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by

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ABSTRACT

Solutions to source-excited field problems are frequently represented as superpositions of source-free field solutions. In the thermo-viscoelastic medium in $(\vec{R};\omega)$ domain, there are, in general, two types of modes: normal modes (eigenmodes) and dissipative modes (non-eigenmodes). The normal modes are everywhere finite and comprise a complete orthonormal set. The dissipative modes are not in general members of a complete orthonormal set; they contribute to the presence of damped resonances due to the spatial attenuation.

In this investigation, directly integrating in the complex $\vec{k}$ domain, it is shown that the longitudinal and the transverse part of the tensor Green's function of a thermo-viscoelastic medium can be expressed in terms of an appropriate undamped scalar Green's function (this represents the contributions of the normal modes) which is always associated with a separate attenuated part (this represents the contributions of the dissipative modes).

INTRODUCTION

Solutions to source-excited field problems may be represented as superpositions of source-free field solutions. In the thermo-viscoelastic medium in $(\vec{R};\omega)$ domain, there are in general, two types of modes: normal modes (eigenmodes) and dissipative modes (non-eigenmodes). The normal modes which are finite everywhere form a complete orthonormal set. The dissipative modes are not in general members of a complete orthonormal set; they contribute to the presence of damped resonances due to the spatial attenuation. However, in the finite regions, a particular field solution may be represented by a superposition of source free solutions as they generally form a complete orthonormal set. Such source free solutions are defined as the normal modes of the given region; they possess a discrete spectrum, and each mode is finite and satisfies the field equations and the described boundary conditions.

In the unbounded regions, as we shall see here, there exists a continuous spectrum which may be associated with a discrete spectrum to allow an appropriate field representations of an arbitrary function. In the presence of a continuous spectrum, there may also exist a set of non-modal solutions of the source free field equations other than non-eigenmodes which contribute to the presence of leaky waves [13]. Since for a conservative (Hermitian) system real resonant frequencies represent eigenmodes, complex resonant frequencies prescribe the non-modal solutions. It should be clarified that the complex frequencies in a Hermitian (non-dissipative) system do not contribute to the presence of the attenuation, whereas in the dissipative system, as indicated here and in [1], the complex frequencies cause the attenuation. In the case of a thermo-viscoelastic medium, which is a dissipative (non-conservative) system, both the eigenmodes (normal modes) and the non-eigenmodes frequencies are in general complex.

Here and in the subsequent investigations, we deal with the acoustic field problems associated with the two-dimensional cross sections of open structures. Such a structure with axis along z for cylindrical or rectangular coordinates (r for spherical) possess translational invariance along the axial direction, and so has an $\exp(ik_z z)$ dependence in the z-direction. For a Hermitian (non-dissipative) medium the normal mode solutions correspond to real or imaginary values of k_z and are representatives of waves either propagating with undecreased amplitude or evanescent with unchangeable phase fluctuation along the z-axis. And, the non-modal solutions correspond to complex values of k_z and are the signs of leaky waves either propagating with decreased amplitude or evanescent with changeable phase fluctuation along the z-axis. The amplitudes of these waves increase indefinitely at certain directions at large distances from the excitations.

The plane wave properties of an unbounded viscoelastic medium have been investigated by Kolsky [11], Ewing, Jardetzky and Press [10]. The propagation of plane Rayleigh waves in a Voigt-solid (viscoelastic solid) was discussed by Caloi [12] who gave a generalization of Rayleigh's dispersion relations. Ewing, et al [10], have also considered the source response in a liquid overlying a homogeneous, non-dissipative elastic medium. In the present report, the tensor Green's evaluation is evaluated. This will give us a solid foundation to prescribe the characteristics of the acoustic response in a liquid layer overlying a multilayered viscoelastic rectangular, cylindrical or spherical structures. One more advantage of this formulation is that the temperature effects will come into the picture through reversible and irreversible thermodynamics in a multilayered medium. In subsequent applications, we shall consider the temperature effects in the usual boundary value problems by evaluating the response due to a point source in a liquid layer overlying a mulitlayered thermo-viscoelastic medium.

METHOD OF APPROACH

The space-frequency domain representation of the tensor Green's function of a thermo-viscoelastic medium is obtained by applying the inverse spatial transformation of the tensor Green's function in $(\vec{k},\omega)$ domain which we discussed in [1].

$$G_{jm}(\vec{R}=\vec{r}-\vec{r}\,';\omega) = \iiint \frac{d^3k}{(2\pi)^3} G_{jm}(\vec{k};\omega)e^{-i\vec{k}\cdot\vec{R}} = \iiint \frac{d^3k}{(2\pi)^3}\{[(\delta_{jm}+\partial_j\partial_m/k^2)\cdot$$

$$\cdot(1/C_T^2k^2+i\omega D_Tk^2-\omega^2)] = [(\partial_j\partial_m/k^2)(1/C_L^2k^2+i\omega D_Lk^2-\omega^2)]e^{-i\vec{k}\cdot\vec{R}}\} \qquad (1)$$

Although each of these integrands has its own structural parameters, the method of their evaluation is quite similar. The most straightforward approach utilizes a spatial representation in spherical coordinates for the arbitrary vector $\vec{k}$ is related to the orthogonal coordinate directions k_1, k_2, k_3 by the relations:

$$k_1 = k\sin\theta\cos\phi,\quad k_2 = k\sin\theta\sin\phi,\quad k_3 = k\cos\theta \qquad (2)$$

and the elemental volume in terms of these variables is $k^2\sin\theta dk d\theta d\phi$.

Eq. (1) reduces to

$$G_{jm}(\vec{R};\omega) = \int_{-\infty}^{\infty}\frac{dk}{(2\pi)^3}\left[\frac{\delta_{jm}k^2+\partial_j\partial_m}{C_T^2k^2+i\omega D_Tk^2-\omega^2} - \frac{\partial_j\partial_m}{C_L^2k^2+i\omega D_Lk^2-\omega^2}\right]\int_0^{\pi} d\theta e^{-ik|\vec{R}|\cos}\sin\theta\int_0^{2\pi}d\phi \qquad (3)$$

After performing the integration in θ,ϕ, and k domains and separating the real eigenvalues from the complex ones, the tensorial Green's function may also be represented by the following spectral representation (see A-I):

$$G_{jm}(\vec{R};\omega) = \frac{L_{jm}}{(2\pi i)^2}\oint d\lambda_{\zeta_1}^{T}\oint d\lambda_{\eta_1}^{T}\, g_{\zeta_1}^{T}(\zeta_1,\zeta_1',\lambda_{\zeta_1})g_{\eta_1}^{T}(\eta_1,\eta_1',\lambda_{\eta_1})g_{\xi_1}^{T}(\xi_1,\xi'_1,\lambda_{\xi_1})$$

$$+\frac{D_{jm}}{(2\pi i)^2}\oint d\lambda_{\zeta_2}\oint d\lambda_{\eta_2}\, g_{\zeta_2}^{L}(\zeta_2,\zeta_2',\lambda_{\zeta_2})g_{\eta_2}^{L}(\eta_2,\eta_2',\lambda_{\eta_2})g_{\xi_2}^{L}(\xi_2,\xi'_2,\lambda_{\xi_2}) \qquad (4a)$$

where $L_{jm} = [\frac{\rho(\mu'-i\omega\mu'')}{(\mu'^2+\omega^2\mu''^2)}\delta_{jm} - \frac{\partial_j\partial'_m}{\omega^2}]e^{-\gamma_T|\vec{R}|}$, and $D_{jm} = \frac{\partial_j\partial_m}{\omega^2}e^{-\gamma_L|\vec{R}|}$ (4b)

After integrating with respect to the eigenvalues $\lambda^T_{\zeta_1}$, $\lambda^T_{\eta_1}$, $\lambda^L_{\zeta_1}$, and $\lambda^L_{\eta_1}$, we obtain G_{jm} in terms of the transverse and longitudinal scalar Green's functions given as:

$$G_{jm}(\vec{R},\omega) = L_{jm}G^T(\vec{R};\omega) + D_{jm}G^L(\vec{R};\omega),\ \text{where}\ G^{T,L} = e^{-ik_{T,L}|\vec{R}|}/4\pi|\vec{R}|\ ,$$

$$k_T = (\omega/C_T)\left\{\frac{[1+\omega^2(\mu''/\mu')^2]^{\frac{1}{2}}+1}{2[1+\omega^2(\mu''/\mu')^2]}\right\}^{\frac{1}{2}},\ \gamma_T = (\omega/C_T)\left\{\frac{[1+\omega^2(\mu''/\mu')^2]^{\frac{1}{2}}-1}{2[1+\omega^2(\mu''/\mu')^2]}\right\}^{\frac{1}{2}},$$

$$k_L = (\omega/C_L)\left\{\frac{[1+\omega^2(\lambda''_T+2\mu''/\lambda'_t+2\mu')^2]^{\frac{1}{2}}+1}{2[1+\omega^2(\lambda''_T+2\mu''/\lambda'+2\mu')^2]}\right\}^{\frac{1}{2}},\ \gamma_L = (\omega/C_L)\left\{\frac{[1+\omega^2(\lambda''_T+2\mu''/\lambda'_t+2\mu')^2]^{\frac{1}{2}}-1}{2[1+\omega^2(\lambda''_T+2\mu''/\lambda'_t+2\mu')^2]}\right\}^{\frac{1}{2}}$$

$$\lambda''_T = \lambda''_v+\lambda''_t,\quad \lambda''_t = \kappa\alpha^2{}_1T\rho(K'_{ad}+\tfrac{4}{3}\mu')(1+\sigma)^2/9C^2{}_p(1-\sigma)^2\ ,$$

$$K'_{ad}+\frac{4}{3}\mu' = \lambda'_t+2\mu'\ ,\ \text{and}\ 1/K'_{ad} = 1/K-T\alpha^2{}_1/C_p \qquad (4c)$$

The transverse wave number k_T and the transverse attenuation constant γ_T derived here agrees with Ewing, et al [10], p. 273, Eqs. (5-100) and (5-101). They did not derive k_L or γ_L.

Evaluations of Eq. (4a) with or without boundaries is straight forward in the rectangular, cylindrical and the spherical coordinate system. Hence, in the spherical coordinate system, we express the tensor Green's function (see references [2] and [3]) through the spectral representation by substituting the generalized eigenvalues λ_{ζ_1} to S_1 and λ_{η_1} to S_2 and the formal evaluations of the integrals in Eq. (4a) gives us the following representations:

$$\left(\frac{1}{2\pi i}\right)^2 \oint_\gamma dS_1 \oint_\gamma dS_2\ G_r(r,r';S_1)G_\theta(\theta,\theta';S_2,S_1)G_\phi(\phi,\phi';S_2)$$

$$= \sum_i \sum_j \Theta_i(\theta,S_j)\Theta_j(\theta',S_j)\Phi_j(\Phi)\Phi_j(\Phi')G_r(r,r';S_i)$$

$$= \left(\frac{1}{2\pi i}\right)^2 \oint_\gamma dS_1 \oint_\gamma dS_2\ G_r(r,r';S_1)G_\theta(\theta,\theta',S_1,S_2)G_\phi(\phi,\phi';S_2)$$

$$= \sum_i \sum_j R_i(r)R_i(r')\chi_j(\theta,S_i)\chi_j(\theta',S_i)G_\phi(\phi,\phi';S_j)$$

$$= -\left(\frac{1}{2\pi i}\right)^2 \oint_\gamma dS_1 \oint_\gamma dS_2\, G_r(r,r';S_1)G_\theta(\theta,\theta';S_2,S_1)G_\phi(\phi,\phi';S_2)$$

$$= \sum_i \sum_j R_i(r)R_i(r')\Phi_j(\phi)\Phi_j(\phi')G(\theta,\theta';S_j,S_i) \qquad (5)$$

where the one-dimensional characteristic Green's functions satisfy

$$(d^2/dr^2+k^2-S_1/r^2)G_r(r,r';S_1) = -\delta(r-r');\; r'^2\delta(r-r') = \frac{1}{2\pi i}\oint dS_1\, G_r = \sum_i R_i(r)R_i(r')$$

$$(d^2/d\phi^2+S_2)G_\phi(\phi,\phi';S_2) = -\delta(\phi,\phi');\; \delta(\phi-\phi') = -\frac{1}{2\pi i}\oint dS_2 G_\phi = \sum_i \phi_i(\phi)\phi_i(\phi')$$

$$(d/d\theta\, \sin\theta d/d\theta - S_3/\sin\theta + S_4\sin\theta)G_\theta(\theta,\theta',S_3,S_4) = -\delta(\theta-\theta')\;;$$

if S_4 is fixed, $\sin\theta'\delta(\theta-\theta') = \frac{1}{2\pi i}\oint_C dS_3\, G_\theta = \sum_m \chi_m(\theta,S_4)\chi_m(\theta',S_4)$, if S_3 is fixed,

$$\delta(\theta-\theta')/\sin\theta' = -\frac{1}{2\pi i}\oint_C dS_4\, G_\theta = \sum_m \Theta_m(\theta,S_3)\Theta_m(\theta',S_3). \qquad (6)$$

For G_r finite at $r = 0$ and $[(d/dr)-ik\, G_r \to 0$, $r \to \infty$, the Green's function is found to be

$$G_r(r,r';S_1) = ij_n(kr_<)h_n^{(1)}(kr_>)/k,\; \mathrm{Re}(n+\tfrac{1}{2}) > 0$$

G_r here denotes the solution for a region extending over the entire radial domain $0<r<\infty$. The associated spectral representation of the identity operator is:

$$r'^2\delta(r-r') = \frac{1}{2\pi i}\oint_\gamma G_r(r,r';S_1)dS_1 = \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dn(2n+1)j_n(kr)h_n^{(1)}(kr') \qquad (7)$$

Similarly for G_ϕ when there are no boundaries in ϕ domain

$$G_\phi(\phi,\phi';S_2) = -\cos\lambda(\pi-|\phi-\phi'|)/2\lambda\sin\lambda\pi,\quad S_2 = \lambda^2 \qquad (8)$$

Singularities: simple poles at $S_2 = m^2$, $m = 0,1,2$,

$$\delta(\phi-\phi') = \frac{1}{\pi}\sum_{m=0}^{\infty} \varepsilon_m \cos m(\phi-\phi') \qquad (9)$$

The characteristic Green's function G_θ in θ domain satisfies the associated Legendre differential equation. In this case, there are no boundaries in the θ domain. The boundary conditions on G_θ are finiteness at $\theta = 0, \pi$.

$$G_\theta(\theta,\theta';S_3,S_4) = -\pi\Gamma(n+\lambda+1)p_n^{-\lambda}(\cos\theta_<)p_n^{-\lambda}(\cos\theta_>)/2\sin(n-\lambda)\pi\Gamma(n-\lambda+1) \qquad (10)$$

$S_3 = \lambda^2$, $S_4 = n(n+1)$, $p_n^\lambda(\theta)$ is the associated Legendre polynomial of order n, degree λ, and argument θ. In order to understand the singularities of G_θ in either the S_3 or S_4 planes, the specified values of S_4 and S_3, respectively, are suitably restricted.

a) $n = -\frac{1}{2} + in_i$, n_i real and fixed. $G_\theta \to \tan^\lambda \frac{\theta_<}{2} / (\tan^\lambda \frac{\theta_>}{2})\lambda$, $\lambda = |\lambda|\exp(i\phi)$, $|\phi| < \pi/2$, $|\lambda| \to \infty$. Singularities include the branch point at $S_3 = 0$.

$$\sin\theta'\delta(\theta-\theta') = \frac{1}{2\pi i}\oint_\gamma G_\theta(\theta,\theta';S_3,S_4)dS_3 = -\frac{1}{2I}\int_{-i\infty}^{i\infty} d\lambda\ \lambda\ p_n^{-\lambda}(\cos\theta)p_n^{-\lambda}\frac{(-\cos\theta')\Gamma(n+\lambda+1)}{\Gamma(n-\lambda+1)\sin(n-\lambda)\pi} \qquad (11)$$

θ and θ' in Eq. (11) may be interchanged.

b) $\lambda = \lambda_r > 0$, λ_r real and fixed. $G_\theta \to e^{-|n_i||\theta-\theta'|}/|2|(\sin\theta\sin\theta')^{\frac{1}{2}}$, $n_i = \text{Im}\ n$, $m = |n|e^{i\phi}$, $|n| \to \infty$, $\phi \neq 0$. In this case, singularities are the poles and they are located at $S_4 = (\lambda+n)(\lambda+n+1)$, $n = 0,1,2,\ldots$

$$\delta(\theta-\theta')/\sin\theta' = -(\frac{1}{2\pi i})\oint_\gamma G_\theta(\theta,\theta';S_3,S_4)dS_4 =$$

$$\sum_{m=0}^{\infty} [2(n+\lambda)+1]\Gamma(n+2\lambda+1)p_{n+\lambda}^{-\lambda}(\cos\theta)p_{n+\lambda}^{-\lambda}(\cos\theta')/2n\ ! \qquad (12)$$

In the cylindrical coordinate system, the eigenvalues $\lambda_{\zeta_1} \to C_\mu$ and $\lambda_{\eta_1} \to C_v$. We therefore write the general integrals in Eq. (4a) in the following manner:

$$\oint d\lambda_{\zeta_1}\oint d\lambda_{\eta_1}\ldots \to \oint_{\gamma_u}\oint_{\gamma_v} g_u(u,u';C_u)g_v(v,v';C_v)g_z(z,z';C_z)dC_u dC_v \qquad (13a)$$

$$\to \sum_i \phi_i(v)\phi^*_i(v')\sum_j \phi_j(z)\phi_j(z')g_u(u,u';C_{uij}) \qquad (13b)$$

The above normal mode representation in Eq. (13b) has been obtained upon evaluation the integrals over the contours γ_u and γ_v in Eq. (13a). The

$\phi_j(z)$ denote the eigenfunctions in the z-domain arising from the eigenvalue problem associated with g_z, C_u being the characteristic parameter. Hence, $(d^2/dz^2+C_z)g_z(z,z',C_z) = -\delta(z-z')$, the spectral representation is inferred from an integration of g_z in the C_u-plane as

$$\delta(\bar{z}-z') = \sum_i \phi(z)\phi(z') = \frac{1}{2\pi i} \oint_\gamma g_z(\bar{z},z';C_z)dC_z \tag{14}$$

Alternatively, one may deform the contour γ_v into the contour $\gamma_{v'}$ in the complex C_v-plane to obtain

$$\oint_{\gamma_u}\oint_{\gamma_v'} g_u(\bar{u},u';C_u)g_v(v,v';C_v)g_z'(z,z';C_z)dC_u dC_v \tag{15}$$

$$\rightarrow \sum_i \phi_i(z)\phi_i^*(z') \sum_j \phi(u)\phi_j^*(u')g_v(v,v';\lambda_{vij}) \tag{16}$$

Here $\phi_i(z)$ are the eigenfunctions in the z domain arising from the eigenvalue problem associated with g_z as the characteristic Green's function and C_v as the characteristic parameter. For a radial Green's function formulation in the second line of Eq. (16), g_z is not a function of C_v; instead g_u as noted thereunder is a function of both C_u and C_v. In this instance, the contour γ_v encloses the singularities of g_u in the C_v-plane, with C_u treated as a fixed parameter. The characteristic Green's function in the radial domain is given by

$$[(d/d\rho)\rho\, d/d\rho + C_u\rho - C_v/\rho]g_u(\rho,\rho';C_u,C_v) = -\delta(\rho-\rho') \tag{17}$$

Then one has, instead of (15)

$$\oint_{\gamma_u'}\oint_{\gamma_v'} g_u(u,u';C_u,C_v)g_v(v,v';C_v)g_z(z,z';k^2-C_u)dC_u dC_v \tag{18}$$

EXAMPLES OF VARIOUS NORMAL MODE REPRESENTATIONS IN THE RECTANGULAR, CYLINDRICAL, AND SPHERICAL COORDINATE

Similarly in cylindrical or rectangular coordinates it can be shown that

$$G(\bar{\rho},\bar{\rho}',z,z') = \sum \phi_j(\bar{\rho})\phi_j(\bar{\rho})g_z(z,z';\kappa_j) = \left(\frac{1}{2\pi i}\right)^2 \oint d\lambda_{\zeta_1}\oint d\lambda_{\eta_1}$$

$$g_{\zeta_1}(\zeta_1,\zeta'_1,\lambda_{\zeta_1})g_{\eta_1}(\eta_1,\eta'_1,\lambda_{\eta_1})g_{\xi_1}(\xi_1,\xi'_1,\lambda_{\xi_1}) \tag{19a}$$

If there are not boundaries $g_z(z,z',\kappa_j)$ is evaluated to be

$$g_z(z,z';\kappa) = \exp[-i\kappa\,|z-z'|]/2i\kappa\ ,\quad \kappa = (k^2-\xi^2-\eta^2)^{1/2} \tag{19b}$$

For unbounded space viewed as a rectangular wave guide in the normal mode representation, we can show that in Eq. (19a), the summation will give:

$$\sum_i \phi_i(\bar{\rho})\phi_i(\vec{\rho}\,') \rightarrow \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} d\eta\; e^{-i\xi(x-x')} e^{-i\eta(y-y')} / (k^2-\xi^2-\eta^2)^{1/2} \tag{19c}$$

With Eq. (19b) and (19c), Eq. (19a) becomes

$$G = (1/i8\pi^2)\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\; e^{-i\xi(x-x')} e^{-i\eta(y-y')} e^{-i(k^2-\xi^2-\eta^2)^{1/2}|z-z'|} / (k^2-\xi^2-\eta^2)^{1/2} \tag{20}$$

In order to obtain the cylindrical coordinate representation, we introduce polar coordinates in both ζ and η space and the x-y space becomes:

$$\xi = \zeta \cos a,\ \eta = \zeta \sin a,\ d\xi d\eta = \zeta d\zeta da,\ z = \rho\cos\phi,\ y = \rho\sin\phi,\ x' = \rho'\cos\phi',$$
$$y' = \rho' \sin\phi' \tag{21}$$

with $-\infty < \xi < \infty$, $0 \leq a \leq 2\pi$. Thus, Eq. (20) can be written as

$$G = (-i/8\pi) \sum_{m=0}^{\infty} \varepsilon_m \cos m(\phi-\phi') \int d\zeta\zeta \left\{ \begin{matrix} 2J_m(\zeta\rho) \\ H_m^2(\zeta\rho) \end{matrix} \right\} J_m(\zeta\rho') g(z,z';\zeta) \tag{22}$$

In spherical coordinates, by using Eqs. (5) - (12), the scalar Green's function appropriate to transverse and longitidunal waves may be given by:

$$G = \frac{1}{4\pi} \sum_m \varepsilon_m \cos m(\phi-\phi') \sum_{\ell=m}^{\infty} \frac{(\ell-m)!}{(\ell+m)!} (2\ell+1) j_\ell(kr) h_\ell^{(2)}(kr') P_\ell^m(\cos\phi) P_\ell^m(\cos\phi') \tag{23}$$

It will now be useful to write the complete expansion of G_{jm} in the cylindrical coordinate system (since we shall subsequently employ this specific expansion in the boundary values problems involving the continental shelf) in the region $0 < \xi < \infty$

$$G_{jm}(\vec{R};\omega) = \frac{L_{jm}}{4\pi i} \sum_m \varepsilon_m \cos m(\phi-\phi') \int_0^{\infty} d\zeta\zeta\; J_m(\zeta\rho) J_m(\zeta\rho') e^{-i|z-z'|\sqrt{k_T^2-\zeta^2}} / \sqrt{k_T^2-\zeta^2}$$
$$+ \frac{D_{jm}}{4\pi i} \sum_m \varepsilon_m \cos m(\phi-\phi') \int_0^{\infty} d\zeta\zeta\; J_m(\zeta\rho) J_m(\zeta\rho') e^{-i|z-z'|\sqrt{k_L^2-\zeta^2}} / \sqrt{k_L^2-\zeta^2} \tag{24a}$$

where $\varepsilon_o = 1$, $\varepsilon_m = 2$, $m \geq 1$.

If we let the viscous terms λ''_v, λ''_v and the temperature dependent term λ''_t vanish, then the tensor Green's function given by Eq. (24) reduces to

$$G_{jm}(\vec{R};\omega) = \frac{L_{jm}}{4\pi i} \sum_m \varepsilon_m \cos m(\phi-\phi') \int_0^\infty d\zeta\zeta\, J_m(\zeta\rho) J_m(\zeta\rho') e^{-i|z-z'|\sqrt{k^2_{Te}-\zeta^2}} / \sqrt{k^2_{T_e}-\zeta^2}$$

$$+ D_{jm} \sum_m \varepsilon_m \cos m(\phi-\phi') \int_0^\infty d\zeta\zeta\, J_m(\zeta\rho) J_m(\zeta\rho') e^{-i|z-z'|\sqrt{k^2_{Le}-\zeta^2}} / \sqrt{k^2_{L_e}-\zeta^2} \quad (25)$$

performing the integrations and summing up the series, we obtain:

$$G_{jm}(\vec{R};\omega) = (\delta_{jm} - \partial_j \partial'_m / k^2{}_{Te}) \frac{e^{-ik_{Te}|\vec{r}-\vec{r}'|}}{C^2{}_T 4\pi|\vec{r}-\vec{r}'|} - \frac{\partial_j \partial_m}{k^2{}_{Te}} \frac{e^{-ik_{Le}|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \quad (26)$$

where $k_{Te} = \omega/C_T$ and $k_{Le} = \omega/C_L$. Eq. (26) is in agreement with Eq. (6) of reference [8]. Note that the time dependence and source term in Eq. (26) differ from the one given by Eq. (6) of reference [8].

In the subsequent applications, it will be shown that to determine the response through numerical integrations in a multilayered, cylindrical structure, we have to use Eqs. (24a) and (22). In the case of determining the response in a multilayered media through the saddle point of integrations the appropriate Eqs. (22) and (24b) involves Hankel-Bessel instead of Bessel functions, Hence, in the region $-\infty \leq \zeta \leq \infty$, Eq. (24a) becomes

$$G_{jm}(\vec{R};\omega) = L_{jm} \sum_m \varepsilon_m \cos_m(\phi-\phi') \int_{-\infty}^{\infty} d\zeta\zeta\, H_m^{(2)}(\zeta\rho) J_m(\zeta\rho') e^{-i|z-z'|\sqrt{k_T^2-\zeta^2}} / 8\pi i \sqrt{k_T^2-\zeta^2}$$

$$+ D_{jm} \sum_m \varepsilon_m \cos m(\phi-\phi') \int_{-\infty}^{\infty} d\zeta\zeta\, H_m^{(2)}(\zeta\rho) J_m(\zeta\rho') e^{-i|z-z'|\sqrt{k_L^2-\zeta^2}} / 8\pi i \sqrt{k_L^2-\zeta^2} \quad (24b)$$

APPENDIX I

After performing θ and ϕ integrations, Eq. (3) may be given by

$$G_{jm}(\vec{R};\omega) = \frac{i}{(2\pi)^2|\vec{R}|}\int_{-\infty}^{\infty} dk\; e^{-ik|\vec{R}|}\left\{\frac{k\delta_{jm}+\partial_j\partial_m/k}{C_T^2k^2+i\omega D_T k^2-\omega^2} - \frac{\partial_j\partial_m/k}{(C_L^2k^2+i\omega D_L k^2-\omega^2)}\right\} \quad (A.1)$$

It is useful to define the complex variable z = k, and then to write the following integrals for a closed path of integration, including real axis, or any other path from $-\infty$ to $+\infty$, in the complex z plane. Clearly, these integrals are well-defined and exponentially decreasing for z in the lower half of the complex z plane. Applying the residue theorem the contour integrals give us:

$$i\oint dz\; e^{-iz|\vec{R}|}z/(2\pi)^2|\vec{R}|\,[z^2-(\omega^2/C_T^2+i\omega D_T)] = \rho(\mu'-i\omega\mu'')e^{-\gamma_T|\vec{R}|}G^T(\vec{R};\omega)$$

$$i\oint dz\; e^{-iz|\vec{R}|}/z(2\pi)^2|\vec{R}|\,[z^2-(\omega^2/C_T^2+i\omega D_T)] = e^{-\gamma_T|\vec{R}|}G^T(\vec{R};\omega)/\omega^2 \;;$$

$$i\oint dz\; e^{-iz|\vec{R}|}/z(2\pi)^2|\vec{R}|\,[z^2-\omega^2/C_L^2+i\omega D_L)] = e^{-\gamma_L|\vec{R}|}G^L(\vec{R};\omega)/\omega^2 \quad (A.2)$$

Here, γ_T, γ_L, k_T, k_L, G^T and G^L are defined by Eq. (4c). Since k_T and k_L are real then, G^T(or G^L) can be represented by the following spectral representation through the one-dimensional characteristic Green's functions (see [2]):

$$G^T(\vec{R};\omega) = \oint d\lambda_{\zeta_1}^T \oint d\lambda_{\eta_1}^T\; g_{\zeta_1}^T(\zeta_1,\zeta'_1,\lambda_{\zeta_1})g_{\eta_1}^T(\eta_1,\eta'_1,\lambda_{\eta_1})g_{\xi_1}^T(\xi_1,\xi'_1,\lambda_{\xi_1})/(2\pi i)^2$$

$$G^L(\vec{R};\omega) = \oint d\lambda_{\zeta_1}^L \oint d\lambda_{\eta_1}^L\; g_{\zeta_1}^L(\zeta_1,\zeta'_1,\lambda_{\zeta_1})g_{\eta_1}^L(\eta_1,\eta'_1,\lambda_{\eta_1})g_{\xi_1}^L(\xi_1,\xi'_1,\lambda_{\xi_1})/(2\pi i)^2 \quad (A.3)$$

Substituting (A.2) and (A.3) into Eq. (A.1), we obtain the results given by Eqs. (4a) and (4b).

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*The structure of the differential equation in this paper is similar to the one discussed here.