Appendices for "Multistage Hierarchical Capture-Recapture Models"

Appendix A

To fit the homogeneous CR model using a single-stage MCMC algorithm, we consider the joint posterior distribution for p and ψ. Under the PX-DA framework, this posterior distribution can be written as

$$[p, \psi | \mathbf{y}_{1:n}, \mathbf{y}_{(n+1):M}, n] \propto [\mathbf{y}_{(n+1):M} | p, \psi, \mathbf{y}_{1:n}, n][p, \psi | \mathbf{y}_{1:n}, n] \; , \tag{35}$$

$$\propto [\mathbf{y}_{(n+1):M} | p, \psi, \mathbf{y}_{1:n}, n][\mathbf{y}_{1:n} | p, n][n | p, \psi][p][\psi] \; , \tag{36}$$

$$\propto [\mathbf{y}_{1:n} | p, n][n | p, \psi][p][\psi] \; , \tag{37}$$

where the full-conditional distribution of $\mathbf{y}_{(n+1):M}$ is proportional to one when conditioned on n (and hence drops out of the right hand side) and the conditional distribution of $\mathbf{y}_{1:n}$ is proportional to the product of zero-truncated binomials

$$[\mathbf{y}_{1:n} | p, n] \propto \frac{\prod_{i=1}^{n} [y_i | p]}{(1 - (1 - p)^J)^n} \; . \tag{38}$$

For a given joint proposal distribution $[p, \psi]^*$, the associated Metropolis-Hastings ratio to update p and ψ jointly is

$$r = \frac{[\mathbf{y}_{1:n} | p^{(*)}, n][n | p^{(*)}, \psi^{(*)}][p^{(*)}][\psi^{(*)}][p^{(k-1)}, \psi^{(k-1)}]^*}{[\mathbf{y}_{1:n} | p^{(k-1)}, n][n | p^{(k-1)}, \psi^{(k-1)}][p^{(k-1)}][\psi^{(k-1)}][p^{(*)}, \psi^{(*)}]^*} \; . \tag{39}$$

To implement the model using PPRB following Hooten et al. (2021), we obtain an initial MCMC sample for p and ψ by fitting the CR model to the observed data while conditioning on fixed and known n. The posterior distribution for the first stage is proportional to

$$[\mathbf{y}_{1:n}|p, n][p][\psi] \; , \tag{40}$$

with respect to p and ψ, and the associated first-stage Metropolis-Hastings ratio is

$$r = \frac{[\mathbf{y}_{1:n}|p^{(*)}, n][p^{(*)}][\psi^{(*)}][p^{(k-1)}, \psi^{(k-1)}]^*}{[\mathbf{y}_{1:n}|p^{(k-1)}, n][p^{(k-1)}][\psi^{(k-1)}][p^{(*)}, \psi^{(*)}]^*} \; . \tag{41}$$

At this first stage, we use a temporary proposal distribution $[p, \psi]^*$ that is convenient.

For the second stage of the PPRB implementation, we assume that the proposal distribution is

$$[p, \psi]^* \propto [\mathbf{y}_{1:n}|p, n][p][\psi] \; , \tag{42}$$

which is equivalent to the first-stage posterior, and randomly sample (with replacement) joint first-stage MCMC realizations to use as proposals in the second stage Metropolis-Hastings updates. The resulting second-stage Metropolis-Hastings ratio becomes

$$r = \frac{[\mathbf{y}_{1:n}|p^{(*)}, n][n|p^{(*)}, \psi^{(*)}][p^{(*)}][\psi^{(*)}][p^{(k-1)}, \psi^{(k-1)}]^*}{[\mathbf{y}_{1:n}|p^{(k-1)}, n][n|p^{(k-1)}, \psi^{(k-1)}][p^{(k-1)}][\psi^{(k-1)}][p^{(*)}, \psi^{(*)}]^*} \; , \tag{43}$$

$$= \frac{[n|p^{(*)}, \psi^{(*)}]}{[n|p^{(k-1)}, \psi^{(k-1)}]} \; , \tag{44}$$

because the proposal cancels with the data model and priors. Thus, the second-stage Metropolis-Hastings ratio is merely a quotient involving the conditional model for n and can be evaluated easily using the first-stage MCMC sample.

Appendix B

To demonstrate the recursive implementation of the hierarchical CR model, we fit the model to simulated data using the single-stage and two-stage approaches. To simulate CR data for this example, we set $M = 100$ individuals in the superpopulation, membership probability $\psi = 0.4$, and detection probability $p = 0.25$. Then we used the hierarchical model in Section 2 as a generative process to simulate data based on $J = 3$ occasions which resulted in $N = 39$ individuals in our population, with $n = 19$ individuals observed by our measurement process. The simulated observed data y_i, for $i = 1, \ldots, n$, can be summarized by the values $\mathbf{y} = (1, 1, 1, 1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 2, 2, 1, 1, 2, 1)'$.

In the PX-DA implementation, we assumed $M - n = 81$ augmented individuals with all-zero capture histories. Thus, we assumed the same M as in our data simulation and this allows us to infer the true ψ. We note that the estimation of N is not constrained by M empirically in this example, therefore we could use larger values of M without influencing the inference for parameter p.

We fit the hierarchical CR from Section 2 to our simulated data using two approaches: 1) a standard single-stage MCMC algorithm for the hierarchical model and 2) a two-stage algorithm based on the recursive formulation of the same model. In both cases, we used $K = 200000$ MCMC iterations. The results of our analyses are summarized in Figure 5. The posterior comparison shown in Figure 5 indicates that the two-stage PPRB approach yields the same inference as the conventional single-stage MCMC algorithm. Both approaches fit exactly the same model, but the recursive framework suggests that other specifications for the conditional model for n (e.g., Poisson) are straightforward to implement. Furthermore, in more complicated models, we can benefit from parallel evaluation of the PMF for n which can improve stability and facilitate computation for large data sets.

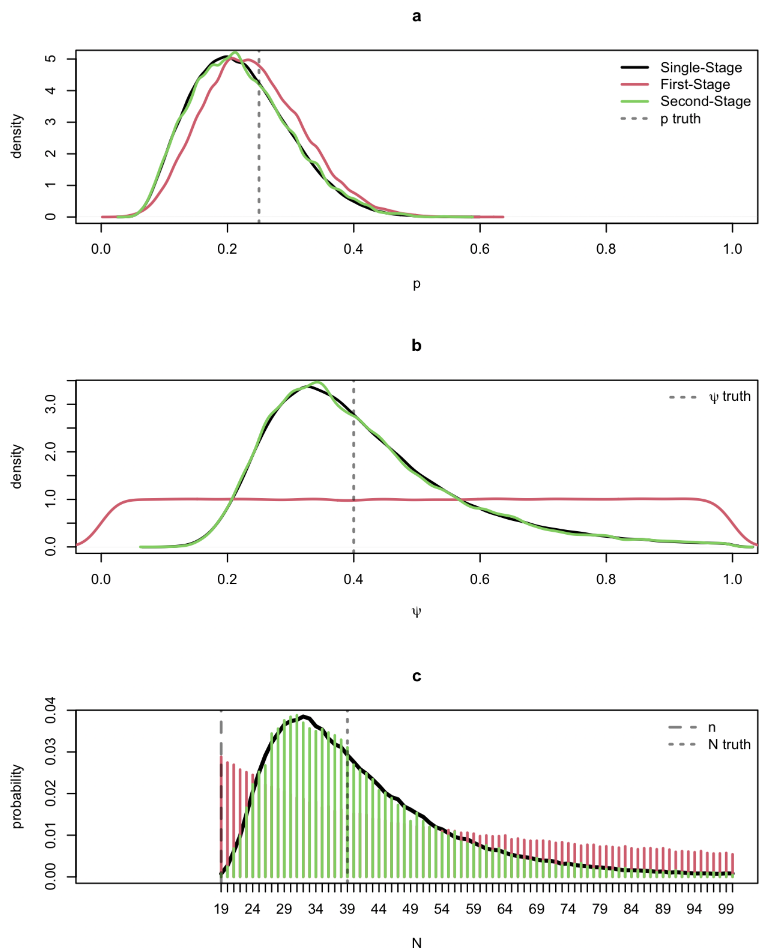

Figure 5: *Marginal posterior distributions for a) p, b) ψ, c) N. Distributions shown are a result of the single-stage MCMC algorithm (black), first-stage of the two-stage MCMC algorithm (red), and second-stage of the two-stage MCMC algorithm (green). Subfigure c shows marginal posterior probability mass functions; black line shown for single-stage for comparison.*

Appendix C

The snowshoe hare SCR data used to fit the model in Section 4 are presented below. The first column is the trap ID (i.e., [1,] indicates trap 1), the second two columns correspond to the $L = 84$ trap locations $\mathbf{X}$ in meters, and the remaining 13 columns contain values that correspond to the sum of detections associated with each individual at each trap.

[1,]	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[2,]	50	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[3,]	100	0	1	0	0	0	0	0	0	0	0	0	0	0	0
[4,]	150	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[5,]	200	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[6,]	250	0	0	1	0	0	0	0	0	0	0	0	0	0	0
[7,]	300	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[8,]	350	0	0	0	1	0	0	0	0	0	0	0	0	0	0
[9,]	400	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[10,]	450	0	0	0	0	1	0	0	0	0	0	0	0	0	0
[11,]	500	0	0	0	0	1	2	0	0	0	0	0	0	0	0
[12,]	550	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[13,]	0	-50	0	0	0	0	0	0	0	0	0	0	0	0	0
[14,]	50	-50	1	0	0	0	0	0	0	0	0	0	0	0	0
[15,]	100	-50	0	0	0	0	0	0	0	0	0	0	0	0	0
[16,]	150	-50	0	0	0	0	0	0	0	0	0	0	0	0	0
[17,]	200	-50	0	0	0	0	0	0	0	0	0	0	0	0	0
[18,]	250	-50	0	0	0	0	0	0	0	0	0	0	0	0	0
[19,]	300	-50	0	0	0	0	0	0	0	0	0	0	0	0	0
[20,]	350	-50	0	0	0	0	0	1	0	0	0	0	0	0	0
[21,]	400	-50	0	0	0	0	0	0	0	0	0	0	0	0	0
[22,]	450	-50	0	0	0	0	0	0	0	0	0	0	0	0	0
[23,]	500	-50	0	0	0	0	0	0	0	0	0	0	0	0	0
[24,]	550	-50	0	0	0	1	0	0	0	0	0	0	0	0	0
[25,]	0	-100	0	0	0	0	0	0	0	0	0	0	0	0	0
[26,]	50	-100	0	0	0	0	0	0	0	0	0	0	0	0	0
[27,]	100	-100	1	0	0	0	0	0	0	0	0	0	0	0	0
[28,]	150	-100	0	0	0	0	0	0	0	0	0	0	0	0	0
[29,]	200	-100	0	0	0	0	0	0	0	0	0	0	0	0	0
[30,]	250	-100	0	0	0	0	0	0	0	0	0	0	0	0	0
[31,]	300	-100	0	0	2	0	0	0	1	0	0	0	0	0	0
[32,]	350	-100	0	0	0	0	0	0	0	0	0	0	0	0	0
[33,]	400	-100	0	0	0	0	0	0	0	0	0	0	0	0	0
[34,]	450	-100	0	0	0	0	0	0	0	0	0	0	0	0	0
[35,]	500	-100	0	0	0	0	0	0	0	0	0	0	0	0	0
[36,]	550	-100	0	0	0	0	0	0	0	0	0	0	0	0	0
[37,]	0	-150	0	0	0	0	0	0	0	0	0	0	0	0	0
[38,]	50	-150	0	0	0	0	0	0	0	0	0	0	0	0	0
[39,]	100	-150	0	0	0	0	0	0	0	1	0	0	0	0	0
[40,]	150	-150	0	0	0	0	0	0	0	0	0	0	0	0	0
[41,]	200	-150	0	0	0	0	0	0	0	0	0	0	0	0	0
[42,]	250	-150	0	0	0	0	0	0	0	0	0	0	0	0	0
[43,]	300	-150	0	0	0	0	0	0	0	0	0	0	0	0	0
[44,]	350	-150	0	0	0	0	0	0	0	0	0	0	0	0	0
[45,]	400	-150	0	0	2	0	0	0	0	0	0	0	0	0	0

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[46,]  450 -150 0 0 0 0 0 0 0 0 0 0 0 0 0
[47,]  500 -150 0 0 0 0 0 0 0 0 0 0 0 0 0
[48,]  550 -150 0 0 0 0 0 0 0 0 0 0 0 0 0
[49,]    0 -200 0 0 0 0 0 0 0 0 0 0 0 0 0
[50,]   50 -200 0 0 0 0 0 0 0 0 0 0 0 0 0
[51,]  100 -200 0 0 0 0 0 0 0 1 0 0 0 0 0
[52,]  150 -200 0 0 0 0 0 0 0 0 0 0 0 0 0
[53,]  200 -200 0 0 0 0 0 0 0 0 0 0 0 0 0
[54,]  250 -200 0 0 0 0 0 0 0 0 0 0 0 0 0
[55,]  300 -200 0 0 0 0 0 0 0 0 0 0 0 0 0
[56,]  350 -200 0 0 0 0 0 0 0 0 0 0 0 0 0
[57,]  400 -200 0 0 0 0 0 0 0 0 1 0 0 0 0
[58,]  450 -200 0 0 0 0 0 0 0 0 0 0 0 0 0
[59,]  500 -200 0 0 0 0 2 0 0 0 0 0 0 0 0
[60,]  550 -200 0 0 0 0 0 0 0 0 0 2 0 0 0
[61,]    0 -250 0 0 0 0 0 0 0 0 0 0 0 0 0
[62,]   50 -250 0 0 0 0 0 0 0 0 0 0 0 0 0
[63,]  100 -250 0 0 0 0 0 0 0 0 0 0 0 0 0
[64,]  150 -250 0 0 0 0 0 0 0 1 0 0 0 0 0
[65,]  200 -250 0 0 0 0 0 0 0 0 0 0 0 0 0
[66,]  250 -250 0 0 0 0 0 0 0 0 0 0 0 0 0
[67,]  300 -250 1 0 0 0 0 0 0 0 0 0 0 0 0
[68,]  350 -250 0 0 0 0 0 0 0 0 2 0 0 0 0
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[73,]    0 -300 0 0 0 0 0 0 0 0 0 0 0 1 0
[74,]   50 -300 0 0 0 0 0 0 0 0 0 0 0 0 1
[75,]  100 -300 0 0 0 0 0 0 0 0 0 0 0 0 0
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[78,]  250 -300 0 0 0 0 0 0 0 0 0 0 0 0 0
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[80,]  350 -300 0 0 0 0 0 0 0 0 0 0 0 0 0
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[83,]  500 -300 0 0 0 0 0 0 0 0 0 0 1 0 0
[84,]  550 -300 0 0 0 0 0 0 0 0 0 0 1 0 0
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