Rigorous estimation of local accuracies revisited

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Abstract. New insights about the concept of local accuracies are elaborated in this article. Recently found evidence supports the mathematical rigor of equations previously published in this journal as the unique alternative to rigorously estimate local accuracies. A mathematical algorithm to compute the averaged local accuracies at a point using the full network statistics of a preselected cluster of surrounding points is introduced. The relationship between eigenvalues and eigenvectors of error ellipsoids among different local frames is also addressed.

Introduction

As far as one can determine, the rigorous formulation for the estimation of local accuracies was introduced twenty years ago. "Local" in this context implies that the final variance-covariance (v-c) matrix (or its error ellipsoid) is referred to the local horizon geodetic frame while incorporating statistics (variances and covariances) from nearby points. The intent is clearly to determine, as much as possible, the influence of observational errors inherent to near-by points to the precision and/or accuracy of any other arbitrary survey mark in the network (horizontal, 3D, Global Navigation Satellite System (GNSS) determined, etc.) to which, in the opinion of the authors, they are observationally connected. The first published rigorous treatment of the subject matter was given explicitly, without derivation, in Appendix A by Geomatics Canada (1996, p.19-20). The same set of equations were reproduced verbatim by Craig and Wahl (2003) supporting the same general methodology concept for cadastral applications. A few years later, identical ideas were posted on the Web by Wallace (2009). The same set of rigorous equations has been used by several authors in practical engineering applications (e.g. Marendić et al. 2011; Lee and Seo 2012).

However, in a book published by Burkholder (2008), perhaps unaware of the aforementioned references, that author introduces a new approximate approach that is not as rigorous as the original written formulations of local accuracies that were previously available in print. The inaccuracy and limitation of this approximation has been confirmed in Soler and Smith (2010) by introducing a novel independent derivation. The controversy was further cleared and settled in a discussion and closure published in this journal (Burkholder 2012; Soler and Smith 2012). Unfortunately, a recent publication by the same author still overlooked the general scientific consensus and a plethora of earlier established facts and again insisted on a similar approximation (Burkholder 2014). To finally settle this issue, we feel obligated to revisit the subject matter and close this chapter once and for all by introducing new alternative mathematical proofs, supported by easy to understand concepts, corroborating to the reader interested in mathematical veracity that Burkholder’s derivation is merely an approximation for computing local accuracies. It is now left to the geospatial engineering community to judge the substance of the two approaches and decide which method is better reinforced by the fundamental principles of theoretical rigor and thoroughness.

Theoretical background

Figure 1 depicts two points denoted 1 and 2 on the surface of a preselected reference ellipsoid. To clarify our arguments, it is assumed that the two points are located along the same meridian which is presumed contained on the plane of the paper. This restriction is introduced to illuminate the comprehension of the concepts although this simplification will not affect the final interpretation of results. A current (e.g. GNSS-defined) global terrestrial geocentric frame (x, y, z), is assumed at the origin of the ellipsoid (not shown in Fig. 1). At point 1 two local (topocentric) frames have been drawn. A local frame (x_1, y_1, z_1) which is parallel to the global terrestrial geocentric frame is identified at point 1 and would be referred herein as the "local terrestrial frame" at point 1. Similarly, the so-called local horizon geodetic frame (e_1, n_1, u_1) at point 1 is also depicted in the figure. By definition (see e.g. Soler 1988), the direction e points towards the geodetic positive east, n points to the geodetic positive north, and u (up) points to the geodetic zenith, positive up. Notice that all local frames are right-handed; furthermore, it follows from the assumptions mentioned above that the e-axis is perpendicular to the plane of the paper, with positive direction pointing towards the reader, the n-axis is also on the plane of the selected meridian and is tangent to the ellipsoid at the point and the u-axis is normal to the ellipsoid forming a right-handed triad. It should be made clear also that the z-axis is on the plane of the meridian while the y- and z-axes are not.

The same logic is applied to point 2 where in a similar way two local frames denoted (x_2, y_2, z_2) and (e_2, n_2, u_2) are pictured. Recall that the axes of the local terrestrial frames, by definition, are parallel to the global geocentric frame and thus, to themselves. However, the (e_i, n_i, u_i), i = 1, 2, local geodetic frames have different spatial orientation anywhere on the surface of the ellipsoid.

It is well-know (e.g. Soler 1988) that the transformation of frames (or its coordinates) between the two described local frames at points 1 and 2 follows immediately from:

$$(x_1, y_1, z_1) \xrightleftharpoons[R_1^T]{R_1} (e_1, n_1, u_1) \ : \ \begin{Bmatrix} e_1 \\ n_1 \\ u_1 \end{Bmatrix} = R_1 \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} \tag{1}$$

$$(x_2, y_2, z_2) \xrightleftharpoons[R_2^T]{R_2} (e_2, n_2, u_2) \ : \ \begin{Bmatrix} e_2 \\ n_2 \\ u_2 \end{Bmatrix} = R_2 \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} \tag{2}$$

where the symbol T indicates matrix transpose. The orthogonal matrix that rotates the local terrestrial frame into the local horizon geodetic frame at any point i can be written explicitly as (see e.g. Soler 1976):

$$R_i = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\cos\lambda\sin\varphi & -\sin\lambda\sin\varphi & \cos\varphi \\ \cos\lambda\cos\varphi & \sin\lambda\cos\varphi & \sin\varphi \end{bmatrix}_i \ ; \ i = 1, 2 \tag{3}$$

With this information, it can be proved that the variance covariance matrix (v-c) of the local horizon geodetic frame (e_i, n_i, u_i) determined as a function of the v-c matrix of the local terrestrial frame (x_1, y_1, z_1) can be written as (Soler and Smith 2010):

$$\Sigma_{(e_1,n_1,u_1)} = R_1 \Sigma_{(x_1,y_1,z_1)} R_1^T \tag{4}$$

and similarly at point 2:

$$\Sigma_{(e_2,n_2,u_2)} = R_2 \Sigma_{(x_2,y_2,z_2)} R_2^T \tag{5}$$

The local accuracies are also referred to in the geodetic-surveying literature as an average of some set of relative accuracies. However, to avoid any possible confusion, in this article one is going to restrict the name of relative local accuracies to the ones referred only to the (*x*, *y*, *z*) frame leaving the nomenclature of "local accuracies" exclusively to the relative accuracies referred to the local geodetic frames (*e*, *n*, *u*).

Let us propagate errors to the basic equation defining the concept of relative local accuracies between two points:

$$\left.\begin{array}{l} \Delta e = e_1 - e_2 \\ \Delta n = n_1 - n_2 \\ \Delta u = u_1 - u_2 \end{array}\right\} \tag{6}$$

The equation written above is also the starting point of the whole mathematical development followed by Burkholder (2012). To facilitate the understanding, and in order to simplify as much as possible the mathematical derivation, the assumption is made that the full v-c matrix of the original terrestrial coordinates (the so-called network accuracies), is restricted to two points and, furthermore, that it is block diagonal (the cross-correlations between points are assumed zero), therefore, one can write explicitly:

$$\Sigma^*_{(x,y,z)} = \left[\begin{array}{c:c} \underset{3\times3}{\Sigma_{(x_1,y_1,z_1)}} & 0 \\ \hdashline 0 & \underset{3\times3}{\Sigma_{(x_2,y_2,z_2)}} \end{array}\right] = \left[\begin{array}{c:c} \Sigma_{11} & 0 \\ \hdashline 0 & \Sigma_{22} \end{array}\right] \tag{7}$$

The asterisk * indicates that the assumption of zero cross-correlations is enforced implying that the v-c matrix is block diagonal. In other words, although there are correlations between the coordinates of each point, nevertheless, the points themselves are not correlated. This situation appears in practice when one combines in a v-c matrix points that belong to two different adjustments with no common observations.

Applying the propagation of errors law to Eq. (6), immediately follows

$$\Sigma^*_{(\Delta e,\Delta n,\Delta u)_{1\to 2}} = \Sigma_{(e_1,n_1,u_1)} + \Sigma_{(e_2,n_2,u_2)} = \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{2\to 1}} \tag{8}$$

This undoubtedly shows that the variances and covariances at points 1 and 2 are scalar quantities that could be added. Finally, substituting Eqs. (4) and (5) in Eq. (8) one arrives, under the stated assumptions, to the rigorous expression to determine the local (relative) accuracies between two arbitrary points 1 and 2 (recall that cross-correlations were assumed zero):

$$Rigorous \Rightarrow \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{1\to 2}} = R_1 \Sigma_{11} R_1^T + R_2 \Sigma_{22} R_2^T = \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{2\to 1}} \tag{9}$$

The above equation can be approximated as follows (Burkholder, 2008). However, as we will see later, this formulation is just an approximation of Eq. (9):

$$Approximate \Rightarrow \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{1\to 2}} = R_1 \Sigma_{11} R_1^T + R_1 \Sigma_{22} R_1^T \neq \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{2\to 1}} \tag{10}$$

The first immediate conclusion comparing Eqs. (9) and (10) is that while Eq. (9) satisfies the commutative property, Eq. (10) does not. It is very instinctive to comprehend that the "relative accuracy" between two points 1 and 2 should be equal to the "relative accuracy" between points 2 and 1 independent of if one is talking about (x, y, z) or (e, n, u) coordinates. Rigorously speaking, they should be identical. Nevertheless, this condition is not enforced by Burkholder's Eq. (10).

Consequences of using the approximation equation

Although in practice, primarily for short distances, Eqs. (9) and (10) may return the same or similar values, mathematically and conceptually speaking, Eq. (10) is an approximation. Let us concentrate further in the differences inherent to Eqs. (9) and (10). The consequences of implementing the rigorous or the approximate equations could be described explicitly by the two procedures outlined below.

Figure 2 schematically shows hypothetical error ellipsoids referred to the local terrestrial frames at points 1 and 2. This is the original information available when using modern three-dimensional GNSS techniques.

In the rigorous derivation of local accuracies (Soler and Smith 2010), the following steps are executed:

1) Transform the v-c matrix referred to the local terrestrial frame at point 1 to the local horizon geodetic frame at point 1 (Eq. (4))
2) Transform the v-c matrix referred to the local terrestrial frame at point 2 to the local horizon geodetic frame at point 2 (Eq. (5))

   or vice versa
3) Add up the v-c matrices obtained in 1) and 2) (see Eq. (9))

The resultant value is a unique v-c matrix termed the v-c matrix of local accuracies between points 1 and 2 or between point 2 and 1, both are identical. This definition and the corresponding final equations are supported by many publications, among them, Geomatics Canada (1996), Craig and Wahl (2003), Wallace (2009), Soler and Smith (2010; 2012) and Soler et al. (2012).

The alternative procedure suggested by Burkholder in his book (Burkholder 2008) although not clearly demonstrated mathematically, in practical terms, performs the following steps:

1) Transform the v-c matrix referred to the local terrestrial frame at point 1 to the local horizon geodetic frame at point 1 (Eq. (4))

2) Transform the v-c matrix referred to the local terrestrial coordinates at point 2 to the local horizon geodetic frame at point 1

3) Add up the v-c matrices obtained in 1) and 2) (see Eq. (10))

However, in this case the resultant relative value of the v-c matrix between two points is not unique as it should be.

Notice nevertheless, that this second approach is mathematically, as well as intuitively, an approximation. Why should the error ellipsoid of the terrestrial coordinates at point 2 be transformed into the local horizon geodetic frame at point 1, when the actual observations and reductions were performed at point 2? For example, the local plumb line (or the normal to the ellipsoid), meridian, etc. at point 2 are generally different to the ones at point 1, furthermore the local environment of point 2 (e.g. GNSS atmospheric corrections, etc.) has nothing to do with point 1. It does not make sense to assume that the local observational errors at point 2 could be transferred to point 1! There exists a unique "combined" local (relative) accuracy value between any two points, period! And this fact is obtained rigorously by propagating errors to Eq. (6). If the definition of local accuracies is to be changed, one first should explain the mathematical rigor of the equations and convince general audience of the intrinsic characteristics of the final product that one wants to propose. That will require a theoretical derivation that starts with propagating errors appropriately from Eq. (6) avoiding the risk of mixing up concepts in the process.

Local accuracies at points in networks

Let us assume a simple spatial network like the one shown in Fig. 3. Considering that every two points produces a single local accuracy value, the total number of unique local accuracies between n points grouped by sets of $m = 2$ points ($n>m$) is:

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \tag{11}$$

In the example of Fig. 3, n = 5 and m = 2. Then, substituting these values in Eq. (11), the total number of unique two-point local accuracies for the network in Fig. 3 will be equal to 10. Written them explicitly:

$$\Sigma^*_{(\Delta e,\Delta n,\Delta u)_{1\to 2}} = R_1 \Sigma_{11} R_1^T + R_2 \Sigma_{22} R_2^T = \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{2\to 1}} \tag{12}$$

$$\Sigma^*_{(\Delta e,\Delta n,\Delta u)_{1\to 3}} = R_1 \Sigma_{11} R_1^T + R_3 \Sigma_{33} R_3^T = \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{3\to 1}} \tag{13}$$

$$\Sigma^*_{(\Delta e,\Delta n,\Delta u)_{1\to 4}} = R_1 \Sigma_{11} R_1^T + R_4 \Sigma_{44} R_4^T = \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{4\to 1}} \tag{14}$$

$$\Sigma^*_{(\Delta e,\Delta n,\Delta u)_{1\to 5}} = R_1 \Sigma_{11} R_1^T + R_5 \Sigma_{55} R_5^T = \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{5\to 1}} \tag{15}$$

$$\Sigma^*_{(\Delta e,\Delta n,\Delta u)_{2\to 3}} = R_2 \Sigma_{22} R_2^T + R_3 \Sigma_{33} R_3^T = \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{3\to 2}} \tag{16}$$

$$\Sigma^*_{(\Delta e,\Delta n,\Delta u)_{2\to 4}} = R_2 \Sigma_{22} R_2^T + R_4 \Sigma_{44} R_4^T = \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{4\to 2}} \tag{17}$$

$$\Sigma^*_{(\Delta e,\Delta n,\Delta u)_{2\to 5}} = R_2 \Sigma_{22} R_2^T + R_5 \Sigma_{55} R_5^T = \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{5\to 2}} \tag{18}$$

$$\Sigma^*_{(\Delta e,\Delta n,\Delta u)_{3\to 4}} = R_4 \Sigma_{44} R_4^T + R_3 \Sigma_{33} R_3^T = \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{4\to 3}} \tag{19}$$

$$\Sigma^*_{(\Delta e,\Delta n,\Delta u)_{3\to 5}} = R_3 \Sigma_{33} R_3^T + R_5 \Sigma_{55} R_5^T = \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{5\to 3}} \tag{20}$$

$$\Sigma^*_{(\Delta e,\Delta n,\Delta u)_{4\to 5}} = R_4 \Sigma_{44} R_4^T + R_5 \Sigma_{55} R_5^T = \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{5\to 4}} \tag{21}$$

Then, the average local accuracy at one point (point 1, for example) in a network could be defined as the average of all local accuracies connecting points radiating from that point.

Therefore, in this particular example, the average local accuracy at point 1 can be computed from the following equation:

$$\bar{\Sigma}^*_{(\Delta e,\Delta n,\Delta u)_{1\to 2,3,4,5}} = \frac{\Sigma^*_{(\Delta e,\Delta n,\Delta u)_{1\to 2}} + \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{1\to 3}} + \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{1\to 4}} + \Sigma^*_{(\Delta e,\Delta n,\Delta u)_{1\to 5}}}{4} \tag{22}$$

And substituting the values of Eqs. (12), (13), (14) and (15) above, after simplification one gets

$$\bar{\Sigma}^*_{(\Delta e,\Delta n,\Delta u)_{1\to 2,3,4,5}} = R_1 \Sigma_{11} R_1^T + \frac{R_2 \Sigma_{22} R_2^T + R_3 \Sigma_{33} R_3^T + R_4 \Sigma_{44} R_4^T + R_5 \Sigma_{55} R_5^T}{4} \tag{23}$$

which clearly makes a lot of sense. The maximum contribution rests on the accuracy of point 1 while the remaining contributions are averaged out. For this main reason, its averaged local accuracy error ellipsoid could be assumed that corresponds to the radiating point, in this case point 1. Consequently, at every point one can assume two error ellipsoids, the original network ellipsoid and the averaged local accuracy error ellipsoid for that point. The averaged local terrestrial error ellipsoid (referred to a frame parallel to the global (x, y, z) frame) is not considered as intuitive as the averaged local accuracy error ellipsoid, because of the difficulty of visualizing in space the x-, y-, and z-axis, therefore, it is neglected in this discussion. This is simply because to know the statistics (variances and covariances) referred to a local north and east in the local horizon plane is more practical and can be easily envisioned. Furthermore, it is clear from Fig. 3 that the averaged local accuracies for points such as number 5, that has other points around it, should get a more realistic value of the quality of the survey at this point in that local area. This is precisely the advantage of providing local accuracies, the observational errors inherent to a points radiating from an arbitrary point also contribute to the final quality of the estimation of its accuracy.

Recall now that equation (23) assumes that the v-c matrix of the points referred to the global terrestrial frame was block diagonal. Otherwise, the contribution of the non-diagonal blocks ($\Sigma_{12}, \Sigma_{13}, \Sigma_{14}, \Sigma_{15}$) and their transposes should be accounted for. For example, the rigorous (complete) average local accuracies at point 1 (see Fig. 3) with only connections to point 2 and 3 takes the form (Soler et al. 2012):

$$\bar{\Sigma}_{(\Delta e,\Delta n,\Delta u)_{1\to 2,3}} = \frac{\Sigma_{(\Delta e,\Delta n,\Delta u)_{1\to 2}} + \Sigma_{(\Delta e,\Delta n,\Delta u)_{1\to 3}}}{2}$$

$$= \frac{R_1\Sigma_{11}R_1^T + R_2\Sigma_{22}R_2^T - R_1\Sigma_{12}R_2^T - R_2\Sigma_{21}R_1^T + R_1\Sigma_{11}R_1^T + R_3\Sigma_{33}R_3^T - R_1\Sigma_{13}R_3^T - R_3\Sigma_{31}R_1^T}{2}$$

$$= R_1\Sigma_{11}R_1^T + \frac{1}{2}[R_2\Sigma_{22}R_2^T + R_3\Sigma_{33}R_3^T - R_1\Sigma_{12}R_2^T - R_2\Sigma_{21}R_1^T - R_1\Sigma_{13}R_3^T - R_3\Sigma_{31}R_1^T] \tag{24}$$

A further clarification is in order: the U.S. Federal Geographic Data Committee (FGDC) (1998) specifies that “local accuracy” be provided as a 95% confidence interval. A practical numerical example of equation (24) applied to three points resulting from a 3D GNSS network computed at the 95% confident levels was explained in Soler et al. (2012).

Computation of the average local accuracies in a network using matrix algebra

A simple matrix procedure to compute the average local accuracies from the original “network accuracy” variance-covariance matrix could be written as follows:

$$\bar{\Sigma}_{(\Delta e,\Delta n,\Delta u)_{1\to 2,3\ldots n-1}} = \frac{1}{n-1} Trace_b \left[[\Im_1][\Re] \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1n} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2n} \\ & & \ddots & \vdots \\ sym. & & & \Sigma_{nn} \end{bmatrix} [\Re]^T[\Im_1]^T \right] \tag{25}$$

In the above equation the following operator has been introduced:

$Trace_b$ = Sum of the diagonal 3x3 blocks of a square matrix formed by 3x3 blocks

The explicit form of the other matrices in equation (25) are:

$$\underset{3(n-1)\times 3n}{[\Im_1]} = \begin{bmatrix} [I] & -[I] & [0] & [0] & \cdots & [0] \\ [I] & [0] & -[I] & [0] & \cdots & [0] \\ [I] & [0] & [0] & -[I] & \cdots & [0] \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ [I] & [0] & [0] & [0] & \cdots & -[I] \end{bmatrix} ; \text{ and } \underset{3n\times 3n}{[\Re]} = \begin{bmatrix} R_1 & [0] & [0] & [0] & [0] \\ & R_2 & [0] & [0] & [0] \\ & & R_3 & [0] & [0] \\ & & & \ddots & \vdots \\ sym. & & & & R_n \end{bmatrix} \tag{26}$$

where $[I]$ is the 3x3 unit matrix. Similarly for point 2,

$$\bar{\Sigma}_{(\Delta e,\Delta n,\Delta u)_{2\to 1,3\ldots n-1}} = \frac{1}{n-1} Trace_b \left[[\Im_2][\Re] \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1n} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2n} \\ & & \ddots & \vdots \\ sym. & & & \Sigma_{nn} \end{bmatrix} [\Re]^T [\Im_2]^T \right] \tag{27}$$

where now:

$$\underset{3(n-1)\times 3n}{[\Im_2]} = \begin{bmatrix} -[I] & [I] & [0] & [0] & \cdots & [0] \\ [0] & [I] & -[I] & [0] & \cdots & [0] \\ [0] & [I] & [0] & -[I] & \cdots & [0] \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ [0] & [I] & [0] & [0] & \cdots & -[I] \end{bmatrix} \tag{28}$$

In general one can write:

$$\underset{3(n-1)\times 3n}{[\Im_i]_{i\neq 1}} = \begin{array}{c} \begin{array}{cccccccc} & & & & i\ column & & & \\ & & & & \downarrow & & & \end{array} \\ \left[\begin{array}{ccccccc} -[I] & & & & [I] & & \\ & -[I] & & & [I] & & \\ & & \ddots & & \vdots & & \\ & & & -[I] & [I] & & \\ \hline & & & & [I] & -[I] & \\ \hline & & & & \vdots & \ddots & \\ & & & & [I] & & -[I] \end{array} \right] \begin{array}{l} \\ \\ \\ \\ \leftarrow i\ row \\ \\ \\ \end{array} \end{array} \tag{29}$$

where the rest of the 3x3 blocks not shown in the above matrix are equal to zero. Another important clarification should be stressed; the matrix written above assumes that the averaged "local accuracies" are computed for all the points using as nearby points the rest of the points contained in the v-c matrix of the network accuracies. Generally speaking this will not be the case, primarily because the local accuracies only need to be determined for stations inside a circle with prescribed radius from the central station where the value of $\overline{\Sigma}_{(\Delta e,\Delta n,\Delta u)}$ is sought. In this case the matrix $\underset{3(n-1)\times 3n}{[\Im_i]_{i\neq 1}}$ takes the form:

$$\underset{3(n-1)\times 3n}{[\Im_i]_{i\rightarrow j,k\ldots}} = \begin{array}{c} \begin{array}{ccccccc} & j\ column & & i\ column & & k\ column & \\ & \downarrow & & \downarrow & & \downarrow & \end{array} \\ \left[\begin{array}{c:c:c:c:c:c:c} \ddots & & & \vdots & & & \\ \hdashline & -[I] & & [I] & & & \\ \hdashline & & \ddots & \vdots & & & \\ & & \ddots & [I] & & & \\ \hdashline & & & [I] & & & \\ \hdashline & & & [I] & \ddots & & \\ & & & \vdots & \ddots & & \\ \hdashline & & & [I] & & -[I] & \\ \hdashline & & & \vdots & & & \ddots \end{array}\right] \begin{array}{l} \\ \leftarrow j\ row \\ \\ \\ \leftarrow i\ row \\ \\ \\ \leftarrow k\ row \\ \\ \end{array} \end{array} \tag{30}$$

Using this matrix in Eq. (25) the resulting algorithm will automatically compute the averaged local accuracy using any arbitrary number of selected points surrounding the "origin point" i. Another possible alternative is to use only site connected directly by observations.

Compendium of useful equations

Assume any 3D network (e.g. a 3D GNSS-determined geocentric network) with given v-c matrix of network accuracies:

$$\Sigma_{(x,y,z)} = \begin{bmatrix} \Sigma_{ii} & \cdots & \Sigma_{ij} & \cdots \\ \vdots & \ddots & \vdots & \\ \Sigma_{ji} & \cdots & \Sigma_{jj} & \cdots \\ sym & & & \ddots \end{bmatrix}; \Sigma_{ii} = \begin{bmatrix} \sigma_{x_i}^2 & \sigma_{x_i y_i} & \sigma_{x_i z_i} \\ & \sigma_{y_i}^2 & \sigma_{y_i z_i} \\ sym & & \sigma_{z_i}^2 \end{bmatrix}; \Sigma_{ij} = \begin{bmatrix} \sigma_{x_i x_j} & \sigma_{x_i y_j} & \sigma_{x_i z_j} \\ & \sigma_{y_i y_j} & \sigma_{y_i z_j} \\ sym & & \sigma_{z_i z_j} \end{bmatrix}$$

(31)

"Relative network accuracies" between two points i and j are defined through the mathematical model:

$$\left.\begin{aligned} \Delta x = x_i - x_j = -\left(x_j - x_i\right) \\ \Delta y = y_i - y_j = -\left(y_j - y_i\right) \\ \Delta z = z_i - z_j = -\left(z_j - z_i\right) \end{aligned}\right\} \tag{32}$$

$$\begin{aligned} \Sigma_{(\Delta x, \Delta y, \Delta z)_{i \to j}} &= [[I] \vdots [-I]] \Sigma_{(x,y,z)_{ij}} \left[\begin{array}{c} [I] \\ \hdashline [-I] \end{array}\right] = [[I] \vdots [-I]] \left[\begin{array}{c:c} \Sigma_{ii} & \Sigma_{ij} \\ \hdashline \Sigma_{ji} & \Sigma_{jj} \end{array}\right] \left[\begin{array}{c} [I] \\ \hdashline [-I] \end{array}\right] \\ &= \Sigma_{ii} + \Sigma_{jj} - \Sigma_{ij} - \Sigma_{ji} = \Sigma_{(\Delta x, \Delta y, \Delta z)_{j \to i}} \end{aligned} \tag{33}$$

And the explicit form of the above equation can be written:

$$\begin{aligned} \Sigma_{(\Delta x, \Delta y, \Delta z)_{i \to j}} &= \begin{bmatrix} \sigma_{x_i}^2 - 2\sigma_{x_i x_j} + \sigma_{x_j}^2 & \sigma_{x_i y_i} - \sigma_{x_i y_j} - \sigma_{x_j y_i} + \sigma_{x_j y_j} & \sigma_{x_i z_i} - \sigma_{x_i z_j} - \sigma_{x_j z_i} + \sigma_{x_j z_j} \\ & \sigma_{y_i}^2 - 2\sigma_{y_i y_j} + \sigma_{y_j}^2 & \sigma_{y_i z_i} - \sigma_{y_i z_j} - \sigma_{y_j z_i} + \sigma_{y_j z_j} \\ sym. & & \sigma_{z_i}^2 - 2\sigma_{z_i z_j} + \sigma_{z_j}^2 \end{bmatrix} \\ &= \Sigma_{(\Delta x, \Delta y, \Delta z)_{j \to i}} \end{aligned}$$

(34)

The original v-c matrix of network accuracies could be referred to the local horizon frames at each point as was introduced in Soler and Smith (2010):

$$\Sigma_{(e,n,u)} = \begin{bmatrix} \Sigma_{(e,n,u)_{ii}} & \cdots & \Sigma_{(e,n,u)_{ij}} & \cdots \\ \vdots & \ddots & \vdots & \\ \Sigma_{(e,n,u)_{ji}} & \cdots & \Sigma_{(e,n,u)_{jj}} & \cdots \\ sym. & & & \ddots \end{bmatrix} = \begin{bmatrix} R_i \Sigma_{ii} R_i^T & \cdots & R_i \Sigma_{ij} R_j^T & \cdots \\ \vdots & \ddots & \vdots & \\ R_j \Sigma_{ji} R_i^T & \cdots & R_j \Sigma_{jj} R_j^T & \cdots \\ sym. & & & \ddots \end{bmatrix} \tag{35}$$

The general form of the rotation matrix R_i is given by Eq. (3). This equation is one of the most important developments in the theory of local accuracies introduced by Soler and Smith (2010). As we will see below, this matrix equation is critical for the development of the rigorous form of other types of relative accuracies. Then, the (relative) local accuracies, by definition, are derived propagating errors from the mathematical model:

$$\left.\begin{aligned} \Delta e &= e_i - e_j = -\left(e_j - e_i\right) \\ \Delta n &= n_i - n_j = -\left(n_j - n_i\right) \\ \Delta u &= u_i - u_j = -\left(u_j - u_i\right) \end{aligned}\right\} \tag{36}$$

Written in compact matrix algebra the equation that should be used to compute local accuracies between two points *i* and *j* is:

$$\Sigma_{(\Delta e,\Delta n,\Delta u)_{i\to j}} = \left[[I] \vdots [-I]\right]\left[\begin{array}{c:c} \Sigma_{(e,n,u)_{ii}} & \Sigma_{(e,n,u)_{ij}} \\ \hdashline \Sigma_{(e,n,u)_{ji}} & \Sigma_{(e,n,u)_{jj}} \end{array}\right]\left[\begin{array}{c} [I] \\ \hdashline [-I] \end{array}\right] = \left[[I] \vdots [-I]\right]\left[\begin{array}{c:c} R_i\Sigma_{ii}R_i^T & R_i\Sigma_{ij}R_j^T \\ \hdashline R_j\Sigma_{ji}R_i^T & R_j\Sigma_{jj}R_j^T \end{array}\right]\left[\begin{array}{c} [I] \\ \hdashline [-I] \end{array}\right]$$

$$= \Sigma_{(e,n,u)_{ii}} + \Sigma_{(e,n,u)_{jj}} - \Sigma_{(e,n,u)_{ij}} - \Sigma_{(e,n,u)_{ji}} = R_i\Sigma_iR_i^T + R_j\Sigma_jR_j^T - R_i\Sigma_{ij}R_j^T - R_j\Sigma_{ji}R_i^T = \Sigma_{(\Delta e,\Delta n,\Delta u)_{j\to i}} \tag{37}$$

Finally, using the equality in Eq. (35), the explicit form of Eq. (37) takes the form:

$$\Sigma_{(\Delta e,\Delta n,\Delta u)_{i\to j}} = \begin{bmatrix} \sigma_{e_i}^2 - 2\sigma_{e_ie_j} + \sigma_{e_j}^2 & \sigma_{e_in_i} - \sigma_{e_in_j} - \sigma_{e_jn_i} + \sigma_{e_jn_j} & \sigma_{e_iu_i} - \sigma_{e_iu_j} - \sigma_{e_ju_i} + \sigma_{e_ju_j} \\ & \sigma_{n_i}^2 - 2\sigma_{n_in_j} + \sigma_{n_j}^2 & \sigma_{n_iu_i} - \sigma_{n_iu_j} - \sigma_{n_ju_i} + \sigma_{n_ju_j} \\ sym. & & \sigma_{u_i}^2 - 2\sigma_{u_iu_j} + \sigma_{u_j}^2 \end{bmatrix}$$

$$= \Sigma_{(\Delta e,\Delta n,\Delta u)_{j\to i}} \tag{38}$$

Notice the remarkable similarities between the notations of Eqs. (34) and (38). It simply amounts to a change in the subscripts. This equation, as mentioned above, was originally published by Geomatics Canada (1996). The derivation of local accuracies using the explicit form of Eq. (38) is an accepted practice supported by international investigators who appropriately cite the Geomatics Canada report (e.g. Marendić et al. 2011, Eq. (2); Lee and Seo 2012, Eq. (21)).

Although some authors represent the local accuracies error ellipses (or ellipsoids) at the center of the line connecting two arbitrary points i and j, this practice is not recommended. In the first place because it could be confused with the error ellipse (ellipsoid) computed at the middle point of the line connecting two arbitrary points. As the reader will see below, the local accuracies error ellipsoid and the middle point of the line error ellipsoid are not the same.

Variance-covariance matrix at the average (middle) point of a spatial segment when the stochastic information at the end points is available

The mathematical model is:

$$\left.\begin{aligned} x_m &= \frac{x_i + x_j}{2} \\ y_m &= \frac{y_i + y_j}{2} \\ z_m &= \frac{z_i + z_j}{2} \end{aligned}\right\} \tag{39}$$

As usual, the network full v-c matrix is given by:

$$\Sigma_{(x,y,z)} = \begin{bmatrix} \Sigma_{ii} & \cdots & \Sigma_{ij} & \cdots \\ \vdots & \ddots & \vdots & \\ \Sigma_{ji} & \cdots & \Sigma_{jj} & \cdots \\ sym & & & \ddots \end{bmatrix} \tag{40}$$

Then, propagating errors:

$$\Sigma_{(x_m,y_m,z_m)_{ij}} = J \Sigma_{(x,y,z)_{ij}} J^T \tag{41}$$

where

$$J = \frac{\partial(x_m, y_m, z_m)}{\partial(x_i, y_i, z_i, x_j, y_j, z_j)} = \frac{1}{2}\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = \frac{1}{2}\left[[I] \,\vdots\, [I]\right] \tag{42}$$

and substituting (42) into (41), finally:

$$\Sigma_{(x_m,y_m,z_m)_{ij}} = \frac{1}{4}[\,\Sigma_{ii} + \Sigma_{jj} + \Sigma_{ij} + \Sigma_{ji}\,] \tag{43}$$

This is an interesting result. The v-c matrix at the middle point of a spatial line between points i and j is equal to one fourth of the sum of the four matrices (two v-c diagonal block matrices and two non-diagonal cross-covariance matrices) related to the points.

If one compares Eq. (43) with Eq. (33) immediately follows:

$$\Sigma_{(\Delta x,\Delta y,\Delta z)_{i\to j}} = 4\Sigma_{(x_m,y_m,z_m)_{ij}} - 2[\,\Sigma_{ij} + \Sigma_{ji}\,] \tag{44}$$

Using Eq. (31) the explicit form of Eq. (43) easily follows:

$$\Sigma_{(x_m,y_m,z_m)_{ij}} = \frac{1}{4}\begin{bmatrix} \sigma^2_{x_i} + 2\sigma_{x_i x_j} + \sigma^2_{x_j} & \sigma_{x_i y_i} + \sigma_{x_i y_j} + \sigma_{x_j y_i} + \sigma_{x_j y_j} & \sigma_{x_i z_i} + \sigma_{x_i z_j} + \sigma_{x_j z_i} + \sigma_{x_j z_j} \\ & \sigma^2_{y_i} + 2\sigma_{y_i y_j} + \sigma^2_{y_j} & \sigma_{y_i z_i} + \sigma_{y_i z_j} + \sigma_{y_j z_i} + \sigma_{y_j z_j} \\ sym. & & \sigma^2_{z_i} + 2\sigma_{z_i z_j} + \sigma^2_{z_j} \end{bmatrix} \tag{45}$$

To get the value of Eq. (43) referred to the local horizon plane (e, n, u), in other words $\Sigma_{(e_m,n_m,u_m)_{ij}}$, following the logic developed from our first paper about local accuracies and recalled herein, Eq. (43) takes the form:

$$\Sigma_{(e_m,n_m,u_m)_{ij}} = \frac{1}{4}[\,R_i\Sigma_{ii}R_i^T + R_j\Sigma_{jj}R_j^T + R_i\Sigma_{ij}R_j^T + R_j\Sigma_{ji}R_i^T\,] \tag{46}$$

and after replacing the values from Eq. (35) immediately follows:

$$\Sigma_{(e_m,n_m,u_m)_{ij}} = \frac{1}{4}\begin{bmatrix} \sigma^2_{e_i} + 2\sigma_{e_i e_j} + \sigma^2_{e_j} & \sigma_{e_i n_i} + \sigma_{e_i n_j} + \sigma_{e_j n_i} + \sigma_{e_j n_j} & \sigma_{e_i u_i} + \sigma_{e_i u_j} + \sigma_{e_j u_i} + \sigma_{e_j u_j} \\ & \sigma^2_{n_i} + 2\sigma_{n_i n_j} + \sigma^2_{n_j} & \sigma_{n_i u_i} + \sigma_{n_i u_j} + \sigma_{n_j u_i} + \sigma_{n_j u_j} \\ sym. & & \sigma^2_{u_i} + 2\sigma_{u_i u_j} + \sigma^2_{u_j} \end{bmatrix} \tag{47}$$

This corroborates, as before, that the symbolic notation equivalence between Eqs. (45) and (47) is retained. The derivation of Eq. (47) would have been very difficult to compute directly from the initial math model defined by Eq. (39) after it has been expressed in the (e, n, u) frame without the introduction of Eq. (35). This validates, once more, that our equations to determine accurate local accuracies are generally rigorous and correct.

Similarly to Eq. (44) it can be written:

$$\Sigma_{(\Delta e,\Delta n,\Delta u)_{i\to j}} = 4\Sigma_{(e_m,n_m,u_m)_{ij}} - 2[\,R_i\Sigma_{ij}R_j^T + R_j\Sigma_{ji}R_i^T\,] \tag{48}$$

On error ellipsoids

As Fig. 3 shows there is a unique error ellipsoid at each point that can be determined from the original network v-c matrix $\Sigma_{(x_i,y_i,z_i)}$ for any arbitrary point i. For simplicity, only error ellipsoids at points 1, 2, and 3 have been drawn in the figure. Let's assume that the network v-c matrix of points 1 and 2 is:

$$\mathbf{\Sigma}_{(x,y,z)} = \mathbf{\Sigma}_{(x_1,y_1,z_1,x_2,y_2,z_2)} = \begin{bmatrix} \begin{bmatrix} 3.003 & 3.508 & -0.743 \\ 3.508 & 34.460 & -17.864 \\ -0.743 & -17.864 & 16.151 \end{bmatrix} & \begin{bmatrix} 1.222 & 0.961 & 0.606 \\ 0.962 & 9.045 & -4.527 \\ 0.593 & -4.537 & 7.046 \end{bmatrix} \\ \begin{bmatrix} 1.222 & 0.962 & 0.593 \\ 0.961 & 9.045 & -4.537 \\ 0.606 & -4.527 & 7.046 \end{bmatrix} & \begin{bmatrix} 4.755 & 6.882 & -2.792 \\ 6.882 & 57.717 & -31.039 \\ -2.792 & -31.039 & 25.470 \end{bmatrix} \end{bmatrix} (\mathrm{cm}^2) \tag{49}$$

Point 1 has the following geodetic curvilinear coordinates: λ_1 = 262° 53' 22.1562", φ_1 = 31° 34' 39.7778", h_1 = 101.712 m referred to the ITRF2000 frame and GRS80 ellipsoid. Then, if one computes the eigenvalues and eigenvectors of the first 3x3 diagonal block in Eq. (49) one obtains the following diagonal matrix of eigenvalues:

$$\Lambda_{(x_1,y_1,z_1)} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_{(x_1,y_1,z_1)} = \begin{bmatrix} 45.647 & 0 & 0 \\ 0 & 5.710 & 0 \\ 0 & 0 & 2.258 \end{bmatrix} \quad (\mathrm{cm}^2) \tag{50}$$

with diagonal elements $\lambda_1 > \lambda_2 > \lambda_3$ and the matrix of column eigenvectors:

$$S_{(x_1,y_1,z_1)} = \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}_{(x_1,y_1,z_1)} = \begin{bmatrix} -0.0791 & 0.3706 & 0.9254 \\ -0.8518 & 0.4571 & -0.2559 \\ 0.5179 & 0.8085 & -0.2795 \end{bmatrix} = \begin{bmatrix} s_{1_x} & s_{2_x} & s_{3_x} \\ s_{1_y} & s_{2_y} & s_{3_y} \\ s_{1_z} & s_{2_z} & s_{3_z} \end{bmatrix} (\mathrm{cm}) \tag{51}$$

The square roots of the diagonal elements of Eq. (50) are the values of the three "principal axes" of the error ellipsoid with semi-axes $a = \sqrt{\lambda_1} =$ 6.830cm; $b = \sqrt{\lambda_2} =$ 2.389cm; $c = \sqrt{\lambda_3} = 1.503$cm. Notice that the semi-axes of the error ellipsoid are not equal to the standard deviations at the point (square roots of the diagonal elements in the v-c of Eq. (49)), namely, $\sigma_{x_1} =$ 1.733cm; $\sigma_{y_1} =$ 5.870cm; and $\sigma_{z_1} = 4.019$cm.

The angles defining the orientations of the three principal axes in the *x*-*y*-*z* frame are:

$$\tan \bar{\lambda}_k = \frac{s_{k_y}}{s_{k_x}}; \quad \tan \varphi_k = \frac{s_{k_z}}{\sqrt{s_{k_x}^2 + s_{k_y}^2}} \quad k = 1,2,3 \ \textit{principal axes}$$

$\bar{\lambda}_1 = 84.6959^\circ$; $\varphi_1 = 31.1898^\circ$; $\bar{\lambda}_2 = 50.9696^\circ$; $\varphi_2 = 53.9499^\circ$; $\bar{\lambda}_3 = 344.5452^\circ$; $\varphi_3 = 16.2314^\circ$.

Now, as mentioned above, the v-c matrix of point 1 referred to the (e_1, n_1, u_1) local horizon frame can be computed as follows:

$$\Sigma_{(e_1 n_1 u_1)} = \begin{bmatrix} \sigma_{e_1}^2 & \sigma_{e_1 n_1} & \sigma_{e_1 u_1} \\ & \sigma_{n_1}^2 & \sigma_{n_1 u_1} \\ sym. & & \sigma_{u_1}^2 \end{bmatrix} = R_1 \Sigma_{(x_1 y_1 z_1)} R_1^T = \begin{bmatrix} 2.624 & 1.013 & 1.167 \\ 1.013 & 5.377 & -0.291 \\ 1.167 & -0.291 & 45.613 \end{bmatrix} \quad (\mathrm{cm}^2) \tag{52}$$

Similarly, computing the eigenvalues of the above symmetric matrix, one arrives at:

$$\Lambda_{(e_1,n_1,u_1)} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_{(e_1,n_1,u_1)} = \begin{bmatrix} 45.647 & 0 & 0 \\ 0 & 5.710 & 0 \\ 0 & 0 & 2.258 \end{bmatrix} \quad (\text{cm}^2) \tag{53}$$

Therefore, as expected, we get exactly the same eigenvalues implying that the network error ellipsoid is a unique estimating surface although it could be referred to different local frames. In other words, the magnitudes of the semi-axes of the error ellipsoid are the same independent of the frame used. Hence, the differences between the matrices in the first diagonal block in Eq. (49) and the matrix in Eq. (52) are merely due to the fact that the values in Eq. (49) refer to the local terrestrial frame (x_1, y_1, z_1) while the elements in the matrix of Eq. (52) refer to the local geodetic horizon frame (e_1, n_1, u_1). Consequently, the magnitude of the semi-axes of the error ellipsoid obtained from the two v-c matrices, being scalar quantities, are invariant under rotations and therefore their size is the same in both local frames, although they are taken along the corresponding eigenvectors referred to each frame. Obviously, the components of the eigenvectors look different because they are referred to two different frames. However, only a rotation is involved.

Using the notation introduced above, the analytical proof that the eigenvalues referred to the two frames are the same follows:

$$\Sigma_{(e_1n_1u_1)} = R_1\Sigma_{(x_1y_1z_1)}R_1^T = R_1\left(S_{(x_1y_1z_1)}\Lambda_{(x_1y_1z_1)}S_{(x_1y_1z_1)}^T\right)R_1^T = \left(R_1S_{(x_1y_1z_1)}\right)\Lambda_{(x_1y_1z_1)}\left(S_{(x_1y_1z_1)}^TR_1^T\right) = S_{(e_1n_1u_1)}\Lambda_{(e_1n_1u_1)}S_{(e_1n_1u_1)}^T \tag{54}$$

and consequently,

$$S_{e_1n_1u_1} = R_1S_{x_1y_1z_1} \text{ and } \Lambda_{e_1n_1u_1} = \Lambda_{x_1y_1z_1}\,. \tag{55}$$

In other words, the components of each eigenvector defining the orientation of the semi-axes of the error ellipsoid referred to the local horizon frame (e_1, n_1, u_1) could be obtained from the eigenvectors originally computed and referred to the local terrestrial frame (x_1, y_1, z_1) through a rotation matrix as follow:

$$\{s_1\}_{(e_1,n_1,u_1)}=R_1\{s_1\}_{(x_1,y_1,z_1)};\quad \{s_2\}_{(e_1,n_1,u_1)}=R_1\{s_2\}_{(x_1,y_1,z_1)};\quad \{s_3\}_{(e_1,n_1,u_1)}=R_1\{s_3\}_{(x_1,y_1,z_1)} \tag{56}$$

or, equivalently, using a single matrix multiplication, for point 1, one can write:

$$\begin{aligned} S_{(e_1,n_1,u_1)} = RS_{(x_1,y_1,z_1)} &= \begin{bmatrix} 0.9923 & -0.1238 & 0 \\ 0.0648 & 0.5196 & 0.8519 \\ -0.1055 & -0.8454 & 0.5237 \end{bmatrix} \begin{bmatrix} -0.0791 & 0.3706 & 0.9254 \\ -0.8518 & 0.4571 & -0.2559 \\ 0.5179 & 0.8085 & -0.2795 \end{bmatrix} \\ &= \begin{bmatrix} -0.0270 & -0.3112 & -0.9500 \\ 0.0065 & -0.9504 & 0.3111 \\ -0.9996 & 0.0022 & 0.0277 \end{bmatrix} = \begin{bmatrix} s_{1_e} & s_{2_e} & s_{3_e} \\ s_{1_n} & s_{2_n} & s_{3_n} \\ s_{1_u} & s_{2_u} & s_{3_u} \end{bmatrix} (cm) \end{aligned} \tag{57}$$

The two angles defining the directions of the orientation in space of the three principal axes in the *e*-*n*-*u* frame could be computed as follows (α = geodetic azimuth; v = geodetic vertical angle):

$$\tan\alpha_k = \frac{s_{k_e}}{s_{k_n}};\quad \tan v_k = \frac{s_{k_u}}{\sqrt{s_{k_e}^2 + s_{k_n}^2}}\qquad k = 1,2,3\ \ \textit{principal axes} \tag{58}$$

$$\alpha_1 = 283.6495^0; v_1 = -88.4098^0; \alpha_2 = 18.1289^0; v_2 = 0.1242^0; \alpha_3 = 288.1323^0; v_3 = 1.5853^0 \tag{59}$$

Comparing now Eqs. (52) and (53) it is very clear that looking into the magnitudes of the semi-axes and the variances (diagonal elements of Eq. (52)) that the principal axes of the error ellipsoid (axes *a*, *b*, and *c*) are almost aligned with the up, north, and east directions where a□ along the up direction, b□ along the north direction, and c□ along the east direction, clearly showing that the maximum error is along the height (up) component as it is generally the case when processing GNSS observations. The actual values are $\sigma_{u_1} = 6.754$cm $\approx a$; $\sigma_{n_1} = 2.319$cm $\approx b$; $\sigma_{e_1} = 1.612$cm $\approx c$. This is corroborated by the direction of the first principal axis that was determined to be: $\alpha_1 = 283.6495^0; v_1 = -88.4098^0$. Notice that the other two semi-axes are practically on the plane of the local geodetic horizon (very small v_2 and v_3 angles).

From Eq. (54) other interesting relationships could be discussed. Taking traces of Eq. (54) one can write:

$$Trace\ \Sigma_{(e_1 n_1 u_1)} = Trace\ [R_1 \Sigma_{(x_1 y_1 z_1)} R_1^T] = Trace\ [S_{(e_1 n_1 u_1)} \Lambda_{(e_1 n_1 u_1)} S_{(e_1 n_1 u_1)}^T] \tag{60}$$

By incorporating the cyclic permutation rule into the above equation and reordering terms:

$$Trace\ \Lambda_{(e_1 n_1 u_1)} = Trace\ \Sigma_{(x_1 y_1 z_1)} = Trace\ \Sigma_{(e_1 n_1 u_1)} \tag{61}$$

Or explicitly:

$$\lambda_1 + \lambda_2 + \lambda_3 = a^2 + b^2 + c^2 = \sigma_{x_1}^2 + \sigma_{y_1}^2 + \sigma_{z_1}^2 = \sigma_{e_1}^2 + \sigma_{n_1}^2 + \sigma_{u_1}^2 = \sigma_p^2 \tag{62}$$

where the scalar σ_p^2 receives the name of *point variance*. The above expression indicates that σ_p, *point standard deviation*, is the magnitude of a vector (invariant with respect to rotations) that could be obtained from the components of any of the vectors: (a, b, c), $(\sigma_{x_1}, \sigma_{y_1}, \sigma_{z_1})$, or $(\sigma_{e_1}, \sigma_{n_1}, \sigma_{u_1})$. The orientation of these four vectors is generally not the same but their magnitude is identical. Therefore, the point variance is unique at any point of a network and thus independent of local frame selection.

The above concepts also apply to the discussion of an "averaged local accuracies error ellipsoid" except that now the orientation of this error ellipsoid resulting from equations such as Eq. (27) is always referred to the local frame *e-n-u* (see Fig. 4). This is one of the great advantages of using "local accuracies", the v-c matrix at each point *i* refers to the more intuitive local horizon geodetic frame although it was directly computed from the original full network (absolute) v-c matrix as written mathematically in Eq. (31). Modern GNSS technology permits the computation of full network v-c matrices including the variances (diagonal elements), covariances between coordinates (non-diagonal elements of the diagonal 3 x 3 blocks and cross-covariances between points (non-diagonal 3 x 3 blocks). This is emphasized because "local accuracy" results are way off the mark when the cross-covariances are assumed zero ($\Sigma_{ij} = [0]\ if\ i \neq j$) in the computations. This important but often ignored fact was already numerically shown in Soler et al. (2012). However, restricting ourselves now to the example selected here shown in Eq. (49), if one implements Eq. (8) the resultant eigenvalues are $\lambda_1^* =$ 122.9090 cm^2; $\lambda_2^* = 12.7917$ cm^2; $\lambda_3^* = 5.8556$ cm^2. Therefore, if the rigorous Eqs. (37) or (38) are used for the calculation, the resulting eigenvalues are $\lambda_1 =$ 97.9413 cm^2; $\lambda_2 = 4.6943$ cm^2; and $\lambda_3 = 4.2999$ cm^2, a significant difference. The eigenvalues of a v-c local accuracy matrix based on a block diagonal network accuracy matrix are larger than the eigenvalues obtained using the full network v-c matrix. Consequently the availability of the non-diagonal blocks of the network v-c matrix is essential to obtain rigorous results. Only recently with the incorporation of GNSS methods and 3D least-squares models has this important achievement been made routinely available to the engineers and surveyors.

Figure 4 shows schematically the parameters involved in the final "averaged local accuracies error ellipsoid" at an arbitrary point *i* resulting from implementing the general Eq. (25), denoted symbolically by $\bar{\Sigma}_{(\Delta e,\Delta n,\Delta u)_{i\to j,k...}}$. From this v-c matrix the following parameters are obtained (see Fig. 4): the averaged standard deviations of the averaged local accuracies along the *e*, *n*, and *up*-axes ($\bar{\sigma}_{\Delta e}, \bar{\sigma}_{\Delta n}, \bar{\sigma}_{\Delta u}$); the three semi-axes *a*, *b*, and *c* of the error ellipsoid, and the three orthonormal eigenvectors $\vec{s}_1$, $\vec{s}_2$, and $\vec{s}_3$ (they are perpendicular to each other and with modulus equal to one) defining the orientation of the principal axes. $\bar{\Sigma}_{(\Delta e,\Delta n,\Delta u)_{i\to j,k...}}$ contains all the information including cross-correlations between points (see Fig.2 and 3) to determine the most accurate result as a function of the primary statistical information of nearby survey marks. The values of $\bar{\sigma}_{\Delta e}$, $\bar{\sigma}_{\Delta n}$, and $\bar{\sigma}_{\Delta u}$ could be converted to the 95% confidence interval suggested by FGDC (1998, 2008) using the logic described in Soler et al. (2012). Although some scientists advocate the use of a bi-normal radial error (Leenhouts 1985), in the opinion of the authors it will be more rigorous and it takes the same effort computationally speaking to report the corresponding error ellipses, although the values of $\bar{\sigma}_{\Delta e}$, $\bar{\sigma}_{\Delta n}$, and $\bar{\sigma}_{\Delta u}$ is all the information that is practically needed.

Conclusions

On the basis of the standard definition of "local accuracies" as announced by scientists more than 20 years ago, new insights about their rigorous definition and their differences with an alternative characterization proposed by Burkholder (2008) are detailed. Attention to the calculation of the so-called "mean (averaged) local accuracy" at a survey point is emphasized by presenting a didactic mathematical discussion with theoretical examples.

According to the authors, "averaged local accuracies" is the best practical way to indicate the quality of geodetic and/or engineering surveyed points on a particular area. With the advent of GNSS technology the availability of variance-covariance matrices between coordinates and cross-covariance matrices between points have improved the rigorous determination of averaged local accuracies at any arbitrary point *i* as a function of the accuracies of its selected surrounding points *j*, *k*,... sharing common observations. This possibility was not attainable before GNSS hardware and 3D methodologies were fully developed. Up until recently, the absence of the knowledge of cross-covariances between points (the non-diagonal 3x3 blocks in the network (absolute) v-c matrix) has impeded the rigorous determination of local accuracies. Furthermore, as explained in previous sections, the assumption that the non-diagonal blocks are zero introduces inaccurate positioning estimates to the results. As Fig. 2 and 3 indicates, presently, we have at our disposal all the information that is need it to compute averaged local accuracies at any point by considering all existing information about the errors implicit in its surrounding survey marks. The approach delineated herein is without any doubt the most accurate way to have a grasp of the overall point-by-point survey quality in local projects where, clearly, the final accuracy of every point is directly affected by the observational errors of their connected neighboring stations. Equation (27) presents a simple mathematical algorithm to implement numerically these concepts using matrix algebra by starting from the original full v-c matrix of the network accuracies given in the usual form of Eq. (31).

Subsequently, some ideas related to error ellipsoids and their inherent eigenvalues and eigenvectors are exploited by explaining the different relationships with respect to different (topocentric) local frames and how to transform between them using their corresponding rotation matrices. At this point the different parameters related to the averaged local error ellipsoid are described (see Fig. 4). In the opinion of the authors this is the type of ellipsoid that should be provided in order to determine the best set of local accuracies.

Acknowledgments

The authors thank three anonymous reviewers for critically reading the manuscript and for their valuable constructive comments and suggestions.

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