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To Nonstationary Random Excitation

by

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## ABSTRACT

The mean square wave response of a lightly damped viscoelastic medium (here we consider Voigt viscoelastic solid model) to a special type of non-stationary random excitation is determined. The excitation function on the viscoelastic medium is taken in the form of a product of a well-defined, slowly varying envelope function, and a part which prescribes the statistical characteristics of the excitation. The latter is assumed to be white or correlated as a wide band process. By taking into consideration the slow variation of the envelope function and the wave characteristics of the lightly damped viscoelastic medium, the mean square response (for the various types of excitation and damping parameters) is evaluated.

## INTRODUCTION

A number of recent papers (1,2,3,4) have considered the response of dynamic systems to random excitation. However, the appropriate theory is well-known for calculating the mean square response of linear systems to both stationary and non-stationary random excitation. We consider here the mean square response of waves of a viscoelastic medium to non-stationary random excitation.

The non-stationary random excitation is of the form:

$$s(t) = e(t)\alpha(t)$$

where  $e(t)$  is a well-defined envelope function and  $\alpha(t)$  is Gaussian narrow band stationary statistical part of the excitation which has zero mean. The non-stationary process is generated by multiplying the sample functions from a stationary process  $\alpha(t)$  and the deterministic function  $e(t)$ .

The motivation for this work grew out of an investigation of the acoustical excitation of a viscoelastic soil for the purpose of classification of ocean subbottom soil sediments. The mean square response is developed in terms of the viscoelastic medium frequency response function (or Green's function in frequency domain) and generalized spectral density function of the input excitation. Both white noise and noise with an exponentially decaying harmonic correlation function are extended to include a rectangular step envelope function.

## GLOSSARY OF SYMBOLS

$a = k[(\lambda' + 2\mu')/\rho]^{1/2} [1 - \frac{k^2(\lambda'' + 2\mu'')^2}{4\rho(\lambda' + 2\mu')}]^{1/2}$  is the damped natural frequency of

the viscoelastic medium.

$b = \frac{(\lambda'' + 2\mu'')k^2}{2\rho}$  is the temporal attenuation of the system.

$c = (\frac{\lambda' + 2\mu'}{\rho})^{1/2}k$  is the natural frequency of the viscoelastic medium

$E[ ] =$  expected value of  $[ ]$

$e(t) =$  envelope function

$\alpha(t) =$  noise function

$b/c =$  compressional wave system damping factor of the viscoelastic medium

$\beta/b$  is the relative comparison between the exponential decay coefficient of the noise correlation function and the decay coefficient associated with the compressional wave system of the viscoelastic medium.

$a/\Omega$  is the relative comparison between the natural damped frequency of the compressional wave system and the frequency of the noise correlation function.

$ct =$  number of response cycles of the compressional wave system of a viscoelastic medium.

$Q = \frac{1}{2(b/c)}$  is the quality factor of the compressional wave system

$g(t-t')$  is the retarded response Green's function for the compressional wave system of the viscoelastic medium in time domain.

$g'(t-t')$  is the real part (even) of  $g(t-t')$ .

$g''(t-t')$  is the imaginary (odd) part of  $g(t-t')$ .

$G(\omega)$  is the Green's function for the compressional wave system of the viscoelastic medium in frequency domain.

$G'(\omega)$  is the even (real) part of  $G(\omega)$ .

$G''(\omega)$  is the odd (imaginary) part of  $G(\omega)$

$\mu'$  is the shear modulus of the viscoelastic medium (Lamé parameter).

$\mu''$  is the shear viscosity of the viscoelastic medium.

$\lambda'$  is the compressional modulus of the viscoelastic medium (Lamé parameter).

$\lambda''$  is the compressional viscosity of the viscoelastic medium.

$\rho$  is the density of medium.

$k$  is the Fourier transform parameter (wave number).

$\beta$  is the correlation function decay constant.

$\Omega$  is the harmonic frequency of the correlation function.

$r(t)$  is the response of the compressional wave system of the viscoelastic medium.

I. EVALUATION OF GREEN'S FUNCTION APPROPRIATE FOR VISCOELASTIC COMPRESSIONAL WAVES

The displacement equation for the compressional wave system for the viscoelastic medium with a forcing term  $\vec{f}$  can be written as [10]

$$\rho \partial_t^2 \vec{u} - [(\lambda' + \mu') + (\lambda'' + \mu'') \partial_t] \nabla \nabla \cdot \vec{u} - (\mu' + \mu'' \partial_t) \nabla^2 \vec{u} = \vec{f} \quad (1)$$

We can write an appropriate Green's function for Equation (1) by taking its divergence. Hence, we have the Green's function for  $\nabla \cdot \vec{u}$  or  $\nabla \cdot \vec{v}$  as

$$[\partial_t^2 - \left(\frac{\lambda' + 2\mu'}{\rho}\right) \nabla^2 - \left(\frac{\lambda'' + 2\mu''}{\rho}\right) \nabla^2 \partial_t] g(\vec{r} - \vec{r}'; t - t') = \delta(\vec{r} - \vec{r}') \delta(t - t') \quad (2)$$

After the time and space Fourier transformation and some algebra, the Green's function can be written as

$$G(\omega) = \frac{1}{-\omega^2 + c^2 + i\omega 2b} \quad (3)$$

Taking the time Fourier transform with the kernel  $\exp[i\omega(t-t')]$ , we obtain the Green's function appropriate for the viscoelastic compressional waves as (see Figure (1))

$$g(t-t') = \eta(t-t') e^{-b(t-t')} \frac{\sin[a(t-t')]}{a} \quad (4)$$

where  $\eta(t-t') = 1$  when  $t > t'$  and 0 when  $t < t'$ .

For real  $\omega$ , the Green's function is usually divided into two parts: a dissipative part and a reactive part. In this case, these are given respectively by the imaginary and real parts of  $G(\omega)$ , and are denoted as  $G''(\omega)$  and  $G'(\omega)$  (see Figure (5) and Figure (6)). Defining  $G(\omega) = G'(\omega) + iG''(\omega)$ .

$$G''(\omega) = \frac{-2b\omega}{[c^2 - \omega^2]^2 + [\omega 2b]^2} \quad (5a)$$

$$G'(\omega) = \frac{c^2 - \omega^2}{[c^2 - \omega^2]^2 + [\omega 2b]^2} \quad (5b)$$

By taking the Fourier transform of Equations (5a) and (5b), we obtain

$$g''(t-t') = e^{-b|t-t'|} \frac{\sin[a(t-t')]}{2a} \quad (6a)$$

$$g'(t-t') = e^{-b|t-t'|} \frac{\sin[a|t-t'|]}{2a} \quad (6b)$$

which are illustrated in Figures (2) and (3).

Since the response is causal, the real and imaginary parts of  $G(\omega)$  are related by Hilbert transform relations.

Defining the external input excitation as  $s(t) = e(t)\alpha(t)$ , we can write the compressional wave system response as

$$r(t) = \int_{-\infty}^{\infty} dt' s(t')g(t-t') \quad (7)$$

We shall assume the system is initially at rest and the input excitation is given by  $s(t)$ .

In this paper, we shall determine the mean-square response  $E[r^2(t)]$  when  $e(t)$  is a unit and a rectangular step function and  $\alpha(t)$  has the correlation functions:

$$R_{\alpha}(\tau) = 2\pi K_0 \delta(\tau) \quad (8)$$

for the white noise; and

$$R_{\alpha}(\tau) = K_0 e^{-\beta|\tau|} \cos \Omega \tau \quad (9)$$

for the correlated noise. Here,  $\tau$  is the time difference  $t_2 - t_1$ .

The compressional wave system response  $r(t)$  can be expressed as

$$r(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G(\omega) S(\omega) e^{i\omega t} \quad (10)$$

where  $S(\omega)$  is Fourier transform of  $s(t)$ . We assume the autocorrelation function of the system to non-stationary input force is given by

$$R_r(t_1, t_2) = E[r(t_1)r(t_2)] \quad (11)$$

Substituting Equation (10) into Equation (11), we obtain

$$R_r(t_1, t_2) = \iint P_r(\omega_1, \omega_2) e^{-i(\omega_1 t_1 - \omega_2 t_2)} d\omega_1 d\omega_2 \quad (12)$$

where

$$P_r(\omega_1, \omega_2) = G^*(\omega_1) G(\omega_2) P_s(\omega_1, \omega_2) \quad (13)$$

Now, defining the mean-square response as

$$E[r^2(t)] = R_r(t, t) \quad (14)$$

we have from Equation (12)

$$E[r^2(t)] = \iint G^*(\omega_1) G(\omega_2) P_s(\omega_1, \omega_2) e^{i(\omega_1 - \omega_2)t} d\omega_1 d\omega_2 \quad (15)$$

Since the generalized spectrum of the input excitation can be written as

$$P_s(\omega_1, \omega_2) = \iint \frac{dt_1 dt_2}{(2\pi)^2} R_s(t_1, t_2) e^{i(\omega_1 t_1 - \omega_2 t_2)} \quad (16)$$

where  $R_s(t_1, t_2) = e(t_1)e(t_2)R_\alpha(\tau)$  and  $R_\alpha(\tau)$  has the Fourier transform  $P_\alpha(\omega)$ , the final form of Equation (16) becomes

$$P_s(\omega_1, \omega_2) = \int \frac{d\omega}{(2\pi)^2} P_\alpha(\omega) S_e(\omega - \omega_1) S_e(\omega_2 - \omega) \quad (17)$$

where the envelope transformation functions are

$$S_e(\omega - \omega_1) = \int \frac{dt_1}{2\pi} e(t_1) e^{-i(\omega - \omega_1)t_1}$$

$$S_e(\omega_2 - \omega) = \int \frac{dt_2}{2\pi} e(t_2) e^{-i(\omega_2 - \omega)t_2} \quad (18)$$

Noting the functions in (18) to be conjugate pairs when  $\omega_1 = \omega_2$ , the substitution of Equation (18) into (14) gives the mean-square response of the compressional wave system

$$E[r^2(t)] = \int_{-\infty}^{\infty} P_\alpha(\omega) |\Lambda(t, \omega)|^2 d\omega \quad (19)$$

where

$$\Lambda(t, \omega) = \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} G(\omega_2) S_e(\omega_2 - \omega) e^{i\omega_2 t} \quad (20)$$

The desired general information for inputs of amplitude modulated stationary noise is given by Equation (19). Hence, our formulation of the compressional

wave system is complete.

## II. UNIT STEP ENVELOPE FUNCTION

When the envelope function  $e(t)$  is a unit step function defined by  $\eta(t)$ , the integral representation of the unit step is defined in [8], page 1358, as the following expression:

$$\begin{aligned} \eta_{\pm}(t-t') &= \pm i \int \frac{d\omega}{2\pi} \mathcal{S}_e(\omega) e^{-i\omega(t-t')} \\ &= \begin{cases} 1 & \text{for } t > t' \quad \text{and } 0 & \text{for } t < t' \\ 0 & \text{for } t > t' \quad \text{and } 1 & \text{for } t < t' \end{cases} \end{aligned}$$

where

$$\mathcal{S}_e(\omega) = \frac{1}{\omega \pm i\epsilon} = P(1/\omega) \mp i\pi\delta(\omega) . \quad (21a)$$

Then the frequency shifted unit step envelope transformation function becomes

$$\begin{aligned} \mathcal{S}_e(\omega_2 - \omega) &= \int e^{-i\omega_2 t} \eta(t) e^{i\omega t} dt \\ \mathcal{S}_e(\omega_2 - \omega) &= \pi\delta(\omega_2 - \omega) + \frac{1}{i(\omega_2 - \omega)} . \end{aligned} \quad (21b)$$

Substitution of Equation (21b) into Equation (20) and the evaluation of the resultant integral gives

$$|\Lambda(t, \omega)|^2 = |G(\omega)|^2 M(t, \omega) \quad (22)$$

where

$$M(t, \omega) = 1 + \Gamma_1(t) + \Gamma_2(t) \left[ \frac{b^2 - a^2 + \omega^2}{a^2} \right] - 2\Gamma_3(t) \cos \omega t - 2\Gamma_4(t) \frac{\omega}{a} \sin \omega t \quad (23)$$

with

$$\Gamma_1(t) = e^{-2bt} \left[ 1 + \frac{b}{a} \sin 2at \right]$$

$$\Gamma_2(t) = e^{-2bt} \sin^2 at$$

$$\Gamma_3(t) = e^{-bt} \left[ \cos at + \frac{b}{a} \sin at \right]$$

$$\Gamma_4(t) = e^{-bt} \sin at \tag{24}$$

Hence, our mean-square response via Equation (19) becomes

$$E[r^2(t)] = \int |G(\omega)|^2 M(t, \omega) P_\alpha(\omega) d\omega \tag{25}$$

It should be noted that in Equation (25), in the limit as  $t \rightarrow \infty$ ,  $M(t, \omega) \rightarrow 1$ , so the last expression reduces to the mean-square response formulation for stationary inputs.

White Noise Inputs: If the input noise is assumed white, then the spectral density function  $P_\alpha(\omega)$  becomes a constant  $P_o$ . So, the mean-square response becomes

$$E[r^2(t)] = P_o \int_{-\infty}^{\infty} |G(\omega)|^2 M(t, \omega) d\omega \tag{26}$$

The result of the last expression is:

$$E[r^2(t)] = \frac{\pi P_o}{2bc^2} \left[ 1 - e^{-2bt} \left( 1 + \frac{b}{a} \sin 2at + 2 \frac{b^2}{a^2} \sin^2 at \right) \right] \tag{27}$$

A normalized plot of Equation (27) is shown in Figure (9).

Correlated Input Excitation: If the input excitation is assumed correlated as stated in Equation (9), then the spectral density becomes

$$P_{\alpha}(\omega) = \frac{K_0}{\pi} \frac{\beta(\beta^2 + \Omega^2 + \omega^2)}{(\omega^2 - \omega_3^2)(\omega^2 - \omega_4^2)} \quad , \quad (28)$$

where  $\omega_3 = \Omega + i\beta$  and  $\omega_4 = -\Omega + i\beta$ . We should note that for white noise  $P_0 = \lim_{\beta \rightarrow \infty} \beta P_{\alpha}(\omega) = \frac{K_0}{\pi}$  and this expression is useful for checking the consistency of our work, as it will be shown later.

Upon substitution of the spectral density for correlated noise in (28) into expression (25), the mean square becomes

$$E[r^2(t)] = K_0 [R_1 T_1(t) + I_1 T_2(t) + R_3 T_3(t) - I_3 T_4(t)] \quad (29)$$

where

$$\begin{aligned} T_1(t) &= \frac{a}{2b} [1 - \Gamma_1(t)] \quad ; \quad T_2(t) = \Gamma_2(t) \quad ; \\ T_3(t) &= [1 + \Gamma_1(t) + \frac{b^2 - a^2 + \Omega^2 - \beta^2}{a^2} \Gamma_2(t) - 2[\Gamma_3(t) + \\ &\quad + \frac{\beta}{a} \Gamma_4(t)] e^{-\beta t} \cos \Omega t - 2 \frac{\Omega}{a} \Gamma_4(t) e^{-\beta t} \sin \Omega t] \quad ; \\ T_4(t) &= [2 \frac{\beta \Omega}{a^2} \Gamma_2(t) - 2[\Gamma_3(t) + \frac{\beta}{a} \Gamma_4(t)] e^{-\beta t} \sin \Omega t + \\ &\quad + 2 \frac{\Omega}{a} \Gamma_4(t) e^{-\beta t} \cos \Omega t] \quad . \end{aligned} \quad (30)$$

and

$$\begin{aligned}
 R_1 &= \operatorname{Re} \left[ \frac{\Omega^2 + \beta^2 + \omega_1^2}{\omega_1 (\omega_1^2 - \omega_3^2) (\omega_3^2 - \omega_2^2)} \right] \frac{\beta}{a^2} \\
 R_3 &= \operatorname{Re} \left[ \frac{1}{(\omega_3^2 - \omega_1^2) (\omega_3^2 - \omega_2^2)} \right] \\
 I_1 &= \operatorname{Imag} \left[ \frac{\Omega^2 + \beta^2 + \omega_1^2}{\omega_1 (\omega_1^2 - \omega_3^2) (\omega_3^2 - \omega_2^2)} \right] \frac{\beta}{a^2} \\
 I_3 &= \operatorname{Imag} \left[ \frac{1}{(\omega_3^2 - \omega_1^2) (\omega_3^2 - \omega_2^2)} \right] \tag{31}
 \end{aligned}$$

With little algebra, it can be shown from the limiting process  $\lim_{\beta \rightarrow \infty} \beta E[r^2(t)]$  the mean-square response for a correlated noise given in Equation (29) reduces to the one for a white noise expression given in Equation (27).

The above expression, (29), indicates the compressional wave system's response is dependent upon variables which involve a Lamé parameter  $\lambda$ , that is  $\lambda'$  and  $\lambda''$ , the shear modulus  $\mu'$ , the shear viscosity  $\mu''$ , the viscoelastic soil medium density  $\rho$ , the wave number  $k$ , the correlation function decay constant  $\beta$  and the correlation frequency  $\Omega$ . We further note that for a large number of response cycles at the exponential decay terms in the correlated noise in Equation (29) tend to zero and the mean-square response reduces to the stationary value

$$E[r^2(t)] \Big|_{t \rightarrow \infty} = K_0 \left[ \left( \frac{a}{2b} \right) R_1 + R_3 \right] .$$

### III. RECTANGULAR STEP ENVELOPE FUNCTION

For a rectangular step envelope function of duration  $t'$ , we have  $e(t) = \eta(t) - \eta(t-t')$ . Upon substitution into (18), we obtain the rectangular step envelope transformation function defined as

$$S_e(\omega_2 - \omega) = [1 - e^{-i(\omega_2 - \omega)t'}] [\pi\delta(\omega_2 - \omega) + \frac{1}{i(\omega_2 - \omega)}] \quad (32)$$

Substitution of the last expression into Equation (20), we obtain

$$\begin{aligned} |\Lambda(t, \omega)|^2 &= |G(\omega)|^2 \{ M(t, \omega)\eta(t) + (\Gamma_1(t) - M(t, \omega) + \\ &+ \Gamma_1(t-t') + [\frac{b^2 - a^2 + \omega^2}{a^2}] [\Gamma_2(t) - \Gamma_2(t-t')] - 2[\Gamma_3(t)\Gamma_3(t-t') + \\ &+ \frac{\omega^2}{a^2} \Gamma_4(t)\Gamma_4(t-t')] \cos\omega t' + 2 \frac{\omega}{a} [\Gamma_3(t-t')\Gamma_4(t)] \sin\omega t' \} \eta(t-t') \} \quad (33) \end{aligned}$$

Hence, from Equation (19) the mean-square response becomes

$$\begin{aligned} E[r^2(t)] &= \int_{-\infty}^{\infty} |G(\omega)|^2 P_{\alpha}(\omega) M(t, \omega) \quad \text{for } 0 \leq t \leq t' \\ E[r^2(t)] &= \int_{-\infty}^{\infty} |G(\omega)|^2 P_{\alpha}(\omega) M_r(t, \omega) \quad \text{for } t \geq t' \quad (34) \end{aligned}$$

where  $M(t, \omega)$  is given by Equation (23) and

$$\begin{aligned} M_r(t, \omega) &= \Gamma_1(t) + \Gamma_1(t-t') + \frac{b^2 - a^2 + \omega^2}{a^2} [\Gamma_2(t) + \Gamma_2(t-t')] - \\ &- 2[\Gamma_3(t)\Gamma_3(t-t') + \frac{\omega^2}{a^2} \Gamma_4(t)\Gamma_4(t-t')] \cos\omega t' + \\ &+ 2 \frac{\omega}{a} [\Gamma_3(t)\Gamma_4(t-t') - \Gamma_3(t-t')\Gamma_4(t)] \sin\omega t' \quad (35) \end{aligned}$$

White Noise Input: If the input excitation is assumed white, then

$$E[r^2(t)] = P_o \int_{-\infty}^{\infty} |G(\omega)|^2 M(t, \omega) d\omega \quad \text{for } 0 \leq t \leq t'$$

$$E[r^2(t)] = P_o \int_{-\infty}^{\infty} |G(\omega)|^2 M_I(t, \omega) \quad \text{for } t \geq t' \quad (36)$$

The first integral is exactly Equation (27) and the second integral is

$$E[r^2(t)] = \frac{\pi P_o}{2bc^2} \{ \Gamma_1(t) + \Gamma_1(t-t') + 2 \frac{b^2}{a^2} [ \Gamma_2(t) - \Gamma_2(t-t') ] -$$

$$- 2 [ \Gamma_3(t)\Gamma_3(t') + \frac{c^2}{a^2} \Gamma_4(t)\Gamma_4(t') ] \Gamma_3(t-t') + 2 \frac{c^2}{a^2} [ 2 \frac{b}{a} \Gamma_4(t)\Gamma_4(t') -$$

$$- \Gamma_4(t)\Gamma_3(t') + \Gamma_3(t)\Gamma_4(t') ] \Gamma_4(t-t') \} \quad \text{for } t \geq t' \quad (37)$$

No plots are done for the first integral in (36), since it is exactly the same as that for a unit step envelope function. A normalized plot of Equation (37) is shown in Figure (14). In the graph, the duration of the rectangular step function  $t'$  was taken to be  $10/c$ . We observe from the graphs, that the response is a square of an exponentially decaying harmonic function. For more details, refer to Section IV.

Correlated Input Excitation: If the input excitation is assumed correlated as in Equation (9), then  $P_\alpha(\omega)$  is given by Equation (28). Upon substitution of Equation (28) into Equation (34) and the evaluation of the resultant integral, we get

$$E[r^2(t)] = K_o [ R_1 T_1(t) + I_1 T_2(t) + R_3 T_3(t) - I_3 T_4(t) ] \quad \text{for } 0 \leq t \leq t'$$

$$E[r^2(t)] = K_0 [ R_1 T_{11}(t) - I_1 T_{22}(t) + R_3 T_{33}(t) - I_3 T_{44}(t) ] \quad \text{for } t \geq t' \quad (38)$$

where

$$\begin{aligned} T_{11}(t) = & \frac{a}{2b} \{ [ \Gamma_1(t) + \Gamma_1(t-t') ] - 2 [ (\Gamma_3(t) + \frac{b}{a} \Gamma_4(t)) \Gamma_3(t-t') - \\ & - (\frac{b}{a} \Gamma_3(t) + \frac{b^2-a^2}{a^2} \Gamma_4(t)) \Gamma_4(t-t') ] \Gamma_3(t') + \\ & + 2 [ (\frac{b}{a} \Gamma_3(t) + \frac{b^2-a^2}{a^2} \Gamma_4(t)) \Gamma_3(t-t') - (\frac{b^2-a^2}{a^2} \Gamma_3(t) + \\ & + \frac{b(b^2-3a^2)}{a^3} \Gamma_4(t)) \Gamma_4(t-t') ] \Gamma_4(t') \} \end{aligned}$$

$$\begin{aligned} T_{22}(t) = & \frac{a}{b} \{ \frac{b}{a} [ \Gamma_2(t) + \Gamma_2(t-t') ] + [ \Gamma_4(t) \Gamma_3(t-t') - (\Gamma_3(t) + \\ & + \frac{2b}{a} \Gamma_4(t)) \Gamma_4(t-t') ] \Gamma_3(t') - [ (\Gamma_3(t) + \frac{2b}{a} \Gamma_4(t)) \Gamma_3(t-t') - \\ & - (\frac{2b}{a} \Gamma_3(t) + \frac{3b^2-a^2}{a^2} \Gamma_4(t)) \Gamma_4(t-t') ] \Gamma_4(t') \} \end{aligned}$$

$$\begin{aligned} T_{33}(t) = & \Gamma_1(t) + \Gamma_1(t-t') + (\frac{b^2-a^2+\Omega^2-\beta^2}{a^2}) [ \Gamma_2(t) + \Gamma_2(t-t') ] - \\ & - 2 [ (\Gamma_3(t) + \frac{\beta}{a} \Gamma_4(t)) \Gamma_3(t-t') - (\frac{\beta}{a} \Gamma_3(t) + \frac{\beta^2-\Omega^2}{a^2} \Gamma_4(t)) \Gamma_4(t-t') ] \times \\ & \times e^{-\beta t'} \cos \Omega t' - \frac{2\Omega}{a} [ \Gamma_4(t) \Gamma_3(t-t') - (\Gamma_3(t) + \frac{2\beta}{a} \Gamma_4(t)) \times \\ & \times \Gamma_4(t-t') ] e^{-\beta t'} \sin \Omega t' \end{aligned}$$

$$\begin{aligned}
T_{44}(t) = & 2 \left\{ \frac{\beta\Omega}{a^2} [ \Gamma_2(t) + \Gamma_2(t-t') ] - [ (\Gamma_3(t) + \frac{\beta}{a} \Gamma_4(t))\Gamma_3(t-t') - \right. \\
& - \left. ( \frac{\beta}{a} \Gamma_3(t) + \frac{\beta^2 - \Omega^2}{a^2} \Gamma_4(t))\Gamma_4(t-t') ] e^{-\beta t'} \sin \Omega t' \right. + \\
& \left. + \frac{\Omega}{\beta} [ \Gamma_4(t)\Gamma_3(t-t') - (\Gamma_3(t) + \frac{2\beta}{a} \Gamma_4(t))\Gamma_4(t-t') ] e^{-\beta t'} \cos \Omega t' \right\}
\end{aligned}$$

#### IV. DISCUSSION OF THE GRAPHS

On all of the graphs, the Figures (9) through (18), the ordinate axis represents the normalized rms response of a viscoelastic medium given by  $(a^4 E [r^2(t)] / K_0)^{1/2}$  and the abscissa axis represents the number of response cycles of the compressional wave system given by  $ct$ .

Figure (9): We note that as the damping factor of the system increases, the rms response also increases. It should be noted that  $b/c$  does not represent the attenuation of the system, whereas  $b$  prescribes the attenuation of the compressional wave system. Therefore, there is no contradiction when the response increases when the damping factor  $b/c$  increases.

Figure (10-13): These figures show the behavior of the system rms plotted for various curves in  $a/\Omega$  for specific values of the quality factor  $Q$  and for specific values of  $\beta/b$ . These figures indicate that the damping values  $b/c$  of the compressional wave system effect the stationary value of the response as well as how quickly stationarity is achieved. The larger damping values of lower  $Q$  values result in lower stationary values and the mean-square response becomes stationary in a shorter duration.

Figure (14): The graph is a family of curves in the damping factor of the compressional wave system given by  $\beta/c$ , where the response to the white noise is modulated by a rectangular step function. The explanation of this is the same as the one for Figure (9).

Figures (15-18): In these figures, we note that the middle curve has the smallest value of the harmonic part of the correlated noise. In Figures (15-18), we note that for a constant  $Q$  of the system, the normalized rms increases as  $\beta/b$  increases.

#### ACKNOWLEDGEMENTS

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Figure (1) Plot of  $g(t-t')$

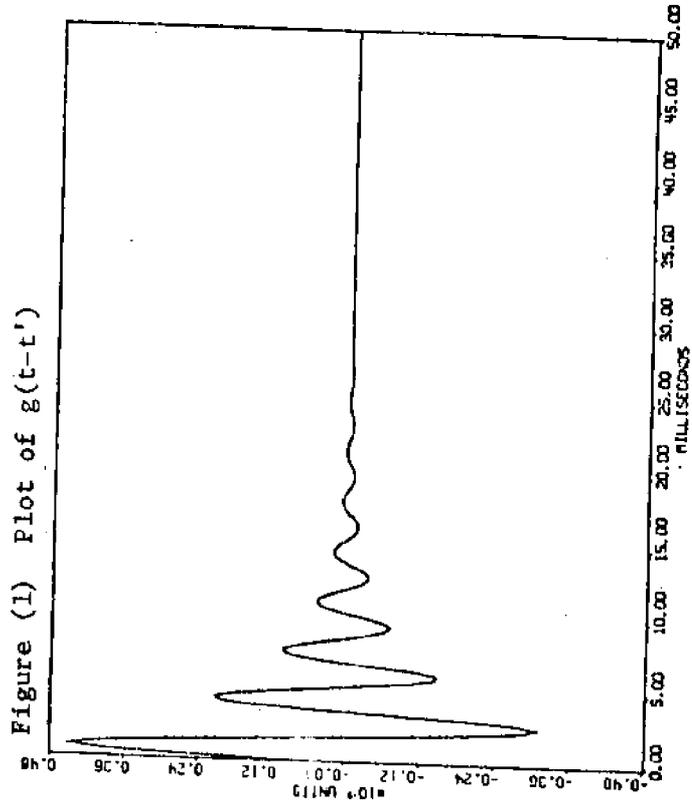


Figure (2) Plot of  $g'(t-t')$

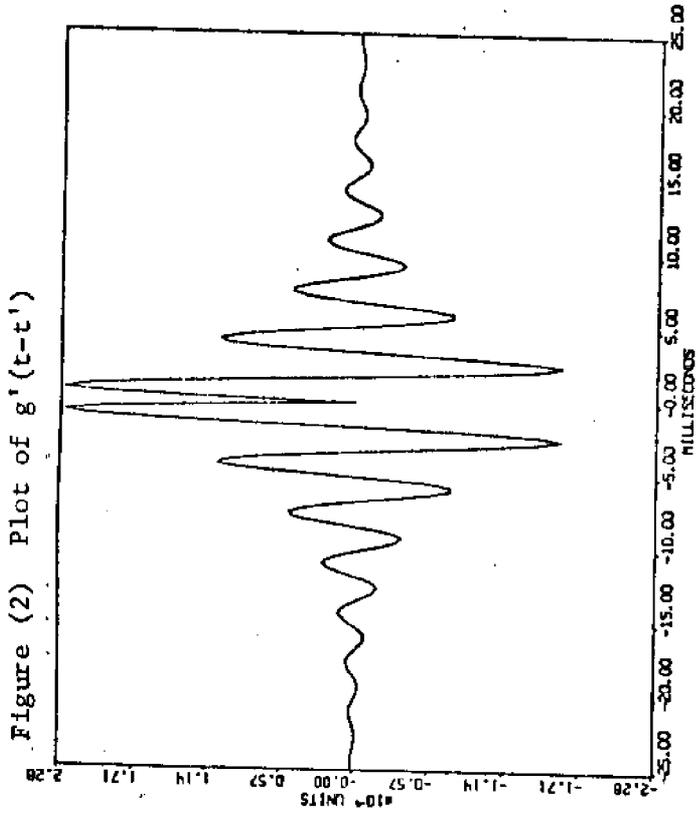


Figure (3) Plot of  $g''(t-t')$

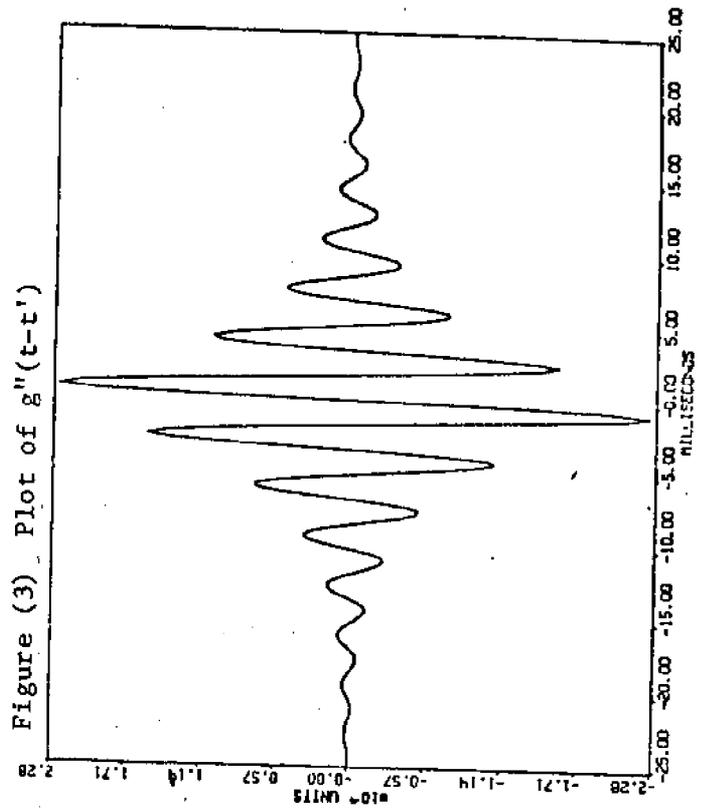


Figure (5) Plot of  $G''(\omega)$

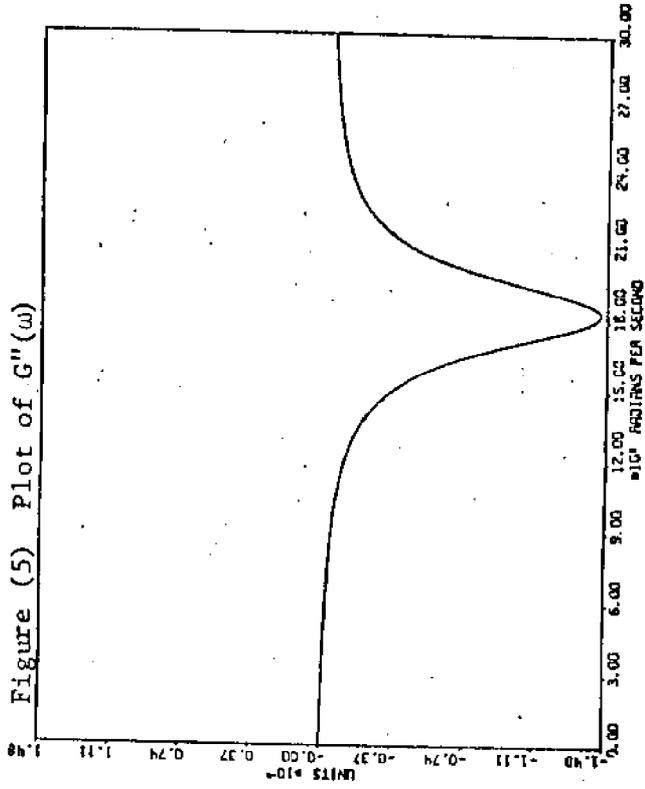


Figure (6) Plot of  $G'(\omega)$

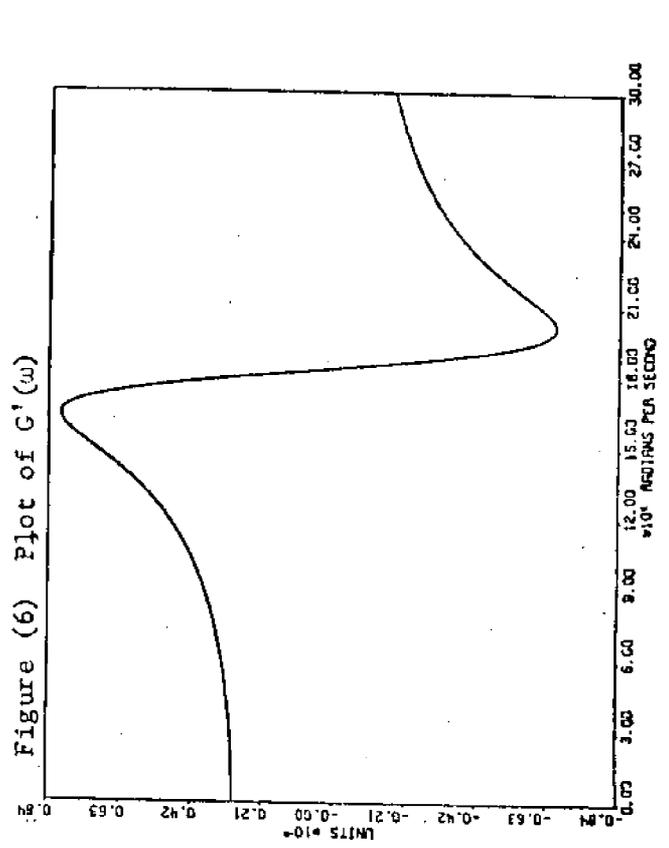


Figure (4) Absolute Value of  $G(\omega)$

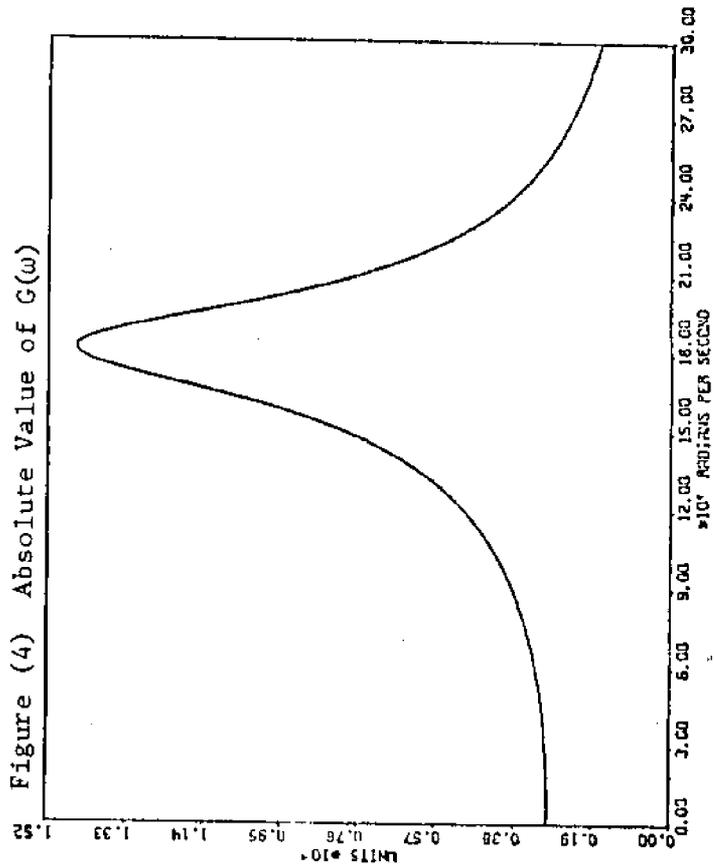


Figure (7)

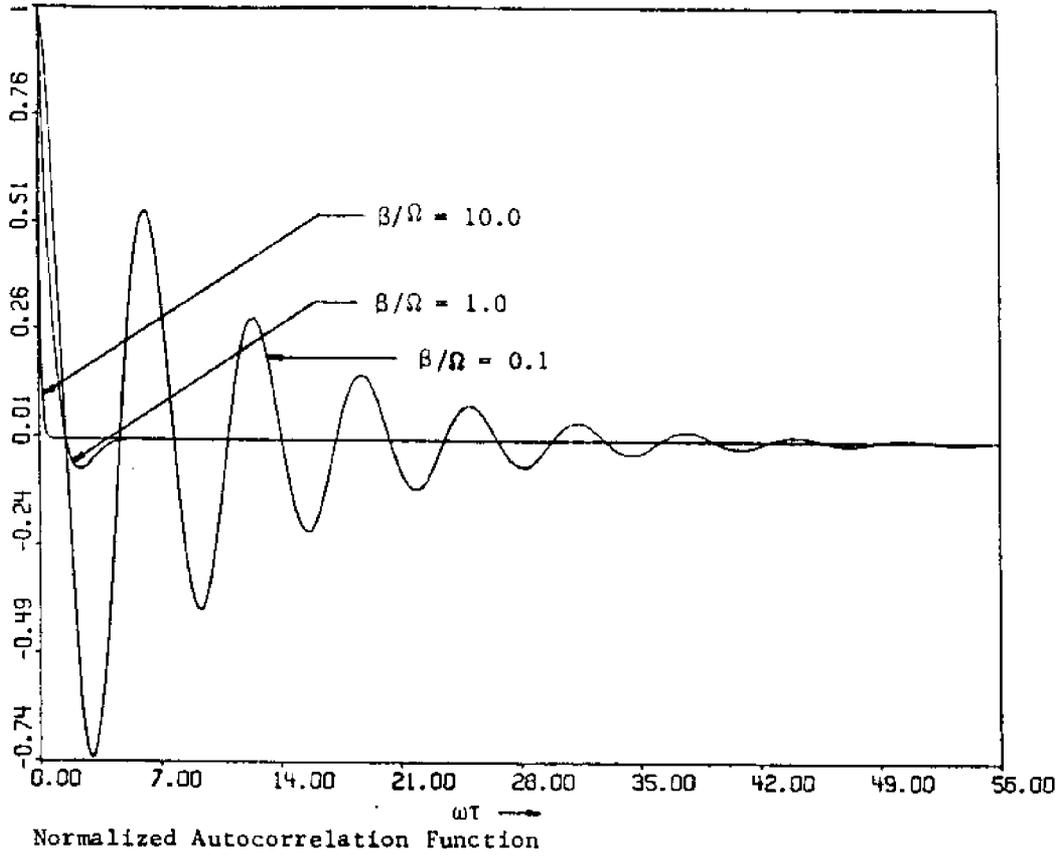
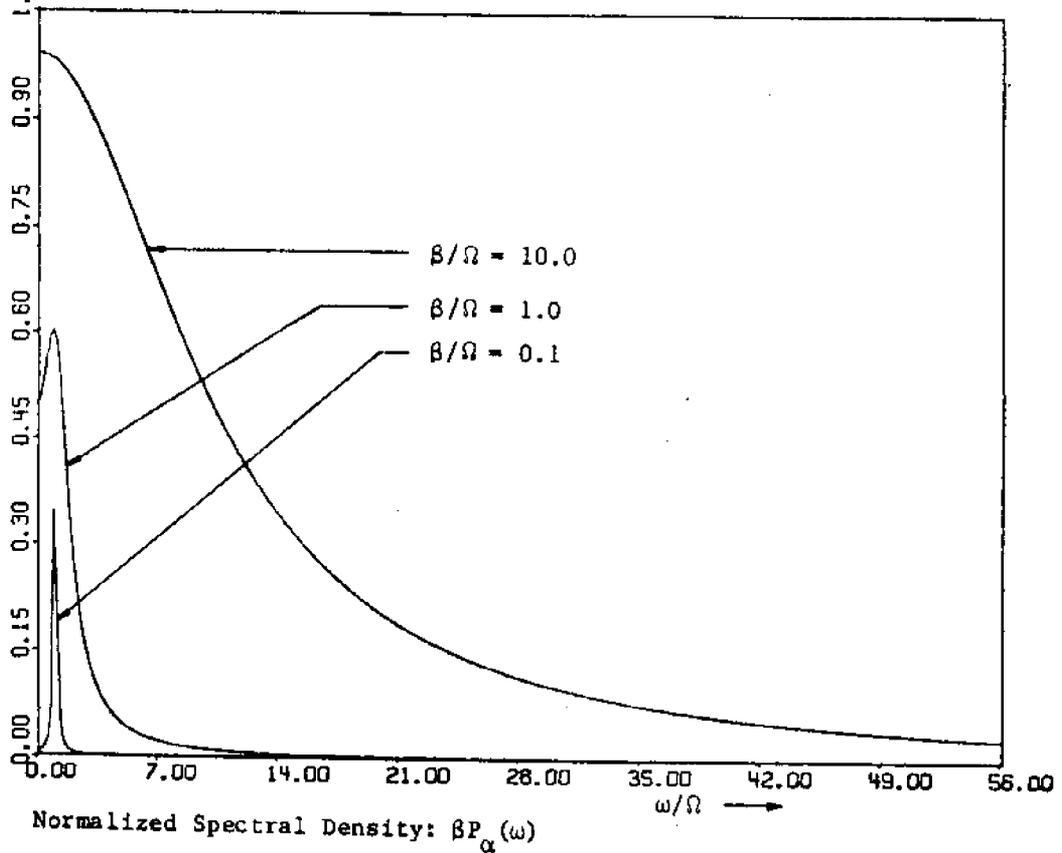
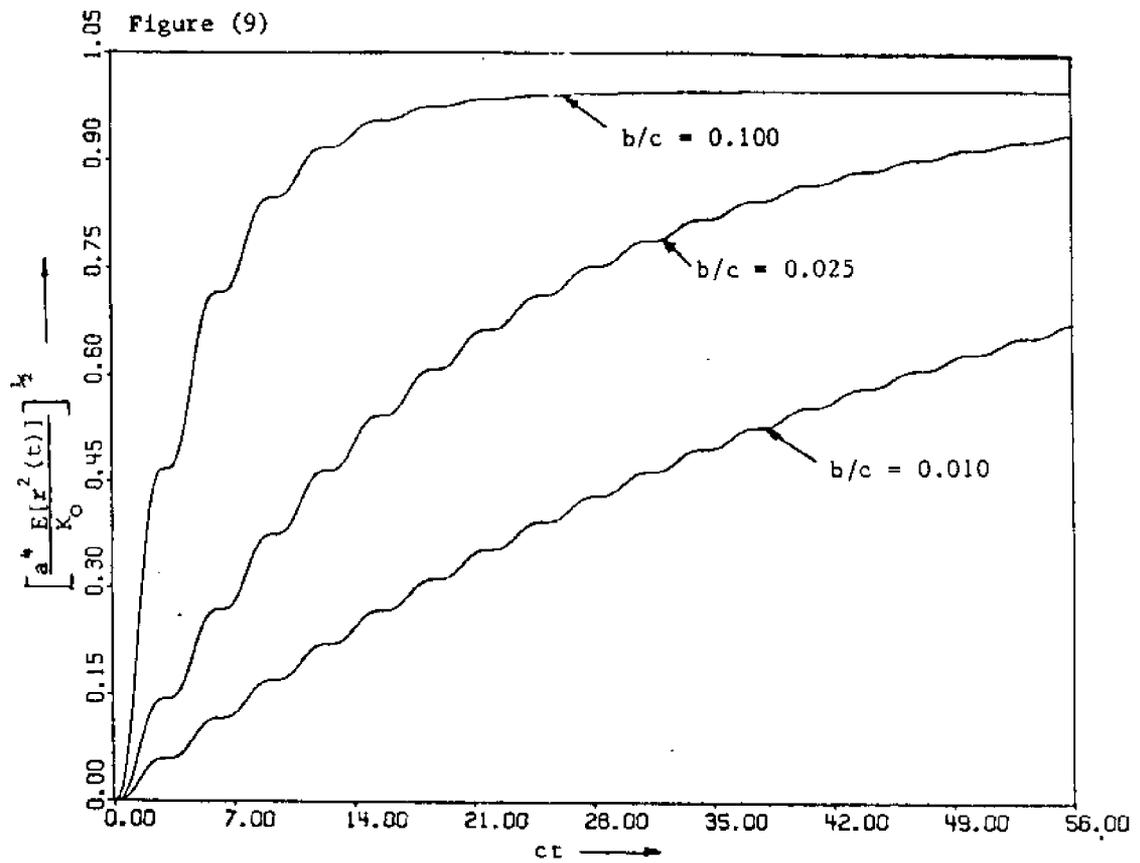
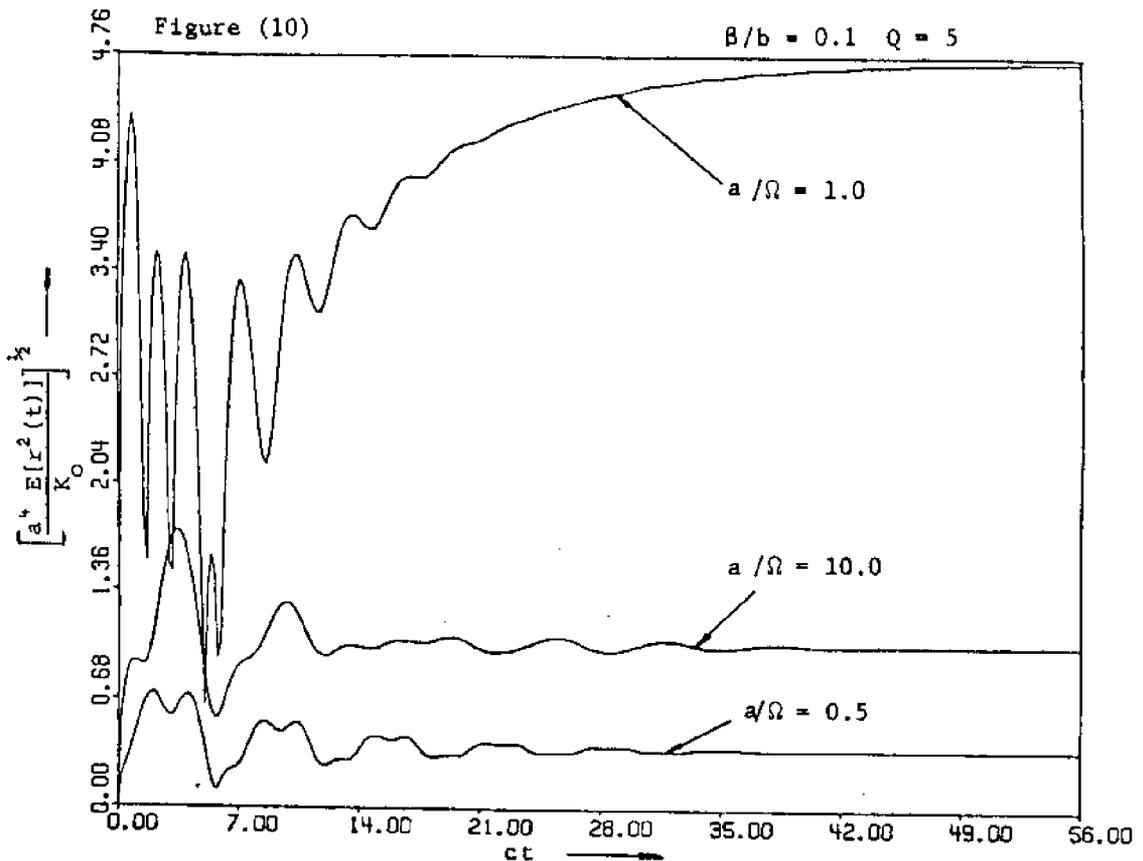


Figure (8)





Normalized rms response to the white noise modulated by a unit step function



Normalized rms response to the correlated noise modulated by a unit step

