Resonant interactions between Rossby modes in a straight coast and a channel

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We study the possibility of having resonant interactions between three Rossby modes on a coast or channel of arbitrary orientation. A Rossby mode comprises two propagating Rossby waves (RWs) to satisfy the no normal flow through the boundary(ies). In each geometry, we state the conditions, degrees of freedom, and RWs of the primary two modes that could force a third mode. We discuss differences between zonal and non-zonal orientation. Resonant interactions are only possible if all RWs participate in the zonal case, while only three RWs in the non-zonal case. The non-zonality reduces the degrees of freedom to solve the resonance conditions, and solutions are more restrictive for more meridional orientations. In particular, there are no solutions if the coast or channel is meridional. For the non-zonal coast, we find a family of solutions for given periods T_1 and T_2 of the primary modes. Using multiple scales, we obtain a uniformly valid solution of the QG potential vorticity equation (QGPVE), with the resonant modes exchanging energy in space. There are no degrees of freedom for the non-zonal channel, and we develop a graphical method to seek resonant solutions, finding some. We provide a bounded solution of the QGPVE in case the primary modes excite one RW, not a channel mode, and the modes do not exchange energy either in time or space. Regarding possible oceanographic applications, we show solutions for the Hawaiian Ridge and inquire if there are solutions in the Mozambique Channel, Tasman Sea, Denmark Strait, and the English Channel.

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1. Introduction

The interaction of a triad of dispersive waves is a fundamental process in the dynamics of fluid flows; in particular, for geophysical flows, its significance is well established (Craik 1988). In weakly nonlinear wave theories, there is considerable interest in studying resonant interactions because they produce the largest amplitudes when compared to all non-resonant interactions (Pedlosky 2013; Graef 1993; García & Graef 1998). In forced problems, out of all the modes that are excited with an imposed forcing, the dominant mode, ie the one that exhibits the largest response, is the resonant mode (Graef 2016).

Our general interest is to investigate whether or not there is resonance in the weakly nonlinear interaction of Rossby normal modes in different geometries on a β-plane. That is, we are interested in bounded domains. Specifically, in this article, we study the possibility of finding resonant triads of Rossby *modes* in two domains whose orientation is arbitrary:

(i) A straight coast, ie, a domain being infinite in one horizontal direction and semi-infinite in the other horizontal direction;

(ii) A rectilinear channel, ie, a domain being infinite in one horizontal direction and bounded in the other horizontal direction.

The key question to answer here is: Does the nonlinear interaction between two Rossby modes can excite a *third mode*? In other words, is it possible to find resonant triads of Rossby modes in these geometries?

It is essential to distinguish between the self-interaction of a Rossby mode and the interaction between Rossby modes. For instance, in the classical reflection problem of Rossby waves at a straight coast (Pedlosky 2013), a mode is defined as an incident plus the reflected wave, ie, a mode is composed of two propagating Rossby waves. The self-interaction of a mode is the nonlinear interaction between an incoming and outgoing wave (as in Graef 1993; Graef & Magaard 1994). In contrast, the interaction between modes would be, in the simplest case, the nonlinear interaction between two modes, ie, between four propagating waves (two of each mode). In a channel, a Rossby mode is also composed of two propagating Rossby waves (RWs), whereas in a gulf or closed basin, four propagating RWs comprise a mode. Therefore, if the weakly nonlinear interaction between two Rossby modes excites a *third mode*, ie there is resonance among the three modes, two RWs must be excited in the coast or channel, and four RWs in the gulf or closed basin.

Longuet-Higgins & Gill (1967) work on resonant interactions between RWs on the infinite β-plane set the tone for studying this type of interaction between planetary or RWs. Although in previous works Stern (1961) and Kenyon (1964) discussed some special cases of resonant interactions between these waves, Longuet-Higgins & Gill (1967) were the first to establish the general conditions for three waves to resonantly interact. The study of these interactions in an infinite ocean or open regions of the ocean is valid if the wave scales are small compared to the size of the domain, and the waves can travel for a long time before finding a boundary. One could also think that the waves in an open region were generated elsewhere or maybe the product of reflection at one or several boundaries. However, when one or more boundaries limit the flow domain, new restrictions on the motion must be imposed to satisfy the boundary conditions. The boundaries restrict the degrees of freedom in the search for solutions to the resonant conditions. An essential aspect of these problems that has received little attention in the literature is the geometry orientation. Graef (1993) and García & Graef (1998) dealt with resonance in the self-interaction of a single Rossby mode in the reflection problem at a straight wall and a channel, respectively. In these studies, the boundary's orientation plays a crucial role: resonance is possible only if $0 < |\sin\alpha| \leqslant 1/3$, where α is the angle that the coast or channel makes with the circles of latitude (positive clockwise). In the case of a rectangular basin with coasts oriented east-west and north-south, Serrano *et al.* (1995) showed that the self-interaction of a Rossby normal basin mode could not produce resonant forcing, whereas LaCasce & Pedlosky (2004) demonstrated that these modes are vulnerable to baroclinic instability.

As far as we know, the study of resonant interactions between free Rossby modes, which are solutions of the linear problem of reflection at a straight coast or wall, has not been reported. If there are two primary Rossby modes nonlinearly interacting, we could ask the following two questions regarding resonance (aside from their self-interaction). What if the nonlinear interaction between the RWs of modes 1 and 2 produces (A) a free RW?; or (B) a third Rossby mode? It should be evident that problem (A) is less restrictive than (B) and even the self-interaction problem. Indeed, in principle, it is always possible to excite a free RW when considering the interaction between two Rossby modes, regardless of the coastal orientation. However, the Fourier space of the resonance conditions' solutions does vary with α (one could find a few cases, for certain ambient parameters and vertical mode numbers, for which there are no solutions). On the other hand, for problem (B), which is the one we study in this paper, we may anticipate that there will be constraints on the RWs' parameters of the primary modes and α.

The occurrence of resonance between barotropic Rossby modes in a *zonal* channel was studied by Plumb (1977), while Mysak (1978) studied resonant interactions between topographic planetary waves in a continuously stratified fluid in a channel of arbitrary orientation. The first-order linear solution in Mysak's study does not consider the planetary vorticity gradient (the β-effect is zero) and so the solution to this order is valid on the f-plane. Therefore, to our knowledge, the question of whether or not there are resonant interactions between Rossby modes in a channel of arbitrary orientation on the β-plane is still open. To this end, we must first establish the resonance conditions, and after that, we need to investigate if there are solutions.

Furthermore, there have been no studies to analyze the occurrence of resonance between Rossby modes in a gulf or in a rectangular basin arbitrarily oriented on the β-plane. Actually, in their seminal paper, Longuet-Higgins & Gill (1967) said as a final conclusion: "For application to the ocean it is generally desirable to consider planetary waves in closed basins. We know . . . in a rectangular basin on a β-plane . . . construct solutions which consist of the sum of four progressive planetary waves . . . . The possibility exists that for basins of certain size and orientation there may be resonance between three modes of low order. An investigation of this possibility is in progress." It is remarkable that after more than 50 years, the problem of finding resonant modes in a rectangular basin has not been tackled, or at least reported in the literature. The results of this article will hopefully contribute or shed some light on it.

In table 1, we summarize all results regarding the existence of resonance in either the nonlinear self-interaction of a Rossby mode or in the nonlinear interaction among Rossby modes in different geometries. It includes those cases reported in the literature (providing at least one reference), those not done to our knowledge, indicated by a question mark (?), and, finally, the cases that we have done in this article. This exercise, hopefully, serves to put our work in a more general context.

For the coast or channel, a Rossby mode is the superposition of two propagating RWs. Thus, the nonlinear interaction between two Rossby modes in each geometry produces 12 forcing terms, which come about as follows. There are 4 RWs, so 6 interactions since each one's self-interaction is null, and each interaction produces two terms, one with the sum and the other with the difference of the wave phases. For the rectangular gulf or basin, a Rossby mode is the superposition of four propagating RWs. Therefore, two modes' nonlinear interaction involves 8 RWs, so there will be 28 interactions and 56 forcing terms. Of course, if the orientation is zonal, many forcings will vanish. One question is: which of the forcing terms should we consider to form a third Rossby mode? This question is non-trivial because we will need to analyze, among all possible interactions, those that could excite two RWs (or four in the case of a gulf or basin) that precisely form a free Rossby mode for each one of the geometries.

Geometry	Orientation	One mode	Among modes
Unbounded		No	Yes, Longuet-Higgins & Gill (1967)
Coast	Zonal	No	Yes
	Non-zonal	Yes, Graef (1993)	Yes, **this work**
Channel	Zonal	No	Yes, Plumb (1977)
	Non-zonal	Yes, García & Graef (1998)	Yes, **this work**
Gulf	Zonal	No, García & Graef (1998)	?
	Non-zonal	Yes, García & Graef (1998)	?
Basin	Zonal	No, Serrano *et al.* (1995)	?
	Non-zonal	?	?

Table 1. Resonant interactions of Rossby modes in different geometries and their orientation. There is no reference for the zonal coast among modes because the problem is exactly as in Longuet-Higgins & Gill (1967), but this fact was overlooked.

We organize the paper as follows. In the next section, we present general considerations of the problem that apply equally to the straight coast and the channel. In section 3, we analyze which of the forcing terms could produce a third mode for both geometries, pointing out the differences between zonal and non-zonal orientations. The solution of the resonance conditions between three Rossby modes in a non-zonal straight coast is presented in Section 4, both analytically and graphically. Section 5 is devoted to finding solutions to the resonance conditions between three Rossby modes in a non-zonal channel. In these last two sections, we inquire if there are restrictions on the coast(s)' orientation α and comment on possible oceanographic applications. In section 6, we show the QG potential vorticity equation (QGPVE)'s solution for the resonant forcing terms in the coast, where we need to use multiple scales to obtain bounded solutions. In the channel, we could only find a solution in the case of problem (A), in which a coastal mode is excited. Finally, the last section provides a discussion and conclusions.

2. General considerations

Consider a β-plane with a coordinate system (x, y, z) in which x is parallel, y perpendicular to the coast or channel and z vertically upwards (figure 1). For the coast, there is a vertical wall at the plane $y = 0$ and for the channel of width W, there is another vertical wall at the plane $y = W$. The origin is somewhere in a mid-latitude region. The governing equation is the QGPVE, which in this coordinate system reads

$$\left\{\left[\partial_t + J(\psi, \cdot)\right]\left[\nabla^2 + \partial_z(\Gamma^2\partial_z)\right] + \beta(\cos\alpha\,\partial_x + \sin\alpha\,\partial_y)\right\} \quad = 0\ , \tag{2.1}$$

where α is the angle that the coast makes with the circles of latitude (positive clockwise), $J(a, b) \equiv \partial_x a\,\partial_y b - \partial_x b\,\partial_y a$ the Jacobian operator, $\nabla^2 = \partial_x\partial_x + \partial_y\partial_y$, t is the time, ψ is the QG streamfunction, β is the northward gradient of the planetary vorticity and $\Gamma^2(z) \equiv f_0^2/N^2(z)$, where f_0 is the Coriolis parameter and $N(z)$ is the Brunt-Väisälä frequency.

For the coast, the kinematic boundary condition of no normal flow is $\partial_x\psi = 0$ at $y = 0$; and for the channel it is $\partial_x\psi = 0$ at $y = 0, W$. Since the domain is partially open, an explicit mass conservation constraint or time-independent circulation is not required (Pinardi & Milliff 1989). Besides, for the type of solutions we will be considering (a sum of Rossby modes), the coasts' condition implies $\psi = 0$ there. The boundary conditions in z are those for a flat bottom and a rigid lid, ie $\left[\partial_t + J(\psi, \cdot)\right]\partial_z\psi = 0$ at $z = -H, 0$, where H is the constant water depth. These conditions will be automatically satisfied,

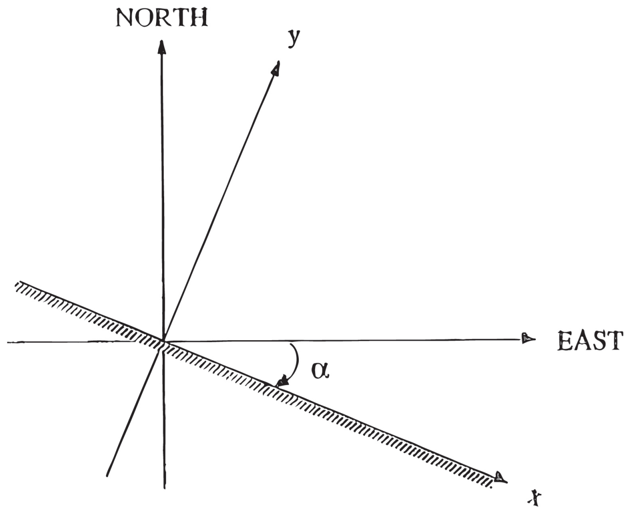

FIGURE 1. Coordinate system. The rotated coordinate system has x parallel and y perpendicular to the coast; α is measured positive clockwise. For the channel of width W, there is another coast at $y = W$.

since the z-dependence of the Rossby modes is given in terms of eigenfunctions $\varphi_{n_j}(z)$ of the familiar vertical Sturm-Liouville problem (Pedlosky 2013).

Without going into the details, the general approach to study the weakly nonlinear interaction between two Rossby modes of a coast or a channel is as follows. One first obtains the non-dimensional version of the QGPVE (2.1) by choosing suitable scaling parameters. There appears a parameter $\varepsilon = U\beta^{-1}L^{-2}$ multiplying the nonlinear terms, which is the β-Rossby number, where U and L are the scales for the horizontal velocity and length. One then assumes $\varepsilon \ll 1$ and writes the solution as a perturbation expansion $= \psi^{(0)} + \varepsilon\psi^{(1)} + \ldots$.

Therefore, mathematically, the problem is to solve the (dimensional) equation:

$$\mathcal{L}\psi^{(1)} = -J\left(\psi^{(0)}, \nabla^2\psi^{(0)} + \partial_z\left[\mathit{\Gamma}^2\partial_z\psi^{(0)}\right]\right) \,, \tag{2.2}$$

where

$$\mathcal{L} \equiv \partial_t\left(\nabla^2 + \partial_z\left[\mathit{\Gamma}^2\partial_z\right]\right) + \beta\left(\cos\alpha\partial_x + \sin\alpha\partial_y\right) \,, \tag{2.3}$$

and $\psi^{(0)}$ is the leading order solution, chosen to be the superposition of any two free Rossby modes for a straight coast or a channel:

$$\begin{aligned} \psi^{(0)} &= \psi_1^{(0)} + {}_2^{(0)} \\ &= \sum_{j=1}^{2} A_j\varphi_{n_j}(z)\left[\cos\left(\theta_{1j}\right) - \cos\left(\theta_{2j}\right)\right] \\ &\equiv {}_{11}^{(0)} - \psi_{21}^{(0)} + {}_{12}^{(0)} - \psi_{22}^{(0)} \,. \end{aligned} \tag{2.4}$$

In the last expression, we have defined the streamfunctions of the four RWs, two of each mode, given by

$$\psi_{ij}^{(0)} = A_j \varphi_{n_j}(z) \cos(\theta_{ij}) \equiv A_j \varphi_{n_j}(z) \cos(k_j x + l_{ij} y - \omega_j t + \vartheta_j)\,, \ j = 1,2;\ i = 1,2\,, \tag{2.5}$$

where for the jth mode, A_j and ϑ_j are (real) amplitude and phase, respectively, k_j is the wavenumber parallel to the coast or channel and ω_j is the frequency; and l_{ij} is the wavenumber perpendicular to the coast or channel of the ith RW of the jth mode.

Our interest is to study the possibility of having resonant interactions between three Rossby modes on a coast or channel of arbitrary orientation. Therefore, we ask whether the forcing of (2.2), ie its RHS, with $\psi^{(0)}$ given by (2.4), could produce a *third mode*, namely,

$$\psi_3^{(1)} = A_3 \varphi_{n_3}(z) \left[\cos(\theta_{13}) - \cos(\theta_{23})\right]\,, \tag{2.6}$$

which is a solution (or free Rossby mode) in the geometry considered.

Of course, each Rossby mode, including the forced mode, must satisfy the relationships

$$2\omega_j l_{0j} + \beta \sin\alpha = 0 \tag{2.7}$$

$$\omega_j \left(k_j^2 + l_{0j}^2 + \Delta_j^2 + \hat{a}_{n_j}^{-2}\right) + \beta\left(k_j \cos\alpha + l_{0j} \sin\alpha\right) = 0, \tag{2.8}$$

or, in compact form, the relation

$$\Delta_j^2 = f_{n_j}(k_j, \omega_j) \equiv \frac{\beta^2}{4\omega_j^2} - \hat{a}_{n_j}^{-2} - \left(k_j + \frac{\beta\cos\alpha}{2\omega_j}\right)^2, \tag{2.9}$$

for $j = 1,2,3$, where $\hat{a}_{n_j}$ is the baroclinic Rossby radius of the n_j vertical mode. We know that the component of the wavenumber vector perpendicular to the wall(s) that form each of the modes, is determined by

$$l_{1j,2j} = l_{0j} \pm \Delta_j\,, \quad j = 1,2,3, \tag{2.10}$$

with l_{0j} given by (2.7). In what follows, we will call l_{1j} the incident wave and l_{2j} the reflected wave of the jth mode [this holds true for all orientations of the straight coast if $\Delta_j > 0$—(see Graef & Magaard 1994)]. Obviously in the case of a channel the terms incident and reflected make no sense; however, this denomination helps us not to introduce new terms and clearly does not lead to confusions.

Finally, we note that upon using some trigonometric identities, the streamfunction of the jth mode [see (2.4)] can be written as

$$\psi_j^{(0)} = -2A_j \varphi_{n_j}(z) \sin(k_j x + l_{0j} y - \omega_j t + \vartheta_j) \sin(\Delta_j y) \tag{2.11}$$

ie the mode is “sort of” a standing wave in the direction perpendicular to the coast or channel (y-direction), but still propagating in the (k_j, l_{0j}) horizontal direction. Also, for a channel, it is $\Delta_j = m_j \pi / W$, where $m_j = 1,2,3,\ldots$ and it is easy to see from (2.11) that $\psi_j^{(0)}$ satisfies the boundary condition at $y = 0$ for the coast, or at $y = 0, W$ for the channel.

3. Which forcings could produce a third mode?

We know that the nonlinear interaction between two waves produces forcing terms with the sum and difference of the wave phases, and that to form a mode we need to have two RWs, of equal wavenumber in the x-direction, same frequency and identical vertical structure. We will now see which of the forcings (produced by the interaction of the waves of the "initial" or primary modes) should we consider to form a third Rossby mode. For both problems (coast and channel), we will point out the difference between the zonal and non-zonal orientation.

3.1. *Forcings produced by the self-interaction of one or both modes*

This case only applies when the geometries are not zonally oriented. First we analyze the forcings produced by the self interaction of both primary modes. As the forced mode must be the sum of two RWs of equal frequency and equal wavenumber component in the x-direction, we obtain that $\omega_3 = 2\omega_1 = 2\omega_2$, and $k_3 = 2k_1 = 2k_2$. Therefore, the modes "initially" considered or primary modes are equal, and this has already been studied by Graef (1993) for the straight coast and by García & Graef (1998) for the channel.

Now we analyze the case in which one of the forcings is produced by the self-interaction of one mode, and the other forcing is produced by the interaction of one of the RWs of one mode with one of the RWs of the other mode. In such situation we get

$$\left.\begin{array}{rcl} \omega_3 &=& 2\omega_1 = \omega_1 \pm \omega_2 \implies \omega_2 = \pm\omega_1 \ , \\ k_3 &=& 2k_1 = k_1 \pm k_2 \implies k_2 = \pm k_1 \ , \end{array}\right\} \tag{3.1}$$

where the $\pm$ sign indicates the sum or difference of the wave phases in the forcing terms produced by the interacting waves. Again the primary modes match, and we are in the previous case. Another possibility from (3.1) arises if we exchange ω_1 and ω_2, so that we consider the self-interaction of mode 2. In such case

$$\left.\begin{array}{rcl} \omega_3 &=& 2\omega_2 = \omega_1 \pm \omega_2 \implies \omega_1 = 3\omega_2 \ , \\ k_3 &=& 2k_2 = k_1 \pm k_2 \implies k_1 = 3k_2 \ , \end{array}\right\} \tag{3.2}$$

where we chose the waves' phase difference, otherwise we are in the case in which the primary modes match. Let's call $\omega_2 = \omega$, then $\omega_1 = 3\omega$ and $\omega_3 = 2\omega$. Then the wavenumbers perpendicular to the coast or channel of mode 3 are:

$$\left.\begin{array}{rcl} l_{13} &=& l_{12} + l_{22} = 2l_{02} \ \text{(self-interaction of mode 2)} \\ l_{23} &=& l_{11} - l_{12} \ . \end{array}\right\} \tag{3.3}$$

If it is a mode, necessarily $l_{13} + l_{23} = 2l_{03} = -\beta \sin\alpha/(2\omega) = l_{02}$, since $\omega_3 = 2\omega$ [in fact from (2.7) it follows that $3l_{01} = l_{02} = 2l_{03}$]. Thus $l_{23} = -l_{02}$, which in combination with the second equation of (3.3) yields $l_{01} = \mathit{\Delta}_2 - \mathit{\Delta}_1$, upon using (2.10). Also $l_{13} - l_{23} = 2\mathit{\Delta}_3 = 3l_{02}$. Thus, between the variables $\mathit{\Delta}_j$, only one is independent, say $\mathit{\Delta}_2$. Therefore, for this particular case in which the frequencies are multiples of ω, we have three equations, one for each mode, ie (2.9) for $j = 1, 2, 3$, and three unknowns: ω, k and $\mathit{\Delta}_2$. If there is a solution for the coast, it is unique (there are no degrees of freedom). For the channel, since $\mathit{\Delta}_j = m_j\pi/W$ must be prescribed, there are two unknowns, the system is incompatible, and there are no solutions. We will not consider this particular case in any further analysis in what follows in this paper. Note, however, that only three RWs participate in exciting, in principle, a third mode.

Thus, it follows from the above considerations that: For a channel, a third Rossby mode can never be excited if we consider the forcing produced by the self-interaction of anyone of the Rossby modes.

3.2. *Forcings produced by the interaction of the four RWs*

Let us take, without loss of generality, the forcing produced by the interaction of the incident waves of each mode and the forcing produced by the interaction of the reflected waves of each one. Thus, the four waves, two of each mode, participate in the formation of a third mode, whose wave parameters are given by

$$\left.\begin{array}{rcl}\omega_3 &=& \omega_1 \pm \omega_2\\ k_3 &=& k_1 \pm k_2\\ l_{13} &=& l_{11} \pm l_{12}\\ l_{23} &=& l_{21} \pm l_{22}\ .\end{array}\right\} \tag{3.4}$$

The sum of the last two relations of (3.4) establishes that

$$l_{03} = l_{01} \pm l_{02}, \tag{3.5}$$

which is trivially satisfied if the coast or channel is zonal ($\sin\alpha = 0$). On the other hand, if the coast or channel are not zonally oriented, (3.5) yields, upon substituting (2.7):

$$(\omega_2 \pm \omega_1)(\omega_1 \pm \omega_2) - \omega_1\omega_2 = 0, \tag{3.6}$$

which is satisfied only if

$$\omega_2 = \frac{1}{2}\left(-1 \pm i\sqrt{3}\right)\omega_1, \tag{3.7}$$

if the sum of the phases is considered; or

$$\omega_2 = \frac{1}{2}\left(1 \pm i\sqrt{3}\right)\omega_1, \tag{3.8}$$

if the difference of the phases is considered (in these solutions for ω_2, the $\pm$ refers obviously to the two roots). From (3.7) or (3.8), product of the sum or difference of the wave phases, one can see that if the frequency of one of the modes is real (as it must be), the frequency of the other is complex, which does not constitute a free Rossby mode. The case $\omega_1 = \omega_2 = 0$ is not possible because we are in the non-zonal orientation $\sin\alpha = 0$, in which stationary currents cannot be solutions of the QGPVE without an external forcing.

Therefore, *for a non-zonally oriented coast or channel, the forcings produced by the interaction between the four RWs of the primary modes can never excite a third mode.*

3.2.1. *Zonal case*

We already saw that the sum $l_{13} + l_{23}$ from (3.4) is trivially satisfied if the coast or channel is zonal. However the difference $l_{13} - l_{23}$ yields $\Delta_3 = \Delta_1 \pm \Delta_2$, which means that a new horizontal structure is produced by the resonant interactions, ie there is "barotropic transfer". Therefore, for the zonal case, the kinematic conditions that must be satisfied for resonance to occur between three Rossby modes are:

$$\left.\begin{array}{rcl}\omega_j\left(k_j^2 + \Delta_j^2 + \hat{a}_{n_j}^{-2}\right) + \beta k_j &=& 0\ , \qquad j = 1, 2, 3\\ \omega_3 &=& \omega_1 \pm \omega_2\\ k_3 &=& k_1 \pm k_2\\ \Delta_3 &=& \Delta_1 \pm \Delta_2\end{array}\right\} \tag{3.9}$$

These conditions are identical to those posed by Longuet-Higgins & Gill (1967) in their study on resonant interactions between barotropic planetary waves. However, our case is a generalization of that work, since here we consider a continuously stratified ocean and the coupling between the vertical structure of the modes. Incidentally, we should mention the work by Vanneste (1995), who treated the nonlinear interaction among normal modes in a multilayer QG (zonal) channel.

In general, there are six equations and twelve variables: ω_j, k_j, Δ_j and n_j. The last three (the n_j) must be specified, and therefore we end up with a system with three degrees of freedom. It is convenient to note that the variables that define the third Rossby mode, except for its vertical structure n_3, may not be taken into account to determine the degrees of freedom of the resonance conditions. In such case the last three relations of (3.9) are eliminated, to obtain the system

$$\left.\begin{array}{rcl} \omega_1\left(k_1^2+\Delta_1^2+a_{n_1}^{-2}\right)+\beta k_1 &=& 0 \\ \omega_2\left(k_2^2+\Delta_2^2+a_{n_2}^{-2}\right)+\beta k_2 &=& 0 \\ (\omega_1\pm\omega_2)\left[(k_1\pm k_2)^2+(\Delta_1\pm\Delta_2)^2+a_{n_3}^{-2}\right]+\beta\,(k_1\pm k_2) &=& 0\,. \end{array}\right\} \tag{3.10}$$

Now we have three equations and nine unknowns, but when we specify the discrete variables n_j, we get a system with three degrees of freedom.

For a channel of constant width W, however, the variables $\Delta_1 = m_1\pi/W$ and $\Delta_2 = m_2\pi/W$ need to be specified. Thus, the system (3.10) has only one degree of freedom. This case is similar to the study of Plumb (1977).

Finally, we note the following fact. In the *zonal* case, and this is true for the coast or channel, if the nonlinear interaction between one RW of mode 1 and one RW of mode 2 excites a free RW, ie if for example $\{\psi_{11}^{(0)},\psi_{12}^{(0)},\psi_{13}^{(0)}\}$ form a resonant triad, then it follows that the interaction between the other RW of mode 1 and the other RW of mode 2, also forces another free RW, ie $\{\psi_{21}^{(0)},\psi_{22}^{(0)},\psi_{23}^{(0)}\}$ also form a resonant triad; and further, these two new waves form a third mode. In other words, the forcing of a third mode occurs automatically. This does not happen in the non-zonal case. Therefore, the zonal orientation is less restrictive to find resonance among modes.

3.3. *Forcings produced by the interaction of three RWs*

Let us now consider the forcing that is produced by the interaction of one of the RWs of one mode with the two RWs of the other mode. In that case, without loss of generality, we have

$$\left.\begin{array}{rcl} \omega_3 &=& \omega_1\pm\omega_2 \\ k_3 &=& k_1\pm k_2 \\ l_{13} &=& l_{11}\pm l_{12} \\ l_{23} &=& l_{11}\pm l_{22} \end{array}\right\} \tag{3.11}$$

The sum of the last two relations of (3.11) yields

$$l_{03} = l_{11} \pm l_{02} \tag{3.12}$$

$$= l_{01} + \Delta_1 \pm l_{02}\ , \tag{3.13}$$

which in terms of the frequencies, ie using (2.7), is

$$\Delta_1 = \left(\frac{\pm\omega_3^2-\omega_1\omega_2}{2\omega_1\omega_2\omega_3}\right)\beta\sin\alpha\ . \tag{3.14}$$

Equation (3.14) that relates ω_1, ω_2 and Δ_1, is additional to the three equations (one for each Rossby mode), and distinguishes the non-zonal case from the zonal case. It also reduces the degrees of freedom.

If the coast or channel is zonally oriented, from (3.14) it follows that $\Delta_1 = 0$, but this implies that $l_{11} = l_{21} = 0$, ie only one RW with the group velocity parallel to the coast and whose solution is $\sim y\cos(kx-\omega t)$, physically there is no reflection; and for the channel it means that there is no mode 1 (see Graef 2017). Thus, the interaction of three RWs cannot produce a third mode in the zonal case.

On the other hand, the difference of the last two relations of (3.11) yields

$$l_{13} - l_{23} = \pm\,(l_{12}-l_{22}) \quad\Longrightarrow\quad \Delta_3 = \pm\Delta_2\ . \tag{3.15}$$

Therefore, the horizontal structure of the "standing" part of the forced mode is identical to that of the mode whose two RWs participate in the interaction (mode 2 in this case). Resonant interactions do not produce new horizontal structure in the non-zonal case.

From the results obtained above it follows that:

(i) If the coast or channel is zonally oriented, we need the participation or interaction of the four RWs, two of each mode, to excite a third Rossby mode that can resonantly interact with the modes that originate it.

(ii) If the coast or channel is not zonally oriented, only three waves (of the four RWs) can participate in exciting, in principle, a third mode that can resonantly interact with the modes that originate it.

(iii) Only in the zonal case a new horizontal structure is created, ie there is "barotropic transfer".

In the non-zonal case, the kinematic conditions for resonance to occur between three Rossby modes can be written as:

$$\left(k_1 + \frac{\beta\cos\alpha}{2\omega_1}\right)^2 + \mathit{\Delta}_1^2 - \frac{\beta^2}{4\omega_1^2} + \hat{a}_{n_1}^{-2} = 0 \tag{3.16}$$

$$\left(k_2 + \frac{\beta\cos\alpha}{2\omega_2}\right)^2 + \mathit{\Delta}_2^2 - \frac{\beta^2}{4\omega_2^2} + \hat{a}_{n_2}^{-2} = 0 \tag{3.17}$$

$$\left[(k_1 \pm k_2) + \frac{\beta\cos\alpha}{2\left(\omega_1 \pm \omega_2\right)}\right]^2 + \mathit{\Delta}_2^2 - \frac{\beta^2}{4\left(\omega_1 \pm \omega_2\right)^2} + a_{n_3}^{-2} = 0 \tag{3.18}$$

$$\mathit{\Delta}_1^2 - \frac{\left[\left(\omega_1 \pm \omega_2\right)^2 \mp \omega_1\omega_2\right]^2}{4\omega_1^2\omega_2^2\left(\omega_1 \pm \omega_2\right)^2}\beta^2\sin^2\alpha = 0 \tag{3.19}$$

Thus, unlike the zonal case, in the non-zonal case we have a system with nine unknowns: k_1, k_2, $\mathit{\Delta}_1$, $\mathit{\Delta}_2$, n_1, n_2, n_3, ω_1, and ω_2, but four equations. Once we specify the n_j, we have a system with two degrees of freedom. For a channel of width W, where $\mathit{\Delta}_1 = m_1\pi/W$ and $\mathit{\Delta}_2 = m_2\pi/W$ need to be specified, the system (3.16)–(3.19) is compatible and determined; that is to say, there are no degrees of freedom. If a solution exists, it is unique.

The solutions of (3.16)–(3.19), for both geometries, will be discussed in the next two sections.

4. Resonant interactions of Rossby modes in a straight coast

We will only treat the non-zonal orientation since, as discussed before, the case of a zonal coast is identical to the work done by Longuet-Higgins & Gill (1967). The resonant conditions (3.16)–(3.19) can be rewritten as:

$$\mathit{\Delta}_1^2 = f_{n_1}\left(k_1, \omega_1\right) \tag{4.1}$$

$$\mathit{\Delta}_2^2 = f_{n_2}\left(k_2, \omega_2\right) \tag{4.2}$$

$$\mathit{\Delta}_2^2 = f_{n_3}\left(k_1 \pm k_2, \omega_1 \pm \omega_2\right) \tag{4.3}$$

$$\mathit{\Delta}_1^2 = g\left(\omega_1, \omega_2\right), \tag{4.4}$$

where

$$f_n\left(k, \omega\right) \equiv \frac{\beta^2}{4\omega^2} - \hat{a}_n^{-2} - \left(k + \frac{\beta\cos\alpha}{2\omega}\right)^2 \tag{4.5}$$

and

$$g(\omega_1,\omega_2) \equiv \frac{\left[(\omega_1 \pm \omega_2)^2 \mp \omega_1\omega_2\right]^2}{4\omega_1^2\omega_2^2(\omega_1 \pm \omega_2)^2}\,\beta^2 \sin^2\alpha\ . \tag{4.6}$$

Equating (4.1) and (4.4) to eliminate $\mathit{\Delta}_1$, we get a quadratic in k_1:

$$\begin{aligned} &4\omega_1^2\omega_2^2\omega_3^2k_1^2 + 4\omega_1\omega_2^2\omega_3^2\beta(\cos\alpha)\,k_1 + \omega_3^4\beta^2\sin^2\alpha + \omega_2\omega_3^2\times \\ &\quad\left[4\omega_1^2\omega_2\hat{a}_{n_1}^{-2} - (\omega_2 \pm 2\omega_1)\,\beta^2\sin^2\alpha\right] + \omega_1^2\omega_2^2\beta^2\sin^2\alpha = 0\ , \end{aligned} \tag{4.7}$$

where the variable ω_3 has been left in (4.7) for simplicity. Solving for k_1, after substituting ω_3 by $\omega_1 \pm \omega_2$, and some algebra and simplifications, we obtain

$$k_1^{(1,2)} = -\frac{\beta\cos\alpha}{2\omega_1} \pm \frac{1}{2}\left[\beta^2\left(\frac{\cos^2\alpha}{\omega_1^2} - \frac{\sin^2\alpha}{\omega_2^2}\right) - 4\hat{a}_{n_1}^{-2} - \frac{\beta^2\sin^2\alpha}{(\omega_1 \pm \omega_2)^2}\right]^{1/2}\ . \tag{4.8}$$

Thus, there are two roots or solutions: $k_1^{(1)}$ and $k_1^{(2)}$, corresponding to the $+$ and $-$ in front of $\frac{1}{2}[\ldots]^{1/2}$, respectively, for the phase sum $(\omega_1 + \omega_2)$, or for the phase difference $(\omega_1 - \omega_2)$. We could not find a condition that only involves the coast orientation α to have $k_1^{(1,2)}$ real. However, it is easy to see that there are no real solutions for a meridional coast $(\alpha = \pi/2)$. The real solutions are restricted to more zonally oriented coasts. We need real wavenumbers parallel to the coast, otherwise, the solution blows up as $x \longrightarrow \pm\infty$. A necessary condition to have $k_1^{(1,2)}$ real is:

$$|\sin\alpha| \leqslant \left[\frac{(1 \pm r)^2\,r^2}{(1 + r^2)(1 \pm r)^2 + r^2}\right]^{1/2} \tag{4.9}$$

where $r = \omega_2/\omega_1 = T_1/T_2$ and $T_1 = 2\pi/\omega_1$, $T_2 = 2\pi/\omega_2$ are the primary modes' periods. This condition is in terms of $|\sin\alpha|$ as in previous works (Graef 1993; García & Graef 1998), and one can easily see special cases. For example, if $r = 1$ (initial modes have equal frequency) it reduces to $|\sin\alpha| \leqslant 2/3$ [see (4.11) below] and if $r = 2$ (ie $\omega_2 = 2\omega_1$) $|\sin\alpha| \leqslant 6/7$.

Figure 2 shows the function $X_\pm(r,\alpha) = |\sin\alpha|^2 - (1 \pm r)^2\,r^2/[(1 + r^2)(1 \pm r)^2 + r^2]$ in which the yellow regions are prohibited $(X_\pm > 0)$; note the region around a meridional coast $(\alpha = 90°)$. If $k_1^{(1,2)}$ are real then r and α must be in the green and blue regions where $X_\pm < 0$. Large values of r or $T_1 \gg T_2$ favour real solutions for more meridionally oriented coasts [$\alpha \in (70, 85)$ or $\alpha \in (95, 110)$ degrees].

To complete the story, however, we still need to calculate the wavenumber k_2 of the second mode. This is accomplished by equating (4.2) and (4.3) to eliminate $\mathit{\Delta}_2$, but this time the term k_2^2 drops out, and we get a linear equation in k_2:

$$\begin{aligned} \left(\pm 2k_1 \pm \frac{\beta\cos\alpha}{\omega_1 \pm \omega_2} - \frac{\beta\cos\alpha}{\omega_2}\right)k_2 &= \frac{\beta^2\sin^2\alpha}{4}\left[\frac{1}{(\omega_1 \pm \omega_2)^2} - \frac{1}{\omega_2^2}\right] + \\ &\qquad \hat{a}_{n_2}^{-2} - \hat{a}_{n_3}^{-2} - k_1^2 - \frac{\beta\cos\alpha}{\omega_1 \pm \omega_2}k_1\ . \end{aligned} \tag{4.10}$$

From (4.10) we can easily solve for k_2 and substitute the roots $k_1^{(1,2)}$ to obtain $k_2^{(1,2)}$ for either the sum or phase difference. It is worth remarking that both (4.8) and (4.10) are necessary conditions to have solutions of the system (4.1)–(4.4). That is, with the roots $k_1^{(1,2)}$ we have to go back to (4.1) to calculate $\mathit{\Delta}_1^2$; similarly with $k_2^{(1,2)}$ we go back to (4.2) or (4.3) to calculate $\mathit{\Delta}_2^2$. Thus, the whole solution is obtained.

In the previous section, we showed that we have two degrees of freedom in this problem.

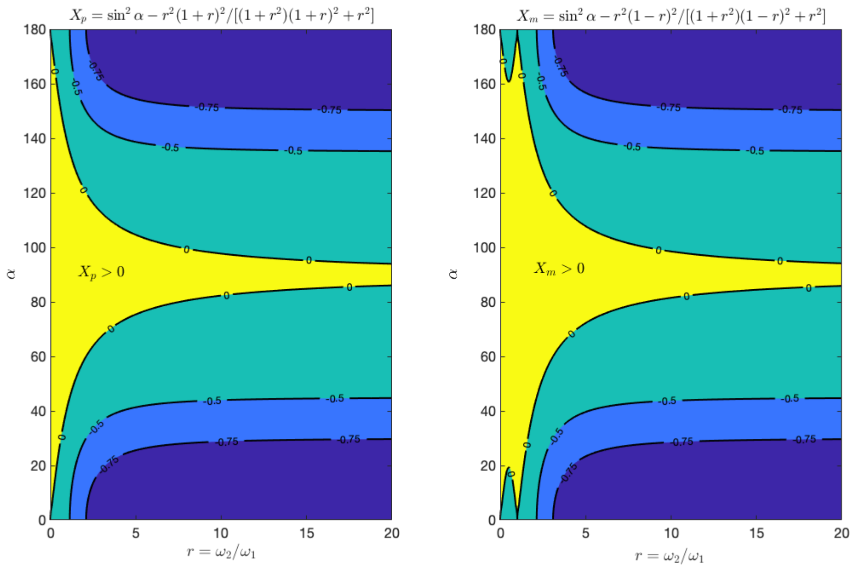

FIGURE 2. The function $X_{\pm}(r,\alpha)$, where $r = \omega_2/\omega_1$ and α is the angle between the eastern direction and the coast (in degrees). If $k_1^{(1,2)}$ are real, then r and α must be in the green and blue regions $X_{\pm} < 0$. Yellow regions have $X_{\pm} > 0$, for which $k_1^{(1,2)}$ are complex. Left panel is X_+; right panel is X_-.

Given the frequencies of the primary modes ω_1 and ω_2, we can get the wavenumbers along the coast of the first mode $k_1^{(1,2)}$ and second mode $k_2^{(1,2)}$, for either the sum or phase difference of the interacting RWs. Thus, for each ω_1 and ω_2, there are two solutions $k_{1p}^{(1,2)}$ for the phase sum and two solutions $k_{1m}^{(1,2)}$ for the phase difference.

In figure 3 we show the real solutions $k_{1p,m}^{(1,2)}$ as a function of the modes' periods T_1 and T_2 for values appropriate for the Hawaiian Ridge: reference latitude $\phi_0 = 21°$ and $\alpha = 25°$; we choose a first baroclinic mode $n_1 = 1$ for Rossby mode 1. Note that the (T_1,T_2) space of real solutions is more restrictive ($T_1 > T_2$) for the phase difference than for the phase sum. Due to (4.10), if k_1 is complex, then k_2 is complex. Thus the white regions of figure 3 will be exactly the same for the wavenumber k_2 of the second mode.

To give an idea of the Rossby waves of each mode of the resonant triad, we calculate their wavelengths as a function of T_1 and T_2 for values of the Hawaiian Ridge and vertical mode numbers $n_1 = 1$, $n_2 = 1$ and $n_3 = 2$ (see figures 4, 5, 6 and 7). A few notes about these four figures are in order. First, the allowed (T_1,T_2) space is reduced further for the wavelengths (as compared to the one for k_1 of figure 3) because we only permit solutions that yield real wavenumber components perpendicular to the coast (otherwise the solution blows up as $y \longrightarrow \infty$). That is, the fact that the k's are real does not guarantee that the l's are real, so when calculating the l's, we must require $\Delta_2^2 > 0$ [see (2.9) and (2.10)]; note that $\Delta_1^2 > 0$ by (4.4) and (4.6) and we have $\Delta_3^2 = \Delta_2^2$. Therefore, the approach to correctly understand figures 4–7 is to choose the periods (T_1, T_2) such that they fall on coloured regions in *all 6 panels* of each figure. Figures 4 and 5 show the wavelengths of the incident and reflected RW of the three modes corresponding to the solution $k_{1p}^{(1)}$ and $k_{1m}^{(1)}$, respectively. For the phase sum $\omega_1 + \omega_2$ (figure 4), the range of wavelengths for the first mode is $\lesssim 1000$ km for the incident RW (note the white wedge

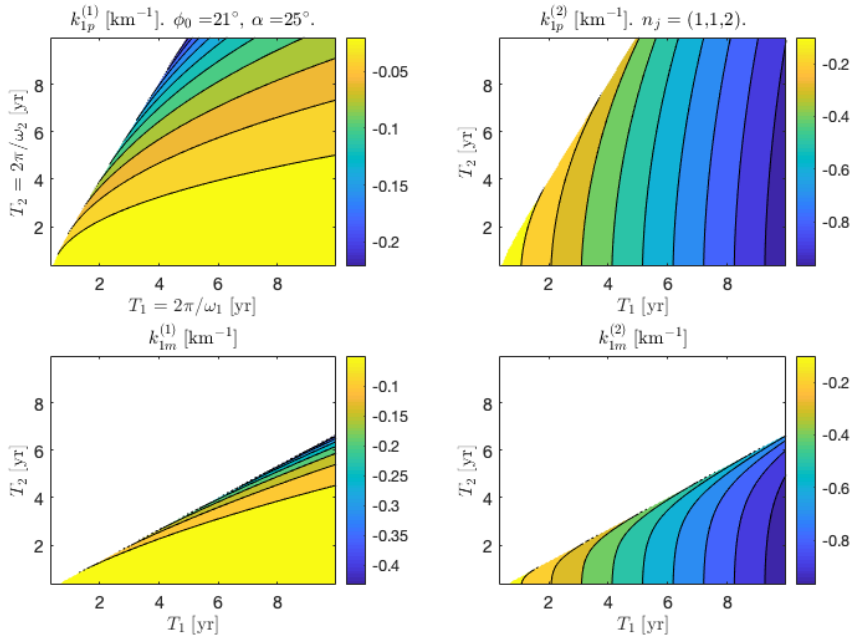

FIGURE 3. The solutions for the wavenumbers $k_1^{(1,2)}$ from (4.8) as a function of the mode periods T_1 and T_2 in years. Upper (lower) panels correspond to the phase sum (difference), left (right) panels are $k_1^{(1)}$ ($k_1^{(2)}$). The white regions yield complex solutions. Reference latitude $\phi_0 = 21°$, $\alpha = 25°$, which are values appropriate for the Hawaiian Ridge; $n_1 = 1$.

in modes 2 and 3) and $\lesssim 50$ for the reflected RW; for the second mode the range is [100,240] km and [20,120] km, respectively; and for the third mode it is [100,1400] km and [$\lesssim 50$,200] km, respectively. Note, though, that in general the space for the larger wavelengths is squeezed in a very small region. For the phase difference $\omega_1 - \omega_2$ (figure 5), the range of wavelenghts is: $\lesssim 1000$ (note the small white wedge in modes 2 and 3 for very small T_2) and [$\lesssim 20$,100]; [$\lesssim 50$,200] and [20,140]; and [$\lesssim 100$,2000] and [$\lesssim 20$,120], for the incident and reflected and for modes 1, 2 and 3, respectively.

Figures 6 and 7 show the wavelengths corresponding to the solution $k_{1p}^{(2)}$ and $k_{1m}^{(2)}$, respectively. It is noteworthy the dramatic reduction in allowable (T_1,T_2) space for the solution superscript (2). This is mainly due to the fact that for western coasts facing north, such as the Hawaiian Ridge, $\alpha \in (0, 90)$ degrees, $\cos\alpha > 0$ and $|k_1^{(2)}| > |k_1^{(1)}|$ [see (4.8)], so that in general $|k_2^{(2)}| > |k_2^{(1)}|$, making Δ_2^2 negative in a much larger region of the (T_1,T_2) space, thus reducing the space for real l's. The real solutions for both k and l lie only within the very tiny region (resembling a slice of a pie), with $T_2 > T_1$ for solution $k_{1p}^{(2)}$ and $T_1 > T_2$ for $k_{1m}^{(2)}$. In both figures all the wavelengths are small: they range approximately between 20 and 200 km.

We produced figures 4–7 for a reference latitude $\phi_0 = 21°$ and a coastal orientation $\alpha = 25°$, which are values appropriate for the Hawaiian Ridge. We conclude that, in this case, the nonlinear interaction between two $n_1 = 1$ (first-mode baroclinic) annual Rossby modes cannot excite a semi-annual $n_3 = 2$ Rossby mode. However, if instead, we consider that the third or excited mode is barotropic with a free surface $n_3 = 0$ (depth $H = 4000$ m), then those annual modes can resonantly interact to force a semi-annual mode (not shown here).

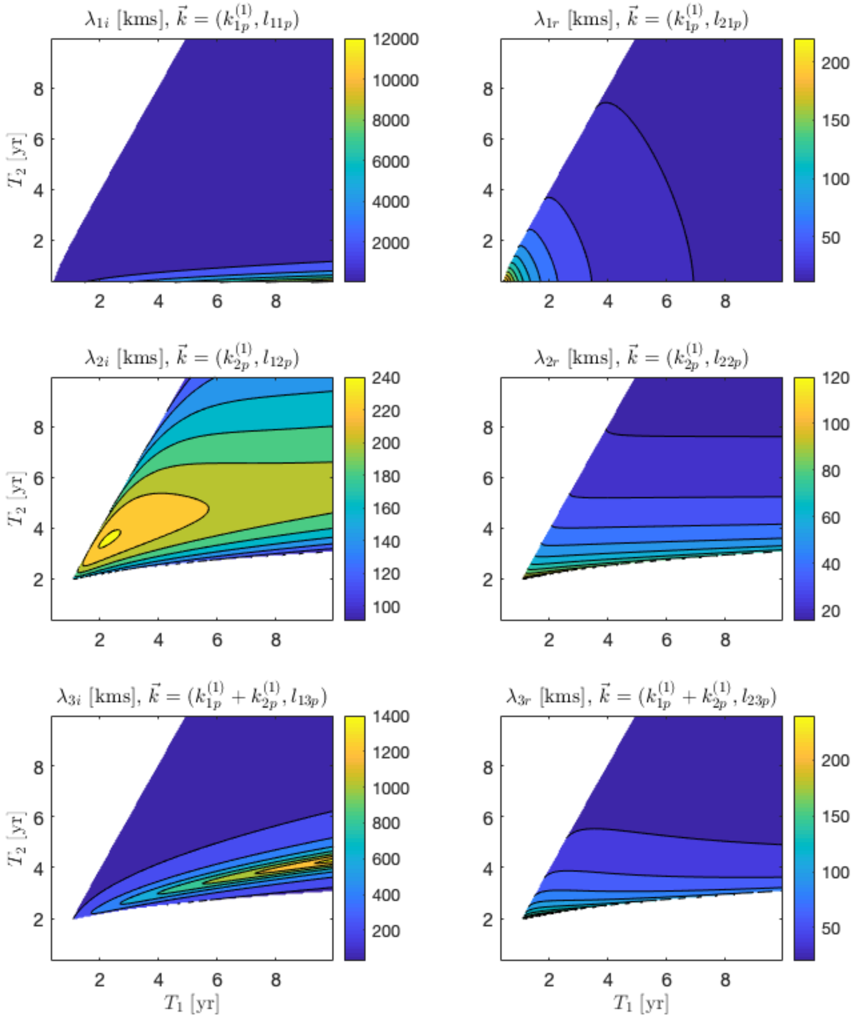

FIGURE 4. Wavelengths (in km) of the incident (left panels) and reflected (right panels) Rossby waves of mode 1 (upper panels), mode 2 (middle panels) and mode 3 (lower panels) corresponding to the solution $k_{1p}^{(1)}$ as a function of the mode periods T_1 and T_2 in years. ϕ_0 and α appropriate for the Hawaiian Ridge and the vertical mode numbers are $n_1 = 1$, $n_2 = 1$, $n_3 = 2$.

A general characteristic emerges by looking at different coastal orientations: the (T_1,T_2) space of real solutions is smaller for the phase difference than for the phase sum.

4.1. *Modes of equal frequency*

If the initial modes have equal frequencies, the number of variables is reduced by one (from 6 to 5), but the number of equations remains the same (4). There is still one degree of freedom, and we can exploit it to examine the possibilities to find resonance easily. This case is compelling because of its similarity to resonance occurring in the self-interaction of a Rossby mode (Graef 1993).

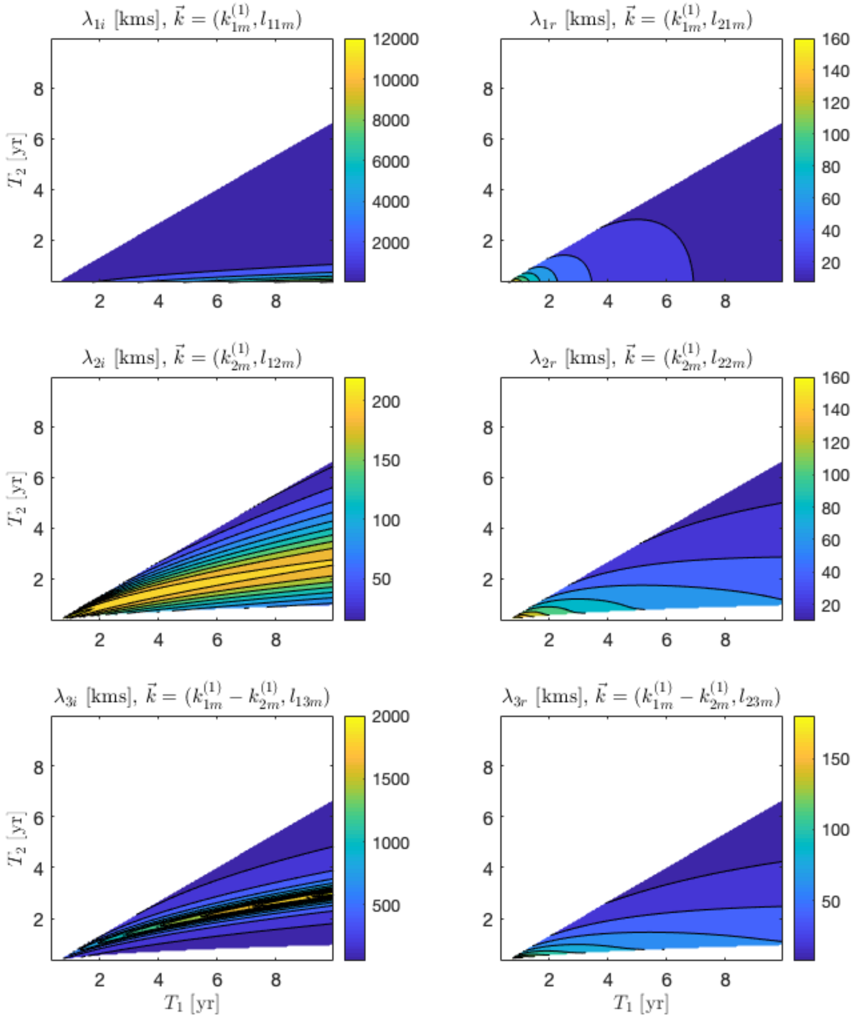

FIGURE 5. As in figure 4, but for the solution $k_{1m}^{(1)}$.

For $\omega_1 = \omega_2 = \omega$, the solution (4.8), which only makes sense for the sum of the phases, is given by

$$k_1^{(1,2)} = -\frac{\beta \cos \alpha}{2\omega} \pm \left[\frac{\beta^2}{4\omega^2} \left(1 - \frac{9}{4} \sin^2 \alpha \right) - \hat{a}_{n_1}^{-2} \right]^{1/2} . \tag{4.11}$$

It is obvious that to have $k_1^{(1,2)}$ real it is necessary that $|\sin \alpha| \leqslant 2/3$. Again, the orientation of the coast or wall imposes a restriction for resonance to occur. We note that this value (of $|\sin \alpha|$) is twice that obtained by Graef (1993) when considering the self-interaction of a Rossby mode in a coast.

As can be observed from figure 4, there are solutions for $T_1 = T_2$ (ie $\omega_1 = \omega_2$) because a good part of the diagonal straight line lies within the coloured regions of all panels.

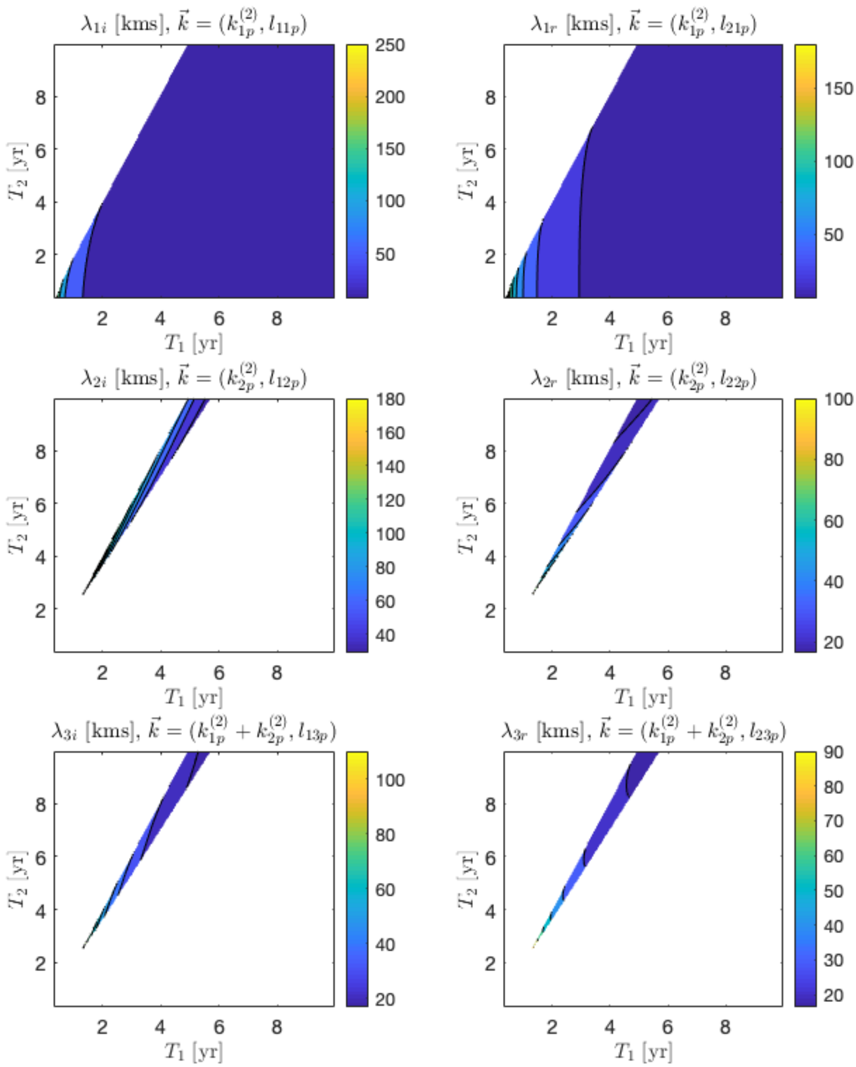

FIGURE 6. As in figure 4, but for the solution $k^{(2)}_{1p}$.

But there are no solutions $\omega_1 = \omega_2$ for figure 6, since the diagonal is outside the coloured regions for modes 2 and 3.

5. Resonant interactions of Rossby modes in a channel

In a channel, we already showed that there are no degrees of freedom. Once the 5 discrete variables (ie the three vertical mode numbers n_j, $j = 1, 2, 3$ and the two horizontal mode numbers m_1 and m_2) are specified, the kinematic conditions (3.16)–(3.19) or (4.1)–(4.4) form a closed system for the four unknowns: ω_1, ω_2, k_1 and k_2. If

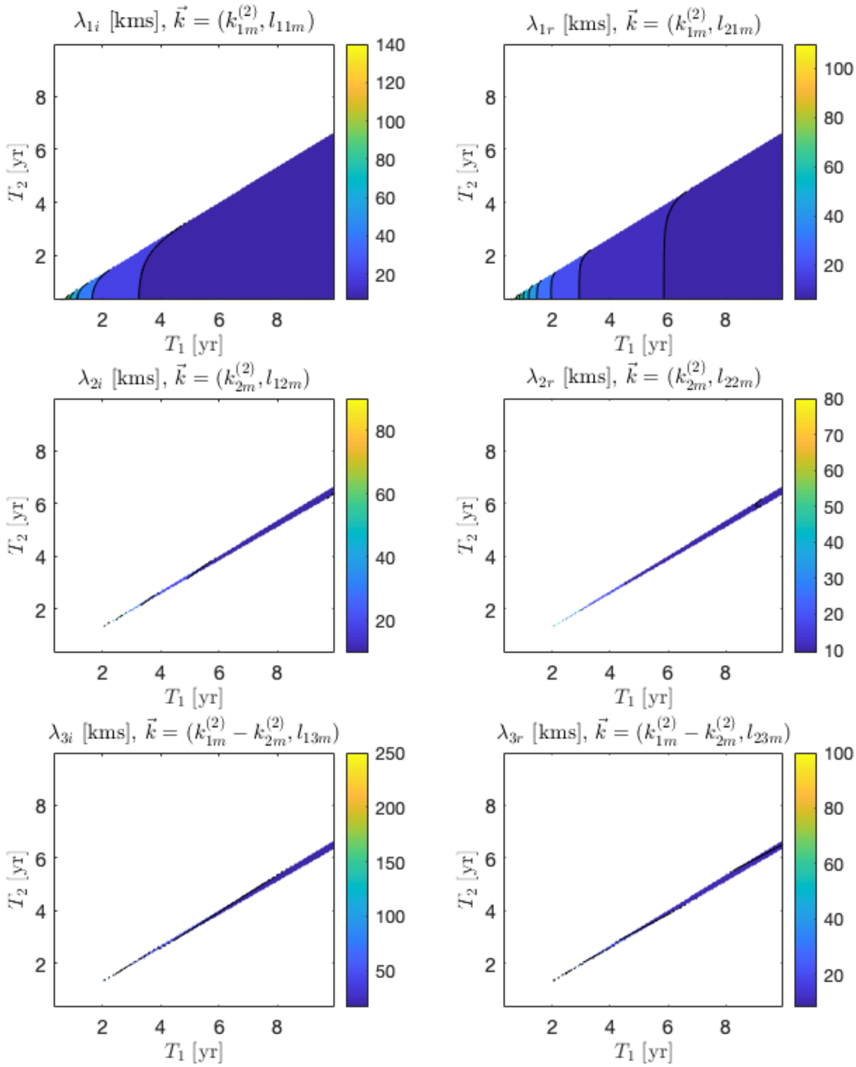

FIGURE 7. As in figure 4, but for the solution $k_{1m}^{(2)}$.

a solution exists, it is unique. The presence of a second boundary, as compared to the straight coast case (only one boundary), makes it a much more restrictive problem.

We tried but did not succeed in arriving at a single equation for any one of the four unknowns. However, using the solutions for the straight coast (4.8) and (4.10), we developed the following graphical method to seek for solutions:

(1)First, we give the mode number m_1 (ie Δ_1) and ω_1. Then from (4.4) we solve for ω_2, yielding:

$$\pm\omega_2^2 + \omega_1\omega_2 \pm \omega_1 \left(\frac{1}{\omega_1} - \frac{2\Delta_1}{\beta \sin\alpha}\right)^{-1} = 0\,, \tag{5.1}$$

whose solution is

$$\omega_2 = \mp\frac{\omega_1}{2} \pm \left[\frac{\omega_1^2}{4} - \omega_1\left(\frac{1}{\omega_1} - \frac{2\Delta_1}{\beta\sin\alpha}\right)^{-1}\right]^{1/2} \tag{5.2}$$

in which, as usual, the $\mp$ in front of $\omega_1/2$ corresponds to the RWs' phase sum (upper sign) and difference (lower sign), and the $\pm$ in front of the square root refers to the roots of ω_2. A necessary and sufficient condition to have the frequency ω_2 real is $2\Delta_1\omega_1 > \beta\sin\alpha$, ie $T_1 < 4\pi\Delta_1/(\beta\sin\alpha)$. This condition [which could be derived by noting that for a non-zonal channel, $\alpha \in (0,\pi)$ covers all possible orientations so that $\sin\alpha > 0$] imposes a restriction on large periods for the first mode, but at the same time from the Rossby mode dispersion relation, equations (4.1) and (4.5), we need to have $\beta > 2\omega_1\Delta_1$ or $T_1 > 4\pi\Delta_1/\beta$. The conditions are opposed, showing us how restrictive it would be to find real solutions.
Now, using (4.8), upon substituting (5.2), we draw the curves $k_1 = \mathcal{F}(\omega_1)$ [there will be four curves corresponding to the two roots $k_1^{(1,2)}$ and the two roots of (5.2) for the phase sum, and other four curves for the phase difference, eight curves total].
(2)From (4.10) we have k_2 as a function of k_1. Draw the curve $k_2 = \mathcal{G}(k_1) = \mathcal{G}[\mathcal{F}(\omega_1)]$, ie k_2 as a function of ω_1 only.
(3)Now k_2 of step 2, for it to be a solution, must also satisfy (4.2) or (3.17), which is the equation for mode 2, quadratic in k_2. That is, given m_2 (ie Δ_2) and substituting ω_2 from (5.2) of step 1 into (3.17), we could draw the curve $f_{n_2}(k_2,\omega_2) = \Delta_2^2$ of this mode for each ω_1.
(4)The intersections of the curves of step 2 and step 3, if there are, are the solutions for k_2 (it could be for more than one frequency ω_1 if there is more than one intersection).
(5)The solutions for k_1 would correspond to the same abscissas ω_1 at which the curves for k_2 intersect, but on the curve of step 1: $k_1 = \mathcal{F}(\omega_1)$.

In figures 8, 9 and 10 we show an example of the graphical method just described, where we have chosen the period of the first mode T_1 as the independent variable instead of the frequency ω_1. The chosen parameters are: $\phi_0 = 20°$, $\alpha = 15°$, channel width $W = 500$ km, horizontal mode numbers $m_1 = 2$, $m_2 = 1$ (recall $m_3 = \pm m_2$) and vertical mode numbers $n_j = (0,0,0)$, ie a fully barotropic case with a free surface and a depth $H = 4000$ m. Figure 8 shows solution (5.2) in terms of the periods, ie T_2 as a function of T_1. There are four curves, two in each panel, which correspond to the positive (blue) and negative (red) root of ω_2 (or T_2). The upper (lower) panel refers to the RWs' phase sum (difference). Note that, for the chosen parameters, T_1 cannot be larger than 0.9 years [recall the restriction $4\pi\Delta_1/\beta < T_1 < 4\pi\Delta_1/(\beta\sin\alpha)$].

As regards to the solution of the resonance conditions, we observe that for the phase sum (figure 9), there is only one solution, since the k_2-curves of steps 2 and 3 (blue and red, respectively) intersect in one panel only (upper left). Such solution corresponds to the along channel wavenumbers $k_{1p1}^{(1)}$ of mode 1 and $k_{2p1}^{(1)}$ of mode 2, where the additional subscript (1 or 2) in k_1 and k_2 refers to the (+ or −) root of ω_2 in (5.2).

On the other hand, for the phase difference (figure 10), there are three solutions, since the k_2-curves of steps 2 and 3 (blue and red, respectively) intersect in three panels (upper left, upper right and lower right), corresponding to solutions $(k_{1m1}^{(1)}, k_{2m1}^{(1)})$, $(k_{1m1}^{(2)}, k_{2m1}^{(2)})$ and $(k_{1m2}^{(2)}, k_{2m2}^{(2)})$, respectively. However, the solutions of the upper and lower right panels represent the same Rossby modes (same mode parameters), but with mode 2 in one panel being mode 3 in the other panel, and vice versa. This can be seen by realizing that the solutions of these panels have identical T_1 (the blue and red curves intersect at the same

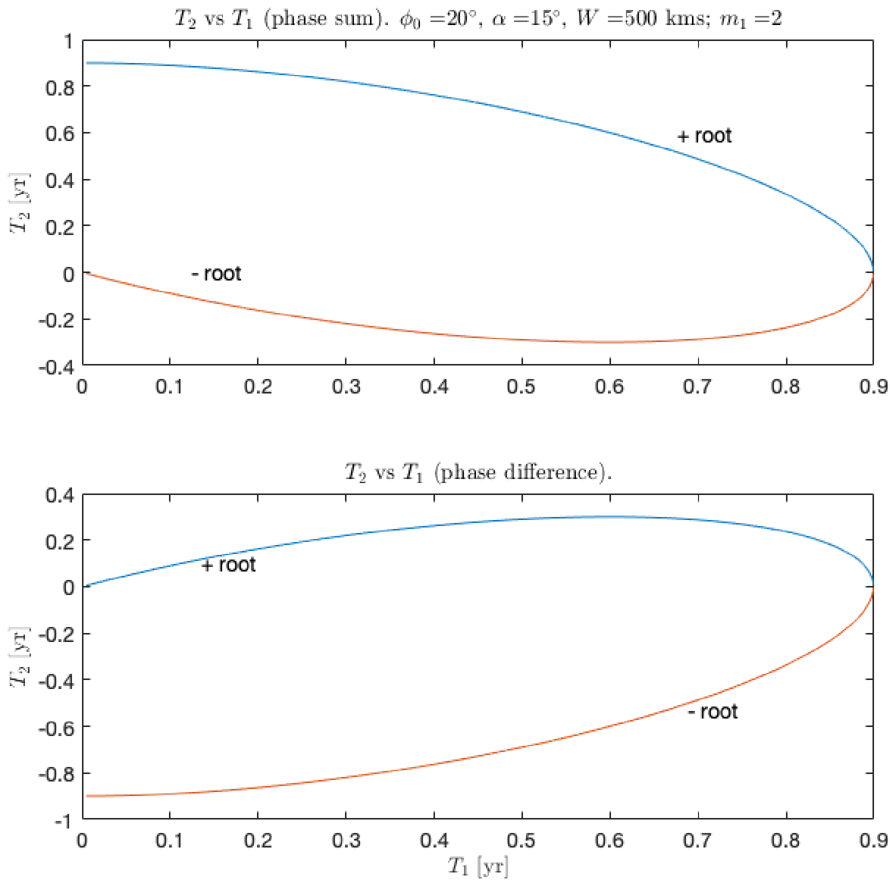

FIGURE 8. Periods T_2 of the second mode as a function of T_1 (years) from solution (5.2). $\phi_0 = 20°$, $\alpha = 15°$, channel width $W = 500$ km, horizontal mode number $m_1 = 2$ and vertical mode number $n_1 = 0$ (free surface, depth $H = 4000$ m). Upper (lower) panel for the phase sum (difference). Blue (red) curve refers to the positive (negative) root of ω_2.

abscissa) and identical k_1, so both solutions have equal first mode parameters. Also, the solution of the upper right panel $(k^{(2)}_{1m1}, k^{(2)}_{2m1})$ has $m_2 = 1$, $k_2 \approx -0.02$ km^{-1} from the graph, $m_3 = -1$ (recall $\Delta_3 = -\Delta_2$ for the phase difference) and $k_3 = k_1 - k_2 \approx 0$; whereas the solution of the lower right panel has $m_2 = -1$, $k_2 \approx 0$, $m_3 = 1$ and $k_3 = k_1 - k_2 \approx -0.02$ km^{-1}. Thus, mode 2 of the upper right panel is mode 3 of the lower right panel and vice versa; they are symmetric solutions with respect to modes 2 and 3.

Curiously enough, the only solution of the phase sum (upper left panel of figure 9) and the solution of the phase difference corresponding to the upper left panel of figure 10, also represent the same Rossby modes, but with the parameters of mode 2 in one panel (or solution) being equal to minus the parameters of mode 3 in the other panel, and vice versa. We call these anti-symmetric solutions concerning modes 2 and 3. We explain. One solution is phase sum (subscript p) and the other is phase difference (subscript m), thus we have $k_{2p} = -k_{3m}$, $\omega_{2p} = -\omega_{3m}$ and $l_{12,22} = -l_{13,23}$. Now, if one computes the eastward phase speed $C_E = \omega/k_E$ of the RWs of each mode (2 and 3), where $k_E = k\cos\alpha + l\sin\alpha$

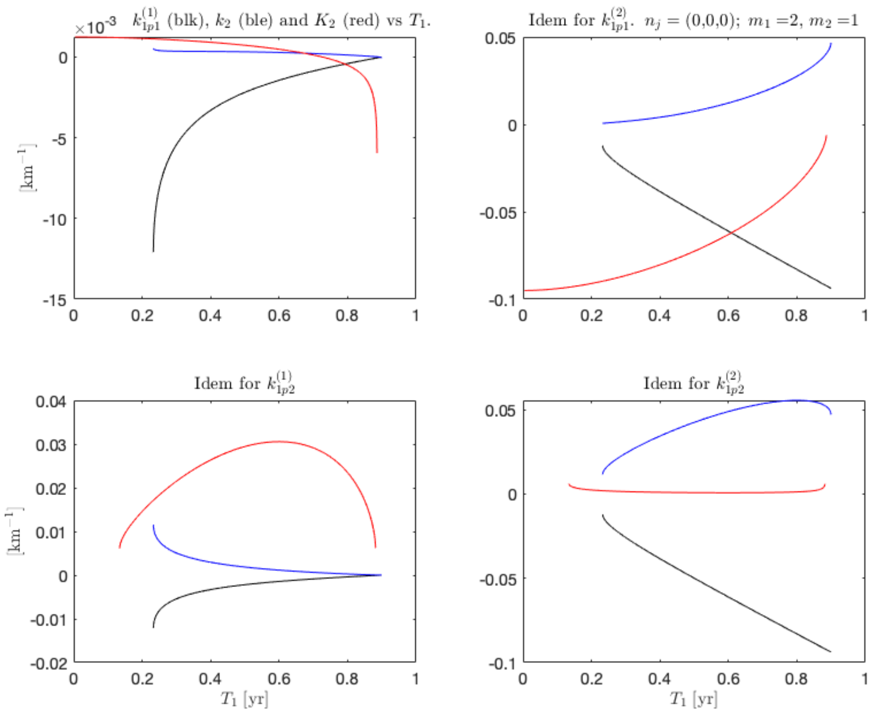

FIGURE 9. Along channel wavenumbers [km^{-1}] k_1 (black) from step 1 and k_2 from steps 2 (blue) and 3 (red) of the graphical method (see text) as a function of T_1 (years). Upper left panel is $k_{1p1}^{(1)}$, where the additional subscript (1 or 2) in k_1 refers to the (+ or -) root of ω_2 in (5.2), obtained from (4.8) and (5.2), and the corresponding k_2 from (4.10) (blue) and from (3.17) (red). Upper right panel is for $k_{1p1}^{(2)}$ and lower left (right) panel is for $k_{1p2}^{(1)}$ ($k_{1p2}^{(2)}$). If the blue and red curves intersect (step 4), there is a real solution (as in the upper left panel). Parameters as in figure 8, with $n_1 = 0$, $n_2 = 0$, $n_3 = 0$ (free surface, $H = 4000$ m) and $m_1 = 2$, $m_2 = 1$.

is the eastward wavenumber, the result is that the C_E of mode 2 of the solution p are equal to the C_E of mode 3 of the solution m and vice versa, and negative, ie all RWs have westward phase speed, as it should be. Thus, the anti-symmetric solutions with identical Rossby mode 1 and Rossby modes 2 and 3 exchanged have one of the modes (2 or 3) with the slowness circle on the $k_E < 0$ space (if the frequency is positive) and the other mode (3 or 2) on the $k_E > 0$ space (if the frequency is negative).

The graphical method of searching for the intersections of the k_2-curves of steps 2 and 3 (ie a change of sign of the difference between the k_2-curves) proved efficient in finding the solutions numerically. By choosing a sufficiently small time step of 10^{-5} year for the period T_1, we achieved numerical errors in the solutions for modes 1 and 2 of $O(10^{-18})$ and mode 3 of $O(10^{-10})$. The solution corresponding to the upper left panel of figure 9 is: $(T_1, T_2, T_3) = (0.67, 0.52, 0.29)$ years, $(k_1, k_2, k_3) = (-0.0010, 0.0002, -0.0008)$ km^{-1} and the wavelengths are: (1894,286) km, (6254,464) km and (2713,604) km for modes 1, 2 and 3, respectively. And the solution corresponding to the lower right panel of figure 10 is: $(T_1, T_2, T_3) = (0.26, -0.84, 0.200)$ years, $(k_1, k_2, k_3) = (-0.0203, -0.0011, -0.0192)$ and the wavelengths are: (283,242) km, (349,1139) km and (296,322) km for modes 1, 2 and 3, respectively.

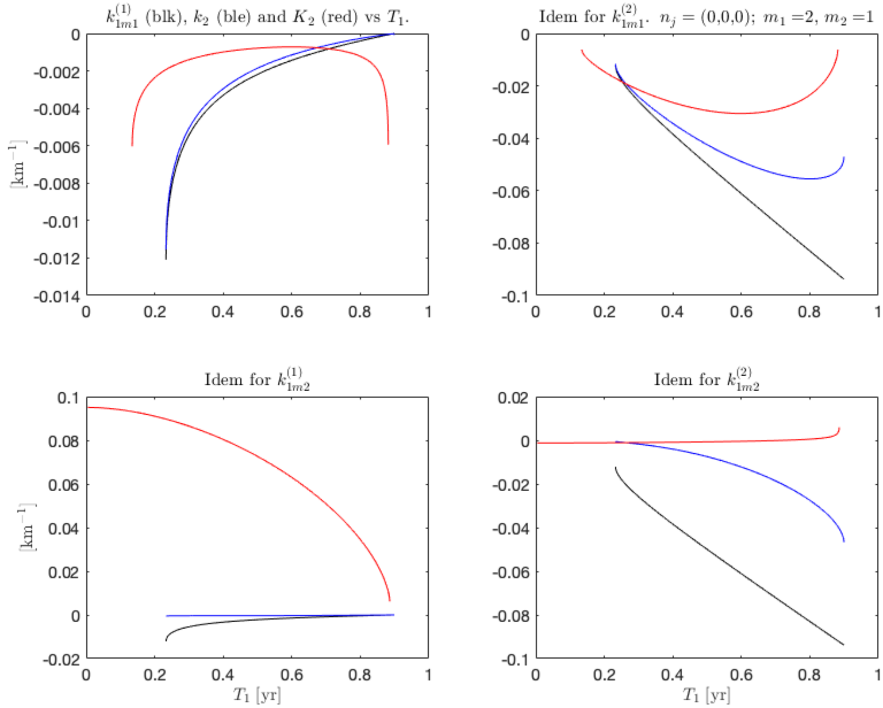

FIGURE 10. As in figure 9, but for the phase difference, ie $k^{(1)}_{1m1}$ and $k^{(2)}_{1m1}$ for the upper left and right panel, respectively, and $k^{(1)}_{1m2}$ and $k^{(2)}_{1m2}$ for the lower left and right panel, respectively. Note that the blue and red curves intersect in the upper left, upper right and lower right panels, so there are real solutions.

If we just change the inclination of the channel to $\alpha = 5°$, ie a more zonally oriented channel, and leave the rest of the input parameters used in figures 8, 9 and 10 unchanged, we get intersections (solutions) in the same four panels. However, the periods are larger than the case $\alpha = 15°$, but the wavelengths are similar.

As a possible oceanographic application, we searched for solutions in four channels with parameters resembling the Mozambique Channel ($\phi_0 = 19.5°$S, $\alpha = 115°$, $W = 750$ km, $H = 3292$ m), the Tasman Sea ($\phi_0 = 38°$S, $\alpha = 110.5°$, $W = 1750$ km, $H = 2500$ m), the Denmark Strait ($\phi_0 = 67°$N, $\alpha = 146.5°$, $W = 300$ km, $H = 400$ m) and the English Channel ($\phi_0 = 49°$N, $\alpha = 157°$, $W = 150$ km, $H = 63$ m) (Graef 2017) and for the vertical and horizontal mode numbers used to produce figures 8, 9 and 10, namely $n_j = (0, 0, 0)$ (all three modes barotropic, free surface) and $m_1 = 2$, $m_2 = 1$. There were no (real) solutions for the Mozambique Channel and the Tasman Sea because these channels are too inclined with respect to the eastern direction. However we found solutions for the Denmark Strait and the English Channel. Again there were four solutions (two and their mirror or symmetric or anti-symmetric solution with identical Rossby mode 1 and Rossby modes 2 and 3 exchanged) in each case, although the intersections of the curves (solutions) were for $k^{(2)}_{1p1}$ and its mirror or anti-symmetric $k^{(2)}_{1m1}$, and for $k^{(1)}_{1m1}$ and its mirror or symmetric $k^{(1)}_{1m2}$ (ie in different panels than in figures 9 and 10). The Rossby mode periods for the Denmark Strait are between 0.54 and 1.30 years, and the wavelengths between 167 and 2724 km. The second mode period of solution for $k^{(2)}_{1p1}$ is

1.00 year with wavelengths of 273 and 2724 km, which is also the period and wavelengths of the third mode of the solution $k^{(2)}_{1m1}$. Thus, if barotropic Rossby modes get excited in the Strait, out of all possible nonlinear interactions among them, the annual Rossby mode $m_2 = 1$ would have a larger amplitude since it is in resonance with two other modes of periods 0.56 and 1.24 years. The periods range between 0.79 and 2.47 years for the English Channel, and the wavelengths range between 79 and 1696 km.

After obtaining solutions for other parameters, in particular for various α's, ie, for a diversity of channel orientations, the following picture emerges:

•There were always four solutions: one for the RWs' phase sum and three for the RWs' phase difference. The solutions came in pairs: a solution and its anti-symmetric or symmetric companion.

•The solution and its anti-symmetric or symmetric companion always correspond to the same root of k_1, either $k_1^{(1)}$ or $k_1^{(2)}$. They represent the same Rossby modes, but with modes 2 and 3 exchanged.

•The anti-symmetric solution arises from solutions corresponding to the RWs' phase sum and phase difference, ie k_{1p} and k_{1m}.

•The last two characteristics of the solutions are because $\Delta_3 = \pm\Delta_2$, which is a consequence of the non-zonal orientation and our choice that wave 1 of mode 1 (ie l_{11}) be the one that interacts with the two waves of mode 2 to produce a third channel mode. Had we chosen that the single wave is one of mode 2, then $\Delta_3 = \pm\Delta_1$, and the solution pair would come with modes 1 and 3 exchanged.

Therefore, we have found real solutions of the resonance conditions for three Rossby modes in a non-zonal channel, for both the RWs' phase sum or difference. Because of the symmetric solutions, we could say that there are only two independent solutions for the waves' phase difference. However, we must realize that even though the symmetric solutions represent the same channel Rossby modes (with modes 2 and 3 exchanged), the amplitudes of modes 2 and 3 are different if we calculate the resonant solutions of the QGPVE at $O(\varepsilon)$.

Finally, we note that in a non-zonal channel, the interaction of two Rossby modes of equal frequency can never excite a third Rossby mode. This is simply because when two unknowns of the system (3.16)–(3.19) or (4.1)–(4.4) are made equal, the number of unknowns is reduced by one (from 4 to 3), but the number of equations remains the same (4). For instance, if $\omega_1 = \omega_2 = \omega$, the solution of (5.1) for $\omega = 0$ is $\omega = 3\beta \sin\alpha/(4\Delta_1)$, which can be plugged into (4.11) to get $k_1^{(1,2)} = F(\Delta_1)$. Up to here, (3.16) and (3.19) would be satisfied, but we are left with two equations, (3.17) and (3.18), and only one remaining unknown k_2. Now since Δ_2 is given (by virtue of having to specify m_2), $k_2 = G(\Delta_2, \Delta_1)$ could be computed from (3.17), but this k_2 will not satisfy in general (3.18). Thus, it is generally impossible to satisfy the resonance conditions.

The last result that there is no resonance between three channel modes if two of them have equal frequencies has the following implication. The self-interaction of a gulf Rossby mode (which is the superposition of two channel modes of equal frequency and vertical mode number) can never excite a third channel mode. Also, it corroborates one result obtained by García & Graef (1998).

6. Solution in the resonant case

In this section, we show the solution for the resonant forcings, based upon the works of Graef (1993), García & Graef (1998), and Graef (2017).

The streamfunctions of the three RWs of the initial modes 1 and 2 that nonlinearly interact in exciting the third mode, for both the straight coast and the channel in the non-zonal case, are, upon dropping the superscript (0) for simplicity:

$$\psi_{11} = A_1\varphi_{n_1}(z)\cos\theta_{11} \quad \text{and} \quad \psi_{i2} = A_2\varphi_{n_2}(z)\cos\theta_{i2}\ , i = 1, 2\ , \tag{6.1}$$

where recall that $\theta_{ij} = k_j x + l_{ij} y - \omega_j t + \vartheta_j$, $j = 1, 2, 3$. The difference between the coast and the channel is that in the latter, Δ_1 and Δ_2 are fixed, ie, wavenumbers perpendicular to the channel take on discrete values. Therefore the resonant forcings are:

$$\begin{aligned} F_{res} &= J(\psi_{11}, q_{12}) + J(\psi_{12}, q_{11}) - J(\psi_{11}, q_{22}) - J(\psi_{22}, q_{11}) \\ &= -\varphi_{n_1}(z)\varphi_{n_2}(z)\Big\{\mathcal{B}_{112}\left[\cos(\theta_{11} - \theta_{12}) - \cos(\theta_{11} + \theta_{12})\right] - \\ &\qquad \mathcal{B}_{122}\left[\cos(\theta_{11} - \theta_{22}) - \cos(\theta_{11} + \theta_{22})\right]\Big\}\ , \end{aligned} \tag{6.2}$$

where $q_{ij} \equiv \left[\nabla^2 + \partial_z(\Gamma^2\partial_z)\right]\psi_{ij}$, the minus sign in the last two Jacobians is due to the minus sign of RW 2 of mode 2: $\psi_2 = \psi_{12} - \psi_{22}$, and the coupling coefficients are, for $i = 1, 2$:

$$\mathcal{B}_{1i2} = \frac{1}{2}A_1A_2\left(k_2^2 + l_{i2}^2 + \hat{a}_{n_2}^{-2} - k_1^2 - l_{11}^2 - \hat{a}_{n_1}^{-2}\right)(k_1 l_{i2} - k_2 l_{11})\ . \tag{6.3}$$

We studied both possibilities: (i) the forced mode corresponding to the phase sum of the RWs, ie $\sim \cos(\theta_{11} + \theta_{12})$ and $\sim \cos(\theta_{11} + \theta_{22})$; and (ii) the forced mode corresponding to the phase difference of the RWs, ie $\sim \cos(\theta_{11} - \theta_{12})$ and $\sim \cos(\theta_{11} - \theta_{22})$. Note that unless $l_{12} = l_{22}$, which implies that $\Delta_2 = 0$, the coefficients of the forced RWs of mode 3 are different. But $\Delta_2 = 0$ means that there is no reflection or the group velocity of the single RW in this case is parallel to the coast, and there is no mode 2 for the channel (see Graef 2017).

6.1. *The straight coast*

Taking here the barotropic case for simplicity, we need a solution for

$$\mathcal{L}\psi^{(1)} = -\mathcal{B}_{112}\cos(\theta_{11} + \theta_{12}) = -\mathcal{B}_{112}\cos\theta_{13}\ , \tag{6.4}$$

where $\mathcal{L}$ is given by (2.3), but replacing the operator $\partial_z\left(\Gamma^2\partial_z\right)$ by $-\hat{a}_0^{-2}$ (where $\hat{a}_0$ is the barotropic Rossby radius).

Following Graef (2017), we put the ansatz $\psi^{(1)} = G_1(y)\cos\theta_{13}$ in (6.4) and since $\omega_3 = \sigma_0(k_3, l_{13})$, where $\sigma_0(k, l) \equiv -\beta(k\cos\alpha + l\sin\alpha)/(k^2 + l^2 + \hat{a}_0^{-2})$ is the RW dispersion relation, ie the forcing is resonant (a free RW), we end up with:

$$(2\omega_3 l_{13} + \beta\sin\alpha)\,G_1' = -\mathcal{B}_{112} \quad \Longrightarrow \quad G_1(y) = \frac{-\mathcal{B}_{112}\,y}{2\omega_3 l_{13} + \beta\sin\alpha}\ , \tag{6.5}$$

ie the particular solution grows linearly in the offshore coordinate. Note that the denominator $2\omega_3 l_{13} + \beta\sin\alpha = 0$ because we precisely require that $\Delta_3 = 0$, ie that the forced mode be a mode or $l_{13} = l_{23}$. In an identical way, the solution for the other forced RW of mode 3 proportional to $\cos(\theta_{11} + \theta_{22})$ is:

$$G_2(y) = \frac{\mathcal{B}_{122}\,y}{2\omega_3 l_{23} + \beta\sin\alpha}\ . \tag{6.6}$$

The solution for the forced mode $\psi^{(1)} = G_1(y)\cos\theta_{13} + G_2(y)\cos\theta_{23}$ obviously satisfies the boundary condition at the coast. An analogous procedure can be done for the RWs of the forced mode corresponding to the phase difference.

Therefore, the solution for forced mode 3 is unbounded, and we reject it on physical grounds. To obtain uniformly valid solutions, we need to invoke the method of multiple scales, as was done in Graef (1993) for the resonant case of the self-interaction of a single mode.

6.1.1. *Multiple scales*

The main idea behind multiple scales is that the mode amplitudes are slowly varying functions of the offshore coordinate y, namely $Y_1 = \varepsilon y$. Generalizing the work by Graef (1993), the leading order solution is written as a superposition of the three modes participating in the resonant triad, allowing their otherwise constant amplitudes to be functions of Y_1, ie

$$\begin{aligned}\psi &= \varphi_{n_1}(z)\,[A_{11}(Y_1)\cos\theta_{11} - A_{21}(Y_1)\cos\theta_{21}] + \varphi_{n_2}(z)[A_{12}(Y_1)\cos\theta_{12} - \\ &\quad A_{22}(Y_1)\cos\theta_{22}] + \varphi_{n_3}(z)\,[A_{13}(Y_1)\cos\theta_{13} - A_{23}(Y_1)\cos\theta_{23}] \\ &= \sum_{j=1}^{3}\sum_{i=1}^{2}(-1)^{i+1}\varphi_{n_j}(z)A_{ij}(Y_1)\cos\theta_{ij}\ . \end{aligned} \tag{6.7}$$

With the new dependence on Y_1, there will be additional forcing terms on the RHS of (2.2) besides the Jacobians, namely $-2\partial_t\partial_{yY_1}\psi - \beta\sin\alpha\,\partial_{Y_1}\psi$, to $O(\varepsilon)$. To find a solution to (2.2), $\psi^{(1)}$ is expanded in terms of the complete set of eigenfunctions $\{\varphi_q(z)\}$:

$$\psi^{(1)} = \sum_{q=0}^{\infty}\Phi_q(x,y,t)\varphi_q(z)\ , \tag{6.8}$$

where $\Phi_q = \int_{-H}^{0}\psi^{(1)}\,\varphi_q(z)\,dz$. The equation governing Φ_q is obtained by multiplying (2.2) by $\varphi_q(z)$, integrating over the depth and using the b.c.'s in z; the result is, after substituting (6.7) into the RHS of the QGPVE (2.2):

$$\begin{aligned}\mathcal{L}'\Phi_q = -\sum_{i=1}^{2}\Big\{&(-1)^i\xi_{n_1n_2q}\,\mathcal{B}_{11i2}\,[\cos(\theta_{11}-\theta_{i2}) - \cos(\theta_{11}+\theta_{i2})] + \\ &(-1)^i\xi_{n_1n_3q}\,\mathcal{B}_{11i3}\,[\cos(\theta_{11}-\theta_{i3}) - \cos(\theta_{11}+\theta_{i3})] + \\ &(-1)^i\xi_{n_2n_3q}\,\mathcal{B}_{12i3}\,[\cos(\theta_{12}-\theta_{i3}) - \cos(\theta_{12}+\theta_{i3})] + \\ &(-1)^{i+1}\xi_{n_2n_3q}\,\mathcal{B}_{22i3}\,[\cos(\theta_{22}-\theta_{i3}) - \cos(\theta_{22}+\theta_{i3})]\Big\} + \\ &\sum_{j=1}^{3}\sum_{i=1}^{2}(-1)^i\delta_{n_jq}\,(2\omega_j l_{ij} + \beta\sin\alpha)\,(\partial_{Y_1}A_{ij})\cos\theta_{ij} + \mathrm{NRF}\ ,\end{aligned} \tag{6.9}$$

where

$$\mathcal{L}' \equiv \partial_t\left(\nabla^2 - \hat{a}_q^{-2}\right) + \beta\,(\cos\alpha\,\partial_x + \sin\alpha\,\partial_y)\ , \tag{6.10}$$

$$\xi_{pql} \equiv \int_{-H}^{0}\varphi_p(z)\varphi_q(z)\varphi_l(z)\,dz \tag{6.11}$$

is the interaction between vertical eigenfunctions (Flierl 1977), and the coupling coefficients between the modes' RWs are, for $i = 1, 2$:

$$\left.\begin{aligned}\mathcal{B}_{11i2} &= \tfrac{1}{2}A_{11}A_{i2}\left(k_2^2 + l_{i2}^2 + \hat{a}_{n_2}^{-2} - k_1^2 - l_{11}^2 - \hat{a}_{n_1}^{-2}\right)(k_1l_{i2} - k_2l_{11})\ , \\ \mathcal{B}_{11i3} &= \tfrac{1}{2}A_{11}A_{i3}\left(k_3^2 + l_{i3}^2 + \hat{a}_{n_3}^{-2} - k_1^2 - l_{11}^2 - \hat{a}_{n_1}^{-2}\right)(k_1l_{i3} - k_3l_{11})\ , \\ \mathcal{B}_{12i3} &= \tfrac{1}{2}A_{12}A_{i3}\left(k_3^2 + l_{i3}^2 + \hat{a}_{n_3}^{-2} - k_2^2 - l_{12}^2 - \hat{a}_{n_2}^{-2}\right)(k_2l_{i3} - k_3l_{12})\ , \\ \mathcal{B}_{22i3} &= \tfrac{1}{2}A_{22}A_{i3}\left(k_3^2 + l_{i3}^2 + \hat{a}_{n_3}^{-2} - k_2^2 - l_{22}^2 - \hat{a}_{n_2}^{-2}\right)(k_2l_{i3} - k_3l_{22})\ .\end{aligned}\right\} \tag{6.12}$$

NRF refers to the non-resonant forcing terms, which include the interactions between the RW of amplitude A_{21} (reflected of mode 1) with the other modes' four RWs, and the self-interaction of each mode. The self-interaction gives rise to a steady flow parallel to the coast and a transient flow oscillating at twice the frequency of each mode (Graef & Magaard 1994).

If we consider the phase sum and difference $\theta_{i3} = \theta_{11} \pm \theta_{i2}$, then the secular terms on the RHS of (6.9) [homogeneous solutions of (6.9)] are: $\sim \cos(\theta_{11} \pm \theta_{i2})$ if $q = n_3$ because they are vertical mode n_3 RWs; $\sim \cos(\theta_{11} - \theta_{i3}) = \cos(\mp\theta_{i2})$ if $q = n_2$ because they are vertical mode n_2 RWs; $\sim \cos(\theta_{12} \mp \theta_{i3}) = \cos(\mp\theta_{11})$, for $i = 1$, and $\sim \cos(\theta_{22} \mp \theta_{i3}) = \cos(\mp\theta_{11})$, for $i = 2$, if $q = n_1$ because they are vertical mode n_1 RWs; and for all these we must have $\xi_{n_1 n_2 n_3} = 0$. Finally we have the secular terms with a Kronecker's delta factor, but only when $q = n_j$. The requirement $\xi_{n_1 n_2 n_3} = 0$ physically means that to have resonance, each vertical mode $\varphi_{n_j}(z)$ must have a non-zero projection on the product of the other two vertical modes, which is the vertical structure of the forcing that produces the j-th mode. In summary, we have secular terms only when $q = n_j$, $j = 1, 2, 3$ (all other q's do not produce secular terms).

Therefore, there are six secular terms on the RHS of (6.9) proportional to $\cos\theta_{ij}$, $i = 1, 2$, $j = 1, 2, 3$, with $\theta_{i3} = \theta_{11} \pm \theta_{i2}$, noting that the term $\sim \cos\theta_{11}$ has two contributions: one from the interactions of RW A_{12} with RWs A_{i3}, and other from the interactions of RW A_{22} with RWs A_{i3}.

We note that

$$2\omega_j l_{ij} + \beta \sin\alpha = (-1)^i \omega_j \left(l_{2j} - l_{1j}\right) = (-1)^{i+1}\, \omega_j\, 2\Delta_j \ , \tag{6.13}$$

which follows from (2.7) and (2.10), and which is non-zero if we have a mode (ie, an incident-reflected RW pair) for a non-zonal coast (and also a mode for the channel).

Finally, we remove the secular terms by requiring that the coefficient of any homogeneous solution of (6.9) be zero, leading to the following system of six (actually five) first-order nonlinear ODEs:

$$\left.\begin{array}{rcl}
(2\omega_1 l_{11} + \beta\sin\alpha)\,\partial_{Y_1} A_{11} & = & \pm\xi_{n_1 n_2 n_3}\left[\mathcal{B}_{1213} + \mathcal{B}_{2223}\right] \ , \\
\partial_{Y_1} A_{21} & = & 0 \ , \\
(2\omega_2 l_{i2} + \beta\sin\alpha)\,\partial_{Y_1} A_{i2} & = & \xi_{n_1 n_2 n_3}\mathcal{B}_{11i3} \ , \quad i = 1, 2 \ , \\
(2\omega_3 l_{i3} + \beta\sin\alpha)\,\partial_{Y_1} A_{i3} & = & \mp\xi_{n_1 n_2 n_3}\mathcal{B}_{11i2} \ , \quad i = 1, 2 \ ,
\end{array}\right\} \tag{6.14}$$

where the upper (lower) sign in the equations for A_{11} and A_{i3} refers to the phase sum (difference). The system (6.14) is subject to the boundary conditions $A_{1j} = A_{2j} = A_j$, $j = 1, 2, 3$, at $Y_1 = 0$, ie at $y = 0$, to warrant no normal flow at the coast. The second equation implies that $A_{21} = \text{constant} = A_1$. This system is relatively more complicated than the typical one found in three-wave resonance problems. Here, the coast's non-zonality obliges that only three RWs (not four as in the zonal case) of the primary modes participate in forcing the third mode. That is why five RWs (out of six RWs of the three modes) have their amplitudes slowly varying in the offshore coordinate to have a bounded solution when the modes are in resonance.

After substituting the coupling coefficients, the dispersion relations and (6.13), the system (6.14) becomes

$$\left.\begin{array}{rcl}
\partial_{Y_1} A_{11} & = & \xi_{n_1 n_2 n_3}\frac{\Delta_3}{\Delta_1}\left(\gamma_{12} A_{12} A_{13} - \gamma_{22} A_{22} A_{23}\right) \ , \\
\partial_{Y_1} A_{21} & = & 0 \ , \\
\partial_{Y_1} A_{i2} & = & \xi_{n_1 n_2 n_3}\gamma_{i2} A_{11} A_{i3} \ , \quad i = 1, 2 \ , \\
\partial_{Y_1} A_{i3} & = & -\xi_{n_1 n_2 n_3}\gamma_{i2} A_{11} A_{i2} \ , \quad i = 1, 2 \ ,
\end{array}\right\} \tag{6.15}$$

which is valid for *both* the phase sum and difference, where $\Delta_3 = \pm\Delta_2$, $\gamma_{i2} = \pm\gamma_{i3}$ and

$$\gamma_{i2} = \frac{\frac{1}{2}\left(k_3^2 + l_{i3}^2 + \hat{a}_{n_3}^{-2} - k_1^2 - l_{11}^2 - \hat{a}_{n_1}^{-2}\right)\left(k_1 l_{i3} - k_3 l_{11}\right)}{2\omega_2 l_{i2} + \beta\sin\alpha}$$
$$= \frac{\pm\frac{1}{2}\left(k_2^2 + l_{i2}^2 + \hat{a}_{n_2}^{-2} - k_1^2 - l_{11}^2 - \hat{a}_{n_1}^{-2}\right)\left(k_1 l_{i2} - k_2 l_{11}\right)}{2\omega_3 l_{i3} + \beta\sin\alpha} = \pm\gamma_{i3}\,. \qquad (6.16)$$

The details are given in the appendix.

There are three functionally independent first integrals of system (6.15). For example, the last four equations directly imply that $\partial_{Y_1}\left(A_{i2}^2 + A_{i3}^2\right) = 0$ for $i = 1, 2$ (two integral constraints). Also, multiplying the first equation by $\Delta_1 A_{11}/\Delta_3$, minus the third equation times A_{12}, plus the fourth equation times A_{22} yields $\partial_{Y_1}\left(\Delta_1 A_{11}^2/\Delta_3 - A_{12}^2 + A_{22}^2\right) = 0$; analogously we can obtain $\partial_{Y_1}\left(\Delta_1 A_{11}^2/\Delta_3 + A_{13}^2 - A_{23}^2\right) = 0$. However only three of these four first integrals of system (6.15) are independent.

In figures 11 and 12 we show the numerical solution of the wave amplitudes of the resonant quintet for parameters of the Hawaiian Ridge and for $(n_1, n_2, n_3) = (1, 1, 0)$ and $(T_1, T_2) = (1, 1.7)$ years, corresponding to solution $k_{1p}^{(1)}$ and $k_{1p}^{(2)}$, respectively. The solution (1) with larger wavelengths exhibits a clear periodic behavior in A_{22} and A_{23}, whereas A_{12} and A_{13} vary much more slowly, which is because $\gamma_{12} \ll \gamma_{22}$ in this case, and A_{11} oscillates at a higher frequency but with a lower amplitude. If we extended the integration farther, say to $Y_1 = 10^5$ km, one could see that A_{11}, A_{12} and A_{13} are also periodic. Solution (2) shows clearly that all four RW amplitudes of modes 2 and 3 oscillate with similar frequencies (equal for A_{i2} and A_{i3}) and equal amplitudes, whereas A_{11} displays a rather different behavior as in solution (1), but it is periodic.

We plot the wavenumber vectors and the slowness circles (ie the curves of constant ω_j for given n_j) of the resonant quintet corresponding to figures 11 and 12, in figures 13 and 14, respectively. There we indicate the coastal orientation (parallel to the k-axis) and one can see graphically that indeed $\boldsymbol{k}_{i3} = \boldsymbol{k}_{11} + \boldsymbol{k}_{i2}$ for $i = 1, 2$, and that $\Delta_3 = \Delta_2$.

In general, the envelope of the incident wave packet A_{11} is nowhere zero. The envelopes of incident RW packets (of modes 2 and 3) oscillate around zero out of phase and at the same frequency; this is also true for the reflected RW packets, but with a different frequency. Because we choose the b.c. of zero amplitude of mode 3 at the coast, it starts there and grows approximately linearly near the coast, as indicated by the straightforward expansion (6.5) and (6.6). The incident (reflected) RW packet of mode 3 reaches an extreme when the incident (reflected) packet of mode 2 is zero.

After running several cases, we observe that if the b.c.'s at $Y_1 = 0$ are $A_{11} = A_{12} = A_{22} = A_1$ and $A_{13} = A_{23} = 0$, the solution for another b.c. $A_1' = dA_1$ is simply $A_{ij}'(Y_1) = dA_{ij}(Y_1/d)$. This is because multiplying the b.c. by d means that ε gets multiplied by d, and $Y_1 = \varepsilon y$. Thus, it is convenient to simply set $A_1 = 1$ (in units of km^2/day, appropriate to typical RW length and time scales).

An interesting situation occurs if the primary Rossby modes 1 and 2 have an annual period (the rest of parameters as in figure 11) so $(T_1, T_2, T_3) = (1, 1, \frac{1}{2})$ years. In this case $\gamma_{12} \approx 0$ which implies that $A_{12} \approx A_1$ and $A_{13} \approx 0$, so the incident RW amplitudes of modes 2 and 3 remain almost constant (equal to the b.c.), whereas the reflected RW amplitudes oscillate at the same frequency. The resonant interaction is such that it preferably excites the reflected RWs.

As an aside remark, it can be shown that, unless the coast is zonal, particular solutions $\sim t\cos\theta_{i3}$, $i = 1, 2$, which satisfy the forced QGPVE, cannot satisfy the boundary condition at the coast $y = 0$. The forced or excited mode 3 cannot grow linearly in time, which ultimately is why the wave amplitudes cannot be slowly varying functions of

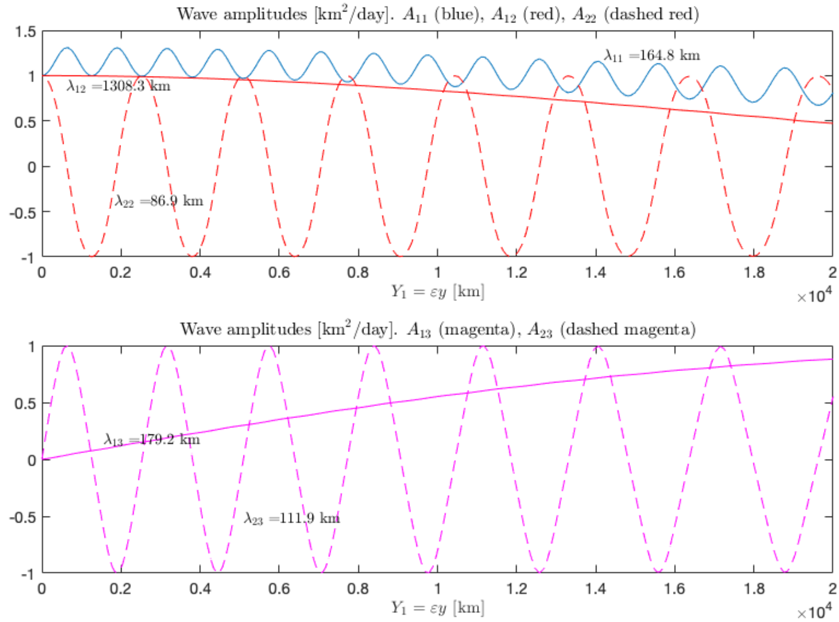

FIGURE 11. Wave amplitudes of a resonant quintet of RWs, which are solution of system (6.15), as a function of $Y_1 = \varepsilon y$. Upper panel: A_{11} (blue); A_{12} (red); A_{22} (dashed red). Lower panel: A_{13} (magenta); A_{23} (dashed magenta). The corresponding wavelengths are indicated on each curve. The amplitude's value at the coast of modes 1 and 2 is 1 km^2/day, which corresponds to a maximum horizontal particle speed of the mode 1 incident RW $U_{11} = 0.038$ km/day and $\varepsilon_{11} = U_{11}|\boldsymbol{k}_{11}|^2/\beta = 0.03$. More realistic values can be adjusted accordingly. Parameters: ϕ_0 and α for the Hawaiian Ridge, vertical mode numbers are $(n_1, n_2, n_3) = (1, 1, 0)$ and the Rossby mode periods are $(T_1, T_2, T_3) = (1, 1.7, 0.63)$ years for solution $k_{1p}^{(1)}$.

time. The speculation of Graef (1993) "on what would happen if three modes are taken, allowing each mode amplitude to be slowly varying in time", failed in the non-zonal case.

In the zonal coast, the incident and reflected RWs' wavelengths of each mode are equal, and their wavenumber vectors satisfy the relations $\boldsymbol{k}_{11} \times \boldsymbol{k}_{12} = -\boldsymbol{k}_{21} \times \boldsymbol{k}_{22}$ and $\boldsymbol{k}_{11} \times \boldsymbol{k}_{22} = -\boldsymbol{k}_{21} \times \boldsymbol{k}_{12}$. Thus the coupling coefficients of the four interactions $\boldsymbol{k}_{11} \leftrightarrow \boldsymbol{k}_{12}$, $\boldsymbol{k}_{11} \leftrightarrow \boldsymbol{k}_{22}$, $\boldsymbol{k}_{21} \leftrightarrow \boldsymbol{k}_{12}$, and $\boldsymbol{k}_{21} \leftrightarrow \boldsymbol{k}_{22}$ are such that the forced mode 3 satisfies the boundary condition at the coast $y = 0$. So, when applying multiple scales, it is sufficient to allow for each mode's amplitude to be a slowly varying function of time.

6.2. *The channel*

For the channel, the solution for the forced mode 3 is uncertain; we could not find it. However, if the resonant forcing given by (6.2) is such that only one RW is excited, ie we do not excite a channel Rossby mode, then we could easily find a solution. Suppose, without loosing generality, that the excited RW is proportional to $\cos(\theta_{13}) = \cos(\theta_{11} \pm \theta_{12})$. This is equivalent to say that the resonant triad is $\{\psi_{11}, \psi_{12}, \psi_{13}\}$. The solution is, adapted from García & Graef (1998) and Graef (2017):

$$\Phi_{n_3} = \mp A_1 A_2 \xi_{n_1 n_2 n_3} \gamma_{13} \,\mathrm{Re}\left[y\, e^{i\theta_{13}} + \frac{W e^{i l_{13} W}}{e^{i\mu W} - e^{i l_{13} W}} \left(e^{i\theta_{13}} - e^{i\theta_{\mu 3}} \right) \right], \tag{6.17}$$

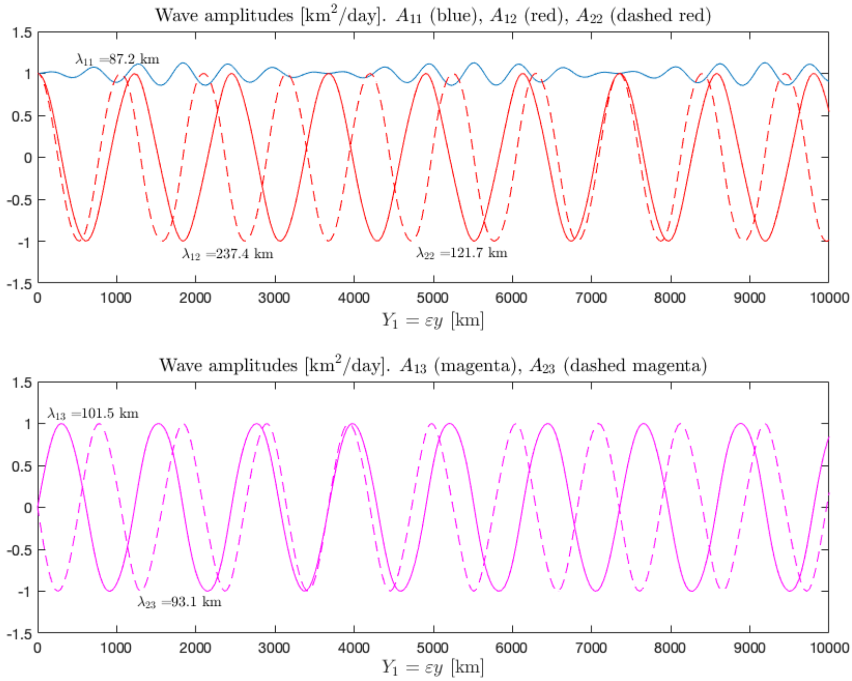

FIGURE 12. As in figure 11, but for solution $k_{1p}^{(2)}$.

where the upper (lower) sign refers to to the phase sum (difference), μ is the other root (besides l_{13}) of the RW dispersion relation $\omega_1 \pm \omega_2 = \sigma_{n_3}(k_1 \pm k_2, \mu)$ or $\omega_3 = \sigma_{n_3}(k_3, \mu)$ and $\theta_{\mu 3} = (k_3 x + \mu y - \omega_3 t + \vartheta_3)$. It is easy to see that $\Phi_{n_3} = 0$ at $y = 0, W$. It is worth remarking that l_{13} is not $-\beta \sin\alpha/(2\omega_3) + m_3\pi/W$, ie the excited RW ψ_{13} is not a wave of a channel mode, or equivalently $\Delta_3 = m_3\pi/W$. But we need the other RW $\sim e^{i\theta_{\mu 3}}$ in order to fulfill the boundary condition at $y = W$. This physically means that a coastal mode gets excited, not a channel mode, because $e^{i\theta_{13}} - e^{i\theta_{\mu 3}}$ is just a coastal mode.

The resonant solution (6.17) is bounded, and there is no need to do multiple scales. It consists of a term proportional to $y\cos\theta_{13}$, plus a term proportional to the real part of $C(e^{i\theta_{13}} - e^{i\theta_{\mu 3}})$, where C is a complex constant, which is a coastal mode (it vanishes at $y = 0$, but not at $y = W$). This solution is reminiscent of the solution when there is resonance in the self-interaction of a channel Rossby mode (García & Graef 1998).

7. Discussion and conclusions

In this paper, we studied whether or not there are resonant interactions between three Rossby modes in two bounded geometries: a coast and a channel, whose orientation is non-zonal. The fact that the boundaries are not along circles of latitude is a new ingredient in these problems, not reported in the literature.

As the superposition of two propagating RWs forms a Rossby mode in a coast or a channel, the nonlinear interaction between two modes produces 12 forcing terms. We first analyzed which of those 12 terms, or which RWs, could excite a third mode. In the zonal case, we need the participation or interaction of the four RWs, two of each mode. However, if the orientation is non-zonal, only three RWs (of the four) can participate in

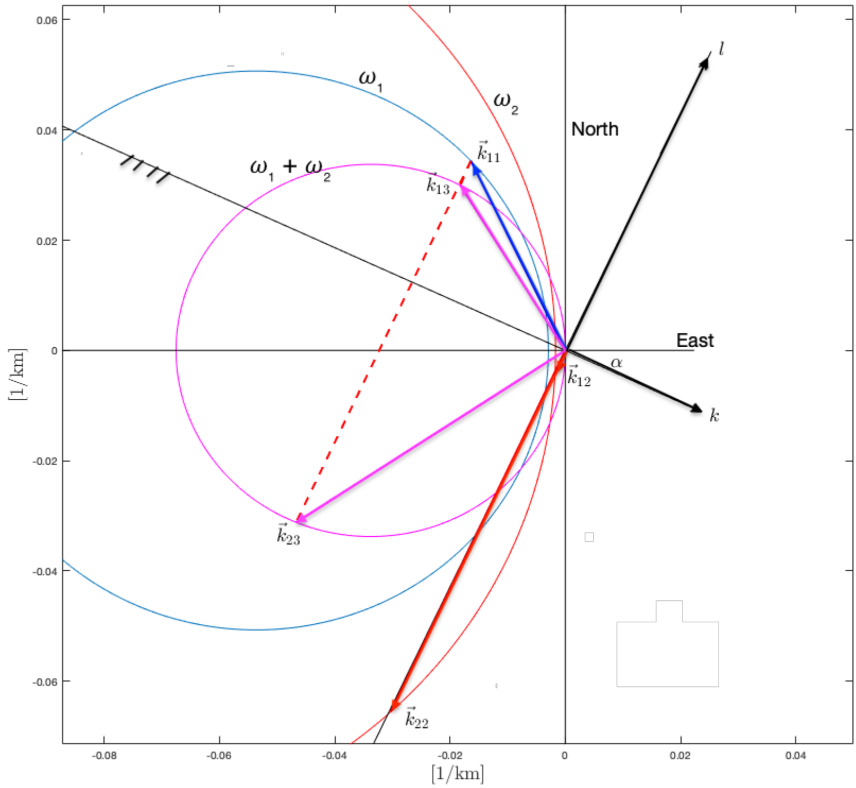

FIGURE 13. The wavenumber vectors and the slowness circles of the resonant quintet of RWs corresponding to figure 11. We indicate the coastal orientation (parallel to the k-axis) making an angle α with respect to the eastern direction. In blue, the RW $(n_1, \omega_1, \boldsymbol{k}_{11})$ of mode 1; in red the RWs $(n_2, \omega_2, \boldsymbol{k}_{i2})$, $i = 1, 2$ of mode 2; and in magenta the RWs $(n_3, \omega_1 + \omega_2, \boldsymbol{k}_{i3})$, $i = 1, 2$ of mode 3. Note that $\boldsymbol{k}_{i3} = \boldsymbol{k}_{11} + \boldsymbol{k}_{i2}$ for $i = 1, 2$, and that $\Delta_3 = \Delta_2$.

forcing, in principle, the third mode. This difference has two significant consequences in the non-zonal case. First, the horizontal structure of the “standing” part of the forced mode proportional to $\sin(\Delta_3 y)$ is identical to the mode whose two RWs participate in the interaction. Second, there appears an additional constraint (equation), which reduces the number of degrees of freedom available to solve the resonance conditions (see table 2). Thus, finding resonant triads is more restrictive in the non-zonal case.

When one considers the interaction between two modes in a zonal coast or channel, the initial modes may have $\Delta_1 = \Delta_2$ or $\Delta_1 = \Delta_2$, but the excited mode is $\Delta_3 = \Delta_1 \pm \Delta_2$ (if $\Delta_1 = \Delta_2$, we can only excite the mode produced by the sum). We always excite a new horizontal structure, so there is “barotropic transfer” in the resonant interaction. This was the case, for example, studied by Plumb (1977), for a zonal channel in a barotropic ocean. However, if we want to excite a third mode in a non-zonal coast or channel, only three RWs can participate, and the excited mode must have the horizontal structure of one of the initial modes ($\Delta_3 = \pm\Delta_2$ or $\Delta_3 = \pm\Delta_1$). One cannot excite a new Δ, and there is no “barotropic transfer”.

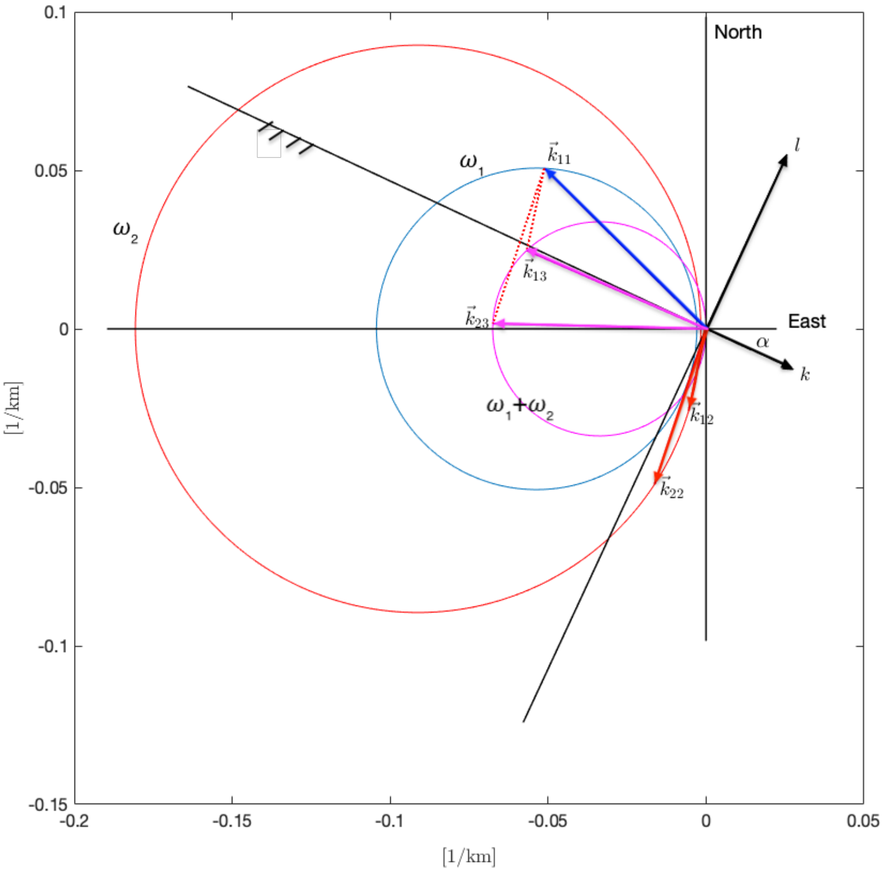

FIGURE 14. As in figure 13, but corresponding to figure 12, ie for solution $k_{1p}^{(2)}$. The frequencies and vertical mode numbers are those of figure 13, but the wavenumbers $\boldsymbol{k}_{11}$, $\boldsymbol{k}_{i2}$ and $\boldsymbol{k}_{i3}$ are different. Note the larger scale here, which is why the whole circles appear in the graph. This graph is a zoom out of figure 13.

As shown in table 2, the non-zonality and the number of boundaries decreases the number of degrees of freedom to solve the resonance or kinematic conditions for the existence of resonant triads. For instance, for a non-zonal coast or wall, the resonance conditions pose a problem with four equations and nine variables: ω_i, k_i, Δ_i, $i = 1, 2$ and n_j, $j = 1, 2, 3$. However, the last three are discrete and must be specified. Thus, we end up with two degrees of freedom: 6 unknowns minus 4 equations. In the case of a non-zonal channel, it is similar but $\Delta_1 = m_1\pi/W$ and $\Delta_2 = m_2\pi/W$ are fixed, thus there are no degrees of freedom.

For the non-zonal coast, we derived analytic expressions for the wavenumbers along the coast k_1 and k_2 of modes 1 and 2, respectively, which are necessary conditions to have solutions of the system (4.1)–(4.4). Although, in general, it is not possible to find a condition to have k_1 real that only involved α, the equation for k_1 reveals that a meridional coast is prohibited, ie there are no real solutions. The more meridionally oriented the coast is, the more restrictive the problem of finding real solutions become.

Geometry	Orientation	Var.	D.V.	Eqs.	D.F.
Coast	Zonal	9	3	3	3
	Non-zonal	9	3	4	2
Channel	Zonal	9	5	3	1
	Non-zonal	9	5	4	0

TABLE 2. The number of variables (Var.), discrete variables (D.V.), equations (Eqs.) and degrees of freedom (D.F.) of the resonance conditions, for each geometry (coast or channel) and its orientation (zonal or non-zonal).

For example, we found that if the period of mode 1 is much larger than the period of mode 2 ($T_1 \gg T_2$), it favors real solutions for the more meridionally oriented coasts [say $\alpha \in (70, 85)$ or $\alpha \in (95, 110)$ degrees for western coasts; or with $\alpha + 180°$ for eastern coasts]. In the particular case $\omega_1 = \omega_2$, a necessary condition to have real solutions is $|\sin\alpha| \leqslant 2/3$, which is twice the value obtained by Graef (1993) when considering resonance in the self-interaction of a Rossby mode at a coast. Therefore, although the orientation of the coast or wall restricts resonance to occur, it is less restrictive in the case of resonance between Rossby modes (with $\omega_1 = \omega_2$) than in the self-interaction of a Rossby mode.

The family of solutions for given mode periods T_1 and T_2 (recall we have two degrees of freedom) was shown by plotting the wavelengths of the six RWs (one incident and one reflected per mode) that participate in the resonant triad of modes. And for each T_1 and T_2, there are two solutions for the initial RWs phase sum ($\omega_3 = \omega_1 + \omega_2$, $k_3 = k_1 + k_2$, $l_{13} = l_{11} + l_{12}$ and $l_{23} = l_{11} + l_{22}$) and two solutions for the phase difference ($\omega_3 = \omega_1 - \omega_2$, $k_3 = k_1 - k_2$, $l_{13} = l_{11} - l_{12}$ and $l_{23} = l_{11} - l_{22}$). By looking at solutions with different coastal orientations, there are two general characteristics of the solutions: a) the larger wavelengths are squeezed in a very small region of the (T_1, T_2)-space; and b) the space of solutions is more limited for the phase difference and it is always $T_1 > T_2$. In fact, even for more zonally oriented coasts, some of the real solutions lie only within a very tiny region (resembling a thin slice of a pie) of the (T_1, T_2)-space.

As a possible oceanographic application and because it has received significant attention since the pioneering work of Mysak & Magaard (1983) regarding the North Hawaiian Ridge Current (White 1983; Oh & Magaard 1984; Sun *et al.* 1988; Price *et al.* 1994; Qiu *et al.* 1997; Firing *et al.* 1999), we showed the solutions for ambient parameters appropriate for the Hawaiian Ridge (figures 4, 5, 6, and 7). The wavelengths of the incident RWs of the first mode corresponding to solutions $k_{1p}^{(1)}$ and $k_{1m}^{(1)}$ are the largest: $\lesssim$ 1000 km, whereas for the third mode, there is a wide range between 100 and 2000 km, and for the second mode they are very short: between less than 50 and 240 km. The wavelengths of the reflected RWs of all modes are short: between 20 and 200 km. There is a significant reduction in the allowable (T_1, T_2)-space (very tiny slices of a pie) for the other solutions, ie for $k_{1p}^{(2)}$ and $k_{1m}^{(2)}$, and all wavelengths (even the incident RWs) are quite short, between 20 and 200 km. We conclude that two annual Rossby modes ($n_1 = n_2 = 1$) cannot resonantly interact to force a semi-annual $n_3 = 2$ Rossby mode. However if we choose $n_3 = 0$ (not shown here), so that the forced mode (mode 3) is barotropic with a free surface (depth $H = 4000$ m), then such resonant interaction is possible. Also, it is not possible to have resonance if one of the initial modes (first mode baroclinic) has a period in the broad peak range from 0.7 to 2.5 years, and the other mode has a period of 6.7 years [these are spectral peak periods of Rossby wave energy for a 5° square east of the Hawaiian Islands (see Magaard 1983)].

For the non-zonal channel, the resonance conditions form a closed system (four equations and four unknowns: ω_i, k_i, $i = 1, 2$), so there are no degrees of freedom. We could not arrive at a single equation for any one of the four unknowns. However, we developed a graphical method to seek solutions using the analytic expressions for k_1 and k_2 derived for the coast, which are also valid for the channel. A meridional channel is prohibited (no real solutions). However we found real solutions for other orientations, like the hypothetical example shown in figures 8, 9 and 10 for a tilted channel with $\alpha = 15°$, width $W = 500$ km, at a reference latitude $\phi_0 = 20°$, horizontal mode numbers $m_1 = 2$, $m_2 = 1$ and vertical mode numbers $n_1 = n_2 = n_3 = 0$ (all barotropic with a free surface and depth $H = 4000$ m). In this example, the mode periods were less than a year, and the RWs' wavelengths of the modes had a wide range: between a few hundreds to more than 6,000 km. As with other examples that we explored, particularly for other α's, there were always four solutions to the resonance conditions: one for the RWs' phase sum and three for the RWs' phase difference. The four solutions were related: two symmetric and two anti-symmetric, with modes 2 and 3 exchanged. The anti-symmetry comes about because $\sigma_n(k, l) = -\sigma_n(-k, -l)$ in the RW dispersion relation.

We pointed out that because there are no degrees of freedom for the resonance conditions in a non-zonal channel, the interaction of two Rossby modes of equal frequency can never excite a third Rossby mode. This result has implications for finding resonant triads in a non-zonal gulf (and by extension in a non-zonal rectangular basin). Since a gulf Rossby mode is the superposition of an incident-reflected channel mode pair at the head of the gulf (Graef 2016), it follows that if there are resonant triads between gulf modes, the excited waves cannot be the product of either mode's self-interaction. In other words, the forced mode cannot have a frequency equal to two times the frequency of either one of the primary modes.

Looking at the worlds oceans, the most conspicuous mid-latitude channels for which planetary wave motion could matter are the Mozambique Channel, the Tasman Sea, the Denmark Strait, and perhaps (because of their irregularity and or size) the South China Sea, the Caribbean Sea, and the English Channel (Graef 2017). As a possible oceanographic application, we searched for solutions of the resonance conditions in four of these channels with $n_j = (0, 0, 0)$ (all three modes barotropic, free surface) and $m_1 = 2$, $m_2 = 1$. There were no solutions for the Mozambique Channel and the Tasman Sea because these channels are too inclined relative to the eastern direction, but we found solutions for the Denmark Strait and the English Channel. Because the annual signal always comes to mind when one thinks about Rossby wave motion, an interesting result for the Denmark Strait was that the second mode period of one solution is 1.00 year with wavelengths of 273 and 2724 km. This solution suggests that if barotropic Rossby modes get excited in the Strait, out of all possible nonlinear interactions among them, the annual Rossby mode $m_2 = 1$ would have a larger amplitude (being in resonance with two other modes of periods 0.56 and 1.24 years). For the English Channel, the smallest and largest of the mode periods were 0.79 and 2.47 years, and of the wavelengths were 79 and 1696 km, respectively, for all modes and the two independent solutions. However, because the lengths of the Denmark Strait and the English Channel are much smaller than some of the mode's wavelengths (≈ 2000 km), most probably we cannot apply our results to these channels.

The solution of the forced QGPVE, when the third mode is in resonance with modes 1 and 2, is unbounded in the coast's case. The pedestrian or straightforward expansion leads to a linear growth in the offshore coordinate y, which we rejected on physical grounds; it is acceptable "near the coast". To obtain a bounded solution in the whole half-plane domain, we used multiple scales, generalizing the work of Graef (1993). First, we wrote the solution of the QGPVE, to leading order in ε, as the superposition of the three Rossby modes in resonance, but allowing the RWs' amplitudes (constant in the straightforward expansion) to be slowly varying functions of the offshore coordinate, namely functions of $Y_1 = \varepsilon y$. Second, we computed all forcing terms that are secular and removed them by requiring that the coefficient of any homogeneous solution of the equation be zero. This requirement led to a system of five first-order nonlinear ODEs for the RWs' amplitudes that participate in the resonant triad (three of the primary modes and two of the forced third mode). In the appendix, we were able to show that the factors multiplying the amplitudes' products, which involve the coupling coefficients, are all related, and only two factors (out of six) are independent. We showed examples (figures 11 and 12) of the wave amplitudes' numerical solution, which exhibit periodic behavior. For parameter values of the Hawaiian Ridge and if the primary modes 1 and 2 have an annual period (so the third mode is semi-annual), the incident RWs' amplitudes of modes 2 and 3 are nearly constant. In contrast, those corresponding to the reflected waves oscillate at the same frequency (in space), indicating that resonant interactions lead to more variability in smaller scales, ie westward intensification. As in Graef (1993), the energies of the modes oscillate in the offshore direction. There is an energy exchange in space with the three resonant modes giving and receiving it, satisfying the boundary condition at the coast, and maintaining the solution bounded as $y \longrightarrow \infty$.

We included two figures (13 and 14) to help the reader locate the resonant modes' incident and reflected waves together with the coastal orientation. We plotted the wavenumber vectors of the resonant quintet on the slowness circles corresponding to the examples of the wave amplitudes' numerical solution. In these figures one could see graphically that $\boldsymbol{k}_{i3} = \boldsymbol{k}_{11} + \boldsymbol{k}_{i2}$ for $i = 1, 2$, and that $\Delta_3 = \Delta_2$.

The solution of the QGPVE for the channel, when the third mode is in resonance with the primary modes 1 and 2, is uncertain, and unfortunately, we could not find it. However, we provided a solution if the nonlinear interaction between a RW of mode 1 and a RW of mode 2 forces or excites a single RW. The excited RW is not a wave belonging to a channel mode (if the channel is zonal, this is impossible: the excited RW is a wave of mode 3 with $m_3 = m_1 \pm m_2$, and also, the other RW of mode 3 gets automatically excited). This resonance is an example of problem (A) mentioned in the introduction. The resonant solution shows that (i) a coastal mode gets excited, needed to satisfy the boundary condition at both coasts; (ii) it is bounded, and there is no need to do multiple scales; (iii) the two channel modes and the coastal mode, although in resonance, do not exchange energy in time or space due to the constraint of the motion imposed by the boundary conditions at the channel's non-zonal coasts or walls; and (iv) it is reminiscent of the solution when there is resonance in the self-interaction of a channel Rossby mode (García & Graef 1998). Why is this lack of energy exchange? First, there is no solution growing linearly in time when there is resonance (this is true if the coast or channel is non-zonal). Second, but this is *speculation*, is that enstrophy is not conserved in a non-zonal channel. Indeed, in the reflection of RWs from a non-zonal wall, enstrophy is not conserved (Pedlosky 2013) since the incident and reflected wave's wavelengths are different. In a non-zonal channel, the RWs that comprise a mode have different wavelengths, and by generalization, enstrophy will not be conserved. It is only for a zonal coast or channel (where enstrophy is conserved) that the resonant triad modes' amplitudes depend slowly on time, so there is energy exchange among the triad members, as shown by Plumb (1977) for a zonal channel.

Regarding possible oceanographic applications, we should keep in mind that our coast or channel is idealized and that bottom topography and irregular coastlines would change these solutions. There is no intention or attempt to compare our solutions with observations. Despite our idealized geometries, the analytical results presented here could provide a dynamic basis to help explain observations. Furthermore, analytical solutions are, in general, a handy tool to test numerical models. Beyond these benefits, we believe in having contributed to the advancement of knowledge in Geophysical Fluid Dynamics.

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Appendix A

In this appendix we show the calculations to go from the ODE's system (6.14) to (6.15) and the relations between the factors multiplying the RW amplitudes' products.

The last four equations of (6.14) are, upon substituting $\mathcal{B}_{11i3}$ and $\mathcal{B}_{11i2}$ given by (6.12):

$$\begin{aligned}\partial_{Y_1} A_{i2} &= \frac{\frac{1}{2} A_{11} A_{i3}\, \xi_{n_1 n_2 n_3} \left(k_3^2 + l_{i3}^2 + \hat{a}_{n_3}^{-2} - k_1^2 - l_{11}^2 - \hat{a}_{n_1}^{-2}\right)(k_1 l_{i3} - k_3 l_{11})}{2\omega_2 l_{i2} + \beta \sin\alpha} \\ &\equiv \gamma_{i2}\, \xi_{n_1 n_2 n_3} A_{11} A_{i3}\ , \quad i = 1, 2\ ,\end{aligned} \tag{A 1}$$

$$\begin{aligned}\partial_{Y_1} A_{i3} &= \frac{\mp\frac{1}{2} A_{11} A_{i2}\, \xi_{n_1 n_2 n_3} \left(k_2^2 + l_{i2}^2 + \hat{a}_{n_2}^{-2} - k_1^2 - l_{11}^2 - \hat{a}_{n_1}^{-2}\right)(k_1 l_{i2} - k_2 l_{11})}{2\omega_3 l_{i3} + \beta \sin\alpha} \\ &\equiv \mp\gamma_{i3}\, \xi_{n_1 n_2 n_3} A_{11} A_{i2}\ , \quad i = 1, 2\ .\end{aligned} \tag{A 2}$$

We now show that $\gamma_{i2} = \pm\gamma_{i3}$, where the $+$ $(-)$ refers to the phase sum (difference). Using the dispersion relations

$$\omega_j \left(k_j^2 + l_{ij}^2 + \hat{a}_{n_j}^{-2}\right) + \beta\,(k_j \cos\alpha + l_{ij} \sin\alpha) = 0\ , \quad i = 1, 2\ , \quad j = 1, 2, 3\ , \tag{A 3}$$

which follows from (2.9), (2.10) and (2.7), and the relation (6.13), we have that

$$\gamma_{i2} = \frac{1}{2} \left[\frac{-\beta\,(k_3 \cos\alpha + l_{i3} \sin\alpha)\,\omega_1 + \beta\,(k_1 \cos\alpha + l_{11} \sin\alpha)\,\omega_3}{\omega_1 \omega_3 \omega_2\, (-1)^{i+1}\, 2\Delta_2}\right](k_1 l_{i3} - k_3 l_{11})\ . \tag{A 4}$$

Substituting the resonance conditions $\boldsymbol{k}_{i3} = \boldsymbol{k}_{11} \pm \boldsymbol{k}_{i2}$, ie $(k_3, l_{i3}) = (k_1 \pm k_2, l_{11} \pm l_{i2})$ [see relations (3.11)], the numerator within square brackets becomes

$$\begin{aligned}-\beta\,(k_1 \cos\alpha + l_{11} \sin\alpha)\,(\omega_1 - \omega_3) \mp \beta\,(k_2 \cos\alpha + l_{i2} \sin\alpha)\,\omega_1 &= \\ \pm\beta\,(k_1 \cos\alpha + l_{11} \sin\alpha)\,\omega_2 \mp \beta\,(k_2 \cos\alpha + l_{i2} \sin\alpha)\,\omega_1&\end{aligned} \tag{A 5}$$

since $\omega_3 = \omega_1 \pm \omega_2$. Finally, note that $\boldsymbol{k}_{11} \times \boldsymbol{k}_{i3} = \pm\boldsymbol{k}_{11} \times \boldsymbol{k}_{i2}$. Thus,

$$\begin{aligned}\gamma_{i2} &= \frac{1}{2} \left[\frac{\pm\beta\,(k_1 \cos\alpha + l_{11} \sin\alpha)\,\omega_2 \mp \beta\,(k_2 \cos\alpha + l_{i2} \sin\alpha)\,\omega_1}{\omega_1 \omega_2 \omega_3\, (-1)^{i+1}\, 2\Delta_2}\right](\pm 1)\,(k_1 l_{i2} - k_2 l_{11}) \\ &= \frac{1}{2} \left[\frac{k_2^2 + l_{i2}^2 + \hat{a}_{n_2}^{-2} - k_1^2 - l_{11}^2 - \hat{a}_{n_1}^{-2}}{\omega_3\, (-1)^{i+1}\, 2\Delta_2}\right](k_1 l_{i2} - k_2 l_{11}) \\ &= \pm\gamma_{i3}\end{aligned} \tag{A 6}$$

because $\Delta_3 = \pm\Delta_2$. Therefore, $\partial_{Y_1} A_{i3} = \mp\gamma_{i3}\xi_{n_1 n_2 n_3} A_{11} A_{i2} = -\gamma_{i2}\xi_{n_1 n_2 n_3} A_{11} A_{i2}$, for both the phase sum and difference.

The first equation of (6.14) takes the form, upon substituting $\mathcal{B}_{12i3}$ for $i = 1$, and $\mathcal{B}_{22i3}$ for $i = 2$, from (6.12):

$$\partial_{Y_1} A_{11} = \xi_{n_1 n_2 n_3} \left(\gamma_{111} A_{12} A_{13} + \gamma_{112} A_{22} A_{23}\right) \ , \tag{A 7}$$

where

$$\gamma_{11i} = \frac{\frac{1}{2}\left(k_3^2 + l_{i3}^2 + \hat{a}_{n_3}^{-2} - k_2^2 - l_{i2}^2 - \hat{a}_{n_2}^{-2}\right)\left(k_2 l_{i3} - k_3 l_{i2}\right)}{2\omega_1 l_{11} + \beta \sin\alpha} \ , \quad i = 1, 2. \tag{A 8}$$

In a similar fashion, using (A 3), (6.13), substituting the resonance conditions, and noting that $\boldsymbol{k}_{i2} \times \boldsymbol{k}_{i3} = \boldsymbol{k}_{i2} \times \boldsymbol{k}_{11}$, we obtain

$$\begin{aligned} \gamma_{11i} &= \frac{1}{2}\left[\frac{\beta\left(k_2\cos\alpha + l_{i2}\sin\alpha\right)\omega_1 - \beta\left(k_1\cos\alpha + l_{11}\sin\alpha\right)\omega_2}{\omega_2\omega_3\omega_1\left(-1\right)^{1+1} 2\Delta_1}\right]\left(k_2 l_{11} - k_1 l_{i2}\right) \\ &= \frac{1}{2}\left[\frac{k_2^2 + l_{i2}^2 + \hat{a}_{n_2}^{-2} - k_1^2 - l_{11}^2 - \hat{a}_{n_1}^{-2}}{\omega_3\, 2\Delta_1}\right]\left(k_1 l_{i2} - k_2 l_{11}\right) \\ &= \frac{\Delta_3}{\Delta_1}\left(-1\right)^{i+1}\gamma_{i3} \ . \end{aligned} \tag{A 9}$$

Therefore, the system of ODE's for the wave amplitudes that is valid for *both* the phase sum and difference is:

$$\left.\begin{aligned} \partial_{Y_1} A_{11} &= \xi_{n_1 n_2 n_3} \tfrac{\Delta_3}{\Delta_1}\left(\gamma_{12} A_{12} A_{13} - \gamma_{22} A_{22} A_{23}\right) \ , \\ \partial_{Y_1} A_{21} &= 0 \ , \\ \partial_{Y_1} A_{i2} &= \gamma_{i2}\, \xi_{n_1 n_2 n_3} A_{11} A_{i3} \ , \quad i = 1, 2 \ , \\ \partial_{Y_1} A_{i3} &= -\gamma_{i2}\, \xi_{n_1 n_2 n_3} A_{11} A_{i2} \ , \quad i = 1, 2 \ , \end{aligned}\right\} \tag{A 10}$$

where $\Delta_3 = \pm\Delta_2$ and $\gamma_{i3} = \pm\gamma_{i2}$.

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