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To Nonstationary Random Excitation

by

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Approved:

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Musa Yildiz - Technical Director

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ABSTRACT

The mean square wave response of a lightly damped thermo-viscoelastic medium to a special type of non-stationary random excitation is determined. The excitation function on the thermo-viscoelastic medium is taken in the form of a product of a well-defined, slowly varying envelope function, and a part which prescribes the statistical characteristics of the excitation. Both the unit step and rectangular step functions are used for the envelope function, and both white noise and noise with an exponentially decaying harmonic correlation function are used to prescribe the statistical property of the excitation. By taking into consideration the slow variation envelope function and the wave characteristics of the lightly damped thermo-viscoelastic medium, the mean square response (as a function of temperature, excitation, and damping parameters with the aid of reversible and irreversible thermodynamics) is evaluated.

INTRODUCTION

A number of recent papers have considered the response of dynamic systems to random excitation. However, the appropriate theory is well-known for calculating the means quare response of linear systems to both stationary and non-stationary random excitation [1,2,3]. We consider here the mean square of waves of a thermoelastic medium to non-stationary random excitation.

The non-stationary random excitation is of the form:

$$s(t) = e(t)\alpha(t)$$

where $e(t)$ is a well-defined envelope function and $\alpha(t)$ is Gaussian narrow band stationary statistical part of the excitation which has zero mean. The non-stationary process is generated by mulitplying the sample functions from a stationary process $\alpha(t)$ and the deterministic function $e(t)$.

In this investigation, it is shown that for the linear approximation, the partial differential equation which prescribes the characteristics of a thermo-viscoelastic medium may be reduced to a well-known classical damped harmonic oscillator model by applying the four-dimensional Fourier-Hilbert transform.

The transport coefficients of a thermo-viscoelastic medium can be classified as (i) compressional wave parameters such as $\overset{t}{C}_L{}^2$ and K'_{ad} (K'_{ad} is expressed in terms of Lame' coefficients as $K'_{ad} = \lambda'_t + \frac{2}{3}\mu'_{ad}$), and (ii) compressional and shear viscous parameters such as λ''_T and μ''_V.

It is further shown that λ''_T has two parts; i.e., λ''_V is the usual well-known temperature independent compressional viscous parameters. The second part, λ''_{te}, is the temperature dependent compressional viscous parameter and is obtained via Onsager relations by employing a technique developed by Eringen [1a].

Eringen's thermo-viscoelastic theory (within the framework of linear irreversible processes is applicable to the Kelvin-Voigt solid) has a different purpose [1c] than what we have in this investigation (We are not interested in heat conduction phenomena, reversible or irreversible). Our purpose in this investigation is to determine the acoustical mean square response of a thermo-viscoelastic medium and how the compressional sound waves dissipated in this medium, by using this information to prescribe the macroscopic properties of the viscoelastic materials.

I. EVALUATION OF GREEN'S FUNCTION APPROPRIATE FOR THERMO-VISCOELASTIC COMPRESSIONAL WAVES

Equation of motion for a thermo-viscoelastic medium with a forcing term f_i can be written as

$$\rho\partial_t^2 u_i - (\mu'+\mu''\partial_t)\partial^2 u_i - (K'_{ad}+\lambda''\partial_t+\mu'/3+\mu''\partial_t)\partial_i\partial_\ell u_\ell = \rho f_i \quad (1)$$

and rearranging the term, we obtain

$$L_{ij}u_j = f_i \text{ where } L_{ij} = [(K'_{ad}+\lambda''_T\partial_t+\frac{\mu'}{3}+\mu''\partial_t)/\rho]\,\partial_i\partial_j+[(\mu'+\mu''\partial_t)\partial^2/\rho+\partial_{t^2}]\delta_{ij} \quad (2)$$

The tensor Green's function for the field equation (2) is written as:

$$L_{ij}G_{jm}(\vec{r}-\vec{r}';t-t') = \delta_{im}\delta(\vec{r}-\vec{r}')\delta(t-t') \quad (3)$$

Using the Fourier temporal and three-dimensional spatial transforms, Eq. (3) is transformed to k,ω domain and reads as follows:

$$G_{jm}(\vec{k};\omega) = P^T_{jm}/(C_T^2k^2+iD_T^tk^2\omega-\omega^2) + P^L_{jm}/(C_L^{t2}k^2+iD_L^tk^2\omega-\omega^2) \quad (4)$$

where $D_T = \frac{1}{2}(\mu''/\rho)$, $D_L^t = \frac{1}{2}(\lambda''_T+2\mu'')/\rho$, $P^T_{jm} = (\delta_{jm}-k_jk_m/k^2)$, $P^L_{jm} = k_jk_m/k^2$, $C_T^2 = (\mu'/\rho)$, and $C_L^{t2} = (K'_{ad}+\frac{4}{3}\mu')/\rho$. We may invert Equation (4) by using inverse Fourier transform and we obtain the retarded tensorial Green's function in the k,t domain:

$$G_{jm}(\vec{k};t-t') = \eta(t-t')\{P^T_{jm}\,e^{-D_T^tk^2(t-t')}\frac{\sin[C_T^tk(1-g^2k^2/\rho)^{\frac{1}{2}}(t-t')]}{C_T^tk(1-g^2k^2)/\rho)^{\frac{1}{2}}}$$

$$+ P^L_{jm}\,e^{-D_L^tk^2(t-t')}\frac{\sin[C_L^tk(1-h^2k^2/\rho)^{\frac{1}{2}}(t-t')]}{C_L^tk(1-h^2k^2/\rho)^{\frac{1}{2}}}\} \quad (5)$$

where $g^2 = \mu''^2/4\mu'$, $h^2 = (\lambda''_T+2\mu'')^2/4(K'_{ad}+4\mu'/3)$

The displacement equation for the compressional wave system for the thermo-viscoelastic medium in k,ω and k,t domain can be deduced from Eqs. (4) and (5). By taking traces of Eq. (4) and (5), we easily separate the compressional responses from the tensor Green's functions which are given in the following:

$$G(k;\omega) = \frac{1}{C_L^{t2}k^2+i\omega D_L^t k^2-\omega^2}\ , \quad G(k,t-t') = \eta(t-t')e^{-D_L^t k^2(t-t')} \ .$$

$$\cdot \frac{\sin[C_L^t k(1-h^2k^2/\rho)^{\frac{1}{2}}(t-t')]}{C_L^t k(1-h^2k^2/\rho)^{\frac{1}{2}}} \tag{6}$$

where $\eta(t-t') = 1$, when $t > t'$ and 0 then $t < t'$ (see Appendix II).

For real ω, the Green's function is usually divided into two parts: a dissipative part and a reactive part. In this case, these are given respectively, by the imaginary and real parts of $G(k;\omega)$ and are denoted as $G''(k;\omega)$ and $G'(k;\omega)$ (as illustrated in Figures (4) and (5), and $G(k,t)$ is plotted in Figure (1)). Defining $G(k;\omega) = G'(k;\omega) + iG''(k;\omega)$.

$$G''(k;\omega) = -D_L^t k^2\omega/(C_L^{t2}k^2-\omega^2)^2+(\omega D_L^t k^2)^2, \quad G'(k;\omega) = (C_L^{t2}k^2-\omega^2)/(C_L^{t2}k^2-\omega^2)^2+(\omega D_L^t k^2)^2 \tag{7}$$

By taking the Fourier transform of Eqs. (7), we obtain

$$G''(k,t-t') = e^{-D_L^t k^2|t-t'|}\ \frac{\sin[C_L^t k(1-h^2k^2/\rho)^{\frac{1}{2}}(t-t')]}{2C_L^t k(1-h^2k^2/\rho)^{\frac{1}{2}}}\ ,$$

$$G'(k,t-t') = e^{-D_L^t k^2|t-t'|}\ \frac{\sin[C_L^t k(1-h^2k^2/\rho)^{\frac{1}{2}}|t-t'|]}{2C_L^t k(1-h^2k^2/\rho)^{\frac{1}{2}}} \tag{8}$$

which are illustrated in Figures (2) and (3).

Defining the external input excitation as $s(t) = e(t)\alpha(t)$, we can write the compressional wave system response as

$$r(t) = \int dt'\ s(t')G(k;t-t') = \int \frac{d\omega}{2\pi}\, G(k;\omega)s(\omega)e^{i\omega(t-t')} \tag{9}$$

In this paper, we shall determine the mean square response $E[r^2(t)]$ when $e(t)$ is a unit step as well as a rectangular step function and $\alpha(t)$ has the correlation functions: $R_\alpha(\tau) = 2\pi K_o\delta(\tau)$ for the white noise; and $R_\alpha(\tau) = K_o e^{-\beta|\tau|}\cos\Omega\tau$ for the correlated noise. Here, τ is the time difference t_2-t_1

We now give the autocorrelation function of the system to non-stationary input force:

$$R_r(t_1,t_2) = E[r(t_1)r(t_2)] = \iint P_r(\omega_1,\omega_2)e^{-i(\omega_1 t_1-\omega_2 t_2)} d\omega_1 d\omega_2 \tag{10}$$

where $P_r(\omega_1,\omega_2) = G^*(k;\omega_1)G(k;\omega_2)P_s(\omega_1,\omega_2)$ (11)

Now, defining the mean square response as

$$E[r^2(t)] = R_r(t;t) = \iint P_r(\omega_1,\omega_2)e^{i(\omega_1-\omega_2)t} d\omega_1 d\omega_2 \tag{12}$$

Since the generalized spectrum of the input excitation can be written as

$$P_s(\omega_1,\omega_2) = \iint \frac{dt_1 dt_2}{(2\pi)^2} R_s(t_1,t_2)e^{i(\omega_1 t_1-\omega_2 t_2)} = \int \frac{d\omega}{(2\pi)^2} P_\alpha(\omega)S_e(\omega-\omega_1)S_e(\omega_2-\omega) \tag{13}$$

where $R_s(t_1,t_2) = e(t_1)e(t_2)R_\alpha(\tau)$ and $R_\alpha(\tau)$ has Fourier transform $P_\alpha(\omega)$, and

$$S_e(\omega-\omega_1) = \int \frac{dt_1}{2\pi} e(t_1)e^{-i(\omega-\omega_1)t_1}, \quad S_e(\omega_2-\omega) = \int \frac{dt_2}{2\pi} e(t_2)e^{-i(\omega_2-\omega)t_2} \tag{14}$$

Noting the functions (14) to be conjugate pairs when $\omega_1 = \omega_2$, the substitution of Eq. (14) into (12) gives the mean square response of the compressional wave system

$$E[r^2(t)] = \int_{-\infty}^{\infty} P_\alpha(\omega) \ |\Lambda(t,\omega)|^2 \ d\omega \tag{15}$$

where $\Lambda(t;\omega) = \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} G(k,\omega_2)S_e(\omega_2-\omega)e^{i\omega_2 t}$ (16)

II. UNIT STEP ENVELOPE FUNCTION

When the envelope function e(t) is a unit step function defined by $\eta(t)$, the integral representation of the unit step is defined in [6], page 1358, as the following expression:

$$\eta_{\pm}(t-t') = \pm i \int \frac{d\omega}{2\pi} S_e(\omega)e^{-i\omega(t-t')} = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega\pm i\epsilon} = \int \frac{d\omega}{2\pi} (P(1/\omega)\mp i\pi\delta(\omega))e^{-i\omega(t-t')} \tag{17}$$

Then the frequency shifted unit step envelope transformation function becomes

$$S_e(\omega_2-\omega) = \pi\delta(\omega_2-\omega) + \frac{1}{i(\omega_2-\omega)} = \int e^{it(\omega-\omega_2)}\, \eta(t)dt \tag{18}$$

Substitution of Eq. (18) into Eq. (16) and the evaluation of the resultant integral gives

$$|\Lambda(t;\omega)|^2 = |G(k,\omega)|^2\, M(t;\omega) \tag{19}$$

where

$$M(t,\omega) = 1 + \Gamma_1(t)+\Gamma_2(t)\left[\frac{B^2-A^2+\omega^2}{A^2}\right] - 2\Gamma_3(t)\cos\omega t - 2\Gamma_4(t)\frac{\omega}{A}\sin\omega t \tag{20}$$

with $\Gamma_1(t) = e^{-2Bt}(1+B/A\ \sin 2At)$, $\Gamma_2(t) = e^{-2Bt}\sin^2 At$,

$$\Gamma_3(t) = e^{-Bt}(\cos At + (B/A)\sin At),\ \Gamma_4(t) = e^{-Bt}\sin At. \tag{21}$$

Hence, the mean square response via Eq. (15) becomes

$$E[r^2(t)] = \int |G(k,\omega)|^2\, M(t,\omega)P_\alpha(\omega)d\omega \tag{22}$$

White Noise Inputs: If the input noise is assumed white, then the spectral density function $P_\alpha(\omega)$ becomes a constant P_o. So, the mean square response becomes

$$E[r^2(t)] = P_o\int_{-\infty}^{\infty}|G(k;\omega)|^2 M(t;\omega)d\omega = \frac{\pi P_o}{2BC^2}\left[1-e^{-2Bt}\left(1+\frac{B}{A}\sin 2At + \frac{2B^2}{A^2}\sin^2 At\right)\right] \tag{23}$$

Correlated Input Excitation: If the input excitation is assumed correlated as indicated by $R_\alpha(\tau) = K_o e^{-\beta|\tau|}\cos\Omega\tau$, this has the spectral density

$$P_\alpha(\omega) = K_o\ \beta(\beta^2+\Omega^2+\omega^2)/\pi(\omega^2-\omega_3^2)(\omega^2-\omega_4^2) \tag{24}$$

where $\omega_3 = \Omega+i\beta$ and $\omega_4 = -\Omega+i\beta$. Upon substitution of the spectral density for correlated noise in (24) into expression (22), the mean square becomes

$$E[r^2(t)] = K_o[F_1L_1(t) + G_1L_2(t) + F_3L_3(t) - G_3L_4(t)] \tag{25}$$

where

$$L_1(t) = [1-\Gamma_1(t)]A/2B; \; L_2(t) = \Gamma_2(t); \; L_3(t) = [1+\Gamma_1(t) + \frac{B^2-A^2+\Omega^2-\beta^2}{A^2}\Gamma_2(t)] -$$

$$-2[\Gamma_3(t) + \frac{\beta}{A}\Gamma_4(t)]e^{-\beta t}\cos\Omega t - 2(\Omega/A)\Gamma_4(t)e^{-\beta t}\sin\Omega t]; \; L_4(t) = 2\beta\Omega\Gamma_2(t)/A^2 -$$

$$-2[\Gamma_3(t) + \beta\Gamma_4(t)/A]e^{-\beta t}\sin\Omega t + 2\Omega\,\Gamma_4(t)e^{-\beta t}\cos\Omega t/A \tag{26}$$

and
$$F_1 = \mathrm{Re}[\frac{\Omega^2+\beta^2+\omega_1^2}{\omega_1(\omega_1^2-\omega_3^2)(\omega_3^2-\omega_2^2)}]\beta/A^2, \quad F_3 = \mathrm{Re}[\frac{1}{(\omega_3^2-\omega_1^2)(\omega_3^2-\omega_2^2)}],$$

$$G_1 = \mathrm{Imag}[\frac{\Omega^2+\beta^2+\omega_1^2}{\omega_1(\omega_1^2-\omega_3^2)(\omega_3^2-\omega_2^2)}]\beta/A^2, \quad G_3 = \mathrm{Imag}[\frac{1}{(\omega_3^2-\omega_1^2)(\omega_3^2-\omega_2^2)}] \tag{27}$$

III. RECTANGULAR STEP ENVELOPE FUNCTION

For a rectangular step envelope function of duration t', we have $e(t) = \eta(t) - \eta(t-t')$. Upon substitution into (14), we obtain the rectangular step envelope transformation function defined as

$$S_e(\omega_2-\omega) = [1-e^{-i(\omega_2-\omega)t'}][\pi\,\delta(\omega_2-\omega) + 1/i(\omega_2-\omega)] \tag{28}$$

Substitution of the last expression into Eq. (16), we obtain

$$|\Lambda(t,\omega)|^2 = |G(k;\omega)|^2\{M(t;\omega)\eta(t) + (\Gamma_1(t)-M(t,\omega) + \Gamma_1(t-t') + [\frac{B^2-A^2+\omega^2}{A^2}]\cdot$$

$$\cdot[\Gamma_2(t)+\Gamma_2(t-t')]-2[\Gamma_3(t)\Gamma_3(t-t') + (\omega^2/A^2)\Gamma_4(t)\Gamma_4(t-t')]\cos\omega t' + 2(\omega/A)\cdot$$

$$\cdot[\Gamma_3(t)\Gamma_4(t-t') - \Gamma_3(t-t')\Gamma_4(t)]\sin\omega t')\eta(t-t') \tag{29}$$

Hence, from Eq. (15) the mean square response becomes

$$E[r^2(t)] = \int_{-\infty}^{\infty} d\omega\;|G(k;\omega)|^2 P_\alpha(\omega)M(t;\omega) \quad \text{for } 0 \le t \le t'$$

$$E[r^2(t)] = \int_{-\infty}^{\infty} d\omega\;|G(k;\omega)|^2 P_\alpha(\omega)M_r(t,\omega) \quad \text{for } t \ge t' \tag{30}$$

where $M(t,\omega)$ is given by Eq. (19) and

$$M_r(t,\omega) = \Gamma_1(t)+\Gamma_1(t-t') + (B^2-A^2+\omega^2)/A^2[\Gamma_2(t)+\Gamma_2(t-t')] -$$

$$- 2[\Gamma_3(t)\Gamma_3(t-t') + (\omega^2/A^2)\Gamma_4(t)\Gamma_4(t-t')]\cos\omega t' + 2(\omega/A) \cdot$$

$$\cdot [\Gamma_3(t)\Gamma_4(t-t') - \Gamma_3(t-t')\Gamma_4(t)]\sin\omega t' \quad (31)$$

White Noise Input: If the input excitation is assumed white, then

$$E[r^2(t)] = P_o \int_{-\infty}^{\infty} d\omega \; |G(k;\omega)|^2 M(t;\omega) \quad \text{for } 0 \le t \le t'$$

$$E[r^2(t)] = P_o \int_{-\infty}^{\infty} d\omega \; |G(k;\omega)|^2 M_r(t,\omega) \quad \text{for } t \ge t' \quad (32)$$

The first integral is exactly Eq. (23) and the second integral is

$$E[r^2(t)] = (\pi P_o/2BC^2)\{\Gamma_1(t)+\Gamma_1(t-t') + 2(B^2/A^2)[\Gamma_2(t)-\Gamma_2(t-t')]$$

$$- 2[\Gamma_3(t)\Gamma_3(t') + (C^2/A^2)\Gamma_4(t)\Gamma_4(t')]\Gamma_3(t-t') + 2(C^2/A^2)[(2B/A)\Gamma_4(t)\Gamma_4(t')$$

$$- \Gamma_4(t)\Gamma_3(t') + \Gamma_3(t)\Gamma_4(t')]\,\Gamma_4(t-t')\} \quad \text{for } t \ge t' \quad (33)$$

Correlated Input Excitation: If the input excitation is assumed correlated then $P_\alpha(\omega)$ is given by Eq. (24). Upon substitution of Eq. (24) into Eq. (30) and the evaluation of the resultant integral, we get

$$E[r^2(t)] = K_o[F_1L_1(t) + G_1L_2(t) + F_3L_3(t) - G_3L_4(t)] \quad \text{for } 0 \le t \_ t'$$

$$E[r^2(t)] = K_o[F_1L_{11}(t) - G_1L_{22}(t) + F_3L_{33}(t) - G_3L_{44}(t)] \quad \text{for } t > t'$$

where $L_{11}(t) = (A/2B)\{[\Gamma_1(t)+\Gamma_1(t-t')] - 2[(\Gamma_3(t)+(B/A)\Gamma_4(t))\Gamma_3(t-t')$

$-((B/A)\Gamma_3(t)+(B^2-A^2)/A^2\Gamma_4(t))\Gamma_4(t-t')]\Gamma_3(t') + 2[((B/A)\Gamma_3(t) +$

$(B^2-A^2)/A^2\Gamma_4(t))\Gamma_3(t-t') - ((B^2-A^2/A^2)\Gamma_3(t) + B((B^2-3A^2)/A^3)\Gamma_4(t))$

$\cdot \Gamma_4(t-t')]\Gamma_4(t')\}$

$L_{22}(t) = (A/B)\{(B/A)[\Gamma_2(t)+\Gamma_2(t-t')] + [\Gamma_4(t)\Gamma_3(t-t') - (\Gamma_3(t)+(2B/A)\Gamma_4(t)) \cdot$

$\cdot \Gamma_4(t-t')]\Gamma_3(t') - [(\Gamma_3(t)+(2B/A)\Gamma_4(t))\Gamma_3(t-t') -((2B/A)\Gamma_3(t)+(3B^2-A^2/A^2) \cdot$

$\cdot \Gamma_4(t))\Gamma_4(t-t')]\Gamma_4(t')\}$

$$L_{33}(t) = \Gamma_1(t) + \Gamma_1(t-t') + \left(\frac{B^2-A^2+\Omega^2-\beta^2}{A^2}\right)[\Gamma_2(t) + \Gamma_2(t-t')] - 2[(\Gamma_3(t) + (\beta/A) \cdot$$

$$\cdot \Gamma_4(t))\Gamma_3(t-t') - ((\beta/A)\Gamma_3(t) + \frac{B^2-\Omega^2}{A^2}\Gamma_4(t))\Gamma_4(t-t')]e^{-\beta t'}\cos\Omega t' - 2(\Omega/A) \cdot$$

$$\cdot [\Gamma_4(t)\Gamma_3(t-t') - (\Gamma_3(t) + (2\beta/A)\Gamma_4(t))\Gamma_4(t-t')]e^{-\beta t'}\sin\Omega t'$$

$$L_{44}(t) = 2\{\frac{\beta\Omega}{A^2}[\Gamma_2(t) + \Gamma_2(t-t')] - [(\Gamma_3(t) + (\beta/A)\Gamma_4(t))\Gamma_3(t-t') - (\frac{\beta}{A}\Gamma_3(t) +$$

$$\frac{\beta^2-\Omega^2}{A^2}\Gamma_4(t))\Gamma_4(t-t')]e^{-\beta t'}\sin\Omega t' + (\Omega/\beta)[\Gamma_4(t)\Gamma_3(t-t') - (\Gamma_3(t) + (2\beta/A) \cdot$$

$$\cdot \Gamma_4(t))\Gamma_4(t-t')]e^{-\beta t'}\cos\Omega t'\}$$

GLOSSARY OF SYMBOLS

$A = k[(K'_{ad} + \frac{4}{3}\mu')/\rho]^{\frac{1}{2}}[(1 - \frac{k^2(\lambda''_T + 2\mu'')^2}{4\rho(K'_{ad} + \frac{4}{3}\mu')}]^{\frac{1}{2}}$ is the damped natural frequency of the thermo-viscoelastic medium.

$B = (\lambda''_T + 2\mu'')k^2/2\rho$ is the temporal attentuation constant of the thermo-viscoelastic medium.

$C = [(K'_{ad} + \frac{4}{3}\mu')k^2/\rho]^{\frac{1}{2}}$ is the natural frequency of the thermo-viscoelastic medium.

$K'_{ad} = Ec_p/3c_p(1-2\sigma) - T\alpha_1^2 E$ is the temperature dependent the modulus of the compression of T.V.E.M.

$\lambda''_T = \lambda''_v + \lambda''_{te}$ is the compressional viscosity of the thermo-viscoelastic medium.

${}^{t}C_L^{\,2} = (K'_{ad} + \frac{4}{3}\mu')/\rho$ is the temperature dependent compressional velocity of T.V.E.M.

μ'' is the shear viscosity of the thermo-viscoelastic medium.

μ' is the modulus of rigidity.

E[ ] is the expected value of [ ].

$e(t)$ is the envelope function.

$\alpha(t)$ is the noise function.

B/C is the damping factor of compressional wave system of the thermo-viscoelastic medium.

β/B is the ratio of the exponential decay coefficient of the noise correlation function to the temporal attenuation coefficient associated with the compressional wave system of the thermo-viscoelastic medium.

A/Ω is the ratio between the natural damped frequency of the compressional wave system to the frequency of the noise correlation function.

Ct is the number of response cycles of the compressional wave system of a thermo-viscoelastic medium.

$Q = \frac{1}{2}(B/C)$ is the quality factor of the compressional wave system of a thermo-viscoelastic medium.

$G(k,t-t')$	is the retarded response thermal Green's function for the compressional wave system of the thermo-viscoelastic medium in time domain.
$G'(k,t-t')$	is the real (even) part of $G(k,t-t')$.
$G''(k,t-t')$	is the imaginary (odd) part of $G(k,t-t')$.
$G(k,\omega)$	is the thermal Green's function for the compressional wave system of the thermo-viscoelastic medium in frequency domain.
$G'(k,\omega)$	is the even (real) part of $G(k,\omega)$.
$G''(k,\omega)$	is the odd (imaginary) part of $G(k,\omega)$.
T.V.E.M.	is a short notation for thermo-viscoelastic medium.
ρ	is the density of the medium.
β	is the decay constant of the correlated noise.
Ω	is the harmonic frequency of the correlated noise.
$r(t)$	is the response of the compressional wave system of the thermo-viscoelastic medium.
α_1	is the thermal expansion coefficient.
κ	is the thermal conductivity of T.V.E.M.
c_p	is the specific heat at constant pressure.
σ	is the Poisson ratio.
E	is the Young's Modulus

$$\lambda''_{te} = \frac{\kappa\alpha_1^2 T\rho (K'_{ad} + \frac{4}{3}\mu')}{9c_p^2} \left(\frac{1+\sigma}{1-\sigma}\right)^2$$

$$\lambda' = \frac{E\sigma}{(1-2\sigma)(1+\sigma)}, \quad \mu' = \frac{E}{2(1+\sigma)}, \quad K' = \frac{E}{3(1-2\sigma)}$$

$$U_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad U_{\ell\ell} = \frac{\partial u_\ell}{\partial x_\ell}$$

F.F.T.	implies fast Fourier transform

DISCUSSION OF THE GRAPHS

Figures (1-6): In this example $G(k;t-t')$ is the Green's function describing the non-equilibrium behavior of the thermo-viscoelastic medium. Specifically, it gives the response to a unit impulsive external force at time t'. $G(k;t-t')$ vanishes until an external force is applied. $G(k;t-t')$ is therefore called the retarded response function, or Green's function. The function that describes the solution which vanishes after the impulse is applied. The retarded response function as a function of temperature is plotted in Figure (1).

It is seen in Figure (1) that the increase of the temperature of the T.V.E.M. (these are retarded responses) corresponds to the decaying oscillations, since λ_T'' takes larger values; in turn, these increase the magnitudes of the attenuations as it is noted in Eq. (A.I-5). In order to observe the underdamped oscillations, the inequality $[k(\lambda_t'+2\mu')^{\frac{1}{2}}/\rho^{\frac{1}{2}}] > [(\lambda_T'' + \mu'')k^2/2\rho]$ has to be satisfied according to Eq. (A.II-4). By using Hilbert and F.F.T. we can separate the real part (this is the even function of time); i.e., $G'(k,t-t')$ as well as the imaginary part (this is the odd function of time); i.e., $G''(k,t-t')$ of the retarded Green's function which are illustrated in Figures (2) and (3). Similarly, for real ω, the Green's function $G(k,\omega)$ is divided into two parts: a dissipative response (this represents the imaginary part of $G(k;\omega)$ and it is an odd function of frequency) denoted by $G''(k,\omega)$,and a reactive response (this represents the real part of $G(k;\omega)$ and it is even function of frequency) $G'(k,\omega)$ are respectively illustrated in Figures (4) and (5). Since $G(k,\omega)$ is complex in ω, the absolute value is illustrated in Figure (6).

On all of the graphs, Figures (7) through (13), the ordinate axis represents the normalized rms response of a thermo-viscoelastic medium given by

$$\frac{k^2(\lambda_t'+2\mu')}{K_0\rho}\left[1-\frac{k^2(\lambda_T''+2\mu_V'')^2}{4(\lambda_t'+2\mu')\rho}\right]E[r^2(t)]^{\frac{1}{2}}$$

and the abcissa axis represents the number of response cycles of the compressional wave system given by Ct.

Figures (7-9): These figures show that the behavior of the system rms plotted for various curves in (A/Ω) as a function of temperature for the specific values of the quality factor Q and for the specific values of β/B. These figures indicate that the damping values B/C of the compressional wave system effect the stationary value of the response as well as how quickly stationarity is achieved. The larger damping values of lower Q values result in lower stationary values and the mean square response obtains stationarity in a shorter duration.

Figures (10-12): In these figures, we note that the middle curve has the smallest value of the harmonic part of the correlated noise. In Figures (10-12) we note that for a constant Q of the system, the normalized rms increases as B/β increases.

REFERENCES

(1a) A. C. Eringen, "Irreversible Thermodynamics and Continuum Mechanics" Physical Review, Vol. 117, No. 5, pp. 1174 (March, 1960).

(1b) A. C. Eringen, "Response of Beams and Plates to Random Loads," Journal of Applied Mechanics, Vol. 24, Trans. ASME Vol. 80, pp. 46-52 (1957).

(1c) A. C. Eringen, MECHANICS OF CONTINUUM, John Wiley & Sons, Inc., New York, N.Y.

(2a) T. K. Caughey and H. J. Stumpf, "Transient Response of a Dynamic System Under Random Excitation," Journal of Applied Mechanics, Vol. 28, No. 4, Trans. ASME Vol. 83, Series E, pp. 563-566 (December, 1961).

(2b) T. K. Caughey, "Non-Stationary Random Input and Responses," Random Vibration, Vol. 2, MIT Press (edited by S. H. Crandall) (1963).

(3) R. L. Barnoski and J. R. Maurer, "Mean Square Response of Simple Mechanical Systems to Non-stationary Random Excitation," Journal of Applied Mechanics, Vol. 36, No. 2, Trans. ASME Vol. 91, Series E, pp. 221-227 (June, 1969).

(4) S. Degroot and P. Mazur, "Non-Equilibrium Thermodynamics," North Holland, Amsterdam (1962).

(5) P. C. Martin, "Statistical Mechanics of Equilibrium and Non-Equilibrium," Proceedings of the International Symposium on Statistical Mechanics and Thermodynamics, Held at Aachen, Germany, 15-20 of June 1964, Editor J. Meixner, North Holland, Amsterdam (1965).

(6) P. C. Martin and J. Schwinger, "Many Particle Systems," Physical Review, Vol. 115, pp. 1342 (1959).

APPENDIX I

It is well-known that for the compressional wave parameters such as $c_L^{t\,2}$ and K'_{ad}, the effects of the temperature will come into the picture through the reversible thermodynamics with the aid of the Maxwell relations. (However, the Lame' shear parameters are not affected by the temperatures.)

It is shown that λ''_T, the compressional viscous parameter, has two parts: one part, i.e., λ''_V is the usual well-known temperature independent compressional viscous parameter; the second part, i.e., λ''_{te} is the temperature dependent compressional viscous parameter and is obtained via Onsager relation employing a technique developed by Eringen and is given by the following expression:

$$\lambda''_{te} = \frac{\kappa\alpha_1^2 T\rho(3K'_{ad} + 4\mu')}{9c_p^2} \left(\frac{1+\sigma}{1-\sigma}\right)^2 \tag{1}$$

First briefly, we indicate that $1/K'_{ad} = 1/K' - T\alpha_1^2/c_p$ results if we use the Maxwell relation within the appropriate Jacobian transformation:

$$-\frac{1}{K'_{ad}} = \left(\frac{\partial V}{\partial P}\right)_S = -\frac{\partial(V,S)}{\partial(T,P)}\frac{\partial(T,P)}{\partial(S,P)} = \left(\frac{\partial V}{\partial P}\right)_T + \frac{T}{c_p}\left(\frac{\partial V}{\partial T}\right)_P^2$$

using the definitions of $\left(\frac{\partial V}{\partial P}\right)_T = -\frac{1}{K'}$ and $\left(\frac{\partial V}{\partial T}\right)_P = \alpha_1$.

Starting with Eqs. (4.6), (4.7),and (6.8) of Eringen (reference (1a), pages 1178-79, and 1180) after some algebraic modifications, we get the following dissipated energy per unit volume and per unit mass:

$$\dot{\varepsilon}_{diss} = -\frac{\kappa}{T}(\nabla T)^2 - 2\mu''_V\left(d_{ik} - \frac{1}{3}\delta_{ik}d_{\ell\ell}\right)^2 - K''_V d_{\ell\ell}{}^2 \tag{2}$$

where $K''_V = \lambda''_V + \frac{2}{3}\mu''_V$.

In order to extract the information which prescribes the temperature dependent viscous parameter λ''_{te}, it is necessary to understand the sound dissipation in the isotropic solids. The thermal conduction part of the energy dissipation on the solid is $\dot{\varepsilon}_{the} = -(\kappa/T)(\nabla T)^2$. On account of viscosity, an amount of energy $2\dot{\varepsilon}_V$ is dissipated per unit time and volume, so that the total viscosity part of $\dot{\varepsilon}_{diss}$ is $-2\dot{\varepsilon}_V$. Using the expression of Eringen (4.18) $\dot{\varepsilon}_V = \mu''_V(d_{ik} - \frac{1}{3}\delta_{ik}d_{\ell\ell})^2 + \frac{1}{2} K''_V d_{\ell\ell}{}^2$, adding these two terms, we obtain Eq. (2).

To calculate the temperature gradient, we use the fact that sound oscillations are adiabatic in the first approximation. Using the expression $S(T) = S_o(T_o) + K'\alpha_1 U_{\ell\ell}$, for the entropy, we can write the adiabatic condition $S_o(T_o) = S_o(T) + K'\alpha_1 U_{\ell\ell}$, where T_o is the temperature in the undeformed state. Expanding the difference $S_o(T) - S_o(T_o)$ in powers of $(T-T_o)$, we have as far as the first-order terms $S_o(T)-S_o(T_o) = (T-T_o)(\partial S_o/\partial T_o)_V = c_V(T-T_o)/T_o$. The derivative of the entropy is taken for $U_{\ell\ell} = 0$; i.e., at constant volume. Thus, $(T-T_o) = -T\alpha_1 K'U_{\ell\ell}/c_V$. Using also the relationship $K' = K'_{iso} = c_V K'_{ad}/c_p$ and $K'_{ad}/\rho = (\frac{\lambda'+2\mu'}{\rho} - \frac{4}{3}\frac{\mu'}{\rho})$, we can write this result as

$$(T-T_o)c_p/T\alpha_1\rho(c_L^2 - \frac{4}{3}c_T^2) = -U_{\ell\ell} \tag{3}$$

Let us now consider the dissipation of compressional sound waves. The sound dissipation coefficient is defined as the ratic of the mean energy dissipation to twice the mean energy flux in the wave. This quantity gives the manner of variation of the wave amplitude with time. The amplitude decreases as $e^{-\gamma|t-t'|}$. Tus, we find the following expression for the temporal attenuation for longitudinal waves.

$$\gamma = \frac{1}{2} < \dot{E}_{diss} > / < E > \tag{4}$$

By substituting for the plane sound waves $U_z = U_o\cos(kz-\omega t)$, $U_x = U_y = 0$.

The straightforward calculations, substituting formulae (2) and (3) into Eq. (4), we obtain

$$\gamma = \frac{k^2}{2\rho} [\lambda_v'' + 2\mu_v'' + \lambda_{te}''] \qquad (5)$$

where $\dot{E}_{diss} = \iiint \dot{\varepsilon}_{diss}\, dV$ and $E = \iiint \varepsilon dV$.

APPENDIX II

Let us now discuss the Fourier Transform longitudinal part of $G(t-t';k)$. Upon transformation, we then have

$$G(t-t';k) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G(\omega;k), \quad G(\omega,k) = \int_{0}^{\infty} d\tau\, e^{i\omega\tau} G(\tau;k) \tag{1}$$

when we put o instead of $-\infty$ in the last transformation in Eq. (1), this reflects the causal nature of $G(\tau,k)$. It is also convenient to define $G(z;k)$ which is a function of complex variable $z = \omega \pm i\varepsilon$ for z, either lower or upper half complex plane according to the choice of the sign of the exponential. The function $G(z;k) \to G(\omega;k)$ as $\varepsilon \to 0$ is clearly analytical and bounded in the defined lower or upper half z-plane. By the inspection of Equation (1) in the text, $G(z;k)$ satisfies ($G(z,k)$ represents longitudinal part of Green's function)

$$[-z^2 + (\frac{\lambda_t' + 2\mu'}{\rho})k^2 + iz(\frac{\lambda_T'' + 2\mu_V''}{\rho})k^2] G(z;k) = 1 \tag{2}$$

We can write Equation (1) with the aid of (2) for $\varepsilon \to 0$ for $\varepsilon > 0$, as

$$G(t-t';k) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{+i\omega(t-t')}}{[-\omega^2 + (\frac{\lambda_t' + 2\mu'}{\rho})k^2 + i\omega(\frac{\lambda_T'' + 2\mu''}{\rho})k^2]} \tag{3}$$

where the integration is along the real axis or any path from $-\infty$ to $+\infty$ in the upper half plane.

Explicitly, the impulse function is given to be

$$G(t-t';k) = \eta(t-t') \exp[-\frac{(\lambda_T'' + 2\mu'')k^2}{2\rho}(t-t')] \cdot$$

$$\cdot \frac{\sin[(t-t')k \frac{(\lambda_t' + 2\mu')^{\frac{1}{2}}}{\rho^{\frac{1}{2}}} (1 - \frac{k^2\ell^2}{\rho})^{\frac{1}{2}}]}{k(\frac{\lambda_t' + 2\mu'}{\rho})^{\frac{1}{2}} [1 - \frac{k^2\ell^2}{\rho}]^{\frac{1}{2}}} \tag{4}$$

where $\ell^2 = \dfrac{(\lambda_T''+2\mu_V'')^2}{4(\lambda_t'+2\mu')}$ and $\eta(t-t')$ when $t>t'$ and 0 when $t< t'$.

For real ω, the Green's function, $G(\omega,k)$ is usually divided into two parts: a dissipative part and a reactive part. In this case, and more generally when the system is stationary, that is time reversal invariant, these are given respectively by the imaginary and real parts of $G(\omega;k)$, and are denoted as $G''(\omega;k)$ and $G'(\omega;k)$, defining $G(\omega,k,\lambda,\mu) = G'(\omega;k,\lambda,\mu) + G''(\omega;k\lambda,\mu)$.

$$G''(\omega;k) = \frac{-[(\lambda_T''+2\mu_V'')k^2/\rho]\omega}{[k^2(\frac{\lambda_t'+2\mu'}{\rho})-\omega^2]^2 + [\omega(\frac{\lambda_T''+2\mu_V''}{\rho})k^2]^2} \tag{5a}$$

$$G'(\omega;k) = \frac{(\lambda_t'+2\mu')k^2/\rho - \omega^2}{[k^2(\frac{\lambda_t'+2\mu'}{\rho})-\omega^2]^2 + [\omega(\frac{\lambda_T''+2\mu_V''}{\rho})k^2]^2} \tag{5b}$$

The Fourier transform of Equation (5a) is the imaginary odd function of time. Hence, we have

$$-iG''(t-t';k,\lambda,\mu) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, G''(\omega;k)e^{-i\omega(t-t')}$$

$$= -i\,\exp[-(\lambda''+2\mu_V'')k^2/2\rho\,|t-t'|\,]\,\cdot$$

$$\cdot\,\frac{\sin[(t-t')k(\lambda_t'+2\mu')^{1/2}/\rho^{1/2}\cdot(1-k^2\ell^2/\rho)^{1/2}]}{2k(\frac{\lambda_t'+2\mu'}{\rho})^{1/2}\,(1-\frac{k^2\ell^2}{\rho})^{1/2}} \tag{6a}$$

Likewise, the Fourier transform of Equation (5b) is the real even function of $(t-t')$.

APPENDIX III

Linear regression equations (of a Voigt solid model) may be adopted for a viscoelastic medium following the procedure developed by De Groot and Mazur [4]

$$\overline{\frac{\partial \alpha(\tau)}{\partial \tau}}^{\alpha_o,\beta_o} = \overline{\beta(\tau)}^{\alpha_o,\beta_o} \tag{1}$$

$$\overline{\frac{\partial \beta(\tau)}{\partial \tau}}^{\alpha_o,\beta_o} = -\omega_L^{\,2}\ \overline{\alpha(\tau)}^{\alpha_o,\beta_o} - D_L k^2\ \overline{\beta(\tau)}^{\alpha_o,\beta_o} \tag{2}$$

Substituting (1) into Eq. (2), we obtain the second order differential equation

$$\frac{\partial^2}{\partial \tau^2}\ \overline{\alpha(\tau)}^{\alpha_o,\beta_o} + D_L k^2 \frac{\partial}{\partial \tau}\ \overline{\alpha(\tau)}^{\alpha_o,\beta_o} + \omega_L^{\,2}\ \overline{\alpha(\tau)}^{\alpha_o,\beta_o} = 0 \tag{3}$$

Multiplying this equation with α_o and averaging over the equilibrium distribution $f(\alpha_o,\beta_o)$, we obtain according to the definition of $R_{\alpha\alpha}$ (see details for De Groot and Mazur, pages 140-143)

$$R_{\alpha\alpha} = \iiiint \alpha_o\alpha\ f(\alpha_o,\beta_o)P(\omega_o,\beta_o \mid \alpha,\beta;\tau)d\alpha_o d\beta_o d\alpha d\beta \tag{4}$$

the differential equation

$$\frac{\partial^2}{\partial \tau^2}\ R_{\alpha\alpha}(\tau) + \beta\frac{\partial}{\partial \tau}\ R_{\alpha\alpha}(\tau) + \omega_L^{\,2}R_{\alpha\alpha}(\tau) = 0,\ \tau>0 \tag{5}$$

where $\beta = D_L k^2$.

Equation (5), which is essentially the equation of the damped harmonic oscillator, has the general solution

$$R_{\alpha\alpha}(\tau) = e^{-\beta\tau}(c_1 \cos\Omega\tau + c_2 \sin\Omega\tau),\ \tau > 0 \tag{6}$$

where c_1 and c_2 are constants, and where

$$\beta = \frac{1}{2} D_L k^2 \qquad \text{and} \quad \Omega = (\omega_L^{\,2} - \beta^2)^{\frac{1}{2}} \tag{7}$$

The formulae are written for the case that Ω is real (the "underdamped case"). If Ω is imaginary (overdamped case), put $\Omega = i\Omega$, and replace in the above equation $\cos i\Omega,\tau$ by $\cosh \Omega_1\tau$ and $\sin i\Omega_1\tau$ by $i \sinh \Omega_1\tau$. By employing the Boundary condition $R_{\alpha\alpha}(0) = R_o$, we find from (6) that

$$c_1 = R_o \tag{8}$$

on the other hand, since

$$\left(\frac{\partial R_{\alpha\alpha}(\tau)}{\partial \tau}\right)_{\tau=0} = R_{\alpha\beta}(0) = <\alpha\beta> 0 \tag{9}$$

It follows from (6) that

$$c_2 = \frac{\beta c_1}{\Omega}$$

The correlation function $R_{\alpha\alpha}(\tau)$ is therefore of the form

$$R_{\alpha\alpha}(\tau) = R_o e^{-\beta\tau}(\cos\Omega\tau + \frac{\beta}{\Omega} \sin\Omega\tau)\ \tau>0 \tag{10}$$

Since $R_{\alpha\alpha}(\tau)$ must be an even function of τ, we have for all times

$$R_n(\tau) = R_{\alpha\alpha}(\tau) = R_o e^{-\beta|\tau|} \cos\Omega\tau \tag{11}$$

Noise correlation $R_n(\tau) = R_{\alpha\alpha}(\tau) = R_o e^{-\beta|\tau|} \cos\Omega\tau$ is given in the text.

$$G'(t-t';k,\lambda,\mu) = \frac{1}{2}[G(t-t';k,\lambda,\mu) + G(t'-t;k,\lambda,\mu)]$$

$$= \exp[-(\frac{\lambda''_T+2\mu''_V}{\rho})k^2|t-t'|\ ] \quad \cdot$$

$$\cdot \frac{\sin[|t-t'|k(\lambda'_t+2\mu')^{\frac{1}{2}}/\rho^{\frac{1}{2}} \quad (1-k^2\ell^2/\rho)^{\frac{1}{2}}]}{k(\frac{\lambda'_t+2\mu'}{\rho})^{\frac{1}{2}}(1-\frac{k^2\ell^2}{\rho})^{\frac{1}{2}}} \tag{6b}$$

Since the response is causal, or equivalent by, since $G(z;k)$, which we have defined to be analytic in the upper of lower half plane, the real and imaginary parts of $G(\omega;k)$ are related by Hilbert Transform according to the relations (see De Groot and Mazur [4]).

$$G'(\omega;k) = P\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\quad\frac{G''(\omega;k,\lambda,\mu)}{\omega'-\omega}$$

$$G''(\omega;k) = -P\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\quad\frac{G'(\omega;k,\lambda,\mu)}{\omega'-\omega}$$

where "P" implies principal value integral; that is, an integral symmetrical about the singularity.

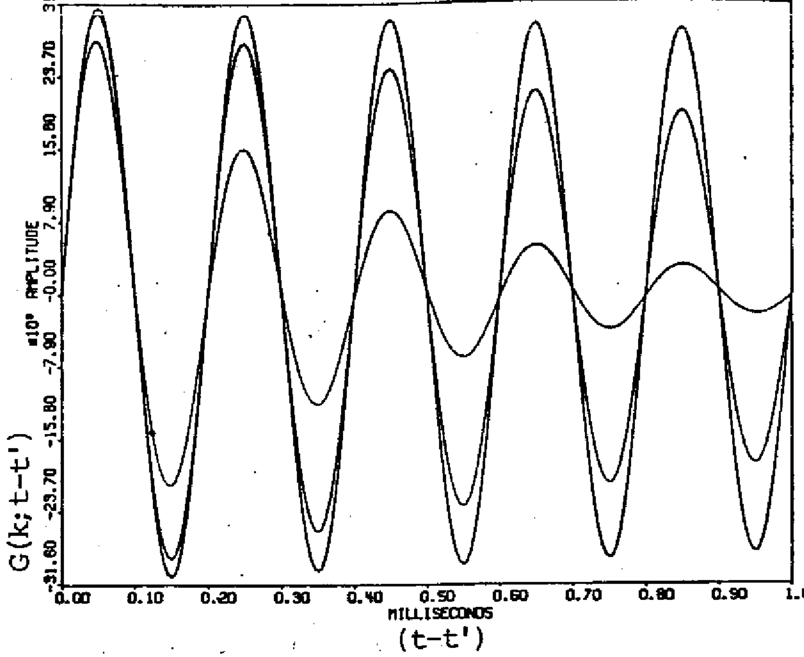
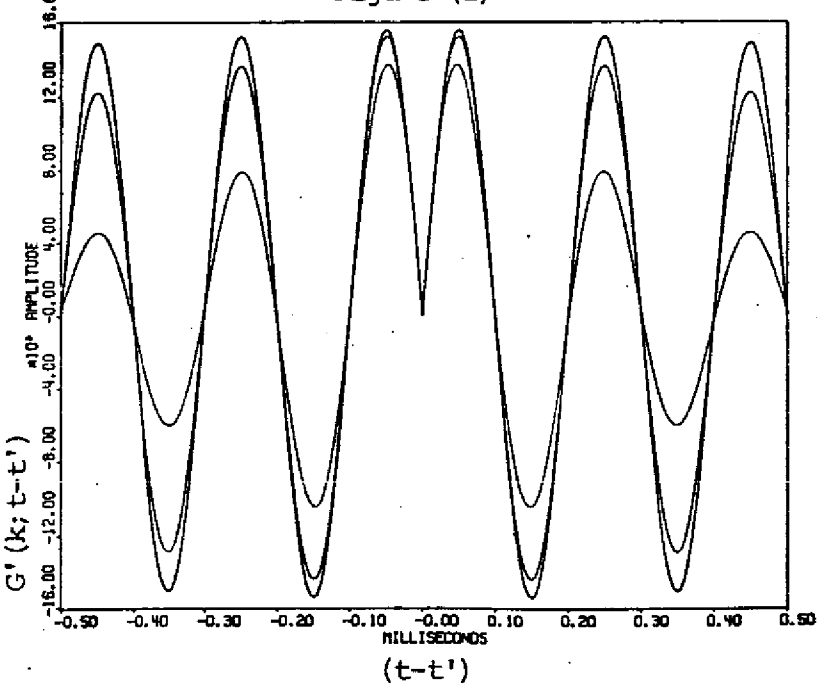


Figure (2)



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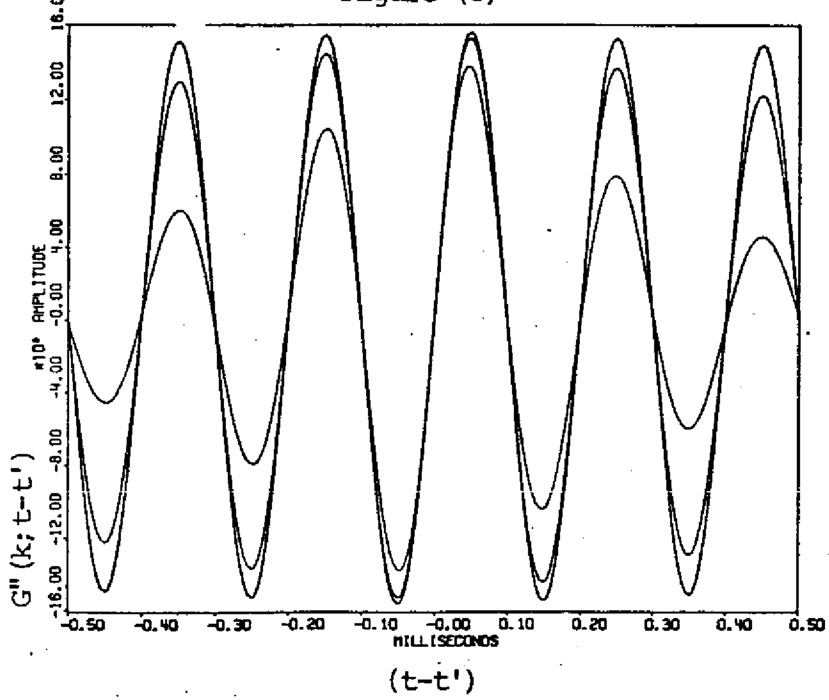

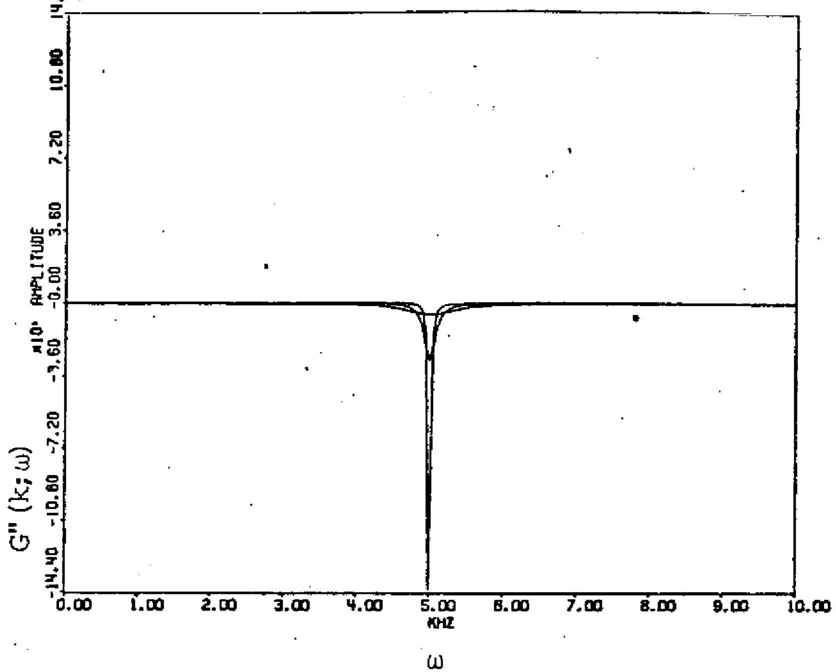

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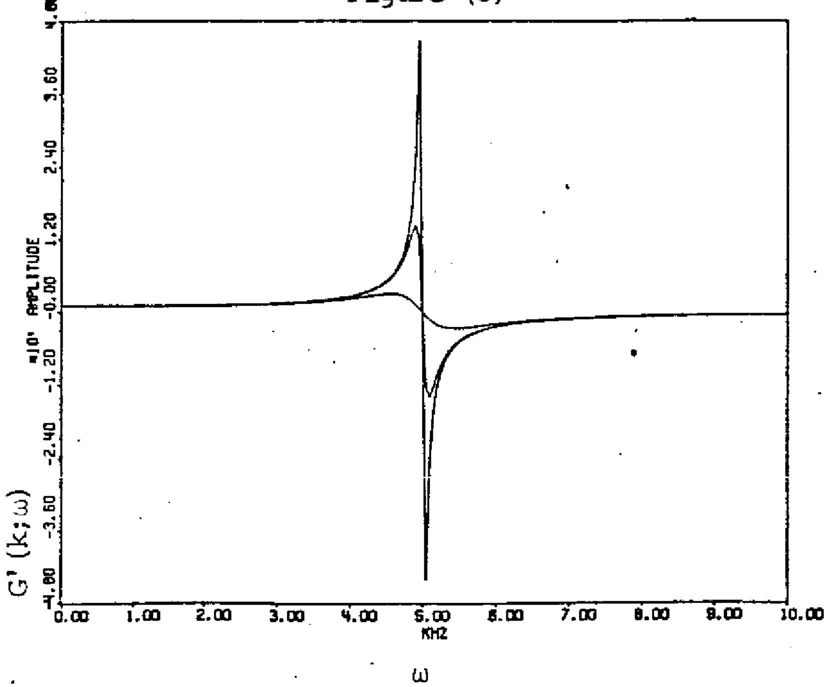
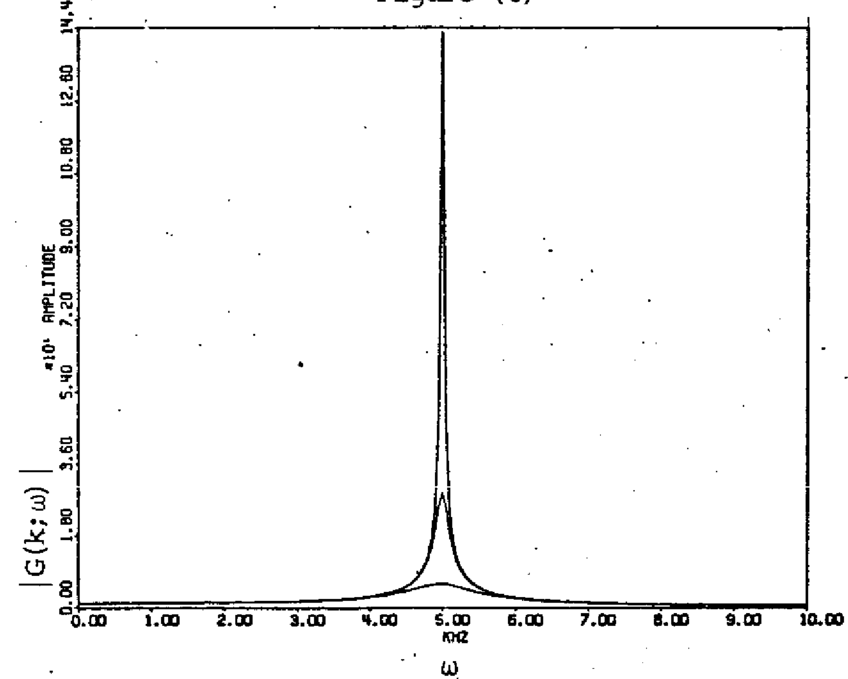


Figure (6)



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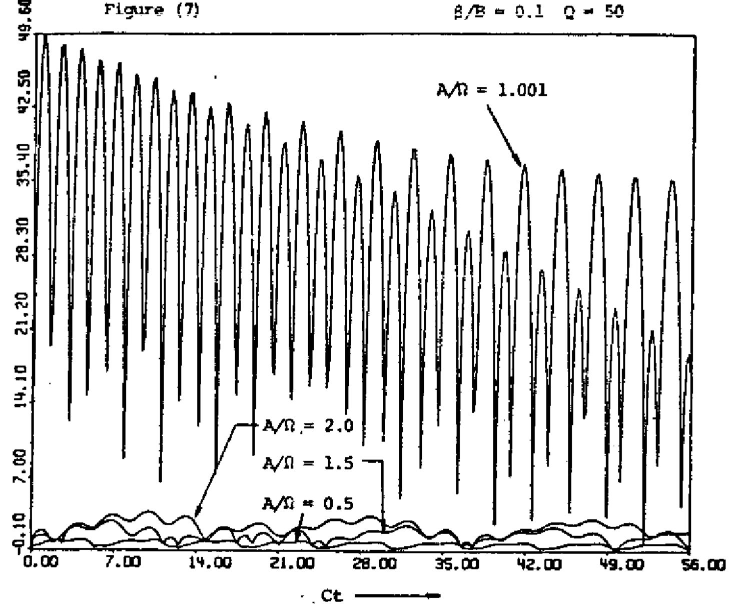
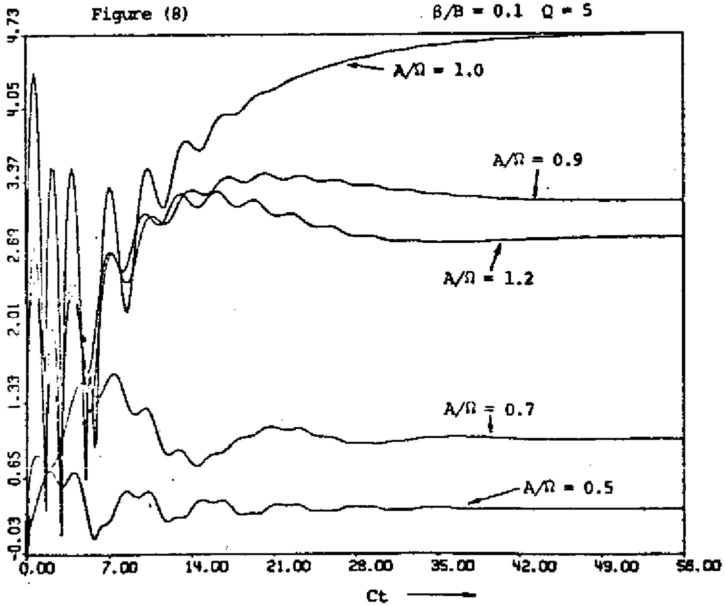


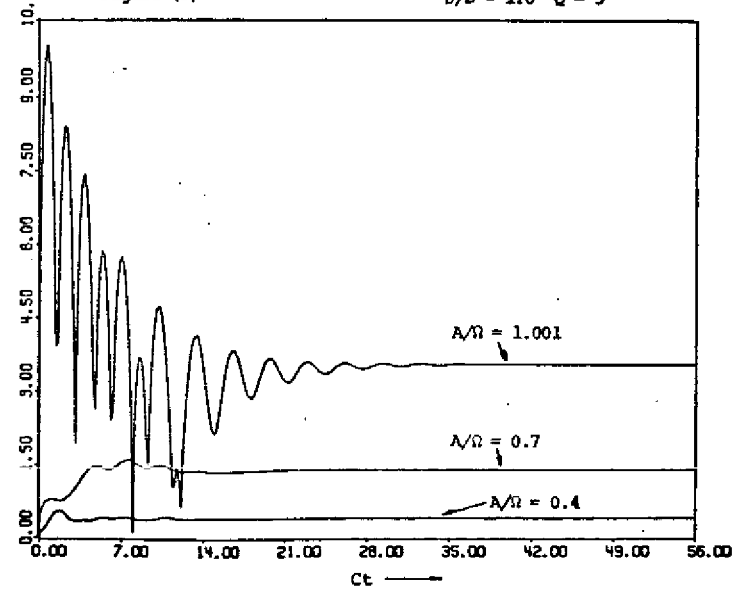
Normalized rms response to the correlated noise modulated by a unit step function.

Figure (8)

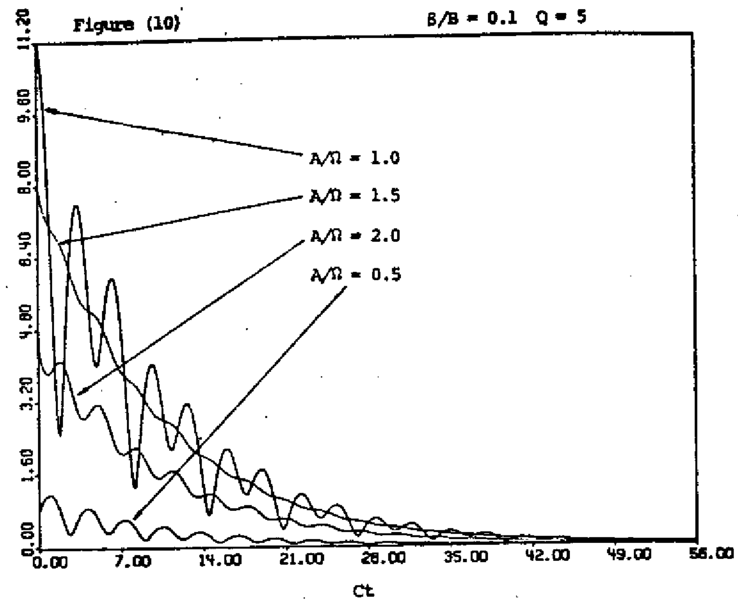


Normalized rms response to the white noise modulated by a unit step envelope function.

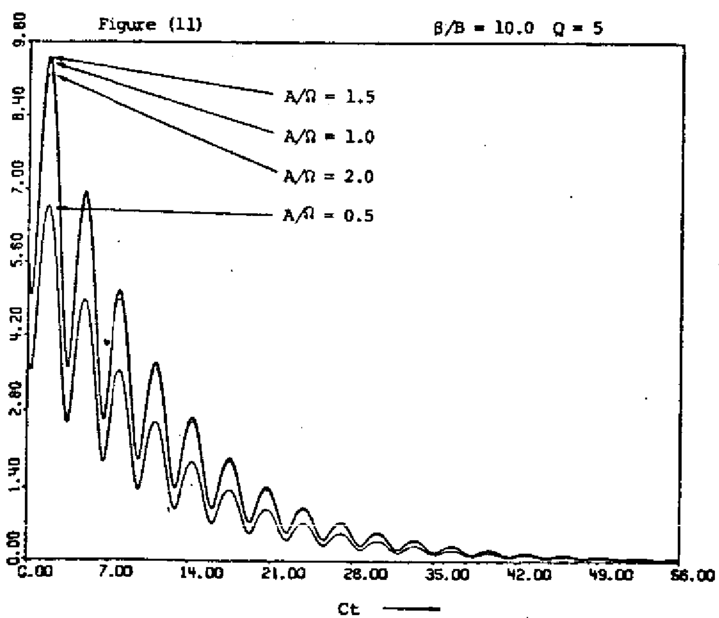
Figure (9)



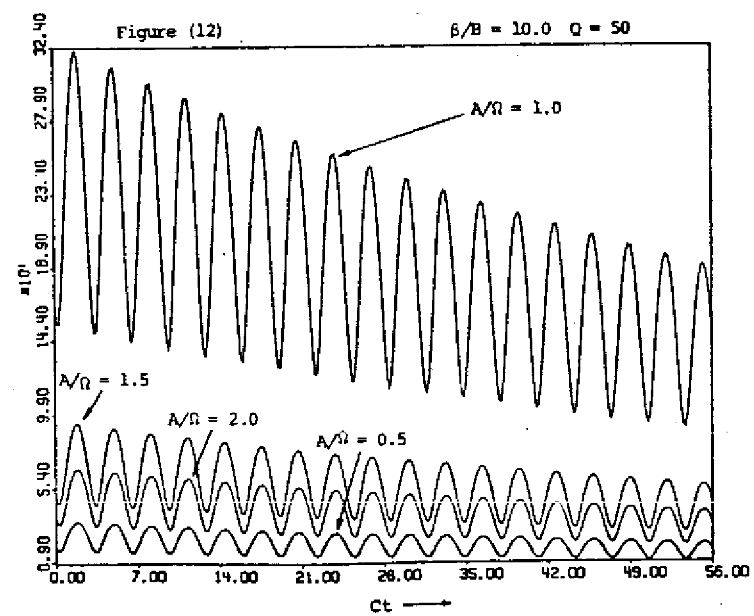
Normalized rms response to the correlated noise modulated by a unit step envelope function.



Normalized rms response to the correlated noise modulated by a rectangular step function.



Normalized rms response to the correlated noise modulated by a rectangular step function.



Normalized rms response to the correlated noise modulated by a rectangular step function.