STABILITY AND NONLINEAR WAVES IN STRATIFIED SHEAR FLOWS

by

D.J. Benney and
S.A. Maslowe

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
MATHEMATICS DEPARTMENT

FOREWARD

This report presents the results of a study initiated in 1972 whose primary objective was to investigate nonlinear aspects of shear flow instability. The Technical Monitor was Dr. Rayford P. Hosker of the NOAA Atmospheric Turbulence and Diffusion Laboratory. The following three specific developments were accomplished and are described herein: (i) formulation of a weakly nonlinear stability theory for stratified shear flows and derivation of the nonlinear amplitude equation; (ii) development of a time-dependent nonlinear critical layer theory for homogeneous shear flows; and (iii) a study of the linear instability of swirling flows to non-axisymmetric perturbations.

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1.0 INTRODUCTION

Clear Air Turbulence is often encountered in the stably stratified mixing layers that develop when warm air masses flow over colder air. A similar phenomenon occurs in the oceanic thermocline, particularly in equatorial waters, where the upper layer may be considerably warmer than the cold deeper water.

It is generally accepted that the theory of hydrodynamic stability is a valuable tool in studying the occurrence of such turbulence [1]. However, the theoretical results obtained to date, although promising in a qualitative sense, leave something to be desired when compared quantitatively with observational data.

With few exceptions, stability investigations have employed the linearized theory. Of particular interest in the linear theory is the result proven by Miles and Howard that a *necessary* condition for instability is that the local Richardson number be somewhere less than 1/4 (see e.g., the survey article by Drazin and Howard [2]). Observations suggest, however that *finite-amplitude* billows can occur for larger values of that parameter. Such events should not really be viewed as surprising, because the importance of nonlinear effects has long been recognized in the transition to turbulence of homogeneous shear flows. Hence, the linear, theory, although both fundamental and important, does have its limitations (these have been discussed, for example, in the survey articles by Stuart [3] and Howard and Maslowe [4]).

The fact that the failure of linearized theory is essentially a localized phenomenon, occurring primarily in the critical layer region (where the mean flow velocity and perturbation phase speed are nearly equal), has been exploited by asymptotic methods in some recent work of the present authors which is discussed in §1.2. By taking into account nonlinear effects within that region, a new class of periodic waves having significant properties was obtained. For example, in the recent computations presented in [5], it was found that with a nonlinear critical layer, solutions of the eigenvalue problem for neutral modes exist for flows having a local Richardson number that is everywhere greater than 1/4. Moreover, the flow pattern within the critical layer of such modes exhibits some of the inherently nonlinear structural details that have been noted in atmospheric radar observations.

Of particular interest, are the existence of thin temperature and velocity boundary layers imbedded within an essentially inviscid flow structure (see Fig. 1) of the Kelvin cats-eye configuration. Viscosity

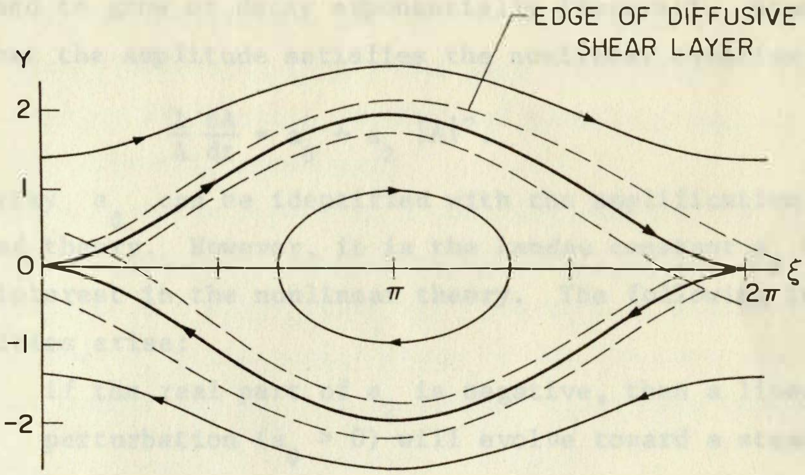

Fig. 1 Flow structure in the nonlinear critical layer.

and heat-conduction are important in these layers even at very high Reynolds numbers and, because of the intense shear there, they are likely sites for small-scale instabilities. The occurrence of these layers for amplifying as well as neutral oscillations has recently been substantiated by the numerical calculations of Patnaik [6], who integrated the full nonlinear Boussinesq equations.

Clearly, there are many interesting nonlinear phenomena occurring in stratified shear flows and that is the aspect that was emphasized in the just-completed study; however, some linear stability computations were also made, the latter in connection with some rotating flows of geophysical interest. The following three specific objectives were pursued each of which is discussed separately in the succeeding sections of this report:

(1) Extension of the weakly nonlinear stability theory of Stuart and Watson to stratified shear flows.

(2) Determination of the time and space-dependent evolution equation for flows with nonlinear critical layers.

(3) Investigation of the linear instability of swirling flows to helical perturbations.

1.1 Weak Nonlinear Stability Theory.

The weakly nonlinear theory of shear flow stability was originally formulated by J.T. Stuart [7] in 1960 for the case of Poiseuille flow. The primary objective of that theory was to obtain a nonlinear equation governing the temporal evolution of the amplitude for a *finite* disturbance. Such a development would remedy several of the inadequacies in a linear theory, e.g., the fact that unsteady infinitesimal disturbances are constrained to grow or decay exponentially (forever). Stuart found in his theory that the amplitude satisfies the nonlinear equation

$$\frac{1}{A}\frac{dA}{dt} = a_0 + a_2 |A|^2 . \tag{1.1}$$

The quantity a_0 can be identified with the amplification factor of linearized theory. However, it is the Landau constant a_2 that is of central interest in the nonlinear theory. The following interesting possibilities arise:

(i) if the real part of a_2 is negative, then a linearly unstable perturbation ($a_0 > 0$) will evolve toward a steady finite-amplitude state having an equilibrium amplitude $|A|^2 = -(a_0/a_2)$. This is termed the supercritical case.

(ii) if the real part of a_2 is positive, modes that would be damped ($a_0 < 0$) in the linearized theory can now amplify if their initial amplitude satisfies the condition $|A_0|^2 > -(a_0/a_2)$. This phenomenon of instability to finite perturbations is termed subcritical instability.

Calculation of the Landau constant requires a substantial numerical effort and it was not until 1967 that Reynolds and Potter [8] made computations for Poiseuille flow. They found $a_2 > 0$ which is qualitatively correct because transition in channel flow occurs experimentally at a much lower Reynolds number than that predicted by linearized theory, i.e., it is the subcritical case that is of interest. Unfortunately, the Reynolds numbers of practical interest are too far below the linear neutral value for Stuart's theory to be applicable.

For stratified shear flows, on the other hand, it seems that there is a large parameter range of practical interest in which the conditions implicit in a weakly nonlinear theory are satisfied. Yet, the only examples of this type of investigation are the parallel studies of Drazin [9] and Maslowe and Kelly [10] on the discontinuous Kelvin-Helmholtz flow. By considering discontinuous flows, one avoids having to contend with the singular behavior that is always associated with inviscid neutral modes that are stability boundaries for continuous stratified shear flows. On the other hand, such results are only meaningful as long wave approximations for smoothly varying profiles. The singularities not only make the problem mathematically interesting, but are also of physical significance as they occur in regions where the gradients in temperature and shear are intense.

The weakly nonlinear theory has been formulated in this study with a specific type of application in mind and numerical calculations of the Landau constant are presently under way.

To illustrate the idea with a concrete example, we consider Holmboe's shear layer model consisting of the velocity profile $\overline{u} = \tanh y$ and density profile $\overline{\rho} = \exp(-\beta \tanh y)$. The neutral eigensolution is given by $c = 0$ and $J_0 = \alpha(1 - \alpha)$, where c is the perturbation phase speed, α the wave number and J_0 is an overall Richardson number.

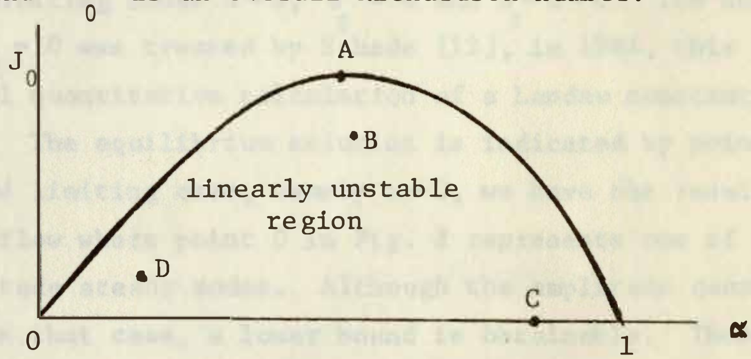

Fig. 2 Linear stability boundary and location of possible finite-amplitude equilibrium states for Holmboe's model.

In the weak nonlinear theory, one can perturb away from any point on the linear neutral stability curve; however, an obvious point of special interest in the present case is the point $\alpha = 1/2$, $J_0 = 1/4$. As shown in Section 2, the theory can be developed by employing expansions of the form

$$J_0 \sim 1/4 + \varepsilon^2 J_2 + O(\varepsilon^3), \text{ and } \alpha \sim 1/2 + \varepsilon^2 \alpha_2 + O(\varepsilon^3), \tag{1.2}$$

where ε is an amplitude parameter. If, as we anticipate (the rationale is given below), the Landau constant turns out to be negative, then the point B in Fig. 2 represents a supercritical finite-amplitude equilibrium state.

Perhaps the most significant result of §2 is that the amplitude evolves according to the 1st-order Landau equation (1.1). It was previously believed that the amplitude equation for stratified shear flows is 2nd-order in time. This belief (see Drazin [9], p. 333) follows from the form of the Taylor-Goldstein equation governing the inviscid theory and the fact that the Kelvin-Helmholtz model leads to a 2nd-order amplitude equation. However, the requirement of adding dissipation to remedy the singularity occurring in the linear inviscid theory for *continuous* flows greatly alters the analysis and results. Because 2nd-order equations are associated with oscillatory phenomena, it has been erroneously concluded by some investigators that steady finite-amplitude modes do not exist in stratified shear flows at high Reynolds numbers. (There is a strong similarity between these results and studies of finite-amplitude baroclinic instability [11]; however, the analogy is not exact and the 2nd-order equation may therefore be relevant in that case.)

As noted above, a_2 is expected to be negative because that result was obtained in the two limiting cases $\alpha = 1$, $J_0 = 0$ and $J_0 = \alpha \to 0$. The homogeneous mixing layer, i.e., $J_0 = 0$ was treated by Schade [12], in 1964, this work being the first actual quantitative calculation of a Landau constant in the Stuart-Watson Theory. The equilibrium solution is indicated by point C in Fig. 2. In the second limiting case, namely $\alpha \to 0$, we have the results for the Kelvin-Helmholtz flow where point D in Fig. 2 represents one of the possible finite-amplitude steady modes. Although the amplitude cannot be determined uniquely in that case, a lower bound is obtainable. These limiting results do not, of course, guarantee that a_2 is negative for $0 < \alpha < 1$, so that its actual calculation is very much of interest as is the equilibrium amplitude should our conjecture be correct that supercritical modes exist.

1.2 The Amplitude Evolution Equation for Shear Flows with Nonlinear Critical Layers.

The perturbation theory outlined above involves an expansion about a linear neutral mode, where the critical layer singularity is removed by adding dissipative effects. Recently, however, Benney and Bergeron [13] have shown that there is an alternative in the case of finite-amplitude waves, namely, to include nonlinear terms in an essentially inviscid critical layer. The principal point that their work emphasizes is that nonlinear effects can be of dominating importance in the critical layer even at amplitudes that elsewhere seem quite small. Hence, at high Reynolds numbers, the weakly nonlinear approach outlined above is somewhat restricted in its applicability. Fortunately, in the nonlinear critical layer theory, a perturbation approach can still be employed to make the analysis tractable.

It was found in [13] that, in contrast to the results of linear viscous theory, there is no phase change across the singular critical point; consequently a new class of neutral modes is possible. For stratified shear flows this result is also true in many circumstances and that fact permits the existence of singular neutral modes in flows having $J(y) > 1/4$, as noted earlier.

The principal limitation of the nonlinear critical layer theory is that it has, thus far, only been developed for neutral modes. In this study, we have attempted to extend the theory to consider waves that are evolving slowly in both space and time. One can, in this way, assess the stability of the periodic solutions obtained previously and also, possibly, gain some insight into the origin of these disturbances. To date, only homogeneous shear flows have been considered, although it seems that the extension to stratified flows with small Richardson numbers should be straightforward. Our principal result is that for most flows with nonlinear critical layers, the amplitude evolves according to the partial differential equation

$$\frac{\partial A}{\partial \tau} + c_g \frac{\partial A}{\partial X} = i\gamma A^2 A^* , \qquad (1.3)$$

where c_g is the linear group velocity and γ is a real constant analogous to the Landau constant in (1.1). Here, X and τ are slow space and time variables defined in Section 3.

While formulating the analysis leading to (1.3), it was realized that the techniques employed were not restricted to this problem, so that many previous theories could be recovered as special cases. For example, by allowing c_g and γ to be complex, Stuart-Watson theory and its recent generalization to wave trains [14] can easily be derived. Other examples, both involving new results and recovering known solutions, are discussed in §3. Of particular interest, are some cases where the amplitude equation becomes second-order in either X or τ.

1.3 Instability of Swirling Flows.

The motivation for this study in terms of atmospheric applications came primarily from the suggestion by Scorer [15] that Clear Air Turbulence may, in some instances, be due to centrifugal instability. A localized analysis using the "parcel method" was employed in [15] to support this contention. It is well-known that instability in a purely rotating flow will occur if the Rayleigh criterion is violated, viz., that the square of the circulation decreases outward. The instability takes the form of toroidal vortices as first observed in the experiments of G.I. Taylor on circular Couette flow. Scorer's idea is that if a flow consists of both an axial and swirl component, then the same type of instability will occur. To extend Rayleigh's criterion, it is then only necessary to employ a frame of reference reflecting the local orientation of the mean flow (the detailed analysis, however, is far from trivial).

The quantitative results obtained in the present study suggest that this idea has some validity for flows having a small Rossby number, i.e., flows where the axial component is small. However, when the Rossby number is large or even O(1), the situation seems to be much more complicated than that and the predictions of Scorer are very much in error.

The present study was limited to rigidly rotating flows and numerical computations of growth rates for unstable perturbations were made for Poiseuille flow (parabolic axial profile). A necessary condition for instability was derived as well. As the detailed analysis and results have already been published [16], only a brief summary is included herein, the latter comprising Section 4 of this report.

2.0 WEAK NONLINEAR STABILITY OF STRATIFIED SHEAR FLOWS

2.1 *Preliminary Considerations.*

The point of view that will be adopted in the following discussion is that the dynamics of the nonlinear modes considered are governed primarily by inviscid processes; hence, the role of viscosity and heat-conduction is to be the essentially mathematical one of remedying the singular behavior of inviscid neutral modes. The Reynolds number, Re, is to be regarded as fixed at some large, but finite, value, i.e., we will not perturb about some critical value of Re as is done in the Stuart-Watson theory. Instead, we will perturb about some point $J_0(\alpha)$ on the linear stability boundary while keeping Re fixed; the stability boundary for Holmboe's model is shown in Fig. 3 for different values of Re, where the numerical values were obtained using the computer program discussed in [17].

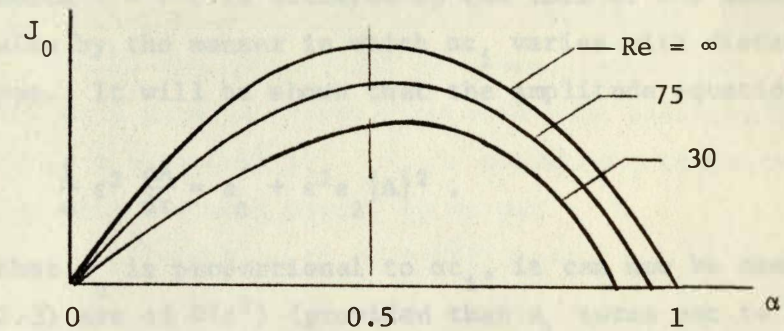

Fig. 3. Stability boundary $J_0(\alpha)$ at fixed Reynolds number.

In the usual linearized theory, one considers the behavior of perturbations consisting of a single Fourier mode proportional to $\exp\{i\alpha(x-ct)\}$, where α is the real wave number, c_r (the real part of the complex quantity c) is the phase speed and αc_i is the amplification factor for growing or decaying modes. It is important to note that the quantity c_i is a concept of linear theory; there is no c_i in a nonlinear theory. Nonetheless, it is conceptually useful in the *weakly* nonlinear theory to identify the amplitude pararmeter ε with the magnitude of αc_i that linear theory would predict at the same values of α, J_0 and Re. It will be seen that the choice $\varepsilon^2 = \alpha c_i$ is appropriate here.

A perturbation approach is to be employed, whereby the stream-function, temperature and density are expressed as power series in ε and expanded about their mean values. These expressions are then substituted into the full Boussinesq equations; the resulting sequence of linear problems obtained by separating out equal powers of ε can then be solved in succession. The O(ε) perturbation to the mean flow satisfies the usual equations of linearized theory; however, the stream function, for example, at this order takes the form

$$\psi^{(1)} = A(\tau)\ \Phi_1(y)\, e^{i\alpha(x-ct)} + A^*(\tau)\ \Phi_1^*(y) e^{-i\alpha(x-ct)}, \quad (2.1)$$

where $\tau = \varepsilon^2 t$ is a slow time scale and the real quantity c is the phase speed. In the "two-timing" method that is employed (see, e.g., the text by Cole [18]), the time derivatives are transformed according to

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau} . \quad (2.2)$$

The choice $\tau = \varepsilon^2 t$ is dictated by the form of the Landau equation (1.1) and also by the manner in which αc_i varies with distance from the neutral curve. It will be shown that the amplitude equation is given by

$$\frac{1}{A}\varepsilon^2 \frac{dA}{d\tau} = a_0 + \varepsilon^2 a_2 |A|^2 . \quad (2.3)$$

Recalling that a_0 is proportional to αc_i, it can now be seen that all terms in (2.3) are of $O(\varepsilon^2)$ (provided that a_2 turns out to be O(1), i.e., the theory is restricted to those values of J_0 and α where that turns out to be the case).

The expansions for J_0 and α can be determined in advance by employing Howard's [19] formula for perturbing away from the neutral curve. These equations (viz., (17) and (19) of [19]) show that near point A, for example, in Fig. 2, αc_i is proportional to $J_{on} - J_0$, where J_{on} is the Richardson number on the neutral curve and J_0 is evaluated at point B. At the point $\{\alpha = 1, J_{on} = 0\}$, on the other hand, Lin's perturbation formula (eq. (3.17) of §3.2.1) shows that αc_i is proportional to $\alpha_n - \alpha$. In order to accomodate both possibilities, we therefore write

$$J_0 \sim J_{on} + \varepsilon^2 J_2 + \ldots \text{ and, } \alpha \sim \alpha_n + \varepsilon^2 \alpha_2 + \ldots . \quad (2.4)$$

With the foregoing assumptions it will be seen that the amplitude equation (2.3) arises from an orthogonality condition that is imposed by the nonlinear forcing that occurs due to the interaction of the fundamental disturbance mode and the 2^{nd} harmonic.

2.2 Analysis

The dimensionless governing equations that result from the Boussinesq approximation are the vorticity equation

$$\frac{D}{Dt}\nabla^2\Psi - \frac{g}{\rho}\frac{\partial\rho}{\partial x} = \frac{1}{Re}\nabla^2(\nabla^2\Psi)\ , \tag{2.5}$$

the energy equation,

$$\frac{DT}{Dt} = \frac{1}{Re\ Pr}\nabla^2 T \tag{2.6}$$

and the equation of state

$$\rho = 1 - \hat{\beta}T_0(T-1)\ . \tag{2.7}$$

Quantities not previously defined are Pr, the Prandtl number, T_0, a reference temperature, and $\hat{\beta}$, the coefficient of thermal expansion.

We consider a parallel shear flow with mean velocity profile $\overline{u}(y)$ and density profile $\overline{\rho} = \exp\{-\beta\overline{r}(y)\}$. It is convenient to introduce a coordinate system moving with the wave speed, so we set

$$\Psi = \int^y\{\overline{u}(y,\tau) - c\}\ dy + \varepsilon\hat{\Psi}(\theta,y,\tau),$$

and

$$\rho = \overline{\rho}(y,\tau) + \varepsilon\hat{\rho}\ (\theta,y,\tau), \tag{2.8}$$

where

$$\theta = \alpha x \text{ and } \tau = \varepsilon^2\alpha t\ .$$

On the fast time scale, the flow is steady in the moving system and we find, after eliminating the temperature by using (2.7), that the perturbation stream function and density satisfy

$$\varepsilon^2\nabla^2\hat{\Psi}_\tau + (\overline{u}-c)\nabla^2\hat{\Psi}_\theta - \overline{u}''\hat{\Psi}_\theta + \frac{J_0\overline{r}'}{\overline{\rho}'}\hat{\rho}_\theta + \varepsilon(\hat{\Psi}_y\nabla^2\hat{\Psi}_\theta - \hat{\Psi}_\theta\nabla^2\hat{\Psi}_y) = \frac{1}{\alpha Re}\nabla^2(\nabla^2\hat{\Psi})\ , \tag{2.9}$$

and

$$\varepsilon^2\hat{\rho}_\tau + (\overline{u}-c)\hat{\rho}_\theta - \overline{\rho}'\ \hat{\Psi}_\theta + \varepsilon(\hat{\Psi}_y\hat{\rho}_\theta - \hat{\Psi}_\theta\hat{\rho}_y) = \frac{1}{\alpha Re\ Pr}\nabla^2\hat{\rho}, \tag{2.10}$$

where $\nabla^2 = \alpha^2 \frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial y^2}$ and $J_0 = g\beta$. To solve (2.9) and (2.10) the perturbation stream function is expanded as

$$\varepsilon\hat{\Psi} \sim \varepsilon\{\phi_1(\tau,y)e^{i\theta} + \phi_1^* e^{-i\theta}\} + \varepsilon^2\{\phi_2(\tau,y)e^{2i\theta} + \phi_2^* e^{-2i\theta}\} + \varepsilon^3\{\phi_{31}(\tau,y)e^{i\theta} + \phi_{31}^* e^{-i\theta} + \phi_{33} e^{3i\theta} + \phi_{33}^* e^{-3i\theta}\} + O(\varepsilon^4) \tag{2.11}$$

with a similar expansion for $\hat{\rho}$.

The quantities J_0 and α are also expanded according to (2.4). In the first-order problem, the variables are separated by writing $\phi_1 = A(\tau)\,\Phi_1(y)$ and $\rho_1(\tau,y) = A(\tau)\,P_1(y)$. It is possible to eliminate P_1 from the resulting two equations so that the problem for a linear neutral mode takes the form

$$L_{(\alpha)}\Phi_1 = (\bar{u}-c)^2\nabla_\alpha^2\Phi_1 - \bar{u}''(\bar{u}-c)\Phi_1 + \frac{i}{\alpha Re}(\bar{u}-c)(1+Pr^{-1})\nabla_\alpha^4\Phi_1 + J_{on}\bar{r}'\Phi_1 - \frac{1}{Pr\,(\alpha Re)^2}\nabla_\alpha^6\Phi_1 + \frac{i}{\alpha\,RePr}(2\bar{u}'\nabla_\alpha^2\Phi_1' - 2\bar{u}'''\Phi_1' - \bar{u}^{IV}\Phi_1) = 0\,, \tag{2.12}$$

where the operator $\nabla_\alpha^2 = \frac{d^2}{dy^2} - \alpha^2$.

The solution of (2.12) with its associated homogeneous boundary conditions constitutes an eigenvalue problem wherein solutions only exist for suitable combinations of the parameters α, c, J_0, Re and Pr. Numerical results for Holmboe's mean flow profiles were obtained by Maslowe and Thompson [17] using a Runge-Kutta procedure; the details and results are discussed in [17].

Proceding now to the $O(\varepsilon^2)$ problem, the variables can again be separated if we take

$$\phi_2 = A^2\Phi_2(y) \quad \text{and} \quad \rho_2 = A^2 P_2(y)\,.$$

After a considerable amount of algebra, a single nonhomogeneous equation for Φ_2 is obtained in the form

$$L_{(2\alpha)}\Phi_2 = (\bar{u}-c)\,(\Phi_1\Phi_1''' - \Phi_1'\Phi_1'') + \tfrac{1}{2}[\Phi_1(\bar{u}'\nabla_\alpha^2\Phi_1 - \bar{u}'''\Phi_1) - \frac{i}{\alpha Re}(\Phi_1'\nabla_\alpha^4\Phi_1 - \Phi_1\nabla_\alpha^4\Phi_1')] + \frac{i}{4\alpha\,Re\,Pr}(\Phi_1\nabla_{2\alpha}^2\Phi_1''' + \Phi_1'\nabla_{2\alpha}^2\Phi_1'' - 2\Phi_1''\nabla_{2\alpha}^2\Phi_1')\,, \tag{2.13}$$

where the operators $L_{(2\alpha)}$ and $\nabla^2_{2\alpha}$ are as defined previously but with α replaced everywhere by 2α, e.g., $\nabla^2_{2\alpha} = \frac{d^2}{dy^2} - 4\alpha^2$.

The numerical solution of (2.13) turns out to be a difficult matter because the "shooting method" used to solve (2.12) was found to be unstable here. Instead, a finite-difference scheme is now being employed; these matters are discussed more fully in §2.3, below.

As a final observation concerning (2.13), we note that while the equation is very complicated due to the inclusion of dissipative term, these are absolutely essential because of the strong singularity at $y = 0$ in the inviscid problem. Near the singularity, the inviscid *linear* neutral mode behaves like $y^{1-\alpha}$; however, the inviscid counterpart of Φ_2 is much more singular, behaving like $y^{-2\alpha}$ as $y \to 0$.

It is the solvability condition associated with the $O(\varepsilon^3)$ term ϕ_{31} that leads to the amplitude equation, and we now proceed to that term. Writing $\phi_{31} = A^2A^*\Phi_{31}$, and eliminating the density, as before, leads to the equation

$$A^2A^*L_{(\alpha)}\Phi_{31} = i\frac{dA}{d\tau}\{2(\bar{u}-c)(\Phi_1'' - \alpha_n^2\Phi_1) - \bar{u}''\Phi_1\}$$
$$+ \{2(\bar{u}-c)^2\alpha_2 - J_2\bar{r}'\}\Phi_1 A - G(y)A^2A^* , \qquad (2.14)$$

where

$$G(y) = [2\Phi_1'^*\nabla^2_{2\alpha}\Phi_2 + \Phi_1^*\nabla^2_{2\alpha}\Phi_2' - \{2\Phi_2\nabla^2_\alpha\Phi_1'^* + \Phi_2'\nabla^2_\alpha\Phi_1\}](\bar{u}-c)$$
$$+ \frac{J_{on}}{\beta}[2(P_2\Phi_1'^* - \Phi_2P_1'^*) + (P_2'\Phi_1^* - \Phi_2'P_1^*)] .$$

The quantity $G(y)$ represents the nonlinear interaction of the second harmonic ($e^{2i\theta}$) and the fundamental mode ($e^{-i\theta}$) terms.

A necessary and sufficient condition for the existence of a solution to (2.14) is that the right side be orthogonal to the solution of the adjoint problem [20; p.874]. The adjoint function $\tilde{\Phi}_1$ satisfies, in the present case, the differential equation (cf. (2.12))

$$(\overline{u}-c)^2\nabla^2_\alpha\tilde{\Phi}_1 + 4\overline{u}'(\overline{u}-c)\tilde{\Phi}_1' + \frac{2i}{\alpha Re}(2+Pr^{-1})\overline{u}'\nabla^2_\alpha\tilde{\Phi}_1'$$

$$+\frac{i}{\alpha Re}\{(1+Pr^{-1})(\overline{u}-c)\nabla^4_\alpha\tilde{\Phi}_1 + 2\overline{u}''\nabla^2_\alpha\tilde{\Phi}_1 + 4(\overline{u}''\tilde{\Phi}')' + \overline{u}^{IV}\tilde{\Phi}\} \qquad (2.15)$$

$$-\frac{1}{Pr(\alpha Re)^2}\nabla^6_\alpha\tilde{\Phi} + \{\overline{u}'(\overline{u}-c) + 2\overline{u}'^2 + J_0\overline{r}'\}\tilde{\Phi}_1 = 0 .$$

For Holmboe's model, $\tilde{\Phi}_1$ satisfies the same boundary conditions as Φ_1, the eigenfunction of the linear problem. Imposing the adjoint orthogonality condition leads to the following result:

$$i\frac{dA}{d\tau}\int_{-\infty}^{\infty}\tilde{\Phi}\,\{2(\overline{u}-c)(\Phi_1'' - \alpha^2\Phi_1) - \overline{u}''\Phi_1\}dy$$

$$+ A\int_{-\infty}^{\infty}\{2(\overline{u}-c)^2\alpha_2 - J_2\overline{r}'\}\Phi_1\tilde{\Phi}\,dy - A^2A^*\int_{-\infty}^{\infty}G(y)\tilde{\Phi}\,dy = 0 , \qquad (2.16)$$

which is equivalent to the Landau equation (2.3).

At the present time, the integrals in (2.16) have not yet been evaluated for Holmboe's flow. However, some computations have been done for the limiting case $\alpha = 1$, $J_0 = 0$, i.e., a homogeneous tanh y velocity profile. An exact solution was found in the limit $Re \to \infty$ for that case by Schade [12] who found that the Landau constant $a_2 = \frac{16}{3\pi}$. However, no calculations have yet been published at finite Reynolds numbers. We have undertaken such calculations in order to isolate the numerical difficulties and, also, because shear layer experiments are usually done at values of the Reynolds number of around 40 (see, e.g., [21]).

2.3 Discussion of Numerical Difficulties.

Boundary-value problems are usually solved by adapting techniques such as the Runge-Kutta method that are really intended for initial-value problems. These "shooting methods" are relatively easy to program, economical in terms of computer storage space and computing time and generally yield highly accurate results. However, one difficulty, rarely discussed in numerical analysis texts, occurs when a small parameter multiplies the highest-ordered derivative of an ordinary differential equation; in that case, the shooting method may be very unstable.

This difficulty occurs, for example, when the Orr-Sommerfeld equation is solved at high Reynolds numbers and is discussed in the text by Betchov and Criminale [22]. The difficulty is essentially that the desired first integrating to y = 0; and (ii) both of these "inviscid solutions" are "contaminated" in the shooting method by the fast-growing "viscous solution" multiplied by B_3 in (2.18).

In order to overcome this difficulty, the finite-difference method has been employed. Although more cumbersome to program than the Runge-Kutta method, this scheme does not seem to be susceptible to the instability noted above. It is also easier to program than the more sophisticated shooting methods. Dr. P.B. Bailey, in his recent review [23] of a new book on shooting methods, has criticized the tendency of the authors to play down the utility of the finite-difference method. Our own experience strongly supports Dr. Bailey's point-of-view. Some of our numerical results are presented in Fig. 4; in addition, programming has been initiated of the 6^{th} order equation (2.13). Some calculations have also been made of the Landau constant a_2 in the homogeneous case and, at Re = 75, it seems to be reduced by nearly 50% from its inviscid value.

2.4 *Distortion of the Mean Flow*

An interesting nonlinear effect that occurs at $O(\varepsilon^2)$ is that the "self-interaction" of the fundamental disturbance mode leads to a distortion of the mean flow profiles. This effect was neglected by Schade, because the distortion vanishes as $Re \to \infty$. It arises mathematically from the product of $e^{i\theta}$ and $e^{-i\theta}$ terms, so that to separate variables, we write (cf. (2.8))

$$\overline{u}(y,\tau) = \tanh y + \varepsilon^2 AA^* f(y) \; . \qquad (2.19)$$

This leads, in the case of a homogeneous shear flow, to the following equation for f(y)

$$\Phi_1''\Phi_1^* - \Phi_1''^*\Phi_1 = \frac{1}{i\alpha Re} f'' \; . \qquad (2.20)$$

The solution of (2.20) can be written

$$f(y) = i\alpha Re \int_{-\infty}^{y} (\Phi_1'\Phi_1^* - \Phi_1'^*\Phi_1)\, dy. \qquad (2.21)$$

It is not believed that this effect is of great importance in flows having inflection points and, therefore, it has not been taken into

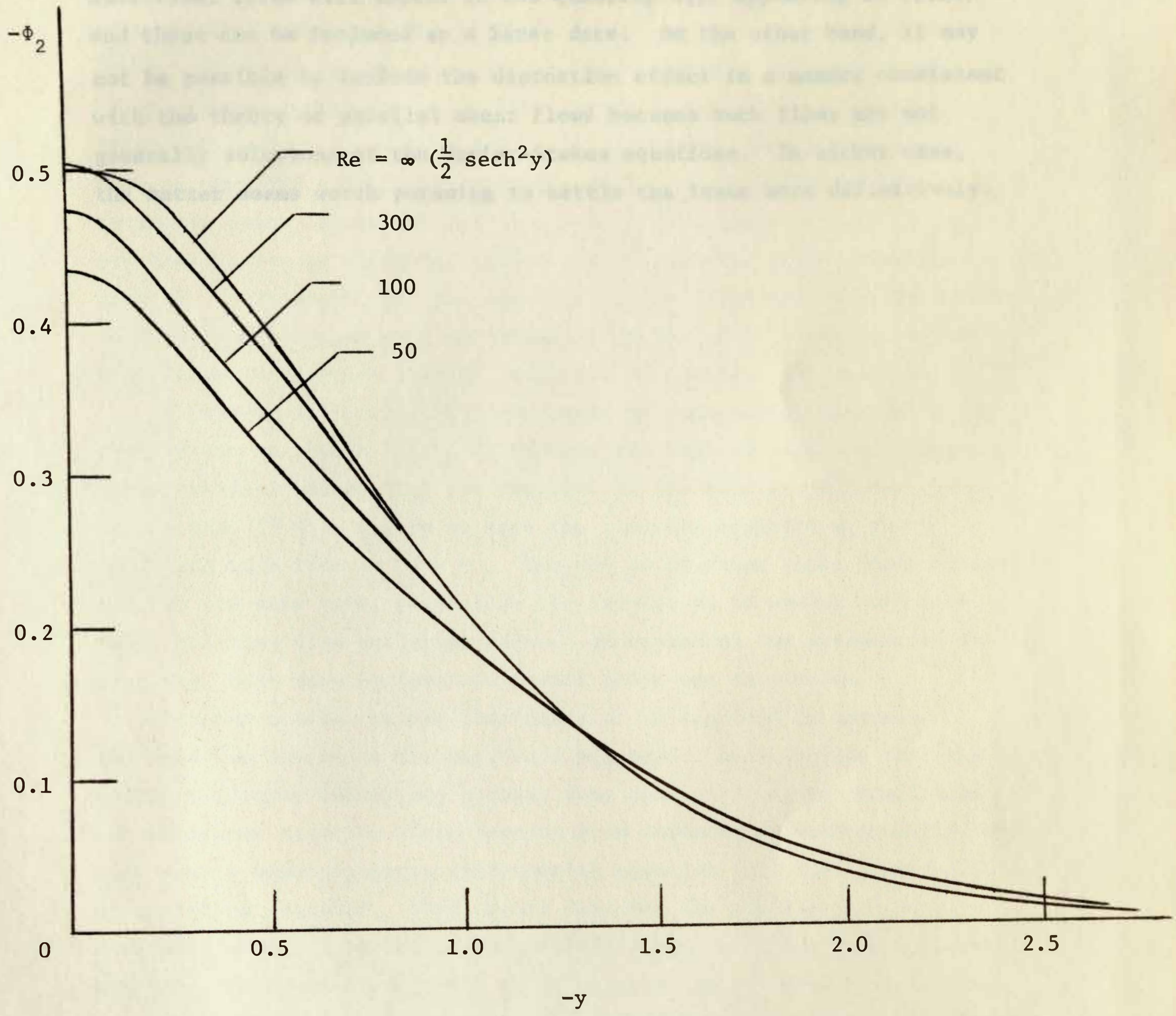

Fig. 4 Comparison of numerical solution for Φ_2 at various Reynolds numbers with the exact result in the inviscid limit.

account in formulating the theory for stratified shear flows. However, f(y) is being computed in the homegeoneous case in order to verify that that is the case. If it turns out that the distortion is significant, additional terms will appear in the quantity G(y) appearing in (2.14) and these can be included at a later date. On the other hand, it may not be possible to include the distortion effect in a manner consistent with the theory of parallel shear flows because such flows are not generally solutions of the Navier-Stokes equations. In either case, the matter seems worth pursuing to settle the issue more definitively.

3.0 SPACE-TIME EVOLUTION OF NONLINEAR WAVES

3.1 The solvability condition.

Our principal objective in this section is to study the evolution of finite-amplitude modes in those parallel shear flows where nonlinear effects are dominant in the critical layer region. As noted previously, Benney and Bergeron [13] have solved the eigenvalue problem for neutral modes; however, important questions remain unresolved concerning the interpretation of these solutions. In particular, one would like to know if they are stability boundaries, i.e., do the adjacent points in parameter space correspond to "nearly-neutral" solutions and, if so, are they stable or unstable? Also, are the neutral modes themselves stable? It is hoped that the theory outlined below will provide answers to these questions as well as insight into the whole limiting process that leads to singular neutral solutions of the Rayleigh equation.

A primary difficulty that one faces in formulating the theory is that we can no longer deduce in advance the type of parameter expansions about critical values that are employed in the weakly-nonlinear theory (e.g., eqs. (2.4)); nor do we know the limiting behavior of the amplitude with time as $\varepsilon \to 0$. One way to overcome these difficulties and, at the same time, generalize the results is to employ both slow space and slow time variables instead of expanding the parameters; in that way, both wave systems and normal modes can be studied.

Another crucial matter that needs to be resolved in advance concerns the source of the amplitude equation. Anticipating that the weakly nonlinear theory may provide some sort of clue, we recall that the amplitude equation there results from imposing an orthogonality condition upon a nonhomogeneous differential equation (cf. eqs. (2.14)-(2.16) of preceding section). That theory presumes the availability of a numerical solution of the Orr-Sommerfeld equation in the zeroth-order problem. However, a *uniformly valid* solution is not available in the nonlinear critical layer approach, because asymptotic methods are employed with the resulting consequence being that the critical layer and outer flow are regarded as asymptotically distinct. Clearly, if one wishes to draw a parallel between the two approaches, the weakly nonlinear theory would have to be reformulated using the asymptotic solution of the Orr-Sommerfeld equation as a starting point.

Looking at the matter from that point of view, one can see, after careful consideration, that the amplitude equation would result from an examination of the outer flow and boundary conditions rather than the inner (critical layer) problem. An orthogonality condition can be obtained, but the integrals that result are singular at the critical point. The role of the critical layer solution is to provide information on the behavior of the integrals near the singularity. One concludes in the viscous theory that the functions involved can be continued analytically into the complex plane and the contour of integration indented so that a specified phase change occurs.

As there is no phase change in the case of a nonlinear critical layer, one might expect that the correct result is obtained by simply interpreting the integral as a principal value. However, for most flows it turns out that the principal value integral does not exist. Therefore, a more refined procedure must be used, i.e., a solvability condition can still be derived and it leads to the amplitude equaltion; however, this condition is *not* that the inhomogeneous terms be orthogonal to the adjoint solution.

To illustrate the procedure, initially we have a (generally singular) solution $\phi(y)$ of the Rayleigh equation

$$\mathcal{L}\phi = \phi'' - \alpha^2\phi - \frac{\bar{u}''}{\bar{u} - c}\phi = 0 \ , \qquad (3.1)$$

where the perturbation stream function in the outer problem is expanded in powers of ε as in (2.11). Now, at $O(\varepsilon^3)$, an equation is encountered having the form

$$\mathcal{L}\,\Theta = f(y) \ , \qquad (3.2)$$

where $f(y)$ is singular at the critical point y_c. Clearly, there will be some restriction on the function $f(y)$ for a solution to exist.

To arrive at the desired solvability condition, we integrate the quantity $\phi\mathcal{L}\Theta - \Theta\mathcal{L}\phi$ between the boundaries y_1 and y_2 and the points $y_c \pm \delta$, where $\delta \ll 1$, on either side of the singularity thereby obtaining

$$[\phi\Theta' - \Theta\phi']_{y_1}^{y_c-\delta} + [\phi\Theta' - \Theta\phi']_{y_c+\delta}^{y_2} = \int_{y_1}^{y_c-\delta}\phi f dy + \int_{y_c+\delta}^{y_2}\phi f dy \ . \qquad (3.3)$$

To actually employ the condition (3.3), the solution for Θ needs to be found and, in that respect, the problem is more difficult than its counterpart in the weakly nonlinear theory.

Let us now wite the solutions to (3.1) and (3.2) as

$$\phi = A\,\phi_a(y) + B\,\phi_b(y) ,$$

and (3.4)

$$\Theta = C\,\phi_a(y) + D\,\phi_b(y) + P(y) ,$$

where $P(y)$ is a particular solution of (3.2). Imposing the boundary conditions, namely,

$$C\,\phi_a(y_1) + D\,\phi_b(y_1) + P(y_1) = 0 ,$$

and

$$C\,\phi_b(y_2) + D\,\phi_b(y_2) + P(y_2) = 0$$

leads to the solvability condition

$$\frac{\phi_a(y_1)}{\phi_a(y_2)} = \frac{\phi_b(y_1)}{\phi_a(y_2)} = \frac{P(y_1)}{P(y_2)} , \tag{3.5}$$

which is the equation equivalent to the orthogonality condition. To actually impose this condition, however, we employ (3.4) in (3.3) to obtain

$$-\,[\phi P' - \phi' P]_{y_c - \delta}^{y_c + \delta} = \int_{y_1}^{y_c - \delta} \phi f dy + \int_{y_c + \delta}^{y_2} \phi f dy , \tag{3.6}$$

where the boundary conditions have been used as well as the fact that $W(\phi_a, \phi_b) = -1$, W being the Wronskian. This procedure is illustrated in the derivation of the amplitude equation for a mixing layer as discussed below.

3.2 Mixing Layer Profile.

The free shear layer is now considered for the two limiting cases in which either viscous or nonlinear effects are dominant in the critical layer. As the velocity profile, we consider $\bar{u}(y) = \tanh y$ which makes this case somewhat atypical because there is a regular neutral solution in the linear problem. However, singularities are encountered nonetheless; these occur at higher orders in the viscous theory, whereas in the nonlinear critical layer theory, singular neutral modes are encountered even in the linear problem associated with the Rayleigh equation. These modes, which have no counterpart in the viscous theory, have nonzero phase speeds, so that the point where $\bar{u}(y) = c$ does not coincide with the inflection point $y = 0$ in the velocity profile.

Our starting point is the inviscid vorticity equation

$$\frac{D}{Dt} \nabla^2\psi = 0 \quad . \tag{3.7}$$

As noted above, only the outer problem is of concern in deriving the amplitude equation, and that is why viscosity is neglected; its role is implicit in treating the singularities arising in the solution of (3.7) as discussed in §3.1. Introducing, again, a coordinate system moving with the wave, we set

$$\psi = \int_0^y \{\bar{u}(y,\tau) - c\}\, dy + \varepsilon\hat{\psi}\,(\theta, X, t, \tau) \tag{3.8}$$

where

$$\Theta = \alpha x,\ X = \varepsilon^2\Theta \quad \text{and} \quad \tau = \alpha\mu(\varepsilon)t\ .$$

Employing the method of multiple scales [18], we transform the derivatives in (3.7) according to

$$\frac{\partial}{\partial t} = \alpha\mu\frac{\partial}{\partial\tau}\ , \quad \frac{\partial}{\partial x} = \alpha\left(\frac{\partial}{\partial\theta} + \varepsilon^2\frac{\partial}{\partial X}\right) \tag{3.9}$$

and

$$\frac{\partial^2}{\partial x^2} = \alpha^2\left(\frac{\partial^2}{\partial\theta^2} + 2\epsilon^2\frac{\partial^2}{\partial\theta\partial X} + \epsilon^4\frac{\partial^2}{\partial X^2}\right) \quad . \tag{3.9}$$

The choice of $\mu(\epsilon)$ represents the point of departure for treating the various cases that are considered below. For a viscous critical layer and also for a nonlinear critical layer with $c \neq 0$, $\mu = \epsilon^2$; on the other hand $\mu = \epsilon$ for a nonlinear critical layer with $c = 0$. It is difficult to provide a definite rationale for these particular choices other than to say that these are the scalings that worked. However, when the linear group velocity either vanishes or becomes infinite, the scaling of X or τ will be influenced accordingly. This point is discussed at greater length in §5.

3.2.1 Viscous Critical Layer.

The expansion for $\hat{\psi}$ that permits separation of variables has the form

$$\hat{\psi} \sim \phi_1(y)\ \{A(X,\tau)e^{i\theta} + A^*e^{-i\theta}\} + \epsilon\phi_2(y)\{A^2e^{2i\theta} + (A^2)^*e^{-2i\theta}\} \tag{3.10}$$
$$+ \epsilon^2\{\phi_{31}(y)\,[A^2A^*e^{i\theta} + A(A^2)^*e^{-i\theta}] + \phi_{33}(y)[A^3e^{3i\theta} + (A^3)^*e^{-3i\theta}]\} + \ldots .$$

Substituting (3.8) - (3.10) into the vorticity equation leads, at $O(\epsilon)$, to the result that ϕ_1 satisfies the Rayleigh equation (3.1). The well-known solution for $\overline{u} = \tanh y$ is given by

$$\phi_1 = \operatorname{sech} y \ , \ c = 0 \text{ and } \alpha = 1 \quad . \tag{3.11}$$

At $O(\epsilon^2)$, ϕ_2 satisfies the nonhomogeneous equation

$$(\overline{u} - c)(\phi_2'' - 4\alpha^2\phi_2) - \overline{u}''\phi_2 = \frac{1}{2}(\phi_1\phi_1''' - \phi_1'\phi_1'')$$
$$= 2\operatorname{sech}^4 y \quad . \tag{3.12}$$

The particular solution of (3.12) satisfying the boundary conditions, as noted in §2.3, is known to be

$$\phi_2 = -\frac{1}{2}\operatorname{sech}^2 y \ . \tag{3.13}$$

We next obtain the desired evolution equation for the amplitude function $A(X,\tau)$ by considering the term ϕ_{31} which satisfies the equation

$$A^2A^*\mathcal{L}\phi_{31} = \{\overline{u} - c\}^{-1}\ \{i\frac{\partial A}{\partial\tau}(\phi_1'' - \alpha^2\phi_1) - A^2A^*G(y)\} - 2i\frac{\partial A}{\partial X}\phi_1 \ , \tag{3.14}$$

where $G(y) = 2\phi_1'(\phi_2'' - 3\phi_2) + \phi_1(\phi_2''' - 3\phi_2') - 2\phi_2\phi_1''' - \phi_2'\phi_1'' = -10 \tanh y \operatorname{sech}^5 y$.

Because the Rayleigh equation is self-adjoint, the adjoint function $\tilde{\phi}_{31} = \phi_1 = \operatorname{sech} y$, so that the orthogonality condition associated with (3.14) is

$$\frac{2}{i}\frac{\partial A}{\partial \tau}\int_{-\infty}^{\infty}\frac{\operatorname{sech}^4 y}{\tanh y}\,dy - 2i\frac{\partial A}{\partial X}\int_{-\infty}^{\infty}\operatorname{sech}^2 y\,dy + 10A^2A^*\int_{-\infty}^{\infty}\operatorname{sech}^6 y\,dy = 0. \quad (3.15)$$

The first integral, being singular, is evaluated by indenting the contour of integration below the singularity; the other integrals are straightforward and lead to the following generalization of Schade's result:

$$\frac{\partial A}{\partial \tau} - \frac{2i}{\Pi}\frac{\partial A}{\partial X} = -\frac{16}{3\Pi}A^2A^* . \quad (3.16)$$

To recover the equation of Stuart-Watson theory, and also gain further insight into (3.16), we restrict attention to a single Fourier mode disturbance and recall that according to Lin's perturbation formula [2, p.42]

$$\alpha c_i = \frac{2\alpha}{\Pi}(1-\alpha) \simeq \frac{2}{\Pi}(1-\alpha) , \quad (3.17)$$

for $\overline{u} = \tanh y$. Considering now an unstable mode, and writing $A(X,\tau) = a(\tau)\exp\{-\frac{1}{2}i\Pi\alpha c_i X\}$, (3.16) becomes

$$\frac{1}{a}\frac{da}{d\tau} = \alpha c_i - \frac{16}{3\Pi}a^2 + O(\varepsilon^2) , \quad (3.18)$$

which is identical to the result obtained by Schade. In other words, from (3.17) it is evident that $(1-\alpha)$, being proportional to αc_i, is $O(\varepsilon^2)$. Therefore, to employ the approach outlined in this report, it can readily be determined in advance, using Lin's perturbation formula, that the appropriate slow space scale is $X = \varepsilon^2 x$.

A second point of interest is that, by extending the usual definition of group velocity so that c_g is complex, stability problems can be treated from the point of view of the general amplitude evolution equation stated in §1.2. Employing the usual definition $c_g = d\omega/d\alpha$, and noting that c_r is independent of α in the present example, (3.17) yields the result

$$c_g = -\frac{2i}{\Pi} . \quad (3.19)$$

It is now clear that (3.15) is of the form given earlier, namely,

$$\frac{\partial A}{\partial \tau} + (c_g - c)\frac{\partial A}{\partial X} = a_2 A^2A^* . \quad (1.3)$$

We can also see that for modes travelling at the group velocity the coefficient of $\partial A/\partial X$ vanishes and a new scaling has to be employed; the result is an equation that is second-order in X, as discussed later in §5.

3.2.2. Nonlinear Critical Layer

The eigenvalue problem associated with the Rayleigh equation was solved numerically in [13] with the condition that there is no phase change across y_c. When $c = 0$, the solution is regular being given simply by (3.11). Unlike the viscous theory, however, singular solutions also exist with $c \neq 0$; the numerical results are reproduced in Fig. 5 below.

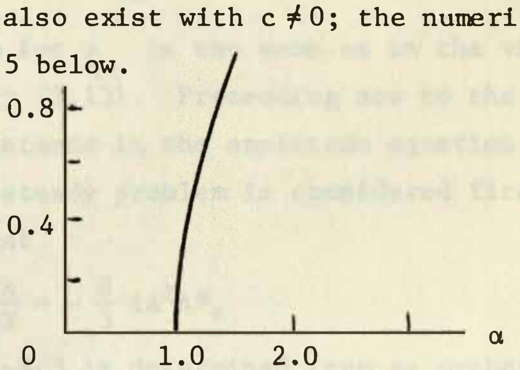

Fig. 5 Neutral stability curve for $\bar{u} = \tanh y$.

The theory for those modes with $c \neq 0$ leads to the amplitude equation (1.3). Computations of the constant a_2 (which is a pure imaginary number when a nonlinear critical layer is employed) must be done numerically and are presently underway. When $c = 0$, however, the problem can be solved in closed form with interesting results.

Although the linear problem in this case is regular, it is found that singularities are encountered at higher orders so that critical layer concepts become relevant. The need for introducing a new time scale into the problem becomes apparent when we consider (3.15); the principal value of the 1st integral is zero so that, with no phase change, the coefficient of $\partial A/\partial \tau$ vanishes.

The difficulty can be traced to the form of the amplitude equation (1.3) when it is observed that $c_g (= d\omega/d\alpha) \to \infty$ as $c \to 0$, i.e., if (1.3) is divided through by c_g, then the coefficient of $\partial A/\partial \tau$ will be zero. In that case, a new slow time scale must be introduced, the appropriate choice being $\tau = \alpha \varepsilon t$. The expansion for $\hat{\psi}$ now takes the form

$$\hat{\psi} \sim \phi_1(y)\{A(X,\tau)e^{i\Theta} + *\} + \varepsilon\{\phi_{21}(A_\tau e^{i\Theta} + *) + \phi_{22}(A^2 e^{2i\Theta} + *)\}$$
$$+ \varepsilon^2\{\phi_{31}(A^2A^* e^{i\Theta} + *) + \phi_{33}(A^3 e^{3i\Theta} + *)\} + . \qquad (3.20)$$

There is now an additional term ϕ_{21} which satisfies the equation

$$(\overline{u} - c)(\phi_{21}'' - \alpha^2\phi_{21}) - \overline{u}''\phi_{21} = i(\phi_1'' - \alpha^2\phi_1) = -2i\,\text{sech}^3 y. \qquad (3.21)$$

This equation is singular at $y = 0$; its solution can be found using variation of parameters and is

$$\phi_{21} = -i\ \text{sech}\ y \int_0^y \{2\cosh^2 y \log|\tanh y| + 1\}\, dy. \qquad (3.22)$$

As the problem for ϕ_{22} is the same as in the viscous theory, the solution is given by (3.13). Proceeding now to the ϕ_{31} term, we remark that the constants in the amplitude equation can be determined most easily if the steady problem is considered first. In that problem, we find that

$$\frac{\partial A}{\partial X} = -\frac{8}{3} iA^2A^*, \qquad (3.23)$$

where the constant $-8/3$ is determined from an orthogonality condition. Having obtained that result, one finds that to achieve separation of variables in the full time-dependent problem, the amplitude equation must take the form

$$\frac{\partial^2 A}{\partial \tau^2} = i\beta\left(\frac{\partial A}{\partial X} - \frac{8}{3} iA^2A^*\right). \qquad (3.24)$$

In order to determine the constant β, one must employ a solvability condition of the form of equation (3.6). This condition first requires that a particular solution be found for ϕ_{31}. That quantity is a solution of the equation

$$\mathcal{L}\phi_{31} = -2i\ \text{sech}\ y + i\beta f(y) \qquad (3.25)$$

where

$$f(y) = \frac{2\ \text{sech}^3 y}{\tanh^2 y} - \frac{2\ \text{sech}^3 y}{\tanh y}\int_0^y \left\{\frac{2\log|\tanh y|}{\text{sech}^2 y} + 1\right\} dy .$$

As the details are somewhat lengthy, we simply remark that the particular solution of (3.25) can be found using the method of variation of parameters in conjunction with the homogeneous solutions

$$\phi_{31}^{(1)} = \text{sech}\ y, \text{ and } \phi_{31}^{(2)} = y\ \text{sech}\ y + \sinh y . \qquad (3.26)$$

By employing T = tanh y as the independent variable, the final result turns out to be

$$\beta = \{\int_0^1 \left(\frac{2 \log T}{1 - T^2} + 1\right)^2 dT - 1\}^{-1} = 0.2377. \tag{3.27}$$

3.3 Broken-Line Profiles

The foregoing theory can be applied most easily to discontinuous flow models because, in such cases, the linear dispersion relation is always available. We can therefore neglect the $\partial A/\partial X$ term, initially, because the value of the Landau constant is independent of linear dispersive effects and can be computed by writing (1.3) as

$$\frac{\partial A}{\partial \tau} = i\gamma A^2 A^* , \tag{3.28}$$

where $\tau = \alpha\varepsilon^2 t$ and the variables to be employed are those defined in (3.8). The $\partial A/\partial X$ term can later be added to (3.28) by evaluating c_g using the dispersion relation.

The particular mean flow that will be used to illustrate the theory is the boundary layer model

$$\bar{u}(y) = \begin{cases} y & 0 \leq y \leq 1 \\ 1 & y > 1 \end{cases} . \tag{3.29}$$

In each region, the perturbation stream function $\hat{\psi}$ must satisfy the vorticity equation

$$\hat{\psi}_{yy} + \alpha^2 \hat{\psi}_{\theta\theta} = 0 , \tag{3.30}$$

while the equation describing the disturbed interface is

$$y = 1 + \varepsilon\eta(\theta,\tau) . \tag{3.31}$$

The boundary conditions imposed at the free surface are firstly the kinematical condition

$$\frac{D\eta}{Dt} = \varepsilon^2 \eta_\tau + \{(\bar{u} - c) + \varepsilon\hat{\psi}_y\}\eta_\theta + \hat{\psi}_\theta = 0 , \tag{3.32}$$

which states essentially that the interface is a streamline, and, secondly, the condition that the pressure is constant across the interface, viz.,

$$[\varepsilon^2 \hat{\psi}_{y\tau} + \{(\bar{u} - c) + \hat{\psi}_y\}\hat{\psi}_{y\theta} - (\bar{u}' + \varepsilon\hat{\psi}_{yy})\hat{\psi}_\theta] = 0 , \tag{3.33}$$

where [ ] denotes the jump in the quantity enclosed within the brackets.

The dependent variables are represented by the following power series expansions:

$$\hat{\psi} \sim \psi^{(1)} + \varepsilon\psi^{(2)} + \varepsilon^2\psi^{(3)} + \ldots .$$
$$\eta \sim \eta^{(1)} + \varepsilon\eta^{(2)} + \varepsilon^2\eta^{(3)} + \ldots . \qquad (3.34)$$

In addition, the quantities $\psi^{(n)}$ and $\bar{u}(y)$, both above and below the interface, are expanded in Taylor series about its mean value $y = 1$. As the resulting equations are quite lengthy, only the principal results will be presented here.

To permit separation of variables, it is found that appropriate forms for the terms in the expansion for η are the following:

$$\eta^{(1)} = A(\tau)e^{i\theta} + A^* e^{-i\theta}$$
$$\eta^{(2)} = A^2 N_2\, e^{2i\theta} + * \qquad (3.35)$$
$$\text{and} \quad \eta^{(3)} = A^2 A^* N_{13}\, e^{i\theta} + A^2 A^* N_{33}\, e^{3i\theta} + * ,$$

where $*$ denotes the complex conjugate. The stream functions in the upper and lower regions, in order to satisfy (3.30), take the following forms:

$$\begin{array}{ll} \underline{y \geq 1} & \underline{y \leq 1} \\ \psi^{(1)} = A a_1 e^{-\alpha y} e^{i\theta} + * & \psi^{(1)} = A b_1 \sinh \alpha y\, e^{i\theta} + * \\ \psi^{(2)} = A^2 a_2 e^{-2\alpha y} e^{2i\theta} + * & \psi^{(2)} = A^2 b_2 \sinh 2\alpha y\, e^{2i\theta} + * \\ \psi^{(3)} = A^2 A^* a_{13}\, e^{-\alpha y} e^{i\theta} + * & \psi^{(3)} = A^2 A^* b_{13} \sinh \alpha y\, e^{i\theta} + * \end{array} \qquad (3.36)$$

Terms proportional to $e^{3i\theta}$ have been omitted because they are not required to the order that will be considered.

If we substitute the above forms for $\hat{\psi}$ and η into the interface condition (3.32) (which is imposed both at $y = 1^+$ and $y = 1^-$) and (3.33), we obtain, at zeroth order, the system

$$M\underline{\phi} \equiv \begin{bmatrix} (1-c) & e^{-\alpha} & 0 \\ (1-c) & 0 & \sinh\alpha \\ 0 & \alpha(1-c)e^{-\alpha} & (1-c)\alpha\cosh\alpha - \sinh\alpha \end{bmatrix} \begin{Bmatrix} 1 \\ a_1 \\ b_1 \end{Bmatrix} = 0 . \qquad (3.37)$$

For a nontrivial solution to exist, the determinant of the matrix M must vanish; this eigenvalue condition leads to the dispersion relation

$$c = 1 - \frac{e^{-\alpha}\sinh\alpha}{\alpha} . \qquad (3.38)$$

Corresponding to the normalization chosen, we also obtain

In a coordinate system moving with the group velocity, the amplitude equation becomes 2^{nd} order in X, as discussed in §5, and instability occurs if $\beta\gamma < 0$ (see e.g., Benney and Newell [24]), where $\beta = -0.5\omega''(\alpha)$. For the flow investigated, it appears that there will be instability to large scale perturbations; the instability involves a resonant interaction and can be interpreted physically as meaning that some incoherence will develop in the primary wave over long periods of space and time.

4.0 INSTABILITY OF RIGIDLY ROTATING SWIRL FLOWS

This investigation concerns flows having the basic velocity components $\{0, \Omega_0 r, W(r)\}$ in a cylindrical coordinate system $\{r, \theta, z\}$. In general, such flows are most unstable to non-axisymmetric disturbances, i.e., to perturbations proportional to $\exp\{i(kz + m\theta - \omega t)$, where k and m are the axial and aximuthal wave numbers, respectively, and ω is a complex quantity whose real part is the frequency, whereas its imaginary part is the amplification factor. All quantities have been non-dimensionalized with respect to a characteristic radial dimension r_0, axial velocity W_0 and a characteristic frequency Ω. The principal dimenionless parameter that emerges is the Rossby number ε defined by

$$\varepsilon = \frac{W_0}{\Omega_0 * r_0} , \tag{4.1}$$

where Ω_0^* is the dimensional angular velocity.

A single second order ordinary differential equation can be derived for the radial perturbation velocity and it develops that a crucial quantity appearing in that equation is the term

$$\gamma(r) = k\varepsilon\Omega_0 W(r) + m\Omega_0 - \omega . \tag{4.2}$$

This quantity seems to play a role analogous to that of the $(\bar{u} - c)$ term appearing in the Rayleigh equation (3.1). In [16], it was proven that a necessary condition for instability is that $m\gamma_r$ must be *positive* somewhere within the range of r (the subscript denotes the real part of γ). On the other hand, for the specific case of a parabolic axial profile, $W = 1 - r^2$, it was proven that $m\gamma_r$ must be *negative* at some value of r for instability to occur. Hence, one concludes that $\gamma_r = 0$, somewhere, is a necessary condition for instability.

This result, which is analogous to the real part of Howard's semicircle theorem [2] for stratified flows, says that all unstable modes are the limits of singular neutral modes as $\omega_i \to 0$. One interpretation of the result is that in a frame of reference moving with the *axial* phase speed of the wave, the mean flow velocity component in the direction of the number vector must vanish. The significance of that interpretation is that it is suggestive of the Tollmien-Schlichting mechanism of instability, whereas previous investigators have speculated that the instability of swirl flows is due to centrifugal instability. Clearly, the question is deserving of further study as the present results seem to apply to a large class of swirl flows, but the forementioned condition has only been *proven* as necessary in one specific case.

A numerical study has also been made of the growth rates of unstable perturbations in the case of rigidly rotating Poiseuille flow. These calculations extend the earlier work of Pedley [25] on the limiting case $\varepsilon << 1$ (rapid rotation) to arbitrary values of the Rossby number. It was found that the greatest instability occurred for $O(1)$ values of ε and that instability persists for large values i.e., $O(10)$ of this parameter. The reader is referred to [16] for further details of this study.

5.0 SUMMARY AND CONCLUSIONS

In §2, the weakly nonlinear stability theory was extended to stratified shear flows and it was shown that this theory leads to the amplitude equation

$$\frac{1}{A}\frac{dA}{dt} = a_0 + \varepsilon^2 a_2 |A|^2 . \tag{5.1}$$

Contrary to what one would expect after examining the equations of linear inviscid theory, eq. (5.1) is not second-order in time. This result can be viewed as a consequence of the fact that inviscid neutral modes for physically realistic flows are singular; therefore, in a linearized or weakly nonlinear theory, there always exist thin regions in which dissipative effects cannot be ignored. Once these effects enter into the theory, an amplitude equation, such as (5.1), that does not yield limit cycle solutions will be the result.

Computations of the Landau constant a_2 are presently being made for the mixing layer profile first studied by Holmboe. It is anticipated that a_2 will turn out to be a negative for that flow. Should that be the case, it will then be possible to calculate the steady finite-amplitude state to which a linearly unstable mode evolves.

It would also be of great interest to study other flows to see if sub-critical instability, $a_2 < 0$, can occur. The significance of such a result would be that flows known to be stable to infinitesimal perturbations could become destabilized when subjected to small but *finite* perturbations; should that occur when the Richardson number is greater than 1/4, it would have important implications in studies of atmospheric turbulence generation.

The original objective of the analysis in §3 was to investigate the temporal evolution of spatially periodic modes in parallel shear flows having nonlinear critical layers. In order to achieve that objective, it was found desirable to adapt some recent ideas from the theory of nonlinear waves. The result is a quite general formulation that appears to unify many aspects of the theories of nonlinear wave motion and hydrodynamic stability. Wave trains can be considered, as well as single Fourier modes, and it is hoped that the analysis provides a sound framework for future investigations of the many subtle questions arising in nonlinear stability theory.

The principal result obtained is that the amplitude evolution equation for nonlinear waves typically has the form

$$\frac{\partial A}{\partial \tau} + \omega'(\alpha)\frac{\partial A}{\partial X} - \frac{1}{2} i\mu\omega''(\alpha)\frac{\partial^2 A}{\partial X^2} = a_2 A^2 A^* , \qquad (5.2)$$

where $X = \mu x$ and $\tau = \mu t$ are slow space and time variables. In the general case, $\mu = \varepsilon^2$, and the dispersive term multiplied by $\omega''(\alpha)$ can be neglected. Interesting exceptional cases arise when $c_g = \omega'(\alpha)$ is either zero or infinite; new scalings are then required as demonstrated by the examples in §3. In the first case, the amplitude equation becomes second-order in X, whereas the case $c_g = \infty$ leads to an equation second-order in τ, as discussed in §3.2.2. Finally, it is evident that if $\omega'(\alpha)$ is real (this would correspond to the fastest growing wave, on a linear basis, in stability problems) then, in a coordinate system moving with the group velocity, (5.2) needs to be rescaled. This can easily be accomplished by introducing the variable $\xi = \varepsilon^{-1}(X-\omega'\tau)$, so that in the (ξ,τ) coordinates, (5.2) becomes

$$\frac{\partial A}{\partial \tau} = i\frac{\omega''}{2}\frac{\partial^2 A}{\partial \xi^2} + a_2 A^2 A^* . \qquad (5.3)$$

Equation (5.3) has arisen previously in a variety of contexts and, as discussed in [24], it describes the side band instability mechanism for dispersive waves.

The occurrence of second-order evolution equations can be predicted once it is recognized that what we are really finding is the Taylor series expansion of $\partial A/\partial \tau$ in the fully nonlinear problem, i.e., eq. (5.2) contains an infinite number of terms higher-order in μ. In the case $c_g = \infty$, the first term in the Taylor expansion is infinite so (5.2) is not the proper equation for pursuing the matter; instead, an equation must be derived that is the Taylor expansion of $\partial A/\partial X$. The $\partial A/\partial \tau$ term in that expansion is multiplied by c_g^{-1} and therefore vanishes, so that the first nonzero time derivative that appears in the amplitude equation is $\partial^2 A/\partial \tau^2$. These ideas seem to have quite general applicability and suggest a number of possible extensions.

The conclusion reached in §3 that a_2 is a pure imaginary number implies that the principal effect of a nonlinear critical layer is simply to alter the frequency so that we cannot say, at the present time, whether such modes are stable or unstable. Surprisingly, it seems that even in the nonlinear theory some viscosity is required to provide a mechanism for transferring energy from the mean flow to an unstable perturbation; that may not be true, however, for stratified shear flows. However, the theory can be extended to that case even though the inner problem there is not tractable; the extension may still be possible if the coefficients in the amplitude equation can be determined from a knowledge of the transition relationships across the critical point, as is the case with homogeneous shear flows.

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