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TECHNICAL REPORT

Estimation of Variance of the Amplitude and Damped Frequency of Compressional Waves in a Subbottom Viscoelastic Soil

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A Report of a
Cooperative University-Industry Research Project
between

University of New Hampshire
Durham, New Hampshire

Raytheon Company
Portsmouth, Rhode Island

UNIVERSITY of NEW HAMPSHIRE
DURHAM, NEW HAMPSHIRE. 03824

Report No. UNH-SG-126

April 1974

A TECHNICAL REPORT TO

THE NATIONAL SEA GRANT PROGRAM

OF

THE NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION

U. S. DEPARTMENT OF COMMERCE

ESTIMATION OF VARIANCE OF THE AMPLITUDE AND DAMPED FREQUENCY OF COMPRESSIONAL WAVES IN A SUBBOTTOM VISCOELASTIC SOIL

by

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This work is a result of research sponsored by NOAA Office of Sea Grant, Department of Commerce, under Grant No. D of C 04-3-158-69. The U.S. Government is authorized to produce and distribute reprints for governmental purposes, notwithstanding any copyright notation that may appear hereon.

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ABSTRACT

Analytical techniques using estimation theory is applied for remote sensing of soil parameters in the presence of noise. Expressions for the variance and the best estimate of the parameters are derived for both the discrete and continuous sampling in the time domain. These derived expressions are applied to evaluate the variance and the best estimate of the amplitude and damped frequency of compressional waves in a viscoelastic medium from the received signal observed over a continuous period of time. The received signal is corrupted by noise, whose complete statistical description is available for all values of time. In our case the noise is assumed to be white Gaussian. The error of estimation of the parameters increases with the damping coefficient of the system.

INTRODUCTION:

The recovery of essential engineering data for the efficient utilization of the sea floor is tedious and expensive. For this reason, remote sensing techniques are used to estimate the soil parameters. Any acoustic signal received from the ocean bottom contains information about the soil along with unwanted noise; so the problem reduces to an estimation of the soil parameters $(U_1; U_2, \ldots U_n)$ from the received signal $R(t; U_1, U_2, \ldots U_n)$ if we assume that a complete statistical description of noise $N(t; U_1, \ldots U_n)$ is known. Then the actual signal $S(t; U_1, \ldots U_n)$ is

$$S(t; U_1, \ldots U_n) = R(t; U_1, \ldots U_n) - N(t; U_1, \ldots U_n) \qquad (1)$$

The well-known techniques from estimation theory can be used in estimating the soil parameters. Ideally, one wishes to find estimates that will yield maximum precision and if one cannot get maximum precision, at the least one has to indicate the precision of estimate. Our measure of the precision of the estimate U^* from the actual value U. The estimate U^* can be either an unbiased or a biased estimate. The biased estimate is a function of the sample size and the parameter U. However, we will sample our data in such a manner as to get an unbiased estimate U^*. In such a case, the measure of the precision of estimate, $E((U^* - U)^2)$ reduces to $Var(U^*)$ and $E(U^*) = U$. $Var(U^*)$ is the variance of U^* and E denotes the expectation value.

In general, the received signal $R(t; U_1, \ldots U_n)$ may contain N parameters. For Convenience of discussion, let us consider the case when the received signal contains only one parameter U, which has to be determined from observation $R(t,U)$. From estimation theory[1,2], we know that the variance of U^* must obey the inequality

$$Var(U^*) > \frac{(\partial E(U^*)/\partial U)^2}{E((\partial(\log P)/\partial U)^2)} \qquad (2)$$

where P denotes the joint probability density of the signal R(t) and the parameter U. The equality sign in Equation (2) is possible only if:

(i) the conditional probability density function R for a given value of U* and U, $P_2(R/U^*; U)$ is independent of U

(ii)
$$\frac{\partial \log P_1(U^*; U)}{\partial U} = K(U)\ (U^*-U)$$

where $P_1(U^*, U)$ is the joint probability density of U* and U. For an unbiased estimate of U, E(U*) = U. Then, Equation (2) simplifies to

$$\text{Var } U^* \geq \frac{1}{E((\partial \log P)/\partial U)^2)} \qquad (3)$$

and equality sign holds if the conditions (i) and (ii) mentioned above are satisfied.

In general, there may be many unbiased estimates of U. But at best, only one efficient estimate of U can exist. In such a case, the unique non-constant solution U* of Equation (4)

$$\partial(\log P)/\partial U = 0 \qquad (4)$$

is the efficient estimate.

In the next two sections, the expressions for an efficient estimate and the precision of the estimate is derived for both discrete and continuous observation times. The results are applied in the final section to find from the received signal the variance and the best estimate of the amplitude and frequency of the compressional waves observed over a continuous period (-T,T) in a viscoelastic medium. The noise in the received signal, which is assumed to be white Gaussian, is due to the inhomogeneities in the medium.

OBSERVATION AT DISCRETE TIMES

Let the received signal be observed at times $t_1, t_2, \ldots t_n$ and let the values of R, S, and N at time t_i be R_i, S_i, and N_i, respectively ($i = 1,2,\ldots n$). One has then

$$R_i = S_i + N_i \quad , \qquad i = 1,2,\ldots n \; . \tag{5}$$

Without loss of generality, one can assume that $E(N(t)) = 0$. The Gaussian noise $N(t)$ is then completely described by its covariance function $\rho(s,t) = E(N(t)N(s))$. For the discrete sample, the noise $N_i(t)$ is described by its covariance matrix $\rho_{ij} = \rho(t_i,t_j)$, which is a nxn matrix for n samples. This matrix possesses n real non-negative eigen values, λ_i, and n real normal eigen vectors $\vec{\phi}_i$ satisfying the relations

$$\rho\vec{\phi}_i = \lambda_i\phi_i \quad , \qquad i = 1,2,\ldots n \tag{6}$$

$$\vec{\phi}_i \cdot \vec{\phi}_j = \delta_{ij} \quad , \quad i,j = 1,2,\ldots n \; . \tag{7}$$

From these two equations, it can also be shown that

$$\sum_i \sum_j \rho_{ij} (\vec{\phi}_k)_i (\vec{\phi}_l)_j = \lambda_k \delta_{kl}, \; k,l = 1,2,\ldots ;n \tag{8}$$

Here $(\phi_k)_i$ denotes the i^{th} component of the k^{th} eigen-vector. It will now be convenient to introduce the vector notations

$$\vec{R} = (R_1, R_2, \ldots R_n)$$

$$\vec{S} = (S_1, S_2, \ldots S_n)$$

$$\vec{N} = (N_1, N_2, \ldots N_n)$$

and their projections on the eigenvector $\vec{\phi}_i$ are r_i, s_i, and n_i, respectively. For example,

$$n_i = \vec{\phi}_i \cdot N = \sum_j (\vec{\phi}_i)_j N_j \quad . \tag{9}$$

The projection of Equation (5) on ϕ_i is then

$$r_i = s_i + n_i, \qquad i = 1,2, \ldots n \tag{10}$$

Consider now the random variables n_i. Since the N_j are by hypothesis separately and jointly Gaussian, Equation (9) indicates that the n_i are also separately and jointly Gaussian. Hence,

$$E(n_i) = \sum_j (\vec{\phi}_i)_j \, E(N_j) = 0 \quad , \qquad i = 1,2, \ldots n$$

and

$$\begin{aligned} E(n_k n_l) &= \sum_i \sum_j (\vec{\phi}_k)_i \, (\vec{\phi}_l)_j \, E(N_i N_j) \\ &= \sum_i \sum_j \rho_{ij} \, (\vec{\phi}_k)_k \, (\vec{\phi}_l)_j = \lambda_k \delta_{kl} \end{aligned} \tag{11}$$

after using Equation (8). The variables n_i are therefore independent Gaussian variates with mean zero and variance $E(n_i^2) = \lambda_i$. Hence, if $\lambda_i \neq 0$, $(i=1,2, \ldots n)$ (a condition which will be assumed in the following unless stated otherwise) the joint distribution function for the n_i is

$$P(n_i) = \frac{1}{(2\pi)^{\frac{m}{2}}} \frac{\exp}{\prod_{i=1}^{m} (\lambda_i)^{\frac{1}{2}}} (-1/2 \sum_j n^2{}_j/\lambda_j) \tag{12}$$

From Equation (9) and the fact that the $\vec{\phi}_i$ are orthonormal, it is seen that the Jacobian of the transformation (9) is unity, so that (12) is the joint density function, P, of the N_i as well. One then has

$$\log p(R_1, \ldots, R_n; U) = -\log((2\pi)^{\frac{n}{2}} \pi(\lambda_i)^{\frac{1}{2}}) - 1/2 \sum_i \frac{n^2{}_i}{\lambda_i} \tag{13}$$

An optimum value of Equation (10) for a given parameter U is $\partial r_i/\partial u = 0$.

Hence:

$$\frac{\partial n_i}{\partial U} = - \frac{\partial s_i}{\partial U} \tag{14}$$

The two basic quantities necessary to find an efficient estimate of U and a measure of precision of estimate are $[\partial(\log p)/\partial U]$ and $E((\partial \log p/\partial U)^2)$. They are given in Equations (15) and (16) after substitution of Equations (13) and (14).

$$\partial \log P/\partial U = \sum_i \frac{1}{\lambda_i} \frac{\partial s_i}{\partial U} \quad (r_i - s_i) \tag{15}$$

$$E\left(\left(\frac{\partial \log P}{\partial U} \right)^2\right) = \sum_i \frac{1}{\lambda_i} \left(\frac{\partial s_i}{\partial U}\right)^2 \tag{16}$$

Equations (15) and (16) can be written in an alternative form not explicitly involving the eigenvectors and eigenvalues of ρ. Let the vector $\vec{F}$ be defined by $\vec{S} = \rho\vec{F}$ and let the projection of $\vec{F}$ on ϕ_i by f_i $(i = 1,2, \ldots, n)$. One has then $s_i = \lambda_i f_i$ and since λ_i is independent of U,

$$\frac{1}{\lambda_i} \frac{\partial s_i}{\partial U} = \frac{\partial f_i}{\partial U} \tag{17}$$

The right side of Equations (12) and (13) can now be written respectively as

$$\sum \frac{\partial f_i}{\partial U} n_i \quad \text{and} \quad \frac{\partial f_i}{\partial U} \frac{\partial s_i}{\partial U}$$

Since the scalar product of two vectors in independent of the coordinate system used, one obtains finally

$$\frac{\partial \log P}{\partial U} = \frac{\partial \vec{F}}{\partial U} \cdot (\vec{R} - \vec{S}) \tag{18}$$

$$E\left(\left(\frac{\partial \log P}{\partial U} \right)^2\right) = \frac{\partial \vec{F}}{\partial U} \cdot \frac{\partial \vec{S}}{\partial U} \tag{19}$$

where

$$\vec{S} = \rho\vec{F} \tag{20}$$

The above calculations can be readily modified to treat the case in which one or more of the eigenvalues of ρ vanishes. For simplicity, only the results for the case of a single vanishing eigenvalue will be given here. Let $\lambda_i = 0$, $\lambda_j \neq 0$, $j \neq i$ if

$$\frac{\partial S_i}{\partial U} \neq 0$$

Then an estimate U which, with probability one is equal to U, can be obtained by solving the equation $\vec{\phi}_i \cdot \vec{S} = \vec{\phi}_i \cdot \vec{R}$ for U. If

$$\frac{\partial S_i}{\partial U} = 0 \quad ,$$

the usual estimation theory applies. In using (15) and (16), however, the terms on the right involving λ_i are to be omitted.

OBSERVATION AT CONTINUOUS TIMES

The results of the preceding section can be extended to treat the problem of estimating u, when R(t) is observed continuously during the time interval (o,T), by passing to the limit as m, the number of times of observation of R(t) in (O,T), approaches infinity. As m increases, the points t_i must partition (O,T) into smaller and smaller intervals as in the usual definition of the Riemann integral.

As $m \to \infty$, the limiting eigenvalue problem Equations (6) and (7) becomes

$$\int_0^T \rho(t,t')\phi_i(t')dt' = \lambda_i\phi_i(t) \quad , \qquad i = 1,2 ,\ldots ,n \tag{21}$$

$$\int_0^T \phi_i(t)\phi_j(t)dt = \delta_{ij}, \qquad i,j = 1,2, \ldots ,n \tag{22}$$

The projection n_i of N on the eigenvectors of ρ is

$$n_i = \int_0^T N(t)\phi_i(t)dt \tag{23}$$

As before, it can be readily shown that the n_i are independent Gaussian variables with mean zero and variance λ_i.

The basic quantities of the estimation theory for observation at continuous time can be now obtained by generalizing Equations (15) and (16). The equation for efficient estimate is

$$\frac{\partial \log P}{\partial U} = \Sigma \frac{1}{\lambda_i} \frac{\partial s_i}{\partial U} (r_i - s_i) = \sum_i \int_0^T \frac{1}{\lambda_i} \frac{\partial s_i}{\partial U} (r_i - s_i)dt \tag{24}$$

and the equation for a measure of the precision of estimate is

$$E\left(\left(\frac{\partial \log \rho}{\partial U}\right)^2\right) = \Sigma \frac{1}{\lambda_i} \left(\frac{\partial s_i}{\partial U}\right)^2 = \sum_i \int_0^T \frac{1}{\lambda_i} \left(\frac{\partial s_i}{\partial U}\right)^2 dt \tag{25}$$

In Equations (24) and (25), however, the λ_i are the eigenvalues of the Fredholm equation

$$\int_0^T \rho(t,t')\phi(t')dt' = \lambda\phi(t) \tag{26}$$

and the r_i and s_i are projections of R and S on the normalized eigenfunctions $\phi_i(t)$, of (6). That is,

$$r_i = \int_0^T R(t)\phi_i(t)dt, \quad s_i = \int_0^T S(t,U)\phi_i(t)dt \tag{27}$$

As in Section I, an alternate formulation can be given, by defining

$$S(t) = \int_0^T \rho(t,t')F(t')dt' \tag{28}$$

which is the analgoms equation for the continuous time of Equation (20) in Section I. Then, in a formal way one can obtain

$$\frac{\partial \log P}{\partial U} = \int_0^T \frac{\partial F(t)}{\partial U} \; [R(t;U) - S(t;U)] \, dt \tag{29}$$

$$E\left(\left(\frac{\partial \log P}{\partial U}\right)^2\right) = \int_0^T \frac{\partial F(t) U}{\partial U} \; \frac{\partial S(t;U)}{\partial U} \, dt \tag{30}$$

which give the basic quantities in closed form. Equations (29) and (30) are certainly valid if the F(t) satisfying Equation (28) possesses a convergent expansion on the eigenfunctions of $\rho(t,t')$. Such, however, is not usually the case. Generally, there does not exist a square-integrable solution F to Equation (28). In certain cases, it has been possible to find families of function F involving delta functions and their derivatives that satisfy Equation (28). The F families in question are substituted in Equations (29) and (30) and families of closed forms for

$$\frac{\partial \log P}{\partial U} \quad \text{and} \quad E\left[\left(\frac{\partial \log P}{\partial U}\right)^2\right] \tag{31}$$

are obtained.

Comparison of these closed forms with the series (24) and (25) can lead to the recognition of closed forms for the series.

Equations (24) and (25) hold, of course, only if all the λ_i are different from zero. If $\lambda_i = 0$, $\lambda_j \neq 0$, $j \neq i$, where i is a single given positive integer, then if

$$\frac{\partial S_i}{\partial U} \neq 0 \tag{32}$$

an estimate of u with zero variance exists. It is obtained by solving

$$\int_0^T \phi_i(t) S(t;U)dt = \int_0^T \phi_i(t) R(t)dt \tag{33}$$

for u. If

$$\frac{\partial S_i}{\partial U} = 0 \quad , \tag{34}$$

then the usual estimation theory holds where the terms in Equations (24) and (25) involving λ_i are to be omitted.

APPLICATION TO A VISCOELASTIC MEDIUM:

In this section, expressions for the efficient estimate and the variance for both the amplitude, A, and frequency of compressional waves, ω_c, of an impulse function in a viscoelastic medium is obtained. The received signal $R(t;\, A,\, \omega_c)$ will consist of the original signal, viz., the impulse function response of the medium due to the point source $g(t;\, A,\, \omega_c)$ and a general noise function $N(t;\, A,\, \omega_c)$. Hence, Equation (1) can be rewritten as

$$R(t;\, A,\, \omega_c) = g(t;\, A,\omega_c) + N(t;\, A,\omega_c) \tag{35}$$

The response of an impulse function or the Green's function for the compressional wave in a viscoelastic medium is given by [3 & 4]

$$\frac{\partial^2 \tilde{g}_u}{\partial t^2} - \frac{\lambda' + 2\mu'}{\rho} \nabla^2 \tilde{g}_u - \frac{\lambda'' + 2\mu''}{\rho} \nabla^2 \left(\frac{\partial \tilde{g}_u}{\partial t}\right) = \delta(\vec{r}-\vec{r}\,')\delta(t) \tag{36}$$

where

$$\tilde{g}_u = \tilde{g}_u(\vec{r}-\vec{r}\,';t)$$

$$\lambda = \lambda' + i\omega\lambda''$$

$$\mu = \mu' + i\omega\mu''$$

λ and μ are the comples Lame' parameters. The shear and bulk modulus is given by μ' and λ', while μ'' and λ'' are the shear and bulk viscosity. A solution to Equation (36) is obtained by taking its space time Fourier transform. The equation then simplifies to

$$((-z^2 + k^2(\lambda + \mu')/\rho + ik^2 z(\lambda'' + 2\mu'')/\rho))g_u(\vec{k},z) = 1$$

or

$$(-z^2 + \omega_n{}^2 + 2i\zeta\omega_n z)\ g(\vec{k},z) = \omega_n{}^2 \tag{37}$$

where ω_n, the natural frequency $= k((\lambda' + 2\mu')/\rho)^{1/2}$ (38)

ζ, the damping constant $= (k/2)\ [(\lambda'' + 2\mu'')^2/\rho(\lambda' + 2\mu')\ ]^{1/2}$ (39)

and

$$g(\vec{k},z) = \omega_n{}^2 g_u(\vec{k},z) = \omega_n{}^2 \int d(\vec{r}-\vec{r}\,') \int dt\ \tilde{g}_u(\vec{r}-\vec{r}\,';t)\exp(i\vec{k} \cdot (\vec{r}-\vec{r}\,') - izt)) \tag{40}$$

Since we are interested in the time domain response of the point source, let us take the inverse time Fourier transform of Equation (37). The time response $g(\vec{k},t)$ is then given by

$$g(\vec{k},t) = \frac{\omega_n^{\,2}}{2\pi} \int_{-\infty}^{\infty} dz \, \exp(izt) \,/\, (-z^2 + 2i\zeta z + \omega_n^{\,2})$$

$$= 0 \qquad\qquad t < 0 \tag{41}$$

$$= \omega_n \exp(-\zeta\omega_n t)\sin(\omega_n t \sqrt{1-\zeta^2}\,) \,/\, (1-\zeta^2)^{\frac{1}{2}} \qquad t > 0$$

$$= A \exp(-\zeta\omega_n t)\sin(\omega_c t)$$

where

$$A = \omega_n / \sqrt{1-\zeta^2} = k((\lambda' + 2\mu')/\rho)^{\frac{1}{2}} (1 - k^2(\lambda'' + 2\mu'')^2/(4\rho(\lambda' + 2\mu')))^{-\frac{1}{2}}$$

$$\omega_c = \omega_n \sqrt{1-\zeta^2} = k(\lambda' + 2\mu'/\rho)^{\frac{1}{2}} (1 - k^2(\lambda'' + 2\mu'')/(4\rho(\lambda' + 2\mu')))^{\frac{1}{2}} \tag{42}$$

$$\zeta\omega_n = (\lambda'' + 2\mu'')k^2/(2\rho)$$

Substituting Equation (41) in Equation (35), the received signal $R(t;A,\omega_c)$ is

$$R(t;\, A,\, \omega_c) = A \sin \omega_c t \exp(-\zeta\omega_n t) + N(t;\, A,\, \omega_c)$$

The noise is assumed to be white Gaussian with mean square value N^2 and a covariance function

$$\rho(t,t') = N^2\delta(t-t') \tag{43}$$

It is assumed that a continuous sampling of the received signal is made from $-T \le t \le T$ and it is desired to estimate A and ω from this signal. From Equation (24) an efficient estimate of any parameter U for observation at continuous times is

$$\int_{-T}^{T} \frac{1}{\lambda} \frac{\partial g}{\partial u} \; (R - g)dt = 0 \tag{44}$$

Here, λ, the eigen value, is given by Equation (26)

$$\int_{-T}^{T} \rho(t,t')\phi(t')\, dt' = \lambda\phi(t)$$

Since $\rho(t,t') = N^2\delta(t-t')$, (Equation (43)), the eigenvalue $\lambda = N^2$.

Substituting λ in Equation (44) and using Equation (40) for g, the efficient estimate $\tilde{A}$ for A and $\tilde{\omega}_c$ for ω_c is

$$\tilde{A} = \frac{\int_{-T}^{T} R(t) \sin \omega_c t \exp(-2\zeta\omega_n t)\, dt}{\int_{-T}^{T} \sin^2\omega_c t \exp(-2\zeta\omega_n t)\, dt} \tag{45}$$

and $\tilde{\omega}_c$ is given by

$$A \int_{-T}^{T} t \sin \omega_c t \cos \omega_c t \exp(-2\zeta\omega_n t)dt = \int_{-T}^{T} t\, R(t) \sin \omega_c t \exp(-\zeta\omega_n t)dt \tag{46}$$

The variance of any parameters U for an efficient estimate is

$$\text{Var } U = \frac{1}{E((\frac{\partial \log p}{\partial U})^2)} \tag{47}$$

where $E\,((\frac{\partial \log p}{\partial U})^2) = \int_{-T}^{T} \frac{1}{\lambda} (\frac{\partial g}{\partial U})^2\, dt$

Substituting the variance of amplitude A and frequency ω are

$$\text{Var } \tilde{A} = \frac{1}{\int_{-T}^{T} \frac{1}{\lambda} (\frac{\partial g}{\partial A})^2 dt} = \frac{N^2}{\int_{-T}^{T} \exp(-2\zeta\omega_n t)\sin^2\omega_c t\, dt}$$

$$= \frac{N^2(\zeta^2\omega_n^{\ 2} + \omega_c^{\ 2})}{\beta \sin 2\zeta\omega_n T - \cos 2\zeta\omega_n T + \omega \cos L\ 2\zeta\omega_n T \sin 2\omega_c t} \tag{48}$$

$$\mathrm{Var}\ \tilde{\omega} = \frac{N^2}{A^2 \int_{-T}^{T} t^2 \exp(-2\omega_n\zeta t) \cos^2 \omega_c t \ dt} \tag{49}$$

The variance of $\tilde{A}$ and ω are plotted in Figures (1) and (2) for various values of ζ. Note that the estimation error increases with increasing value of ζ. However, as the system is lightly damped, the error varies over a narrow range.

REFERENCES:

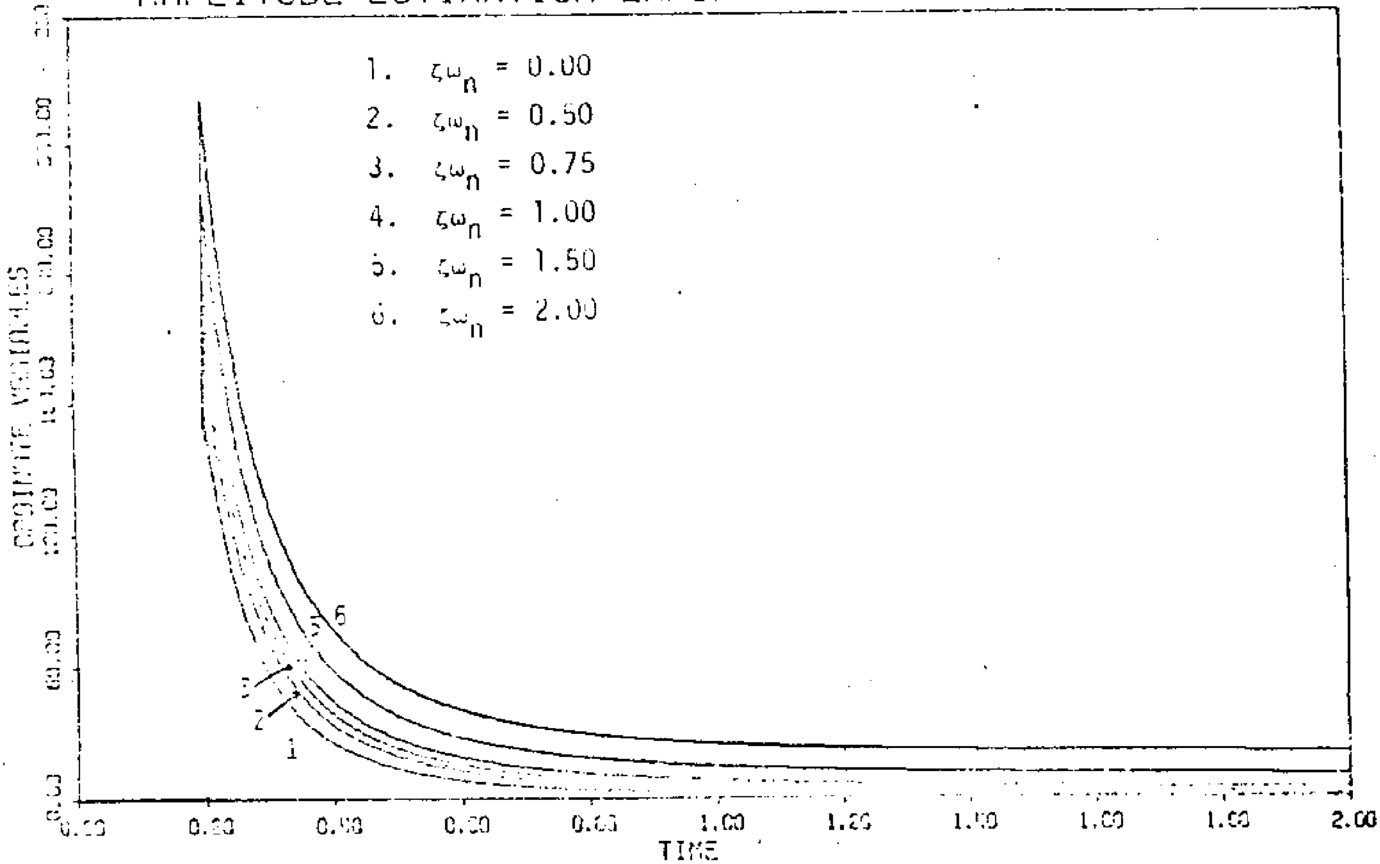
1). Carl W. Helstrom, Statistical Theory of Signal Detection, Pergamon Press,1960.

2). Nasser E. Nahi, Estimation Theory and Applications,John Wiley & Sons, 1969.

3). A. Yildiz, "Wave Propagation in an Elastic Field with Couple Stresses", J. of App. Mech., December, 1972, PP. 1146-1147.

4). M. Yildiz, "Basic Equations of Thermo-visco Elasticity", Memorandum IX, UNH-Sea Grant Project (unpublished).

AMPLITUDE ESTIMATION ERROR VS OBSERVATION TIME



FREQUENCY ESTIMATION ERROR VS OBSERVATION TIME