserved quantities can serve as labels provided that the labeling is unique. Salmon (1988, 1998, 327–329) included the Lin constraints, with (X, Y, S) as the labels and obtained the following compact formulas with $N = 2$:

$$\mathbf{u} = -\nabla\phi + A\nabla X + \Lambda\nabla S, \quad \text{and} \qquad (17)$$

$$\boldsymbol{\omega} = \nabla A\times\nabla X + \nabla\Lambda\times\nabla S. \qquad (18)$$

Even though the variations are still assumed to vanish at the beginning and end times, the initial velocity field enters the analysis as the Lagrange multipliers of the Lin constraints (see section 7). Thus, (18) allows barotropic vorticity and nonzero potential vorticity. However, the physical significance of the apparently nonmeteorological Lagrange multiplier A associated with the X label constraint is unclear, and there are generally insufficient degrees of freedom to enter a parcel's 3D initial position when $N = 2$. Consequently, the formula does not seem useful for following the evolution of a parcel's vorticity.

Dellar (2011) used variational analysis in the Hamiltonian framework to derive equations of motion for generalized β planes that incorporate the vertical and horizontal components of the rotation vector and their changes with latitude. Since they are derived from Hamilton's principle, the equations conserve energy, angular momentum, and potential vorticity.

3. The velocity due to the earth's rotation in a tangent plane

To find a formula for parcel velocity in a nontraditional f plane (i.e., a tangent plane in solid-body rotation with both a horizontal and vertical component) or a nontraditional β plane, we first need a formula for the planetary velocity (the velocity relative to the fixed stars of a point in the atmosphere that rotates with the earth). The planetary velocity is the vector

$$\mathbf{u}_E = \boldsymbol{\Omega} \times \mathbf{r}, \tag{19}$$

where $\boldsymbol{\Omega} \equiv \Omega(\cos\phi\mathbf{j} + \sin\phi\mathbf{k})$ is the earth's angular velocity (with magnitude Ω), $\mathbf{r} \equiv \mathbf{x} + a\mathbf{k}$ is the position vector from the earth's center, $\mathbf{x} \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector from the tangent plane's origin (at $\mathbf{r} = a\mathbf{k}$), ϕ is latitude, and a is the earth's radius. Note that $\mathbf{u}_E$ is a vector potential for $2\boldsymbol{\Omega}$ because $\nabla \times \mathbf{u}_E = 2\boldsymbol{\Omega}$. The derivative of (19) following a parcel is

$$D\mathbf{u}_E/Dt = \Omega \times \mathbf{u}, \tag{20}$$

where D/Dt is the material derivative in the rotating frame of the earth. (Note that $D\mathbf{u}_E/Dt \neq 0$, despite $\partial\mathbf{u}_E/\partial t = 0$.) Thus, the material derivative of $\mathbf{u}_E$ supplies one-half of the Coriolis acceleration in the derivation of the equations of motion.

On a nontraditional f plane, the earth's angular velocity is the constant value $\boldsymbol{\Omega}(0, \cos\phi_0, \sin\phi_0)$, where ϕ_0 is the latitude at the origin. In terms of the Coriolis parameters $f_0 \equiv 2\Omega\sin\phi_0$ and $\kappa_0 \equiv 2\Omega\cos\phi_0$, the planetary velocity on the f plane is

$$\mathbf{u}_f = (\Omega R_0 + \kappa_0 z/2 - f_0 y/2)\mathbf{i} + (f_0 x/2)\mathbf{j} - (\kappa_0 x/2)\mathbf{k}, \tag{21}$$

where $R_0 = a\cos\phi_0$ is the distance from the earth's axis to the origin. This velocity field is one of solid-body rotation with planetary vorticity $(0, \kappa_0, f_0)$. It satisfies (19) and (20).

In Grimshaw's (1975) nontraditional β-plane approximation, the horizontal component κ of the Coriolis parameter is constant ($=\kappa_0$), and the vertical component f varies with latitude according to $f = f_0 + \beta y$, where $\beta \equiv \kappa_0/a$. In Cartesian geometry, $\mathbf{u}_E$ must now have deformation and thus cannot be a field of solid-body rotation. Consequently, it cannot be obtained from (19) or as a solution of (20). It can only be determined up to the additive gradient of an indeterminate potential $\beta\chi(x, y)$ because a β plane is unphysical. For the analysis in section 6, it is convenient to define

$$\begin{aligned}\mathbf{u}_\beta &\equiv \mathbf{u}_f + \beta[-(y^2/4)\mathbf{i} + (xy/2)\mathbf{j}]\\ &= (\Omega R_0 + \kappa_0 z/2 - \mathcal{F}/2)\mathbf{i} + (fx/2)\mathbf{j} - (\kappa_0 x/2)\mathbf{k}\end{aligned} \tag{22}$$

where $F \equiv f_0 y + \beta y^2/2$. The planetary velocity is $\mathbf{u}_P = \mathbf{u}_\beta + \beta\nabla\chi(x, y)$. This choice of planetary velocity is permissible because it reduces to $\mathbf{u}_f$ in the case $\beta = 0$ and yields the correct planetary vorticity $2\boldsymbol{\Omega} = (0, \kappa_0, f)$. Dellar (2011) defined a vector potential for $2\boldsymbol{\Omega}$ [his (B4)] that, for Grimshaw's β plane, is compatible with $\mathbf{u}_P$.

The absolute velocity of a parcel is

$$\mathbf{u}_a \equiv \mathbf{u} + \mathbf{u}_\beta + \beta\nabla\chi(x, y), \tag{23}$$

and its initial value is

$$\mathbf{U}_a = \mathbf{U} + \mathbf{U}_\beta + \beta\hat{\nabla}\chi_0(X, Y), \tag{24}$$

where

$$\begin{aligned}\mathbf{U}_\beta = (\Omega R_0 + \kappa_0 Z/2 - f_0 Y/2 - \beta Y^2/4)\mathbf{i}\\ + (f_0 X/2 + \beta XY/2)\mathbf{j} - (\kappa_0 X/2)\mathbf{k},\end{aligned} \tag{25}$$

and χ_0 is the parcel's initial value of χ. The initial absolute vorticity is

$$\begin{aligned}\boldsymbol{\omega}(\tau_0) = \hat{\nabla} \times \mathbf{U} + (0, \kappa_0, f_0 + \beta Y) = \hat{\nabla}U \times \hat{\nabla}X + \hat{\nabla}V \times \hat{\nabla}Y\\ + \hat{\nabla}W \times \hat{\nabla}Z + \kappa_0\hat{\nabla}Z \times \hat{\nabla}X + (f_0 + \beta Y)\hat{\nabla}X \times \hat{\nabla}Y.\end{aligned} \tag{26}$$

4. Useful matrices

In this section, we introduce some important matrices and investigate their properties. We can think of the Lagrangian coordinates X, Y, and Z either as curvilinear coordinates that are dragged by the flow through location space, the space with Eulerian coordinates (x, y, z) as a Cartesian system, or as Cartesian coordinates in label ($\mathbf{X}$) space (Salmon 1998, p. 5). By the chain rule,

$$d\mathbf{x} \equiv \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \partial x/\partial X & \partial x/\partial Y & \partial x/\partial Z \\ \partial y/\partial X & \partial y/\partial Y & \partial y/\partial Z \\ \partial z/\partial X & \partial z/\partial Y & \partial z/\partial Z \end{bmatrix} \begin{bmatrix} dX \\ dY \\ dZ \end{bmatrix} \equiv \mathbf{J}d\mathbf{X}, \tag{27}$$

where

$$\mathbf{J} \equiv \begin{bmatrix} \partial x/\partial X & \partial x/\partial Y & \partial x/\partial Z \\ \partial y/\partial X & \partial y/\partial Y & \partial y/\partial Z \\ \partial z/\partial X & \partial z/\partial Y & \partial z/\partial Z \end{bmatrix} \tag{28}$$

is the Jacobian matrix of the transformation $\mathbf{T} \equiv \mathbf{x}(\mathbf{X}, \tau)$ from Lagrangian to Eulerian coordinates. For consistency, we use square brackets for matrices throughout this paper. The columns of $\mathbf{J}$, $\partial\mathbf{x}/\partial X$, $\partial\mathbf{x}/\partial Y$, and $\partial\mathbf{x}/\partial Z$, which are tangent to the coordinate curves of X, Y, and Z, respectively, are the covariant basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ (Margenau and Murphy 1956, p. 193). This fact is expressed in the notation

$$\mathbf{J} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix}. \tag{29}$$

Hence, the elements of the column vector $d\mathbf{X}$ are the differentials of the contravariant coordinates. More generally, for a generic vector $\mathbf{A}$ with contravariant components A^1, A^2, and A^3,

$$\begin{bmatrix} \mathbf{A}\cdot\mathbf{i} \\ \mathbf{A}\cdot\mathbf{j} \\ \mathbf{A}\cdot\mathbf{k} \end{bmatrix} = \mathbf{J}\begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3. \tag{30}$$

Thus, the matrix $\mathbf{J}$ ($\mathbf{J}^{-1}$) operating on the column vector of contravariant (Cartesian) components of a vector yields its Cartesian (contravariant) components. The Jacobian of $\mathbf{T}$ is the determinant of $\mathbf{J}$, det$\mathbf{J}$, which $=$ $\mathbf{e}_1\cdot(\mathbf{e}_2\times\mathbf{e}_3)$. The inverse of (27) is

$$d\mathbf{X} = \mathbf{J}^{-1}d\mathbf{x}, \tag{31}$$

where $\mathbf{J}^{-1}$ is the Jacobian matrix of the reverse transformation $\mathbf{T}^{-1} \equiv \mathbf{X}(\mathbf{x}, t)$, and, by the chain rule,

$$\mathbf{J}^{-1} = \begin{bmatrix} \partial X/\partial x & \partial X/\partial y & \partial X/\partial z \\ \partial Y/\partial x & \partial Y/\partial y & \partial Y/\partial z \\ \partial Z/\partial x & \partial Z/\partial y & \partial Z/\partial z \end{bmatrix}. \tag{32}$$

The determinants of $\mathbf{J}$ and $\mathbf{J}^{-1}$ are related to a parcel's specific volume by the Lagrangian continuity equation and its reciprocal as follows:

$$\frac{\alpha}{\alpha_0} = \frac{\partial(x,y,z)}{\partial(X,Y,Z)} = \det\mathbf{J} = \mathbf{e}_1\cdot\mathbf{e}_2\times\mathbf{e}_3, \quad \text{and} \tag{33}$$

$$\frac{\alpha_0}{\alpha} = \frac{\partial(X,Y,Z)}{\partial(x,y,z)} = \det\mathbf{J}^{-1} = \mathbf{e}^1\cdot\mathbf{e}^2\times\mathbf{e}^3, \tag{34}$$

where α_0 is the parcel's initial specific volume at time τ_0, and $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$ are the contravariant basis vectors (Borisenko and Tarapov 1979, p. 25).

We can obtain a different expression for $\mathbf{J}^{-1}$ by expanding (29) into the form

$$\mathbf{J} = \begin{bmatrix} \mathbf{e}_1\cdot\mathbf{i} & \mathbf{e}_2\cdot\mathbf{i} & \mathbf{e}_3\cdot\mathbf{i} \\ \mathbf{e}_1\cdot\mathbf{j} & \mathbf{e}_2\cdot\mathbf{j} & \mathbf{e}_3\cdot\mathbf{j} \\ \mathbf{e}_1\cdot\mathbf{k} & \mathbf{e}_2\cdot\mathbf{k} & \mathbf{e}_3\cdot\mathbf{k} \end{bmatrix}, \tag{35}$$

taking the transpose of the matrix of cofactors of $\mathbf{J}$ and dividing the result by the determinant of $\mathbf{J}$. The identity of Lagrange (Kreyszig 1972, p. 215) is useful here. For example, the cofactor

$$\begin{aligned}(\mathbf{e}_2\cdot\mathbf{j})(\mathbf{e}_3\cdot\mathbf{k}) - (\mathbf{e}_2\cdot\mathbf{k})(\mathbf{e}_3\cdot\mathbf{j}) &= (\mathbf{e}_2\times\mathbf{e}_3)\cdot(\mathbf{j}\times\mathbf{k}) \\ &= (\mathbf{e}_2\times\mathbf{e}_3)\cdot\mathbf{i}.\end{aligned} \tag{36}$$

We thus find that

$$\begin{aligned}\mathbf{J}^{-1} &= \frac{1}{\mathbf{e}_1\cdot\mathbf{e}_2\times\mathbf{e}_3}\begin{bmatrix} \mathbf{e}_2\times\mathbf{e}_3\cdot\mathbf{i} & \mathbf{e}_2\times\mathbf{e}_3\cdot\mathbf{j} & \mathbf{e}_2\times\mathbf{e}_3\cdot\mathbf{k} \\ \mathbf{e}_3\times\mathbf{e}_1\cdot\mathbf{i} & \mathbf{e}_3\times\mathbf{e}_1\cdot\mathbf{j} & \mathbf{e}_3\times\mathbf{e}_1\cdot\mathbf{k} \\ \mathbf{e}_1\times\mathbf{e}_2\cdot\mathbf{i} & \mathbf{e}_1\times\mathbf{e}_2\cdot\mathbf{j} & \mathbf{e}_1\times\mathbf{e}_2\cdot\mathbf{k} \end{bmatrix} \\ &\equiv \begin{bmatrix} \mathbf{e}^1\cdot\mathbf{i} & \mathbf{e}^1\cdot\mathbf{j} & \mathbf{e}^1\cdot\mathbf{k} \\ \mathbf{e}^2\cdot\mathbf{i} & \mathbf{e}^2\cdot\mathbf{j} & \mathbf{e}^2\cdot\mathbf{k} \\ \mathbf{e}^3\cdot\mathbf{i} & \mathbf{e}^3\cdot\mathbf{j} & \mathbf{e}^3\cdot\mathbf{k} \end{bmatrix},\end{aligned} \tag{37}$$

where the rows of $\mathbf{J}^{-1}$, are the reciprocal or contravariant basis vectors, and $\mathbf{e}^1 = \mathbf{e}_2\times\mathbf{e}_3/[\mathbf{e}_1\cdot(\mathbf{e}_2\times\mathbf{e}_3)]$, etc. (Borisenko and Tarapov 1979, p. 25). From (37) and (32), it is apparent that $\mathbf{e}^1 = \nabla X$, $\mathbf{e}^2 = \nabla Y$, and $\mathbf{e}^3 = \nabla Z$. Substituting for the covariant basis vectors in (37) gives

$$\mathbf{J}^{-1} = \frac{\partial(X,Y,Z)}{\partial(x,y,z)}\begin{bmatrix} \dfrac{\partial(y,z)}{\partial(Y,Z)} & \dfrac{\partial(z,x)}{\partial(Y,Z)} & \dfrac{\partial(x,y)}{\partial(Y,Z)} \\ \dfrac{\partial(y,z)}{\partial(Z,X)} & \dfrac{\partial(z,x)}{\partial(Z,X)} & \dfrac{\partial(x,y)}{\partial(Z,X)} \\ \dfrac{\partial(y,z)}{\partial(X,Y)} & \dfrac{\partial(z,x)}{\partial(X,Y)} & \dfrac{\partial(x,y)}{\partial(X,Y)} \end{bmatrix} \tag{38}$$

via (33). We can form the inverse of $\mathbf{J}^{-1}$ similarly. This yields the alternative form for $\mathbf{J}$:

$$\mathbf{J} = \frac{\partial(x,y,z)}{\partial(X,Y,Z)}\begin{bmatrix} \dfrac{\partial(Y,Z)}{\partial(y,z)} & \dfrac{\partial(Z,X)}{\partial(y,z)} & \dfrac{\partial(X,Y)}{\partial(y,z)} \\ \dfrac{\partial(Y,Z)}{\partial(z,x)} & \dfrac{\partial(Z,X)}{\partial(z,x)} & \dfrac{\partial(X,Y)}{\partial(z,x)} \\ \dfrac{\partial(Y,Z)}{\partial(x,y)} & \dfrac{\partial(Z,X)}{\partial(x,y)} & \dfrac{\partial(X,Y)}{\partial(x,y)} \end{bmatrix}. \tag{39}$$

The generic vector $\mathbf{A}$ also has the following expansion with respect to the contravariant basis $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$:

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3, \tag{40}$$

where A_1, A_2, and A_3 are its covariant components. Taking the dot product with the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ gives us

$$\begin{aligned}\begin{bmatrix} \mathbf{A}\cdot\mathbf{i} \\ \mathbf{A}\cdot\mathbf{j} \\ \mathbf{A}\cdot\mathbf{k} \end{bmatrix} &= \begin{bmatrix} \mathbf{e}^1\cdot\mathbf{i} & \mathbf{e}^2\cdot\mathbf{i} & \mathbf{e}^3\cdot\mathbf{i} \\ \mathbf{e}^1\cdot\mathbf{j} & \mathbf{e}^2\cdot\mathbf{j} & \mathbf{e}^3\cdot\mathbf{j} \\ \mathbf{e}^1\cdot\mathbf{k} & \mathbf{e}^2\cdot\mathbf{k} & \mathbf{e}^3\cdot\mathbf{k} \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \\ &= \begin{bmatrix} \partial X/\partial x & \partial Y/\partial x & \partial Z/\partial x \\ \partial X/\partial y & \partial Y/\partial y & \partial Z/\partial y \\ \partial X/\partial z & \partial Y/\partial z & \partial Z/\partial z \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}.\end{aligned} \tag{41}$$

The square matrix in (41) we define as

$$\mathbf{M}[\tilde{\mathbf{x}}(\tau),\tilde{\mathbf{x}}(\tau_0)] \equiv \begin{bmatrix} \partial X/\partial x & \partial Y/\partial x & \partial Z/\partial x \\ \partial X/\partial y & \partial Y/\partial y & \partial Z/\partial y \\ \partial X/\partial z & \partial Y/\partial z & \partial Z/\partial z \end{bmatrix} = \begin{bmatrix} \mathbf{e}^1\cdot\mathbf{i} & \mathbf{e}^2\cdot\mathbf{i} & \mathbf{e}^3\cdot\mathbf{i} \\ \mathbf{e}^1\cdot\mathbf{j} & \mathbf{e}^2\cdot\mathbf{j} & \mathbf{e}^3\cdot\mathbf{j} \\ \mathbf{e}^1\cdot\mathbf{k} & \mathbf{e}^2\cdot\mathbf{k} & \mathbf{e}^3\cdot\mathbf{k} \end{bmatrix} = (\mathbf{J}^{-1})^{\mathrm{T}}, \quad (42)$$

where $\tilde{\mathbf{x}}(\tau) = \mathbf{x}$, $\tilde{\mathbf{x}}(\tau_0) = \mathbf{X}$ from (6), superscript T denotes transpose, and, hereinafter, the arguments of $\mathbf{M}$ (and $\mathbf{P}$ below) will be shortened to (τ, τ_0). Note that $\mathbf{M}$ operating on the column vector of covariant components of a vector yields its Cartesian components, and $\mathbf{M}^{-1}$ performs the reverse transformation. These matrices are important because $\mathbf{M}^{-1}$ converts any gradient $\nabla\sigma$ in location space to the corresponding gradient in label space, and $\mathbf{M}$ transforms the gradient back to location space. This follows from the chain rule, whereby

$$\nabla\sigma \equiv \begin{bmatrix} \partial\sigma/\partial x \\ \partial\sigma/\partial y \\ \partial\sigma/\partial z \end{bmatrix} = \begin{bmatrix} \partial X/\partial x & \partial Y/\partial x & \partial Z/\partial x \\ \partial X/\partial y & \partial Y/\partial y & \partial Z/\partial y \\ \partial X/\partial z & \partial Y/\partial z & \partial Z/\partial z \end{bmatrix} \begin{bmatrix} \partial\sigma/\partial X \\ \partial\sigma/\partial Y \\ \partial\sigma/\partial Z \end{bmatrix}, \quad (43)$$

or

$$\nabla\sigma = \mathbf{M}(\tau,\tau_0)\hat{\nabla}\sigma. \quad (44)$$

The reverse transformation is

$$\hat{\nabla}\sigma = \mathbf{M}^{-1}(\tau,\tau_0)\nabla\sigma, \quad (45)$$

where

$$\mathbf{M}^{-1}(\tau,\tau_0) = \mathbf{J}^{\mathrm{T}} = \begin{bmatrix} \partial x/\partial X & \partial y/\partial X & \partial z/\partial X \\ \partial x/\partial Y & \partial y/\partial Y & \partial z/\partial Y \\ \partial x/\partial Z & \partial y/\partial Z & \partial z/\partial Z \end{bmatrix}. \quad (46)$$

The columns of $\mathbf{M}$ are the contravariant basis vectors, so

$$\nabla\sigma = \frac{\partial\sigma}{\partial X}\mathbf{e}^1 + \frac{\partial\sigma}{\partial Y}\mathbf{e}^2 + \frac{\partial\sigma}{\partial Z}\mathbf{e}^3, \quad (47)$$

which shows that in location space $\partial\sigma/\partial X$, $\partial\sigma/\partial Y$, and $\partial\sigma/\partial Z$ are the covariant components of $\nabla\sigma$.

We will also need to find a matrix operator $\mathbf{P}$ such that

$$\nabla A \times \nabla B = \mathbf{P}(\tau,\tau_0)\hat{\nabla}A \times \hat{\nabla}B, \quad \text{and} \quad (48)$$

$$\mathbf{P}^{-1}(\tau,\tau_0)\nabla A \times \nabla B = \hat{\nabla}A \times \hat{\nabla}B. \quad (49)$$

With the aid of the rule (47) applied to A and B, we obtain

$$\begin{aligned} \nabla A \times \nabla B &= \left(\frac{\partial A}{\partial X}\mathbf{e}^1 + \frac{\partial A}{\partial Y}\mathbf{e}^2 + \frac{\partial A}{\partial Z}\mathbf{e}^3\right) \times \left(\frac{\partial B}{\partial X}\mathbf{e}^1 + \frac{\partial B}{\partial Y}\mathbf{e}^2 + \frac{\partial B}{\partial Z}\mathbf{e}^3\right) \\ &= \frac{\partial(A,B)}{\partial(Y,Z)}\mathbf{e}^2\times\mathbf{e}^3 + \frac{\partial(A,B)}{\partial(Z,X)}\mathbf{e}^3\times\mathbf{e}^1 + \frac{\partial(A,B)}{\partial(X,Y)}\mathbf{e}^1\times\mathbf{e}^2 \\ &= \mathbf{e}^1\cdot\mathbf{e}^2\times\mathbf{e}^3\left[\frac{\partial(A,B)}{\partial(Y,Z)}\mathbf{e}_1 + \frac{\partial(A,B)}{\partial(Z,X)}\mathbf{e}_2 + \frac{\partial(A,B)}{\partial(X,Y)}\mathbf{e}_3\right], \end{aligned} \quad (50)$$

because $\mathbf{e}_1 = \mathbf{e}^2 \times \mathbf{e}^3/[\mathbf{e}^1\cdot(\mathbf{e}^2\times\mathbf{e}^3)]$, etc. This becomes

$$\nabla A \times \nabla B \equiv \begin{bmatrix} \dfrac{\partial(A,B)}{\partial(y,z)} \\ \dfrac{\partial(A,B)}{\partial(z,x)} \\ \dfrac{\partial(A,B)}{\partial(x,y)} \end{bmatrix} = \det(\mathbf{J}^{-1})[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \times \begin{bmatrix} \dfrac{\partial(A,B)}{\partial(Y,Z)} \\ \dfrac{\partial(A,B)}{\partial(Z,X)} \\ \dfrac{\partial(A,B)}{\partial(X,Y)} \end{bmatrix} = \frac{\alpha_0}{\alpha}\mathbf{J}\hat{\nabla}A \times \hat{\nabla}B \quad (51)$$

after use of (29) and (34). Thus,

$$\mathbf{P}(\tau,\tau_0) = \frac{\alpha_0}{\alpha}\mathbf{J} = \begin{bmatrix} \dfrac{\partial(Y,Z)}{\partial(y,z)} & \dfrac{\partial(Z,X)}{\partial(y,z)} & \dfrac{\partial(X,Y)}{\partial(y,z)} \\ \dfrac{\partial(Y,Z)}{\partial(z,x)} & \dfrac{\partial(Z,X)}{\partial(z,x)} & \dfrac{\partial(X,Y)}{\partial(z,x)} \\ \dfrac{\partial(Y,Z)}{\partial(x,y)} & \dfrac{\partial(Z,X)}{\partial(x,y)} & \dfrac{\partial(X,Y)}{\partial(x,y)} \end{bmatrix}, \quad \text{and} \quad (52)$$

$$\mathbf{P}^{-1}(\tau,\tau_0) = \frac{\alpha}{\alpha_0}\mathbf{J}^{-1} = \begin{bmatrix} \dfrac{\partial(y,z)}{\partial(Y,Z)} & \dfrac{\partial(z,x)}{\partial(Y,Z)} & \dfrac{\partial(x,y)}{\partial(Y,Z)} \\ \dfrac{\partial(y,z)}{\partial(Z,X)} & \dfrac{\partial(z,x)}{\partial(Z,X)} & \dfrac{\partial(x,y)}{\partial(Z,X)} \\ \dfrac{\partial(y,z)}{\partial(X,Y)} & \dfrac{\partial(z,x)}{\partial(X,Y)} & \dfrac{\partial(x,y)}{\partial(X,Y)} \end{bmatrix} \quad (53)$$

from (38), (39), (33), and (34). It is shown in the appendix that $\mathbf{P}^{-1}$ is the propagator for a directed material element of area. Hence, $\mathbf{P}$ is related to vortex-tube stretching.

From (51) and (30), it is evident that $\alpha_0\partial(A, B)/\partial(Y, Z)$, $\alpha_0\partial(A, B)/\partial(Z, X)$, and $\alpha_0\partial(A, B)/\partial(X, Y)$ are the contravariant components of $\alpha\nabla A \times \nabla B$ in location space. Even though $\nabla A \times \nabla B$, like vorticity, is an axial vector (or pseudovector) that transforms differently from a true vector (Springer 1962, p. 76), $\alpha\nabla A \times \nabla B$, like α times vorticity, is a true vector.

5. Parcel velocity formula via integration

We now seek an expression for the velocity of a parcel. The equations of motion, continuity, and entropy equations in a frame rotating with the earth are

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\alpha\nabla p - \nabla\Phi + \mathbf{F} = T\nabla S - \nabla(c_p T + \Phi) + \mathbf{F}, \quad (54)$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \quad \text{and} \quad (55)$$

$$\dot{S} \equiv \frac{DS}{Dt} = \frac{J}{T}, \quad (56)$$

where $\Phi \equiv gz - \mathbf{u}_E\cdot\mathbf{u}_E/2$ is the sum of gravitational and centrifugal potentials, $\mathbf{F} \equiv F_1\nabla x + F_2\nabla y + F_3\nabla z$ is the friction force, $\dot{S}$ is the rate of entropy production in the parcel, and J is the diabatic heating rate.

The steps to obtaining a velocity formula are (i) use the matrix $\mathbf{M}^{-1}$ to convert (54) to Lagrangian coordinates, (ii) utilize $\mathbf{M}^{-1}$ as an integrating factor, (iii) integrate over time, and (iv) multiply by the inverse matrix $\mathbf{M}$ to convert the resulting formula back to Eulerian coordinates.

When we pre-multiply (54) by $\mathbf{M}^{-1}$ and use (45), we get

$$\mathbf{M}^{-1}(\tau,\tau_0)\left(\frac{\partial\mathbf{u}}{\partial\tau} + 2\boldsymbol{\Omega}\times\mathbf{u}\right) = T\hat{\nabla}S - \hat{\nabla}(c_p T + \Phi) + F_1\hat{\nabla}x + F_2\hat{\nabla}y + F_3\hat{\nabla}z. \quad (57)$$

By (46),

$$\mathbf{M}^{-1}(\tau,\tau_0)\frac{\partial\mathbf{u}}{\partial\tau} = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial y}{\partial X} & \frac{\partial z}{\partial X} \\ \frac{\partial x}{\partial Y} & \frac{\partial y}{\partial Y} & \frac{\partial z}{\partial Y} \\ \frac{\partial x}{\partial Z} & \frac{\partial y}{\partial Z} & \frac{\partial z}{\partial Z} \end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial\tau} \\ \frac{\partial v}{\partial\tau} \\ \frac{\partial w}{\partial\tau} \end{bmatrix} = \frac{\partial u}{\partial\tau}\hat{\nabla}x + \frac{\partial v}{\partial\tau}\hat{\nabla}y + \frac{\partial w}{\partial\tau}\hat{\nabla}z = \frac{\partial}{\partial\tau}(u\hat{\nabla}x + v\hat{\nabla}y + w\hat{\nabla}z) - \hat{\nabla}\left(\frac{u^2+v^2+w^2}{2}\right), \quad (58)$$

since $\partial\mathbf{x}/\partial\tau = \mathbf{u}$. We also have

$$\mathbf{M}^{-1}(\tau,\tau_0)(2\boldsymbol{\Omega}\times\mathbf{u}) = \mathbf{M}^{-1}(\tau,\tau_0)\begin{bmatrix} -fv + \kappa_0 w \\ fu \\ -\kappa_0 u \end{bmatrix} = \left(\frac{\partial x}{\partial\tau}\hat{\nabla}\mathcal{F} - \frac{\partial\mathcal{F}}{\partial\tau}\hat{\nabla}x\right) + \kappa_0\left(\frac{\partial z}{\partial\tau}\hat{\nabla}x - \frac{\partial x}{\partial\tau}\hat{\nabla}z\right). \quad (59)$$

But

$$\frac{\partial x}{\partial\tau}\hat{\nabla}\mathcal{F} - \frac{\partial\mathcal{F}}{\partial\tau}\hat{\nabla}x = \frac{\partial}{\partial\tau}(x\hat{\nabla}\mathcal{F} - \mathcal{F}\hat{\nabla}x) - x\frac{\partial\hat{\nabla}\mathcal{F}}{\partial\tau} + \mathcal{F}\frac{\partial\hat{\nabla}x}{\partial\tau}, \quad (60)$$

and

$$\frac{\partial x}{\partial\tau}\hat{\nabla}\mathcal{F} - \frac{\partial\mathcal{F}}{\partial\tau}\hat{\nabla}x = \hat{\nabla}\left(\frac{\partial x}{\partial\tau}\mathcal{F} - \frac{\partial\mathcal{F}}{\partial\tau}x\right) + x\hat{\nabla}\frac{\partial\mathcal{F}}{\partial\tau} - \mathcal{F}\hat{\nabla}\frac{\partial x}{\partial\tau} \quad (61)$$

Adding (60) and (61) gives

$$\frac{\partial x}{\partial\tau}\hat{\nabla}\mathcal{F} - \frac{\partial\mathcal{F}}{\partial\tau}\hat{\nabla}x = \frac{1}{2}\frac{\partial}{\partial\tau}(x\hat{\nabla}\mathcal{F} - \mathcal{F}\hat{\nabla}x) + \frac{1}{2}\hat{\nabla}\left(\mathcal{F}\frac{\partial x}{\partial\tau} - x\frac{\partial\mathcal{F}}{\partial\tau}\right). \quad (62)$$

Similarly,

$$\kappa_0\left(\frac{\partial z}{\partial\tau}\hat{\nabla}x - \frac{\partial x}{\partial\tau}\hat{\nabla}z\right) = \frac{\kappa_0}{2}\frac{\partial}{\partial\tau}(z\hat{\nabla}x - x\hat{\nabla}z) + \frac{\kappa_0}{2}\hat{\nabla}\left(x\frac{\partial z}{\partial\tau} - z\frac{\partial x}{\partial\tau}\right) \quad (63)$$

Therefore,

$$\begin{aligned}\mathbf{M}^{-1}(\tau,\tau_0)(2\mathbf{\Omega}\times\mathbf{u}) &= \left(\frac{\partial x}{\partial\tau}\hat{\nabla}\mathcal{F} - \frac{\partial\mathcal{F}}{\partial\tau}\hat{\nabla}x\right) + \kappa_0\left(\frac{\partial z}{\partial\tau}\hat{\nabla}x - \frac{\partial x}{\partial\tau}\hat{\nabla}z\right)\\ &= \frac{1}{2}\frac{\partial}{\partial\tau}(\kappa_0 z\hat{\nabla}x - \mathcal{F}\hat{\nabla}x + fx\hat{\nabla}y - \kappa_0 x\hat{\nabla}z) + \frac{1}{2}\hat{\nabla}(\mathcal{F}u - \kappa_0 zu - fxv + \kappa_0 xw)\\ &= \frac{\partial}{\partial\tau}(u_\beta\hat{\nabla}x + v_\beta\hat{\nabla}y + w_\beta\hat{\nabla}z) - \hat{\nabla}(\mathbf{u}\cdot\mathbf{u}_\beta)\end{aligned} \tag{64}$$

from (59), (62), (63), and (22). Inserting (58) and (64) into (57) then gives

$$\begin{aligned}&\frac{\partial}{\partial\tau}(u_B\hat{\nabla}x + v_B\hat{\nabla}y + w_B\hat{\nabla}z)\\ &\quad= T\hat{\nabla}S - \hat{\nabla}\left(c_pT + \Phi - \frac{\mathbf{u}\cdot\mathbf{u}}{2} - \mathbf{u}\cdot\mathbf{u}_\beta\right) + F_1\hat{\nabla}x\\ &\qquad + F_2\hat{\nabla}y + F_3\hat{\nabla}z,\end{aligned} \tag{65}$$

where $\mathbf{u}_B \equiv (u_B, v_B, w_B) \equiv \mathbf{u} + \mathbf{u}_\beta$. We now integrate (65) over time from the initial time τ_0 to the current time τ, and apply the initial conditions $\mathbf{x} = \mathbf{X}$, $\mathbf{u}_B = \mathbf{U}_B$ at τ_0, where $\mathbf{U}_B \equiv \mathbf{U} + \mathbf{U}_\beta$. We get

$$\begin{aligned}&u_B\hat{\nabla}x + v_B\hat{\nabla}y + w_B\hat{\nabla}z - \mathbf{U}_B = -\hat{\nabla}\psi\\ &\quad + \int_{\tau_0}^{\tau}[T(\tilde{\tau})\hat{\nabla}S(\tilde{\tau}) + F_1(\tilde{\tau})\hat{\nabla}\tilde{x} + F_2(\tilde{\tau})\hat{\nabla}\tilde{y} + F_3(\tilde{\tau})\hat{\nabla}\tilde{z}]\,d\tilde{\tau},\end{aligned} \tag{66}$$

where

$$\psi \equiv \int_{\tau_0}^{\tau}\left(c_pT + \Phi - \frac{\mathbf{u}\cdot\mathbf{u}}{2} - \mathbf{u}\cdot\mathbf{u}_\beta\right)d\tilde{\tau}. \tag{67}$$

After integration by parts and use of (56), this becomes

$$\begin{aligned}&u_B\hat{\nabla}x + v_B\hat{\nabla}y + w_B\hat{\nabla}z - \mathbf{U}_B\\ &\quad= \Lambda(\tau)\hat{\nabla}S(\tau) - \hat{\nabla}\psi(\tau) + \int_{\tau_0}^{\tau}\mathbf{G}(\tilde{\tau})\,d\tilde{\tau},\end{aligned} \tag{68}$$

where Λ is the cumulative temperature defined in section 2, and

$$\mathbf{G}(\tilde{\tau}) \equiv F_1(\tilde{\tau})\hat{\nabla}\tilde{x} + F_2(\tilde{\tau})\hat{\nabla}\tilde{y} + F_3(\tilde{\tau})\hat{\nabla}\tilde{z} - \Lambda(\tilde{\tau})\hat{\nabla}\dot{S}(\tilde{\tau}) \tag{69}$$

is the integrand of a Lagrangian integral (integral over time following a parcel). For frictionless isentropic flow ($\mathbf{G} \equiv 0$) in a nonrotating system, (68) reduces to Serrin's (1959) generalization to isentropic flow of Weber's transformation of the equations of motion for homentropic flow.

Finally, we pre-multiply (68) by $\mathbf{M}(\tau, \tau_0)$. Note first that

$$\begin{aligned}\mathbf{M}(\tau,\tau_0)\mathbf{U}_B &= \begin{bmatrix}\partial X/\partial x & \partial Y/\partial x & \partial Z/\partial x\\ \partial X/\partial y & \partial Y/\partial y & \partial Z/\partial y\\ \partial X/\partial z & \partial Y/\partial z & \partial Z/\partial z\end{bmatrix}\begin{bmatrix}U_a - \beta\partial\chi_0/\partial X\\ V_a - \beta\partial\chi_0/\partial Y\\ W_a\end{bmatrix}\\ &= U_a\nabla X + V_a\nabla Y + W_a\nabla Z - \beta\nabla\chi_0\end{aligned} \tag{70}$$

via (24). We thus obtain

$$\begin{aligned}\mathbf{u}_a &= U_a\nabla X + V_a\nabla Y + W_a\nabla Z - \nabla\Psi + \Lambda\nabla S\\ &\quad + \mathbf{M}(\tau,\tau_0)\int_{\tau_0}^{\tau}\mathbf{G}(\tilde{\tau})\,d\tilde{\tau}\end{aligned} \tag{71}$$

on the Grimshaw β plane where $\Psi = \psi - \beta(\chi - \chi_0)$. On the nontraditional f plane, the potential term is simply $-\nabla\Psi$, where Ψ is now

$$\Psi \equiv \int_{\tau_0}^{\tau}\left(c_pT + gz - \frac{\mathbf{u}_a\cdot\mathbf{u}_a}{2}\right)d\tilde{\tau}. \tag{72}$$

The velocity formula is implicit through the definition of Ψ. However, the implicitness affects only the irrotational part of the wind. If the normal velocity is known at the boundaries of the domain, we may determine Ψ at time τ as the solution of the elliptic partial differential equation obtained by substituting the velocity formula into the continuity equation [(55)] (Hunt and Hussain 1991).

From the curl of (71) or, alternatively, $\mathbf{P}\hat{\nabla}\times$ (68), we obtain the following explicit formula of the form (5) for the absolute vorticity $\boldsymbol{\omega}$ of a parcel on a nontraditional β plane:

$$\begin{aligned}\boldsymbol{\omega} &= \nabla U\times\nabla X + \nabla V\times\nabla Y + \nabla W\times\nabla Z + \kappa_0\nabla Z\times\nabla X\\ &\quad + (f_0 + \beta Y)\nabla X\times\nabla Y + \nabla\Lambda\times\nabla S + \mathbf{P}(\tau,\tau_0)\int_{\tau_0}^{\tau}\hat{\nabla}\times\mathbf{G}(\tilde{\tau})\,d\tilde{\tau}.\end{aligned} \tag{73}$$

6. Vorticity integral via propagator

An integral of the vector vorticity equation circumvents the complication caused by a β plane not having a unique planetary velocity. The vorticity equation is

$$D\boldsymbol{\omega}/Dt - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \boldsymbol{\omega}\nabla\cdot\mathbf{u} = \nabla T\times\nabla S + \nabla\times\mathbf{F} = \nabla p\times\nabla\alpha + \nabla\times\mathbf{F}, \qquad (74)$$

and the corresponding homogeneous equation, obtained by equating the left side to zero, is the barotropic vorticity equation.

Note that

$$\begin{aligned}(\boldsymbol{\omega}\cdot\nabla)\mathbf{u} - \boldsymbol{\omega}\nabla\cdot\mathbf{u} &= \begin{bmatrix}\partial u/\partial x & \partial u/\partial y & \partial u/\partial z\\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z\\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z\end{bmatrix}\boldsymbol{\omega} - \frac{1}{\alpha}\frac{\partial\alpha}{\partial\tau}\boldsymbol{\omega}\\ &= \begin{bmatrix}\partial u/\partial X & \partial u/\partial Y & \partial u/\partial Z\\ \partial v/\partial X & \partial v/\partial Y & \partial v/\partial Z\\ \partial w/\partial X & \partial w/\partial Y & \partial w/\partial Z\end{bmatrix}\begin{bmatrix}\partial X/\partial x & \partial X/\partial y & \partial X/\partial z\\ \partial Y/\partial x & \partial Y/\partial y & \partial Y/\partial z\\ \partial Z/\partial x & \partial Z/\partial y & \partial Z/\partial z\end{bmatrix}\boldsymbol{\omega} - \frac{1}{\alpha}\frac{\partial\alpha}{\partial\tau}\mathbf{J}\mathbf{J}^{-1}\boldsymbol{\omega}\\ &= \left(\frac{\partial\mathbf{J}}{\partial\tau} - \frac{1}{\alpha}\frac{\partial\alpha}{\partial\tau}\mathbf{J}\right)\mathbf{J}^{-1}\boldsymbol{\omega} = \frac{\partial}{\partial\tau}\left(\frac{\alpha_0}{\alpha}\mathbf{J}\right)\frac{\alpha}{\alpha_0}\mathbf{J}^{-1}\boldsymbol{\omega} = \frac{\partial\mathbf{P}(\tau,\tau_0)}{\partial\tau}\mathbf{P}^{-1}(\tau,\tau_0)\boldsymbol{\omega}\end{aligned} \qquad (75)$$

by continuity, the chain rule, (28), (32), (52), and (53). The vorticity equation now becomes

$$\frac{\partial\boldsymbol{\omega}}{\partial\tau} - \frac{\partial\mathbf{P}(\tau,\tau_0)}{\partial\tau}\mathbf{P}^{-1}(\tau,\tau_0)\boldsymbol{\omega} = \nabla T\times\nabla S + \nabla\times\mathbf{F}. \qquad (76)$$

Clearly, $\mathbf{P}(\tau, \tau_0)$ satisfies the homogeneous version of (76) (i.e., the barotropic vorticity equation) and is thus the propagator for the equation (DJ06). As well as satisfying its own homogeneous equation, the propagator has the following properties:

$$\mathbf{P}^{-1}(\tau,\tau_0) = \mathbf{P}(\tau_0,\tau),\quad \mathbf{P}(\tau,\tau_0) = \mathbf{P}(\tau,\tilde{\tau})\mathbf{P}(\tilde{\tau},\ \tau_0),\quad \mathbf{P}(\tau,\tau) = \mathbf{I}, \qquad (77)$$

where $\mathbf{I}$ is the 3×3 unit matrix.

Pre-multiplying (76) by $\mathbf{P}^{-1}(\tau, \tau_0)$ gives us

$$\begin{aligned}\mathbf{P}^{-1}\frac{\partial\boldsymbol{\omega}}{\partial\tau} &= \mathbf{P}^{-1}\frac{\partial\mathbf{P}}{\partial\tau}\mathbf{P}^{-1}\boldsymbol{\omega} + \hat{\nabla}T\times\hat{\nabla}S + \mathbf{P}^{-1}\nabla\times\mathbf{F}\\ &= \frac{\partial(\mathbf{P}^{-1}\mathbf{P})}{\partial\tau}\mathbf{P}^{-1}\boldsymbol{\omega} - \frac{\partial\mathbf{P}^{-1}}{\partial\tau}\mathbf{P}\mathbf{P}^{-1}\boldsymbol{\omega} + \hat{\nabla}T\times\hat{\nabla}S + \mathbf{P}^{-1}(\nabla F_1\times\nabla x + \nabla F_2\times\nabla y + \nabla F_3\times\nabla z)\\ &= -\frac{\partial\mathbf{P}^{-1}}{\partial\tau}\boldsymbol{\omega} + \hat{\nabla}T\times\hat{\nabla}S + \hat{\nabla}F_1\times\hat{\nabla}x + \hat{\nabla}F_2\times\hat{\nabla}y + \hat{\nabla}F_3\times\hat{\nabla}z\end{aligned} \qquad (78)$$

after use of (49). Hence,

$$\frac{\partial(\mathbf{P}^{-1}\boldsymbol{\omega})}{\partial\tau} = \hat{\nabla}T\times\hat{\nabla}S + \hat{\nabla}F_1\times\hat{\nabla}x + \hat{\nabla}F_2\times\hat{\nabla}y + \hat{\nabla}F_3\times\hat{\nabla}z. \qquad (79)$$

Integrating (by parts in some places) over time from τ_0 to τ yields the formula for the initial vorticity:

$$\begin{aligned}\boldsymbol{\omega}(\tau_0) &= \mathbf{P}^{-1}(\tau,\tau_0)\boldsymbol{\omega}(\tau) - \int_{\tau_0}^{\tau}[\hat{\nabla}T(\tilde{\tau})\times\hat{\nabla}S(\tilde{\tau}) + \hat{\nabla}F_1(\tilde{\tau})\times\hat{\nabla}\tilde{x} + \hat{\nabla}F_2(\tilde{\tau})\times\hat{\nabla}\tilde{y} + \hat{\nabla}F_3(\tilde{\tau})\times\hat{\nabla}\tilde{z}]\,d\tilde{\tau}\\ &= \mathbf{P}^{-1}(\tau,\tau_0)\boldsymbol{\omega}(\tau) - \hat{\nabla}\Lambda(\tau)\times\hat{\nabla}S(\tau) - \int_{\tau_0}^{\tau}\mathbf{H}(\tilde{\tau})\,d\tilde{\tau},\end{aligned} \qquad (80)$$

where

$$\mathbf{H}(\tilde{\tau}) \equiv \hat{\nabla}\times\mathbf{G}(\tilde{\tau}) = \hat{\nabla}F_1(\tilde{\tau})\times\hat{\nabla}\tilde{x} + \hat{\nabla}F_2(\tilde{\tau})\times\hat{\nabla}\tilde{y} + \hat{\nabla}F_3(\tilde{\tau})\times\hat{\nabla}\tilde{z} - \hat{\nabla}\Lambda(\tilde{\tau})\times\hat{\nabla}\dot{S} \qquad (81)$$

is the vector integrand of the Lagrangian integral [after use of (56)]. We then pre-multiply (80) by $\mathbf{P}(\tau, \tau_0)$ and utilize (48) to get

$$\boldsymbol{\omega}(\tau) = \mathbf{P}(\tau,\tau_0)\boldsymbol{\omega}(\tau_0) + \nabla\Lambda(\tau)\times\nabla S(\tau) + \mathbf{P}(\tau,\tau_0)\int_{\tau_0}^{\tau}\mathbf{H}(\tilde{\tau})\,d\tilde{\tau}, \qquad (82)$$

where the first term on the right is the barotropic vorticity, the second term is the baroclinic vorticity if the flow is isentropic, and the last term includes the frictional vorticity and an alteration to the baroclinic vorticity owing to diabatic heating. Equation (82) is the same as (3.8b) in DJ06. It is demonstrated in section 8 that all the circulation theorems follow from (82). In inviscid isentropic flow, the integral term vanishes, leaving

$$\boldsymbol{\omega}(\tau) = \mathbf{P}(\tau, \tau_0)\boldsymbol{\omega}(\tau_0) + \nabla\Lambda(\tau) \times \nabla S(\tau), \tag{83}$$

where the first and second terms on the right are the barotropic and baroclinic vorticity, respectively. Unlike the baroclinic and frictional vorticities, the barotropic vorticity is independent of the path that the parcel takes between **X** and **x**.

The barotropic vorticity in (83) may be written in several different forms. By (52),

$$\boldsymbol{\omega}_{\mathrm{BT}}(\tau) = \mathbf{P}(\tau, \tau_0)\boldsymbol{\omega}(\tau_0) = (\alpha_0/\alpha)\mathbf{J}\boldsymbol{\omega}(\tau_0), \tag{84}$$

which is Cauchy's formula. The inverse of (84), $\boldsymbol{\omega}(\tau_0) = \mathbf{P}^{-1}\boldsymbol{\omega}_{\mathrm{BT}}(\tau)$, is (2) in Article 146 of Lamb (1932). Inserting (26) for the initial vorticity into (84) and using (48) produces

$$\boldsymbol{\omega}_{\mathrm{BT}}(\tau) = \nabla U \times \nabla X + \nabla V \times \nabla Y + \nabla W \times \nabla Z + \kappa_0 \nabla Z \times \nabla X + (f_0 + \beta Y)\nabla X \times \nabla Y. \tag{85}$$

We can relate the barotropic vorticity to the initial vorticity by expanding ∇U, ∇V, and ∇W by the chain rule. For example,

$$\nabla U \times \nabla X = \left(\frac{\partial U}{\partial X}\nabla X + \frac{\partial U}{\partial Y}\nabla Y + \frac{\partial U}{\partial Z}\nabla Z\right) \times \nabla X = \frac{\partial U}{\partial Z}\nabla Z \times \nabla X - \frac{\partial U}{\partial Y}\nabla X \times \nabla Y. \tag{86}$$

Substituting this and similar expressions for $\nabla V \times \nabla Y$ and $\nabla W \times \nabla Z$ into (85) gives us

$$\boldsymbol{\omega}_{\mathrm{BT}}(\tau) = \left(\frac{\partial W}{\partial Y} - \frac{\partial V}{\partial Z}\right)\nabla Y \times \nabla Z + \left(\frac{\partial U}{\partial Z} - \frac{\partial W}{\partial X} + \kappa_0\right)\nabla Z \times \nabla X + \left(\frac{\partial V}{\partial X} - \frac{\partial U}{\partial Y} + f_0 + \beta Y\right)\nabla X \times \nabla Y. \tag{87}$$

Inserting (85) into (82) gives us the general formula for the vorticity of a parcel on a Grimshaw β plane:

$$\boldsymbol{\omega}(\tau) = \nabla U \times \nabla X + \nabla V \times \nabla Y + \nabla W \times \nabla Z + \kappa_0 \nabla Z \times \nabla X + (f_0 + \beta Y)\nabla X \times \nabla Y + \nabla\Lambda \times \nabla S + \mathbf{P}(\tau, \tau_0)\int_{\tau_0}^{\tau} \mathbf{H}(\tilde{\tau})\, d\tilde{\tau}, \tag{88}$$

where **P** and **H** are given by (52) and (81). This is (73) again.

7. Velocity formula by calculus of variations

We now show how the isentropic frictionless version of velocity (71) can be derived from an Eulerian form of Hamilton's principle of least action (Salmon 1988, 234–235). The action, a functional, is defined as the integral over volume and time of the Lagrangian density function L. The Lagrangian density is the absolute kinetic energy of the fluid minus its internal and potential energies (all per unit volume). Hamilton's principle in particle dynamics states that the action is stationary with respect to small virtual displacements of the particles from their actual motion (Feynman et al. 1964). Because the Eulerian coordinates are independent variables here, the action is stationary with respect to small changes $\delta\mathbf{X}$ in the Lagrangian coordinates at each point rather than changes in the locations of individual particles (Salmon 1988, p. 234). The analysis has to account for the relationships between $\delta\mathbf{X}$ and the variations $\delta\mathbf{u}$, $\delta\alpha$, and δS in the other dependent variables. Solving $D\mathbf{X}/Dt = 0$ for the velocity field yields $\mathbf{u} = -\mathbf{J}\partial\mathbf{X}/\partial t$, so the variation in $\mathbf{u}$ is dependent on the Lagrangian coordinates as well as $\mathbf{x}$ and t (Salmon 1988, p. 234). The same is true for the density field because it is related to $\mathbf{X}$ via the continuity equation [(34)], and also for entropy, which is a function of $\mathbf{X}$ because it is conserved. However, the variations in α, S, and $\mathbf{u}$ can be considered independent of $\mathbf{X}$ if constraints are added to the Lagrangian density (Hildebrand 1965, 139–142). Therefore, we add constraints of mass and entropy conservation and the Lin constraints to L.

Thus, the variational problem is

$$0 = \delta\int_{t_0}^{t} dt \int_{\Xi} d\Xi\, L(\mathbf{u}, \rho, S, \mathbf{X}) = \int_{t_0}^{t} dt \int_{\Xi} d\Xi\, \delta L(\mathbf{u}, \rho, S, \mathbf{X}), \tag{89}$$

where Ξ is the 3D spatial domain with surface boundary Σ, the time integral is from the initial time t_0 to the current time t, and

$$L(\mathbf{u},\rho,S,\mathbf{X}) = \rho\frac{\mathbf{u}_a\cdot\mathbf{u}_a}{2} - \rho E(\alpha,S) - \rho g z - \Psi^*\left(\frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u}\right) - \Lambda^*\rho\left(\frac{\partial S}{\partial t} + \mathbf{u}\cdot\nabla S\right) - U^*\rho\left(\frac{\partial X}{\partial t} + \mathbf{u}\cdot\nabla X\right) - V^*\rho\left(\frac{\partial Y}{\partial t} + \mathbf{u}\cdot\nabla Y\right) - W^*\rho\left(\frac{\partial Z}{\partial t} + \mathbf{u}\cdot\nabla Z\right). \quad (90)$$

Here, Ψ^*, Λ^*, U^*, V^*, and W^* are the Lagrange multipliers. In the case of a nonrotating atmosphere, (89) and (90) reduce to Bretherton's (1970) (10). We take the variations and use integration by parts where necessary in the manner prescribed by Hildebrand (1965, 135–136). After using (10) and the divergence theorem, we obtain

$$\begin{aligned} 0 = &\int_{t_0}^{t} dt \int_{\Xi} d\Xi \left\{ \begin{array}{l} \delta\mathbf{u}\cdot\rho(\mathbf{u}_a + \nabla\Psi^* - \Lambda^*\nabla S - U^*\nabla X - V^*\nabla Y - W^*\nabla Z) \\ +\delta\rho(\mathbf{u}_a\cdot\mathbf{u}_a/2 - c_p T - gz + D\Psi^*/Dt) + \delta S[\rho(D\Lambda^*/Dt - T)] + \delta\mathbf{X}\cdot(\rho D\mathbf{U}^*/Dt) \end{array} \right\} \\ &- \int_{t_0}^{t} dt \oiint_{\Sigma} d\boldsymbol{\Sigma}[\Psi^*\rho\delta\mathbf{u}\cdot\mathbf{n} + (\Psi^*\delta\rho + \rho\Lambda^*\delta S + \rho\mathbf{U}^*\cdot\delta\mathbf{X})\mathbf{u}\cdot\mathbf{n}] \\ &- \int_{\Xi} d\Xi[\Psi^*\delta\rho + \rho\Lambda^*\delta S + \rho\mathbf{U}^*\cdot\delta\mathbf{X}]_{t_0}^{t}, \end{aligned} \quad (91)$$

where $\mathbf{n}$ is the unit outward normal on Σ, and we have omitted groups of terms that cancel owing to the constraints. Because the variations are arbitrary, we must have

$$\mathbf{u}_a(t) = -\nabla\Psi^* + \Lambda^*\nabla S + U^*\nabla X + V^*\nabla Y + W^*\nabla Z, \quad (92)$$

$$D\Psi^*/Dt = c_p T + gz - \mathbf{u}_a\cdot\mathbf{u}_a/2, \quad (93)$$

$$D\Lambda^*/Dt = T, \quad \text{and} \quad (94)$$

$$D\mathbf{U}^*/Dt = 0. \quad (95)$$

The natural boundary conditions are that the normal velocity vanishes on Σ and that the variations $\delta\rho$, δS, and $\delta\mathbf{X}$ vanish at the beginning and end times. We stipulate that $\Psi^* = 0$ and $\Lambda^* = 0$ initially. Then Λ^* becomes identical to the cumulative temperature Λ, and (92) evaluated at the start gives us

$$\mathbf{u}_a(t_0) = U^*\mathbf{i} + V^*\mathbf{j} + W^*\mathbf{k}, \quad (96)$$

so $\mathbf{U}^*$ is the initial absolute velocity $\mathbf{U}_a$ given by (24) and (25) with $\beta = 0$. Thus, (92) becomes

$$\mathbf{u}_a(t) = -\nabla\Psi^* + \Lambda\nabla S + U_a\nabla X + V_a\nabla Y + W_a\nabla Z. \quad (97)$$

By calculus of variations, we have found for a nontraditional f plane the inviscid isentropic version of (71), the velocity formula obtained by integration.

8. Conservation and circulation theorems

We now show that the formulas are fundamental to conservation of potential vorticity and the circulation theorems. Conservation of Ertel's potential vorticity derives from the vorticity formula. For any scalar fields A, B, and Θ, we have

$$\begin{aligned} \alpha(\nabla A\times\nabla B)\cdot\nabla\Theta &= \alpha_0\frac{\partial(x,y,z)}{\partial(X,Y,Z)}\frac{\partial(A,B,\Theta)}{\partial(x,y,z)} \\ &= \alpha_0\frac{\partial(A,B,\Theta)}{\partial(X,Y,Z)} \end{aligned} \quad (98)$$

with use of (33). Taking the scalar product of (88) with $\alpha\nabla\Theta$, where Θ is any conserved variable, and applying (98) to each term on the right gives

$$\begin{aligned} \alpha\boldsymbol{\omega}(\tau)\cdot\nabla\Theta &= \alpha_0\left[\frac{\partial(\Theta,U)}{\partial(Y,Z)} + \frac{\partial(\Theta,V)}{\partial(Z,X)} + \frac{\partial(\Theta,W)}{\partial(X,Y)} + \kappa_0\frac{\partial\Theta}{\partial Y} + (f_0 + \beta Y)\frac{\partial\Theta}{\partial Z} + \frac{\partial(\Lambda,S,\Theta)}{\partial(X,Y,Z)}\right] \\ &= \alpha_0\left[\left(\frac{\partial W}{\partial Y} - \frac{\partial V}{\partial Z}\right)\frac{\partial\Theta}{\partial X} + \left(\frac{\partial U}{\partial Z} - \frac{\partial W}{\partial X} + \kappa_0\right)\frac{\partial\Theta}{\partial Y} + \left(\frac{\partial V}{\partial X} - \frac{\partial U}{\partial Y} + f_0 + \beta Y\right)\frac{\partial\Theta}{\partial Z} + \frac{\partial(\Lambda,S,\Theta)}{\partial(X,Y,Z)}\right] \\ &= \alpha_0\boldsymbol{\omega}(\tau_0)\cdot\hat{\nabla}\Theta + \alpha_0\frac{\partial(\Lambda,S,\Theta)}{\partial(X,Y,Z)} \end{aligned} \quad (99)$$

when $\mathbf{H} \equiv 0$. In inviscid isentropic flow, setting $\Theta = S$ eliminates the baroclinic term and proves potential vorticity conservation. We may also take the baroclinic term out of the equation by just considering the barotropic vorticity. Setting Θ equal to X, Y, and Z, in turn, gives us the conservation laws

$$\begin{bmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \end{bmatrix} \cdot \alpha \boldsymbol{\omega}_{\mathrm{BT}}(\tau) = \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} \cdot \alpha_0 \boldsymbol{\omega}(\tau_0), \qquad (100)$$

because $\nabla X = \mathbf{e}^1$, etc. Thus, the contravariant components in location space of $\alpha\boldsymbol{\omega}(\tau)$ are equal to the initial components of this vector and hence are conserved following the motion (Salmon 1998, p. 202; Dahl et al. 2014). The effects of vortex-tube stretching and tilting on a parcel are incorporated solely into the changing covariant basis vectors that are attached to the parcel. Incidentally, by (37), we may write (100) as $\mathbf{J}^{-1}\alpha\boldsymbol{\omega}_{\mathrm{BT}}(\tau) = \alpha_0\boldsymbol{\omega}(\tau_0)$, which is simply the inverse of Cauchy's formula.

The baroclinic vorticity term $\nabla\Lambda \times \nabla S$ is a solenoidal vector field and thus has its own vortex tubes. In isentropic flow, the baroclinic vortex lines are the intersections of the isentropic surfaces with the surfaces of constant Λ because $\boldsymbol{\omega}_{\mathrm{BC}}$ is normal to both $\nabla\Lambda$ and ∇S (Davies-Jones 2000). The terms that compose the barotropic vorticity $\boldsymbol{\omega}_{\mathrm{BT}}$ in (85) are all solenoidal and individual solutions of the barotropic vorticity equation. Therefore, these fields [or, more strictly, α times these fields—Salmon (1998, p. 199)] are frozen into the fluid. The vortex lines of the term $\nabla U \times \nabla X$ are the intersections of the surfaces of constant U and constant X, and similarly for the other terms in $\boldsymbol{\omega}_{\mathrm{BT}}$.

Helicity is conserved over a material volume Ξ of homentropic inviscid flow if the volume is made up of closed vortex tubes (zero vorticity flux on the bounding surface Σ). This follows from (71) and (87), which, for homentropic inviscid flow in an inertial frame, reduce to

$$\mathbf{u}(\tau) = U\nabla X + V\nabla Y + W\nabla Z - \nabla\Psi \quad \text{and} \qquad (101)$$

$$\boldsymbol{\omega}(\tau) = \xi_0 \nabla Y \times \nabla Z + \eta_0 \nabla Z \times \nabla X + \zeta_0 \nabla X \times \nabla Y, \qquad (102)$$

where (ξ_0, η_0, ζ_0) is the initial vorticity $\boldsymbol{\omega}(\tau_0)$. The helicity density is

$$\begin{aligned} \mathbf{u}(\tau)\cdot\boldsymbol{\omega}(\tau) &= (U\xi_0 + V\eta_0 + W\zeta_0)\nabla X \cdot (\nabla Y \times \nabla Z) - \nabla\Psi \cdot \boldsymbol{\omega}(\tau) \\ &= \mathbf{U}\cdot\boldsymbol{\omega}(\tau_0)\frac{\partial(X,Y,Z)}{\partial(x,y,z)} - \nabla\cdot[\Psi\boldsymbol{\omega}(\tau)], \end{aligned} \qquad (103)$$

and the helicity is therefore

$$\iiint_{\Xi(\tau)} \mathbf{u}(\tau)\cdot\boldsymbol{\omega}(\tau)\,dx\,dy\,dz = \iiint_{\Xi(\tau_0)} \mathbf{U}\cdot\boldsymbol{\omega}(\tau_0)\,dX\,dY\,dZ - \oiint_{\Sigma(\tau)} \Psi\boldsymbol{\omega}(\tau)\cdot\mathbf{n}\,d\Sigma. \qquad (104)$$

Hence, helicity is conserved if $\boldsymbol{\omega}\cdot\mathbf{n} = 0$ on the bounding surface. The flow then has nonzero helicity only when the vortex tubes of $\nabla u \times \nabla x$, $\nabla v \times \nabla y$, and $\nabla w \times \nabla z$ are interlinked (Moffatt and Tsinober 1992, p. 283).

Eckart's (1960) circulation theorem follows immediately from the velocity (68). For the generic scalar fields A and B,

$$\oint A\hat{\nabla}B\cdot d\mathbf{X} = \oint A\left(\frac{\partial B}{\partial X}dX + \frac{\partial B}{\partial Y}dY + \frac{\partial B}{\partial Z}dZ\right) = \oint A\,dB. \qquad (105)$$

Hence, applying the line-integral operator $\oint_C \cdot d\mathbf{X}$ to (68) and using (105), (23), and (24) yields

$$\oint_{C(\tau)} \mathbf{u}_a \cdot d\mathbf{x} - \oint_{C(\tau_0)} \mathbf{U}_a \cdot d\mathbf{X} = \oint_{C(\tau)} \Lambda\,dS - \int_{\tau_0}^{\tau} d\tilde{\tau}\left[\oint_{C(\tilde{\tau})} \Lambda(\tilde{\tau})\,d\dot{S}(\tilde{\tau}) + \oint_{C(\tilde{\tau})} \mathbf{F}(\tilde{\tau})\cdot d\tilde{x}\right], \qquad (106)$$

which is Eckart's theorem for the circulation around a material curve C (Dutton 1976, p. 374). The barotropic circulation, obtained by setting the right side to zero, is constant, in agreement with Kelvin's circulation theorem.

The circulation theorems can also be obtained from the vorticity (82). The element of vorticity flux is given by the scalar product of the vector element of material area and the vorticity vector. Note that a scalar product is equivalent to a 1×3 row vector times a 3×1 column vector. After integration over a finite material surface area and use of (A4), we obtain

$$\text{Flux} = \iint_{\sigma(\tau)} d\boldsymbol{\sigma}(\tau)\cdot\boldsymbol{\omega}(\tau) = \iint_{\sigma(\tau_0)} d\boldsymbol{\sigma}(\tau_0)\cdot\mathbf{P}^{-1}(\tau,\tau_0)\boldsymbol{\omega}(\tau), \qquad (107)$$

where $d\boldsymbol{\sigma}$ is a row vector, and $\boldsymbol{\omega}$ and $\mathbf{P}^{-1}\boldsymbol{\omega}$ are column vectors. Inserting (82) and then (A4) into (107) yields

$$\iint_{\sigma(\tau)} d\boldsymbol{\sigma}(\tau)\cdot[\boldsymbol{\omega}(\tau)-\nabla\Lambda\times\nabla S] = \iint_{\sigma(\tau_0)} d\boldsymbol{\sigma}(\tau_0)\cdot\left[\boldsymbol{\omega}(\tau_0)+\int_{\tau_0}^{\tau}\mathbf{H}(\tilde{\tau})\,d\tilde{\tau}\right]. \quad (108)$$

Applying (48) at time $\tilde{\tau}$ gives us

$$\nabla A(\tilde{\tau})\times\nabla B(\tilde{\tau}) = \mathbf{P}(\tau,\tau_0)\hat{\nabla}A(\tilde{\tau})\times\hat{\nabla}B(\tilde{\tau}). \quad (109)$$

From (81), $\mathbf{H}$ is the sum of terms of the form $\hat{\nabla}A(\tilde{\tau})\times\hat{\nabla}B(\tilde{\tau})$. By (109) and (A5),

$$\begin{aligned}\iint_{\sigma(\tau_0)} d\boldsymbol{\sigma}(\tau_0)\cdot\int_{\tau_0}^{\tau} d\tilde{\tau}\hat{\nabla}A(\tilde{\tau})\times\hat{\nabla}B(\tilde{\tau}) &= \int_{\tau_0}^{\tau} d\tilde{\tau}\iint_{\sigma(\tau_0)} d\boldsymbol{\sigma}(\tau_0)\cdot\hat{\nabla}A(\tilde{\tau})\times\hat{\nabla}B(\tilde{\tau})\\ &= \int_{\tau_0}^{\tau} d\tilde{\tau}\iint_{\sigma(\tilde{\tau})} d\boldsymbol{\sigma}(\tilde{\tau})\mathbf{P}^{-1}(\tilde{\tau},\tau_0)\cdot\mathbf{P}(\tilde{\tau},\tau_0)\tilde{\nabla}A(\tilde{\tau})\times\tilde{\nabla}B(\tilde{\tau})\\ &= \int_{\tau_0}^{\tau} d\tilde{\tau}\iint_{\sigma(\tilde{\tau})} d\boldsymbol{\sigma}(\tilde{\tau})\cdot\tilde{\nabla}A(\tilde{\tau})\times\tilde{\nabla}B(\tilde{\tau}). \end{aligned} \quad (110)$$

Thus, (108) becomes

$$\begin{aligned}\iint_{\sigma(\tau)} d\boldsymbol{\sigma}(\tau)[\boldsymbol{\omega}(\tau)-\nabla\Lambda\times\nabla S] &= \iint_{\sigma(\tau_0)} d\boldsymbol{\sigma}(\tau_0)\boldsymbol{\omega}(\tau_0)\\ &+\int_{\tau_0}^{\tau} d\tilde{\tau}\iint_{\sigma(\tilde{\tau})} d\boldsymbol{\sigma}(\tilde{\tau})[\tilde{\nabla}\times\mathbf{F}(\tilde{\tau})-\tilde{\nabla}\Lambda(\tilde{\tau})\times\tilde{\nabla}\dot{S}(\tilde{\tau})]. \end{aligned} \quad (111)$$

Stokes' theorem then gives us

$$\oint_{C(\tau)}(\mathbf{u}_a-\Lambda\nabla S)\cdot d\mathbf{x}-\oint_{C(\tau_0)}\mathbf{U}_a\cdot d\mathbf{X} = \int_{\tau_0}^{\tau} d\tilde{\tau}\oint_{C(\tilde{\tau})}[\mathbf{F}(\tilde{\tau})-\Lambda(\tilde{\tau})\tilde{\nabla}\dot{S}(\tilde{\tau})]\cdot d\tilde{\mathbf{x}}, \quad (112)$$

which is (106) again.

In moist convection, dry entropy S is not conserved. It is better to use the $-\alpha\nabla p$ form of the pressure gradient term [see (54)], where α is the reciprocal of the density of the air *and water* system (e.g., Davies-Jones 2015). Then, (112) converts to

$$\oint_{C(\tau)}\mathbf{u}_a\cdot d\mathbf{x}-\oint_{C(\tau_0)}\mathbf{U}_a\cdot d\mathbf{X} = \int_{\tau_0}^{\tau} d\tilde{\tau}\left[\oint_{C(\tilde{\tau})}p(\tilde{\tau})\,d\alpha(\tilde{\tau})+\oint_{C(\tilde{\tau})}\mathbf{F}(\tilde{\tau})\cdot d\tilde{\mathbf{x}}\right]. \quad (113)$$

For shallow convection,

$$\oint_{C(\tilde{\tau})}p(\tilde{\tau})\,d\alpha(\tilde{\tau})\approx\oint_{C(\tilde{\tau})}b(\tilde{\tau})\,dz(\tilde{\tau}), \quad (114)$$

where b is the buoyancy force.

9. Concluding remarks

Equation (71), obtained by integrating the equations of motion, expresses the velocity of a parcel in a general flow with friction and diabatic heating on Grimshaw's (1975) nontraditional β plane. The restriction of this formula to frictionless isentropic flow on a nontraditional f plane is obtained variationally from Hamilton's principle of least action by utilizing the *vital* Lin constraints in an Eulerian framework.

Although the velocity formula is implicit, the rotational velocity is explicit. Thus, the curl of (71) provides the explicit (73) for a parcel's absolute vorticity in general flow. This same formula is obtained by integrating the vector vorticity equation using the propagator $\mathbf{P}$ defined in (52). The propagator method is more powerful than the variational one because it produces a vorticity formula for general flows on a β plane instead of just inviscid isentropic flows on an f plane.

The formulas pertain to individual parcels. If we knew all the trajectories $\mathbf{x}(\mathbf{X},\tau)$, then we would have complete knowledge of the motion of the fluid. Simulations that use the Lagrangian description of fluid would provide this knowledge. However, keeping track of all the parcel trajectories is computationally expensive, and the trajectories may be chaotic, so almost all models use the mathematically simpler Eulerian description. The latter give us a subset of this knowledge: namely, the dependent variables (velocity, entropy, and specific volume) as functions of the Eulerian coordinates. The Eulerian method works with incomplete knowledge (the parcel paths are unknown), because the governing equations are independent of how the parcels are labeled (Salmon 1998, p. 337).

For physical understanding, however, this subset of knowledge is often insufficient. It does not tell us, for instance, how an intense vortex forms in a model simulation. To understand why the vortex is there, we need at least some trajectory information. In particular, we need to find out the paths of the parcels that end up in the vortex and to determine the deformations and the torques that these parcels experience along the way. We can do this by adding passive auxiliary equations to the model. If the diabatic heating and friction is weak, we might be able to diagnose the origin of a concentrated vortex by adding the equations $D\mathbf{X}/Dt = 0$, $D\mathbf{U}/Dt = 0$, $DS/Dt = 0$, and $D\Lambda/Dt = T$ (with $\Lambda = 0$ initially) and marching them forward for a short while from an appropriate starting time using a stable, accurate, high-order, upstream-differencing scheme. The auxiliary equations cannot be integrated for too long, because the surfaces of constant X, etc., become increasingly convoluted. The relative strengths of the solenoids of the variable pairs, Λ and S, U and X, etc., at the time of vortex formation might determine the origin of rotation.

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APPENDIX

The Propagator for a Material Surface Element

Following Dutton (1976, p. 386), we let the material surface be described by the equation $\mathbf{x} = \mathbf{x}(\lambda, \mu, \tau)$ where λ and μ are parameters of the surface, and τ is time. The vector element of area on this surface is

$$d\boldsymbol{\sigma}(\tau) = \frac{\partial \mathbf{x}}{\partial \lambda} \times \frac{\partial \mathbf{x}}{\partial \mu} d\lambda d\mu \quad \text{or}$$

$$d\boldsymbol{\sigma}(\tau) = \left[\frac{\partial(y,z)}{\partial(\lambda,\mu)} \quad \frac{\partial(z,x)}{\partial(\lambda,\mu)} \quad \frac{\partial(x,y)}{\partial(\lambda,\mu)} \right] d\lambda d\mu \qquad \text{(A1)}$$

when written as a row vector. At the initial time τ_0, this area element is given by

$$d\boldsymbol{\sigma}(\tau_0) = \left[\frac{\partial(Y,Z)}{\partial(\lambda,\mu)} \quad \frac{\partial(Z,X)}{\partial(\lambda,\mu)} \quad \frac{\partial(X,Y)}{\partial(\lambda,\mu)} \right] d\lambda d\mu. \qquad \text{(A2)}$$

By the chain rule for Jacobians (Margenau and Murphy 1956, p. 20),

$$\left[\frac{\partial(y,z)}{\partial(\lambda,\mu)} \quad \frac{\partial(z,x)}{\partial(\lambda,\mu)} \quad \frac{\partial(x,y)}{\partial(\lambda,\mu)} \right] = \left[\frac{\partial(Y,Z)}{\partial(\lambda,\mu)} \quad \frac{\partial(Z,X)}{\partial(\lambda,\mu)} \quad \frac{\partial(X,Y)}{\partial(\lambda,\mu)} \right] \times \begin{bmatrix} \frac{\partial(y,z)}{\partial(Y,Z)} & \frac{\partial(z,x)}{\partial(Y,Z)} & \frac{\partial(x,y)}{\partial(Y,Z)} \\ \frac{\partial(y,z)}{\partial(Z,X)} & \frac{\partial(z,x)}{\partial(Z,X)} & \frac{\partial(x,y)}{\partial(Z,X)} \\ \frac{\partial(y,z)}{\partial(X,Y)} & \frac{\partial(z,x)}{\partial(X,Y)} & \frac{\partial(x,y)}{\partial(X,Y)} \end{bmatrix} \qquad \text{(A3)}$$

Thus,

$$d\boldsymbol{\sigma}(\tau) = d\boldsymbol{\sigma}(\tau_0)\mathbf{P}^{-1}(\tau,\tau_0), \qquad \text{(A4)}$$

which shows that $\mathbf{P}^{-1}$ must be the propagator of a material surface element.

At any intermediate time $\tilde{\tau}$,

$$d\boldsymbol{\sigma}(\tau_0) = d\boldsymbol{\sigma}(\tilde{\tau})\mathbf{P}(\tilde{\tau},\tau_0). \qquad \text{(A5)}$$

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