Supplementary material to paper entitled "Tracking time differences of arrival of multiple sound sources in the presence of clutter and missed detections" by Pina Gruden, Eva-Marie Nosal, and Erin Oleson

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1 Derivation of the probability hypothesis density (PHD) filter equations based on separate update of persistent and newborn targets and amplitude information

A PHD filter is a frequently used filter within the random finite set framework [4, 6] and has been used for a variety of multi-target tracking applications. It has been shown that improved tracking performance can be achieved with the PHD filter by incorporating the amplitude information to the measurements [2, 3]. However, the formulation presented in Refs.[2, 3] performs a joint update step for both newly appearing and persistent targets, which can bias the number of estimated targets [1]. In the formulation proposed in

Ref. [1], the newborn and persistent targets are updated separately, but it does not incorporate amplitude information. In this supplementary material, we derive a PHD filter formulation that incorporates the amplitude information for more informative measurements and improved tracking, and updates persistent and newborn targets separately for reduced bias in the number of estimated targets.

Sections 1.1 and 1.2 discuss works by Ref. [1] and Refs. [2, 3], respectively. In Section 1.3 we derive the extended filter equations that incorporate both the amplitude information as well as the separate update.

Note, we do not provide a full background to the PHD filter, but refer the reader to references that discuss PHD filter and its implementations in detail [4, 5, 6, 7]. Also note, that the PHD filter discussed here ignores the target-spawning model.

1.1 Extended PHD filter with separate prediction and update for persistent and newborn targets

This section summarizes findings of Ref. [1], concerning the separate PHD filter recursion for persistent and newborn targets. The standard formulation of the PHD filter assumes that the target birth PHD is known *a priori*. However, when the birth PHD is unknown, it can be based on the received measurements [1]. When initiating the birth PHD based on the measurements, the prediction and update steps need to be carried out separately for the persistent and the newborn targets in order to avoid biasing the number of estimated targets [1].

The PHD, $v_{k|k}(\boldsymbol{x})$, can thus be expressed as two PHDs, one for persisting targets, $v_{k|k,p}(\boldsymbol{x})$, and one for newborn targets, $v_{k|k,b}(\boldsymbol{x})$. Throughout this document the subscripts $k|k-1$ and $k|k$ are used to indicate the predicted and the updated elements, respectively. The subscripts p and b are used to denote the persistent and newborn targets, respectively.

The following assumptions are made [1, 7]:

1. *Probability of target survival*- is state independent and is the same for persistent and new targets, $p_{S,p}(\boldsymbol{x}) = p_{S,b}(\boldsymbol{x}) = p_S$.

2. *Target birth PHD and transition density* - a newborn target becomes a persistent target in the next time step, but persistent targets cannot become newborn targets.

The prediction step of the PHD filter then becomes [1]:

$$v_{k|k-1,p}(\boldsymbol{x}) = p_S \int f_{k|k-1}(\boldsymbol{x}|\boldsymbol{x}') \left(v_{k-1|k-1,p}(\boldsymbol{x}') + v_{k-1|k-1,b}(\boldsymbol{x})\right) d\boldsymbol{x}' \,, \tag{1}$$

$$v_{k|k-1,b}(\boldsymbol{x}) = \gamma_k(\boldsymbol{x}) \,, \tag{2}$$

where $v_{k|k-1}(\cdot)$ denotes the predicted PHD, $f_{k|k-1}(\cdot)$ denotes the single target state transition function, and $\gamma_k(\cdot)$ denotes birth PHD.

The update step of the PHD filter, for persistent targets is [1]:

$$v_{k|k,p}(\boldsymbol{x}) = [1 - p_D(\boldsymbol{x})] v_{k|k-1,p}(\boldsymbol{x}) + \sum_{\boldsymbol{z} \in \boldsymbol{Z}_k} \frac{p_D(\boldsymbol{x}) g_k(\boldsymbol{z}|\boldsymbol{x}) v_{k|k-1,p}(\boldsymbol{x})}{\mathcal{L}(\boldsymbol{z})} \,, \tag{3}$$

where

$$\mathcal{L}(\boldsymbol{z}) = \kappa_k(\boldsymbol{z}) + \int g_k(\boldsymbol{z}|\boldsymbol{x}) v_{k|k-1,b}(\boldsymbol{x}) d\boldsymbol{x} + \int p_D(\boldsymbol{x}) g_k(\boldsymbol{z}|\boldsymbol{x}) v_{k|k-1,p}(\boldsymbol{x}) d\boldsymbol{x} \,, \tag{4}$$

where $v_{k|k}(\cdot)$ denotes the updated PHD, $p_D(\cdot)$ denotes the probability of detection, $g_k(\cdot)$ denotes the single target measurement likelihood function, and $\kappa_k(\cdot)$ denotes clutter PHD.

The update step for the newborn targets is [1]:

$$v_{k|k,b}(\boldsymbol{x}) = [1 - p_D(\boldsymbol{x})] v_{k|k-1,b}(\boldsymbol{x}) + \sum_{\boldsymbol{z} \in \boldsymbol{Z}_k} \frac{p_D(\boldsymbol{x}) g_k(\boldsymbol{z}|\boldsymbol{x}) v_{k|k-1,b}(\boldsymbol{x})}{\mathcal{L}(\boldsymbol{z})} \,. \tag{5}$$

Note that, since the newborn targets are initiated based on the measurements, it is often assumed that newborn targets are always detected, *i.e.* $p_D(\boldsymbol{x}) = 1$ for newborn [1], and thus Eq.(5) reduces to:

$$v_{k|k,b}(\boldsymbol{x}) = \sum_{\boldsymbol{z}\in\boldsymbol{Z}_k} \frac{g_k(\boldsymbol{z}|\boldsymbol{x})v_{k|k-1,b}(\boldsymbol{x})}{\mathcal{L}(\boldsymbol{z})}\,. \tag{6}$$

1.2 Extended measurement model- adding amplitude feature to the measurements

This section summarizes findings of Refs. [2, 3]. We consider an extended measurement model, where measurements include amplitude information, a, in addition to a standard position measurement z, such that the measurement vector $\tilde{\boldsymbol{z}} = [z, a]^T$. In the present study, the position measurement z is the time difference of arrival (TDOA) information and the amplitude measurement a is the amplitude of the cross-correlation function. Moreover, the state vector is expanded to include the expected SNR of a target, d, such that $\tilde{\boldsymbol{x}} = [\boldsymbol{x}^T, d]^T$.

The measurements are obtained by thresholding the amplitudes of the cross-correlogram with a threshold λ. It is assumed that the probability of target detection given a threshold λ is only dependent on the SNR d, and not on position $\boldsymbol{x}$ [2]:

$$\tilde{p}_D^{\lambda}(\tilde{\boldsymbol{x}}) = p_D^{\lambda}(d)\,. \tag{7}$$

Further, we assume that amplitude of the cross-correlation measurement a is independent of the TDOA measurement z. Given a threshold λ, used to obtain these measurements, the likelihood functions for targets, $\tilde{g}_k(\tilde{\boldsymbol{z}}|\tilde{\boldsymbol{x}})$, and clutter, $\tilde{c}_k(\tilde{\boldsymbol{z}})$ can be written as [2, 3]:

$$\tilde{g}_k(\tilde{\boldsymbol{z}}|\tilde{\boldsymbol{x}}) = g_k(z|\boldsymbol{x})g_a^{\lambda}(a|d)\,, \tag{8}$$

$$\tilde{c}_k(\tilde{\boldsymbol{z}}) = c_k(z)c_a^{\lambda}(a)\,, \tag{9}$$

where $g_k(z|\boldsymbol{x})$ and $c_k(z)$ are the target and clutter likelihoods based on z; and where $g_a^{\lambda}(a|d)$ and $c_a^{\lambda}(a)$ are the amplitude probability density functions for target and clutter given a threshold λ, respectively. Moreover we define [2]:

$$g_a(a|d) = g_a^{\lambda}(a|d)p_D^{\lambda}(d)\,. \tag{10}$$

The clutter PHD for the extended measurement can be written as [2]:

$$\tilde{\kappa}_k(\tilde{\boldsymbol{z}}) = \tilde{c}_k(\tilde{\boldsymbol{z}})r_k\,, \tag{11}$$

where r_k denotes the clutter rate.

In practice, the expected SNR d is rarely known, and hence it is useful to marginalize out the parameter d over the range of possible values to find a likelihood g_a and a probability of detection p_D that are not dependent on d [2].

Let $p(d)$ be the the expected probability distribution of SNR values. The amplitude likelihood g_a, and the probability of detection p_D, where d has been marignalized out, are defined as [2]:

$$g_a(a) = \int g_a(a|\delta)p(\delta)d\delta\,, \tag{12}$$

$$p_D^{\lambda} = \int p_D^{\lambda}(\delta)p(\delta)d\delta\,. \tag{13}$$

Further, using Rayeigh distribution to describe $g_a(a|d)$, and defining $p(d)$

to be normalised for the region $[d_1, d_2]$ and proportional to $1/(1+d)$, Eqs. (12) and (13) can be written as [2]:

$$g_a(a) = \frac{2\left(\exp\left(\frac{-a^2}{2(1+d_2)}\right) - \exp\left(\frac{-a^2}{2(1+d_1)}\right)\right)}{a(\ln(1+d_2) - \ln(1+d_1))}\,, \tag{14}$$

$$p_D^\lambda = \int_{d_1}^{d_2} \frac{1}{1+\delta} \exp\left(\frac{-\lambda^2}{2(1+\delta)}\right) d\delta\,. \tag{15}$$

1.3 Derivation of the filter that incorporates separate update and the amplitude information

This Section shows the derivation of the PHD filter equations using separate update for persistent and newborn targets and an extended measurement model.

Prediction step of the PHD filter

The predicted PHDs for persistent and newborn targets accounting for the extended states $\tilde{\boldsymbol{x}}$ can be written as [2]:

$$\tilde{v}_{k|k-1,p}(\tilde{\boldsymbol{x}}) = v_{k|k-1,p}(\boldsymbol{x})p(d)\,, \tag{16}$$

$$\tilde{v}_{k|k-1,b}(\tilde{\boldsymbol{x}}) = v_{k|k-1,b}(\boldsymbol{x})p(d)\,, \tag{17}$$

where $p(d)$, is the expected probability distribution of SNR values, as discussed in the previous section.

Update step of the PHD filter

With the extended states $\tilde{\boldsymbol{x}}$ and measurements $\tilde{\boldsymbol{z}}$, the PHD update for persistent targets in Eq.(3) can be written as:

$$\tilde{v}_{k|k,p}(\tilde{\boldsymbol{x}}) = [1-\tilde{p}_D(\tilde{\boldsymbol{x}})]\tilde{v}_{k|k-1,p}(\tilde{\boldsymbol{x}})+$$

$$\sum_{\tilde{\boldsymbol{z}}\in\tilde{\boldsymbol{Z}}_k} \frac{\tilde{p}_D(\tilde{\boldsymbol{x}})\tilde{g}_k(\tilde{\boldsymbol{z}}|\tilde{\boldsymbol{x}})\tilde{v}_{k|k-1,p}(\tilde{\boldsymbol{x}})}{\tilde{\kappa}_k(\tilde{\boldsymbol{z}}) + \int \tilde{p}_D(\tilde{\boldsymbol{x}})\tilde{g}_k(\tilde{\boldsymbol{z}}|\tilde{\boldsymbol{x}})\tilde{v}_{k|k-1,b}(\tilde{\boldsymbol{x}})d\tilde{\boldsymbol{x}} + \int \tilde{p}_D(\tilde{\boldsymbol{x}})\tilde{g}_k(\tilde{\boldsymbol{z}}|\tilde{\boldsymbol{x}})\tilde{v}_{k|k-1,p}(\tilde{\boldsymbol{x}})d\tilde{\boldsymbol{x}}}\,. \tag{18}$$

Substituting Eqs. (7), (8), (9), (11), (16), and (17) into Eq. (18) and assuming the threshold λ was applied, then gives:

$$v_{k|k,p}(\boldsymbol{x})p(d) = [1-p_D^\lambda(d)]v_{k|k-1,p}(\boldsymbol{x})p(d)+$$

$$\sum_{\tilde{\boldsymbol{z}}\in\tilde{\boldsymbol{Z}}_k^\lambda} \frac{p_D^\lambda(d)g_k(\boldsymbol{z}|\boldsymbol{x})g_a^\lambda(a|d)v_{k|k-1,p}(\boldsymbol{x})p(d)}{\mathcal{L}(\tilde{\boldsymbol{z}})}\,, \tag{19}$$

where

$$\mathcal{L}(\tilde{\boldsymbol{z}}) = r_k c_k(z)c_a^\lambda(a) + \int p_D^\lambda(\delta)g_k(\boldsymbol{z}|\boldsymbol{x})g_a^\lambda(a|\delta)v_{k|k-1,b}(\boldsymbol{x})p(\delta)d\boldsymbol{x}d\delta$$

$$+\int p_D^\lambda(\delta)g_k(\boldsymbol{z}|\boldsymbol{x})g_a^\lambda(a|\delta)v_{k|k-1,p}(\boldsymbol{x})p(\delta)d\boldsymbol{x}d\delta\,. \tag{20}$$

Further, considering Eqs. (10) and (12), Eqs. (19) and (20) become:

$$v_{k|k,p}(\boldsymbol{x}) = [1 - p_D^{\lambda}(d)]v_{k|k-1,p}(\boldsymbol{x}) + \sum_{\tilde{\boldsymbol{z}} \in \tilde{\boldsymbol{Z}}_k^{\lambda}} \frac{g_k(\boldsymbol{z}|\boldsymbol{x})g_a(a|d)v_{k|k-1,p}(\boldsymbol{x})}{\mathcal{L}(\tilde{\boldsymbol{z}})}\,, \tag{21}$$

$$\begin{aligned}\mathcal{L}(\tilde{\boldsymbol{z}}) = r_k c_k(z) c_a^{\lambda}(a) &+ g_a(a) \int g_k(\boldsymbol{z}|\boldsymbol{x}) v_{k|k-1,b}(\boldsymbol{x}) d\boldsymbol{x} \\ &+ g_a(a) \int g_k(\boldsymbol{z}|\boldsymbol{x}) v_{k|k-1,p}(\boldsymbol{x}) d\boldsymbol{x}\,. \end{aligned} \tag{22}$$

Similarly, the PHD update for newborn targets in Eq.(5), with extended states and measurements can be written as:

$$\begin{aligned}\tilde{v}_{k|k,b}(\tilde{\boldsymbol{x}}) &= [1 - \tilde{p}_D(\tilde{\boldsymbol{x}})]\tilde{v}_{k|k-1,b}(\tilde{\boldsymbol{x}}) + \\ &\sum_{\tilde{\boldsymbol{z}} \in \tilde{\boldsymbol{Z}}_k} \frac{\tilde{p}_D(\tilde{\boldsymbol{x}})\tilde{g}_k(\tilde{\boldsymbol{z}}|\tilde{\boldsymbol{x}})\tilde{v}_{k|k-1,b}(\tilde{\boldsymbol{x}})}{\tilde{\kappa}_k(\tilde{\boldsymbol{z}}) + \int \tilde{p}_D(\tilde{\boldsymbol{x}})\tilde{g}_k(\tilde{\boldsymbol{z}}|\tilde{\boldsymbol{x}})\tilde{v}_{k|k-1,b}(\tilde{\boldsymbol{x}})d\tilde{\boldsymbol{x}} + \int \tilde{p}_D(\tilde{\boldsymbol{x}})\tilde{g}_k(\tilde{\boldsymbol{z}}|\tilde{\boldsymbol{x}})\tilde{v}_{k|k-1,p}(\tilde{\boldsymbol{x}})d\tilde{\boldsymbol{x}}}\,. \end{aligned} \tag{23}$$

Following the same steps as for the derivation for the persistent target update, the update for the neborn targets can be written as:

$$v_{k|k,b}(\boldsymbol{x}) = [1 - p_D^{\lambda}(d)]v_{k|k-1,b}(\boldsymbol{x}) + \sum_{\tilde{\boldsymbol{z}} \in \tilde{\boldsymbol{Z}}_k^{\lambda}} \frac{g_k(\boldsymbol{z}|\boldsymbol{x})g_a(a|d)v_{k|k-1,b}(\boldsymbol{x})}{\mathcal{L}(\tilde{\boldsymbol{z}})}\,. \tag{24}$$

Further, since newborn targets are initiated from the measurements, it is assumed that they are always detected, hence $p_D^{\lambda}(d) = 1$ for newborn targets and Eq.(24) reduces to:

$$v_{k|k,b}(\boldsymbol{x}) = \sum_{\tilde{\boldsymbol{z}} \in \tilde{\boldsymbol{Z}}_k^{\lambda}} \frac{g_k(\boldsymbol{z}|\boldsymbol{x})g_a(a|d)v_{k|k-1,b}(\boldsymbol{x})}{\mathcal{L}(\tilde{\boldsymbol{z}})}\,. \tag{25}$$

It should be noted that the update equations for both persistent targets, Eq.(21), and newborn, Eq.(25), still contain terms dependent on the expected SNR d. We use the marginalization across the expected SNR and replace $g_a(a|d)$ and $p_D^\lambda(d)$ with $g_a(a)$ and p_D^λ defined in Eqs. (14) and (15), to obtain the final PHD update equations for persistent and newborn targets:

$$v_{k|k,p}(\boldsymbol{x}) = [1 - p_D^\lambda] v_{k|k-1,p}(\boldsymbol{x}) + \sum_{\tilde{\boldsymbol{z}} \in \tilde{\boldsymbol{Z}}_k^\lambda} \frac{g_k(\boldsymbol{z}|\boldsymbol{x}) g_a(a) v_{k|k-1,p}(\boldsymbol{x})}{\mathcal{L}(\tilde{\boldsymbol{z}})}, \tag{26}$$

$$v_{k|k,b}(\boldsymbol{x}) = \sum_{\tilde{\boldsymbol{z}} \in \tilde{\boldsymbol{Z}}_k^\lambda} \frac{g_k(\boldsymbol{z}|\boldsymbol{x}) g_a(a) v_{k|k-1,b}(\boldsymbol{x})}{\mathcal{L}(\tilde{\boldsymbol{z}})}. \tag{27}$$

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