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Interaction of Stress Waves in a Nonlinear Viscoelastic Medium

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between

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NOMENCLATURE

$\vec{A}$, A_i	Polarization vector
C	Longitudinal wave speed
C_T	Transverse wave speed
I	Action
$\vec{K}_L$	Longitudinal propagation vector
$\vec{K}_T$	Transverse propagation vector
M, N, P, Q	Arguments of various nonlinear wave functions
r	Field variable, radius
R	Work density
$\vec{S}$, S_i	Source vector
t	Time variable
U_i	Component of displacement vector
V	Potential energy density
$\vec{V}$, V_i	Source vector
$\vec{W}$, W_i	Source vector
x,y,z	Field variables
α, β	Identification of primary waves
$\bar{A}$	Ratio of Linear moduli for longitudinal deformations
$\bar{B}$	Ratio of linear moduli for transverse deformations
Y_i	Components of source vector
Γ	Coefficient of source term
γ, δ	Multiplication factors for nonlinear propagation vectors

γ_{ikmn}	Elastic constitutive tensor
δ_{ik}	Kroniker delta
ε_{ik}	Strain tensor
ζ, η	Linear viscous moduli
η_{ikmn}	Viscosity tensor
λ, μ	Linear elastic
ρ_0	Density of medium
σ_{ik}	Stress tensor
ψ	Dissipation function
ω	Angular frequency
$\mathcal{E}$	Total internal energy
V	Potential energy density
$\mathfrak{F}$	Free Energy
$\mathfrak{T}$	Kinetic energy density
$\mathcal{L}$	Lagrangian density

ABSTRACT

INTERACTION OF STRESS WAVES IN A

NONLINEAR VISCOELASTIC MEDIUM

by

PETER M. VOGEL

Nonlinear interaction of monochromatic viscoelastic waves is investigated by calculating the second order displacement field that results when either two distinct waves interact with each other, or a single wave interacts with itself. This is accomplished by studying two types of interaction problems; volume interaction and the interaction of plane waves.

Volume interaction is studied by having two viscoelastic waves of arbitrary propagation direction and polarization interact. Next, the problem is simplified to the elastic case and the radiated nonlinear wave that results from resonant interaction of the primary waves is calculated for the following cases.

1) Interaction of two transverse waves for various polarizations and propagation directions.
2) Interaction of two longitudinal waves for various propagation directions.
3) Interaction of a longitudinal and a transverse wave for various polarizations and propagation directions.

The interaction of colinearally propagating viscoelastic plane waves is considered next. The second order wave resulting when two monochromatic primary waves interact, is calculated for the following cases.

1) Interaction of distinct transverse waves for various polarizations.

2) Interaction of distinct longitudinal waves.

3) Interaction of a transverse and a longitudinal wave.

4) Self-interaction of a transverse wave.

5) Self-interaction of a longitudinal wave.

In addition, a thorough development of the nonlinear equations that govern the displacement field is presented as well as an explanation of the mechanisms through which the nonlinearities arise.

The conclusion contains a listing of all of the nonlinear waves that result for the various interaction cases, as well as a description of an experiment designed to observe the nonlinear wave and measure the parameters necessary to calculate the nonlinear elastic constants. Also, some preliminary experimental observations are reported.

CHAPTER I

INTRODUCTION

The entire theory of linear elastic (or viscoelastic) wave propagation is based on Hooke's law and the linearized strain tensor. The most characteristic property of these linear elastic waves is that any wave can be obtained by simple superposition of separate monochromatic waves. Each of these waves propagates independently of the others without regard to its polarization. Thus one says that these waves do not interact. However, Hooke's law and the linearized strain tensor are only approximations and in certain instances do not appropriately describe the nature of the medium. These instances are the cases of large amplitude waves and an inherently nonlinear medium. Consequently, nonlinear equations must be derived in order to model the characteristics of wave propagation for these cases.

Because the equation of motion for the wave is nonlinear, simple periodic solutions are no longer admissable and the nonlinear interaction of waves must be considered. The anharmonic solutions of the nonlinear equation of motion and the polarization analysis for the interaction of large amplitude viscoelastic waves propagating in a nonlinear medium will be considered in this thesis.

The anharmonic effects of wave propagation in solids are not unlike those of other dynamic problems and are very similar to those effects in wave propagation problems of acoustics and electrodynamics. In general the nonlinear contributions are small when compared to the linear effects except for the cases of large dynamic motions and parametric resonance. Therefore, in this thesis, emphasis will be placed on these two conditions.

CHAPTER II

HISTORICAL DEVELOPMENTS

The anharmonic effects of wave propagation in a continuum can be predicted from the field equations of continuum mechanics, since in general all constitutive relations are nonlinear. The first quantitative detection of these anharmonic effects occurred in acoustics and was published by A. L. Thuras et al (1) in 1934. It was noticed that frequency doubling occurred when an intense sound wave propagated in the atmosphere. Although analysis was presented, the nonlinear equations of acoustics were finally formulated by M. J. Lighthill (2) and these equations were used extensively to solve a variety of problems. A very selective list of some of the work in nonlinear acoustics is listed in references 3 to 15. Nonlinear interaction and the anharmonic effects of electromagnetic and plasma waves have also been investigated by R. Y. Chiao, E. Garmire, and C. H. Townes (16), A. Yildiz (17) (respectively) as well as others.

The anharmonic effects of elastic wave propagation has been investigated both theoretically and experimentally. Theoretical analysis was initiated by Landau and Lifshitz (18), who showed how the nonlinear equation of motion could be derived and discussed the results of interaction of waves. Subsequently Z. A. Goldberg's work (19) in 1960 kindled a new interest in the problem. In his paper, he showed that a second order longitudinal wave was generated when a transverse or a longitudinal primary wave propagated in an elastic media. Shortly thereafter, Jones and Kobett (20) investigated the interaction of two distinct waves of either polarization (longitudinal or transverse) propagating in various directions. This was accomplished by solving the inhomogenious vector equation of motion by using a tensor valued Green's function. The work of Jones and Kobett remains the significant theoretical basis for much of the subsequent work in the field although several deficiencies mar their work. First, there is a term deleted in an important result. Second, the Green's function is obscure in origin. Third, the discussion of quantum mechanical results contradicts the results found by using classical mechanics and this discrepancy is not resolved.

The work of Jones and Kobett was redone by Childress and Hambrich (21) in 1964. In this case the authors used the "wave packet" formalism. The problem was also solved by using quantum mechanics exclusively by Taylor and Rollins (22). While the volume interaction work of Jones and Kobett was being studied in the American school, the plane wave interaction work of Goldberg was being extended in the Soviet school. Papers published in the early 1960's by Gedroits and Kroselnokov (23), Viktorov (24), Pospilov (25), and Stepanov (26) were devoted to the same problem Goldberg studied. However, it is felt that none of the above mentioned authors shared the insight that Goldberg had of the problem. Gedroits et al performed laboratory experiments which were later published (27). Viktorov discussed the effects of boundaries on the interaction phenomenon and he also explained the phenomenon by investigating the nonlinear stress tensor. Pospilov discussed the viscoelastic case; however, he did not solve the problem in general. Stepanov showed some results that are contrary to all previous work and the method used to obtain them, makes them unconvincing. In addition to the above mentioned experimental work by Gedroits et al, observation of nonlinear waves have been reported by Kung-Hsin Jen, L. K. Żarembo and V. A. Krasilnekov (28) and F. R. Rollins et al (29) in laboratory experiments. Although both experiments were well done, the only conclusion to be drawn from either is that observation of nonlinear waves in solids is possible. Recently there was an attempt by Y. M. Chen (30) to investigate the interaction of viscoelastic waves; however, he neglects various polarizations and types of wave interactions adding little insight to the theory.

Thus one observes that the anharmonic effects of elastic waves has been considered both from a classical mechanics and quantum mechanics points of view; however, at no time has the entire problem been formulated in a unified manner. Furthermore dissipation of the waves has not been adequately considered. It is particularly important to incorporate attenuation into the model for two reasons.

1) dissipation is a physical part of all wave propagation phenomena
2) the problem of resonant interaction of waves cannot be completed unless internal damping is introduced.

These two points will be further amplified in the subsequent sections of this thesis.

The author in this thesis has attempted a unified and general formulation of the problem. A complete study of the problem will help design experiments to determine the nonlinear elastic constants that are a vital part of the quantitative understanding of nonlinear wave propagation. Completion of the experiment is beyond the scope of this work; however, the design of an initial experiment and some results are presented.

CHAPTER III

FORMULATION OF THE PROBLEM

1) Thermodynamic Aspects of Wave Propagation.

In general, the equation of motion for elastic waves can be derived by writing Newton's Second Law and the constitutive equation and a brief explanation of the thermodynamic process. However, this is not the case when considering wave propagation in a nonlinear viscoelastic medium. Derivation of the equation of motion must be accomplished by using the most basic approach. In this section an equation of motion that governs the propagation of elastic waves in a nonlinearly elastic, linearly viscous solid medium will be derived. To this end, the thermodynamics of the deformation process will be considered and the appropriate constitutive equation will be derived. The first law of thermodynamics states that the difference between the heat acquired by a unit volume of material and the work done by the internal stresses in that volume is equal to the change in internal energy. For a reversible process, the heat change is given by T dS where T is the temperature and S is the entropy. Thus the first law can be written as:

$$d\mathcal{E} = T\,dS - dR \qquad (1)$$

Here $\mathcal{E}$ is the internal energy density and R is the work density. The free energy density $\mathfrak{F}$ is given by:

$$\mathfrak{F} = \mathcal{E} - TS \qquad (2)$$

Thus:

$$d\mathfrak{F} = -dR - S\,dT \qquad (3)$$

The thermodynamics that governs the deformation process of the wave propagation must be determined. Elastic, (not viscoelastic) waves can be characterized as adiabatic deformations. In this case, it is argued, that temperature changes due to deformations cause only negligible quantities of heat transfer due to the rapidity of these temperature fluctuations and the fact that the temperature gradients occur as often in one direction as in the other. In this case

$$\frac{d}{dt} S = 0$$

or for an elastic medium

$$S = \text{constant}$$

We therefore conclude that the thermodynamic process that characterizes elastic wave propagation is an adiabatic-isentropic process and the elastic constants λ and κ are assumed to be the adiabatic constants. The Lamé parameter, μ, is independent of the thermodynamic processes because it is only associated with deformations that do not involve volume changes. The deformation process of viscoelastic waves cannot be characterized as adiabatic because it must involve mechanical energy dissipation and thus energy loss. However, if the assumption is made that the mechanical energy dissipation (which must result in the increased internal energy of the medium) results for a reasonably short periods of time, only a negligible temperature rise will result. Hence, viscoelastic waves may be characterized as approximately isothermal. In this case the constants λ and κ are referred to as the isothermal moduli. It is because there is no physical deformation that does not dissipate energy and because viscoelastic waves will be a major part of the following work, hereafter the constants λ and κ will be assumed to be the isothermal moduli and the deformation process of viscoelastic wave propagation will be assumed to be an isothermal one.

2) Energy Concepts in Wave Propagation

In order to provide a basis for the nonlinear-viscoelastic formulation, the familiar linear elasticity will be investigated. The force (F_i) due to the internal elastic stresses can be written as:

$$F_i = \partial_k \sigma_{ki} \text{ - - - - - - - - - - - (4)}$$

where σ_{ki} is the stress tensor. Then it can be shown that the work density can be written as:

$$R = 1/2\ \gamma_{ikmn} \varepsilon_{ik} \varepsilon_{mn} \text{ - - - - - (5)}$$

where γ_{ikmn} is the constitutive tensor of Hooke's law and ε_{ik} is the linear strain tensor. By substituting the expression for work as expressed in eq. (5) into eq. (1), it follows that for an adiabatic process:

$$\partial_{(\varepsilon_{ik})} (\mathcal{E}) \Big|_s = \gamma_{ikmn} \varepsilon_{mn} = \sigma_{ik} \text{ - - - - (6)}$$

Similarly when working with nonlinear elasticity, the stress tensor will be obtained by taking the partial derivative of energy with respect to components of the strain tensor. Again this energy will be the total internal energy for an adiabatic process or the free energy for an isothermal process. The crucial difference for nonlinear elasticity is that the strain tensor now contains the second order term $\partial_m U_e\ \partial_n U_e$ which was previously neglected. So in nonlinear analysis one assumes that the second order terms are significant either because of large deformations or because the deformations take place in an inherently nonlinear material. To construct a nonlinear model one first expands the internal (or free) energy in second and third order powers of the strain tensor for which an invariant (scalar) term can be formed. For an isotropic material, the energy becomes:

$$\mathcal{E}\ (\text{or } \mathfrak{F}) = \mu \varepsilon_{ik} + \frac{\lambda}{2} \varepsilon_{ee}\ \varepsilon_{ee} + \frac{A}{3} \varepsilon_{ik}\ \varepsilon_{ie} \varepsilon_{ke}$$
$$+ B \varepsilon_{ik}\ \varepsilon_{ik} \varepsilon_{ee} + \frac{C}{3} \varepsilon_{ee}^3 \text{ - - - - - - - - - - - (7)}$$

The above expansion can be made by observing that the strain tensor is symmetric and that each term must be a scalar. However, the coefficients were chosen so that the first two terms would yield Hooke's law when considered alone and the last three terms would be identical to the results obtained by Murnaghan (31). This expansion was first considered by Murnaghan (who did not use tensor notation) in 1936 and the use of the above is sometimes referred to as Murnaghan's five constant isotropic elasticity.

The internal (or free) energy can be cast into the form:

$$\mathcal{E}\ (\text{or}\ \mathfrak{F}) = \beta_{ikmn}\, \varepsilon_{ik}\, \varepsilon_{mn} \quad \text{------------} \quad (8)$$

where the fourth order tensor is as follows:

$$\beta_{ikmn} = \gamma_{ikmn} + A/3\ \varepsilon_{ke}\, \delta_{im}\, \delta_{em} + B\ \varepsilon_{ie}\, \delta_{im}\, \delta_{km}$$

$$+ \frac{C}{3}\, \varepsilon_{ee}\, \delta_{ik}\, \delta_{mn} \quad \text{---------------} \quad (9a)$$

and where γ_{ikmn} is the constitutive tensor of Hooke's law:

$$\gamma_{ikmn} = \lambda\ (\delta_{ik}\, \delta_{mn}) + \mu\ (\delta_{in}\, \delta_{km} + \delta_{im}\, \delta_{kn}) \quad \text{--------} \quad (9b)$$

Because the argument of second order smallness is no longer valid, one must consider all the terms in the strain tensor.

$$\varepsilon_{ik} = 1/2\ (\partial_i U_k + \partial_k U_i + \partial_i U_e\, \partial_k U_e) \quad \text{----------} \quad (10)$$

Substitution of this strain tensor into the expression for the energy density yields:

$$\mathcal{E}\ (\text{or}\ \mathfrak{F}) = \mu/4\ (\partial_k U_i + \partial_i U_k)^2 + (\lambda/2)\ (\partial_e U_e)^2$$

$$+ (\mu + A/4)\ (\partial_k U_i\, \partial_i U_e\, \partial_k U_e) + (\lambda/2 + B/2)\ (\partial_e U_e)\ (\partial_k U_i)^2$$

$$+ A/12\ \partial_k U_i\, \partial_e U_k\, \partial_i U_e + B/2\ \partial_k U_i\, \partial_i U_k\, \partial_e U_e$$

$$+ C/3\ (\partial_e U_e)^3 \quad \text{--------------------} \quad (11)$$

where terms up to and including third order terms in displacement are retained and all high order terms are neglected.

The form that the stress tensor takes when nonlinear effects are considered can now be derived. For isothermal deformations:

$$-\delta R = \delta \mathfrak{F} \quad \text{---------------} \quad (12)$$

In order to take the variation of the free energy, it is only necessary to note that it is a function of the displacement gradient, $(\partial_k U_i)$. Then:

$$\delta\ (F) = \partial_{(\partial_k U_i)}(F)\ \delta\ (\partial_k U_i) = \partial_k [\partial_{(\partial_k U_i)}(F)\ \delta U_i] - \delta U_i \partial_k [\partial_{(\partial_k U_i)}(F)] \tag{13}$$

Substituting Eq. (13) into Eq. (12) and into the definition of work density:

$$\int \delta\ R dV = \int F_i\ \delta U_i dV \tag{14}$$

and noting that $F_i = \partial_k \sigma_{ki}$, and by equating the coefficients of δU_i, one concludes that for an infinite medium:

$$\sigma_{ki} = \partial_{(\partial_k U_i)}(F) \tag{14a}$$

More explicitly the stress tensor is found to be:

$$\begin{aligned}\sigma_{ki} &= \mu(\partial_k U_i + \partial_i U_k) + \lambda\ \partial_e U_e\ \delta_{ik} \\ &+ (\mu + A/4)\ [\partial_k U_e\ \partial_i U_e + 2\ \partial_e U_e\ \partial_k U_i] \\ &+ (\lambda/2 + B/2)\ [(\partial_m U_n)^2\ \delta_{ki} + 2\ \partial_e U_e\ \partial_k U_i] \\ &+ A/12[(\partial_e U_k)(\partial_i U_e) + (\partial_k U_e\ \partial_i U_e) + \partial_e U_k\ \partial_e U_i] \\ &+ B/2[6\ \partial_e U_e\ \partial_k U_i + \partial_m U_n\ \partial_n U_m\ \delta_{ik}] + C[\partial_e U_e]^2\ \delta_{ik}\end{aligned} \tag{15}$$

Examination of some of the terms will show that unlike the linear stress tensor, this stress tensor is unsymmetric. In the above, only the reversible energy density or the purely elastic energy density has been considered. A perfectly reversible process rarely exists in the physical world and it never exists for a stress wave. Therefore dissipation of the wave must be considered.

The thermodynamic process, characterizing the propagation of a viscoelastic wave, has been considered in a previous section and found to be an isothermal process.

In this case, the finite velocity, internal motion of the deformation process causes dissipation of the energy because of the internal friction (or viscosity) of the medium.

Here, the dissipation of energy will be incorporated into the analysis by constructing a dissipative function from which the non-conservative forces and the dissipative stress tensor can be derived. The introduction of attenuation into the model will be accomplished following the methods developed by Landau and Lifshitz (Ref. 18, Art. 34).

If one has a mechanical system whose motion involves the dissipation of energy, this motion can be described by the ordinary equations of motion, with the forces acting on the system augmented by the dissipative forces or frictional forces, which are linear functions of velocity. These forces can be written as the velocity derivatives of a certain quadratic function ψ of the velocities, and this function is called the dissipative function. The frictional force, f_a, corresponding to a generalized coordinate, q_a, of the system is then given by:

$$f_a = -\partial_{q_a}(\psi)$$

Because ψ is a function of q_a, one writes

$$\delta\ \psi = \partial_{q_a}(\psi) \quad \delta q_a = - f_a \delta q_a \tag{16}$$

or for a continuum:

$$\delta \int_V \psi dv = \int_V f_i \ \delta \dot{U}_i dv \tag{17}$$

where ψ is the dissipative function density.

Because the forces due to dissipation are zero during simple translational or rotational motion of the deformed body, the dissipation function density must be zero when either $\dot{U}_i$ = constant, or when $\dot{U}_i = \varepsilon_{ijk} r_j \Omega_k$.

The first of these conditions requires that the dissipative function density must not be a function of the particle velocity but rather a function of its gradient. The second condition requires that the dissipative function ψ be only a function of the symmetric parts of the velocity gradient. That is:

$$\psi = \psi\ (\dot{\varepsilon}_{ik}) \quad \text{----------- (18)}$$

where ε_{ik} is the rate of strain tensor.

$$\dot{\varepsilon}_{ik} = \frac{1}{2}\ (\partial_i \dot{U}_k + \partial_k \dot{U}_i) \quad \text{----------- (19)}$$

If the dissipative function is constructed in such a way that the dissipative stress that is derived from it has a traceless term, (which has as its coefficient η), the function will take the form:

$$\psi = \eta\ (\dot{\varepsilon}_{ik} - \frac{1}{3}\,\delta_{ik}\ \dot{\varepsilon}_{ee})^2 + \frac{1}{2}\,\zeta(\dot{\varepsilon}_{ee})^2 \quad \text{------ (20)}$$

This can be written in the more convenient form:

$$\psi = \eta(\dot{\varepsilon}_{ik})^2 + (\zeta/2 - \eta/3)\ (\dot{\varepsilon}_{ee})^2 \quad \text{--------- (20a)}$$

The dissipative stress tensor (σ'_{ki}) can be found in a way analogous to that of finding the linear stress tensor. We know that:

$$f_i = \partial_k\ \sigma'_{ki} \quad \text{------------ (21)}$$

and that:

$$-\ \delta \int \psi\ dV = - \int f_i\ \delta\dot{U}_i dV = \int \partial_k \sigma_{ki}\ \delta\dot{U}_i dV \quad \text{--------- (22)}$$

However,

$$\delta(\psi) = \partial\dot{\varepsilon}_{ik}(\psi)\ \delta\ (\dot{\varepsilon}_{ik}) = \partial\dot{\varepsilon}_{ik}(\psi)\frac{1}{2}(\partial_i\delta\dot{U}_k + \partial_k\delta\dot{U}_i) \quad \text{------ (23)}$$

The rate of strain tensor is symmetric and therefore:

$$\delta\ \psi = \partial\dot{\varepsilon}_{ik}(\psi)\ \partial_k\ \delta\dot{U}_i = \partial_k(\partial\dot{\varepsilon}_{ik}(\psi))\ \delta\dot{U}_i - \partial_k(\partial\dot{\varepsilon}_{ik}(\psi)\ \delta\dot{U}_i) \quad \text{- (24)}$$

Combination of Eqs. (22) and (24) will yield the following for an infinite media.

$$\sigma_{ki}' = \partial_{\dot{\varepsilon}_{ik}}(\psi) = (\zeta - \frac{2}{3}\,\eta)\ \dot{\varepsilon}_{ee}\delta_{ik} + 2\eta\dot{\varepsilon}_{ik} \quad \text{----- (25)}$$

Substitution of the rate of strain tensor into Eq. (25) will yield:

$$\sigma'_{ki} = (\zeta - \frac{2}{3}\,\eta)\ (\partial_e\dot{U}_e)\delta_{ik} + \eta(\partial_i\dot{U}_k + \partial_k\dot{U}_i) \quad \text{----- (26)}$$

It is no coincidence that the dissipative stress tensor resembles the elastic stress tensor for an isotropic media. The dissipative function can be put in the form:

$$\psi = \frac{1}{2}\,\eta_{ikmn}\,\dot{\varepsilon}_{ik}\,\dot{\varepsilon}_{mn} \qquad \text{-------------- (27)}$$

where η_{ikmn} is similar to the fourth order tensor of Hooke's law and is called the viscosity tensor. For an isotropic medium this tensor can be reduced to

$$\eta_{ikmn} = (\zeta - \tfrac{2}{3}\eta)\;\delta_{ik}\,\delta_{mn} + \eta(\delta_{in}\delta_{nm} + \delta_{im}\,\delta_{kn}) \qquad \text{-------- (28)}$$

Unlike the elastic stress tensor, the viscous stress tensor contains no nonlinear terms with respect to the displacement. In formulating this stress tensor it was assumed that the dissipative forces were exclusively a function of the linear velocity. Thus viscosity is only a first order phenomenon. The only energy left to consider before the equation of motion can be derived is the Kinetic Energy density, which is simply given by:

$$\mathfrak{T} = \rho_{o/2}\,\dot{U}_i\dot{U}_i \qquad \text{------------------- (29)}$$

for a solid of density ρ_o.

3) Equation of motion for waves in a nonlinear viscoelastic medium

The equation of motion can now be derived by using Hamilton's principle:

$$\delta\, I = \delta \int_1^2 \int_V \mathcal{L}\, dx_k dt = o \qquad \text{----------- (30)}$$

where points 1 and 2 are points in time where the variation of the displacement is zero, and V is a volume that is enclosed by a surface on which the variation of the displacement is zero. For this problem the Lagrangian density takes the form:

$$\mathcal{L} = \mathfrak{T}(\dot{U}_i) - V\,(\partial_k U_i,\ \partial_k \dot{U}_i) \qquad \text{---------- (31)}$$

Where V is the potential energy density and is equal to the sum of the reversible and irreversible parts,

$$V = \mathfrak{F}_{rev}(\partial_i U_k) + \mathfrak{F}_{IRR}(\partial_i \dot{U}_k) \quad \text{-------------- (32)}$$

The reversible part is the free energy as expressed in Eq. (11) and the irreversible part $\mathfrak{F}_{IRR}$ takes into account the effect due to dissipation. Substitution of the Lagrangian density into Hamilton's principle results in:

$$\int_1^2 \int_V [\delta(\mathfrak{T}(\dot{U}_i)) - \delta(\mathfrak{F}(\partial_k U_i))] dx_k dt - \int_1^2 \int_V \delta \mathfrak{F}_{IRR} dx_k dt = 0 \quad \text{---- (33)}$$

The conservative part of the above equation can be integrated in the usual way:

$$\int_1^2 \int_V [\delta(\mathfrak{T}(\dot{U}_i)) - \delta(\mathfrak{F}(\partial_k U_i))] dx_k dt =$$

$$\int_1^2 \int_V [\partial_{(\dot{U}_i)}(\mathfrak{T}(\dot{U}_i))\delta\dot{U}_i - \partial_{(\partial_k U_i)}(\mathfrak{F})\delta(\partial_k U_i)] dx_k dt \quad \text{---- (33a)}$$

where each part is integrated by parts.

$$\int_1^2 \int_V \partial_{(\dot{U}_i)}(\mathfrak{T})\delta\dot{U}_i \; dx_k dt = \int_V \partial_{(\dot{U}_i)}(\mathfrak{T})\delta U_i \Big|_1^2 dx_k - \int_1^2 \int_V \partial_t \partial_{(\dot{U}_i)}(\mathfrak{T})\delta U_i \; dt dx_k$$

$$\int_1^2 \int_V \partial_{(\partial_k U_i)}(\mathfrak{F})\delta(\partial_k U_i) \; dx_k dt = \int_1^2 \partial_{(\partial_k U_i)}(\mathfrak{F})\delta U_i \Big|_S dt -$$

$$\int_V \int_1^2 \partial_k \partial_{(\partial_k U_i)}(\mathfrak{F})\delta U_i dx_k dt \quad \text{-------- (34)}$$

The first terms in the above expressions are zero sinze δU_i does not vary at the end points. Next the irreversible part of the potential energy must be integrated. One notes that variation of potential energy is the variation of work,

$$\int_V \int_1^2 \delta V_{IRR} dx_k dt = \int_V \int_1^2 F_i \; \delta U_i dx_k dt \quad \text{------------- (35)}$$

The force in this case has been shown to be the gradient of the dissipative stress tensor, and therefore the integral on the right becomes:

$$\int_V \int_1^2 \partial_k \sigma'_{ki} \; \delta U_i \; dx_k dt$$

or:

$$\int_V \int_t^2 \partial_k \, \partial_{(\dot{\varepsilon}_{ik})}(\psi) \;\; \delta U_i dx_k dt$$

Hence, the entire equation of motion can be written as:

$$\partial_t \partial_{\dot{U}_i}(\mathfrak{J}) - \partial_k \partial_{(\partial_k U_i)}(\mathfrak{J}) - \partial_k \partial_{(\dot{\varepsilon}_{ki})}(\psi) = o \quad \text{----------} \quad (36)$$

After substitution and rearrangement, the equation of motion takes the form:

$$\rho_o \partial^2{}_t U_i - \partial_t [\eta \partial^2{}_k U_i + (\zeta - \tfrac{2}{3}\eta)\partial_i \partial_k U_k] - \mu \partial^2{}_k U_i - (\lambda + \mu)\partial_i \partial_k U_k = S_i(r,t) \quad \text{--} \quad (37)$$

where the source term contains all of the nonlinear terms and is:

$$S_i(r,t) = (\mu + A/4)[\partial^2{}_k U_e \partial_i U_e + \partial^2{}_k U_e \partial_e U_i + 2\partial_k \partial_e U_i \partial_k U_e]$$

$$+ (\lambda + \mu + A/4 + B)[\partial_i \partial_k U_e \partial_k U_e + \partial_e \partial_k U_k \; \partial_e U_i]$$

$$+ (\lambda + B) \; \partial^2{}_k U_i \partial_e \; U_e + (B + 2C) \; \partial_i \partial_k U_k \partial_e U_e$$

$$+ (A/4 + B)[\partial_e \partial_k U_k \partial_i U_e + \partial_i \partial_k U_e \partial_e U_k] \quad \text{--------------} \quad (38)$$

Before discussing the solution to Eq. (37), it may be instructive to restate the model that this equation is intended to represent. This equation is intended to govern the propagation of waves in a linearly viscous, nonlinearly elastic solid medium. The moduli that compose the coefficients of the displacement terms are the isothermal moduli and thus the deformation process modeled is an isothermal one.

In order to derive this equation, the elastic free energy was expanded to sufficiently high orders in the strain tensor so that all the second order terms in displacement would be retained in the equation of motion. In addition second order terms were also incorporated into the strain tensor. Therefore the above equation of motion has incorporated into it the effects of two types of second order nonlinearities:

1) Anharmonic effects may arise due to the nonlinearity of the medium in which the wave is propagating. The model considers this by

including higher order terms in the energy expansion. A highly nonlinear material would have large numerical values for the constants A, B, and C, and a very linear material would have A = B = C = 0. By the way the values A, B, and C are in general negative (B is positive for metals) and they are approximately an order of magnitude smaller than the isothermal first order moduli for polymers and of the order or one order larger in value for metals and several orders larger than the linear moduli for some crystals. This will be discussed in more detail later.

2) The second type of anharmonic effect is one that arises from large deformations. The model takes this into account by including higher order terms in the energy expansion and by not discarding the second order term in the strain tensor. The relative importance of these two effects will vary considerably with the type of material, and the effect of nonlinearities in general will be very much dependent on the amplitude of the wave. A numerical example will illustrate this point. The maximum stresses due to a one dimensional longitudinal elastic wave propagating in a medium of polystyrene will be calculated. Polystyrene was chosen for two reasons. It is easy to excite a large amplitude wave in this material because of its low density and secondly the second order elastic constants have been computed for this material (32).

$$\lambda = 2.89 \times 10^{10} \quad \text{dyn/cm}^2$$

$$\mu = 1.38 \times 10^{10} \quad \text{"}$$

$$A = -1.00 \times 10^{11} \quad \text{"}$$

$$B = -8.3 \times 10^{10} \quad \text{"}$$

$$C = -1.06 \times 10^{11} \quad \text{"}$$

$$\rho_0 = 1.05 \text{ gm/cm}^3$$

The maximum stress due to the various deformations of a 10K cycle/sec longitudinal wave with amplitude of 5×10^{-9} cm propagating in the above material are as follows:

Linear stress = 7.15 dyne/cm^2

Nonlinear stress with (A=B=C=0) = 1.4×10^{-7} dyne/cm^2

Nonlinear stress due to terms whose coefficient is

A, B, or C = 4.24×10^{-5} dyne/cm^2

These stresses were calculated by using Eq. (15). If a wave of the same frequency but with amplitude of 5×10^{-4} cm were propagating in the same material, the stresses would be 7.15×10^{4} dyne/cm^2, 1.4×10^{3} dyne/cm^2, and 4.24×10^{5} dyne/cm^2 respectively.

It can be seen from this that by increasing the magnitude of the wave by an order of 5, the nature of the stresses is completely changed. If the nonlinear stresses were not significant in the first example, they are very much significant in the second example where the oscillations are well into the nonlinear region. It is interesting to note that polystyrene is not by any means a particularly nonlinear material (some metals and crystals are several orders of magnitude more nonlinear) and even in the laboratory a wave with an amplitude of 5×10^{-4} cm can be generated by using quartz crystals excited by sufficiently high voltages. In addition to high amplitude waves, there is one other instance when nonlinear oscillations become predominant, and that is the case of resonance, which will be discussed later. The effect of the nonlinear deformation, whether it is due to large deformations or a predominantly nonlinear material, will be the same and that is the creation of a second order wave that is not the result of superposition.

4) The Perturbation Technique

The only variable in the equation of motion, Eq. (37), is the $i^{\underline{th}}$ component of the displacement vector and it appears as a linear term on the left side of the equation and as a second order term on the right hand side. The solution to this equation can be obtained by perturbing the variable U_i, in which case the solution of the equation is the sum of the solutions to N distinct equations:

$$U_i(\vec{r},t) = \sum_{N=0}^{\infty} U^{(N)}{}_i(\vec{r},t) \quad \text{------------- (39)}$$

The N^{th} term of the variable, U_i, is the solution to the N^{th} order equation of motion:

$$\mathcal{L}(\vec{r},t)U_i^{(N)}(r,t) = S_i^{(N)}(\vec{r},t) \quad \text{----------- (40)}$$

where $\mathcal{L}(\vec{r},t)$ is the linear operator:

$$\mathcal{L}(\vec{r},t) = \rho_o\partial^2 t - \eta\partial_t\partial^2{}_k - (\zeta - \frac{2}{3}\eta)\partial_t\partial_i\partial_k\delta_{ik}$$

$$-\mu\ \partial_k^2 - (\lambda+\mu)\ \partial_i\partial_k\delta_{ik} \quad \text{---------- (41)}$$

and $S_i^{(N)}(\vec{r},t)$ is the source term:

$$S_i^{(N)}(\vec{r},t) = \sum_{J=1}^{N-1} \{(\mu + A/4)\ (\partial^2{}_k U_e{}^{(N-j)}\partial_i U_e{}^{(j)} + \partial^2{}_k U_e{}^{(N-j)}\partial_e U_i{}^{(j)}$$

$$+ 2\ \partial_e\partial_k U_i{}^{(N-j)}\partial_k U_e{}^{(j)})$$

$$+ (\lambda+\mu+A/4+B)(\partial_i\partial_k U_e{}^{(N-j)}\partial_k U_e{}^{(j)} + \partial_e\partial_k U_k{}^{(N-j)}\partial_e U_i{}^{(j)})$$

$$+ (\lambda+B)(\partial^2{}_k U_i{}^{(N-j)}\partial_e U_e{}^{(j)}) + (B+2C)\ (\partial_i\partial_k U_k{}^{(n-j)}\partial_e U_e{}^{(j)})$$

$$+ (A/4+B)\ (\partial_e\partial_k U_k{}^{(N-j)}\partial_i U_e{}^{(j)} + \partial_i\partial_k U_e{}^{(N-j)}\partial_e U_k{}^{(j)})\} \quad \text{------ (42)}$$

The N = 0 equation corresponds to rigid body motion and is therefore of no interest ($U^{(o)}{}_i$ = constant). When N = 1 is substituted into Eq. (40), the homogeneous equation results.

$$\mathcal{L}(\vec{r},t)\ U^{(1)}{}_i(\vec{r},t) = 0 \quad \text{---------- (43)}$$

The solution to this equation will be discussed for the appropriate cases; however, at this time it will be stated that the solution consists of dilitational or longitudinal displacement fields and deviatoric or transverse displacement fields. These fields are characterized by:

$$\text{div}\ (\vec{U}_L) = \psi_L \qquad \text{curl}\ (\vec{U}_L) = 0$$

$$\text{div}\ (\vec{U}_T) = 0 \qquad \text{curl}\ (\vec{U}_T) = \vec{\psi}_T$$

The waves represented by the solution of the linear equation propagate independent of each other, and because the interaction or self interaction of these waves will be investigated; henceforth, these linear waves will be referred to as the primary waves. The N = 2 order equation is the lowest that results in an inhomogeneous equation, the source term consisting of second order combinations of the primary waves. It is for this reason that we say the N = 2 order is the lowest that results in an interaction. Because

$$U_i^{(N+1)}(\vec{r},t) << U^{(N)}{}_i(\vec{r},t)$$

one can conclude that for all practical purposes, the displacement field is given by a finite sum of the $N^{\underline{th}}$ order displacements. In fact, only the second order displacements (and of course the primary waves) have been observed in the crude laboratory experiments completed thus far. In addition to finding the nonlinear contribution to the displacement field, calculation of the N=2 order displacement is important because doing so gives insight into the physics of wave interaction.

The equation of motion was derived in such a way that the displacement terms of higher order than 2 were disregarded. If the free energy density were expanded further in powers of the strain tensor, and if the appropriate higher order terms were retained in the equation of motion, perturbation of the variable would have allowed calculation of displacements higher than second order. As mentioned before these higher order displacements become increasingly smaller in magnitude and importance. For instance the N = 3 order wave will be due to either interaction of a primary wave with a second order wave or third order primary wave interaction. The N = 4 wave will be due to either interaction of two second order waves, a third order wave and a primary wave, or fourth order primary wave interaction. The types of interaction are easily seen by investigating the source term $S_i^{(N)}$. This thesis is concerned with only the second order interaction and accordingly only second order terms are included in the source term. These terms are of the form:

$$\sum_{J}^{N-1} A^{(N-j)} \quad B^{(j)}$$

Substitution of N = 1 into the above summation will yield zero, however substitution of N = 2 will yield a combination of two primary waves as the source term. Hence we are considering primary wave interactions.

If terms of the third order were considered, the source term would be of the form:

$$\sum_{K}^{N-1} \quad \sum_{J}^{N-K-1} \quad A^{(N-K-J)} \quad B^{(K)} \quad C^{(J)}$$

and if the fourth order terms were considered, the form of the source term would be:

$$\sum_{L}^{N-1} \quad \sum_{K}^{N-L-1} \quad \sum_{J}^{N-K-L-1} \quad A^{(N-K-L-J)} \quad B^{L} \quad C^{K} \quad D^{J}$$

In this manner any order of nonlinearity can be considered, however, consideration of the amount of algebra necessary to complete the analysis make such calculations impractical. Also there is little experimental evidence of the existence of interaction waves of higher order.

CHAPTER IV

VOLUME INTERACTIONS OF ELASTIC WAVES

First the interaction of stress waves in three dimensional space will be modeled. A three dimensional model is especially useful when two primary waves cross each other. The volume of interaction is then approximately spherical and this volume behaves as a spherical radiator of interaction waves. In the case of self interaction it is assumed that a wave crosses a spherical nonlinear element and the self interaction occurs. The spherical nonlinear element then serves as a radiator of interaction waves.

1) The Primary Wave.

As mentioned previously, the Primary Waves will be represented by the real part of the solution of the linear equation (N = 1):

$$\mathcal{L}(\vec{r},t)\ U_i^{(1)}(\vec{r},t) = 0 \qquad \text{------------ (1)}$$

The next step is to Fourier time transform this equation; however, before doing so, the Fourier transformation pairs for both time and space transformation will be listed. The transformation pairs are:

$$\bar{f}(\vec{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\vec{r},t)\ \exp(i\omega t)dt \qquad \text{--------- (2a)}$$

$$f(\vec{r},t) = \int_{-\infty}^{\infty} \bar{f}(\vec{r})\ \exp(-i\omega t)d\omega \qquad \text{--------- (2b)}$$

$$\bar{f}(\vec{\gamma}) = \int_V f(\vec{r})\ \exp(i\vec{\gamma}\cdot\vec{r})dV \qquad \text{--------- (2c)}$$

$$f(\vec{r}) = 1/(2\pi)^3 \int_\gamma \bar{f}(\vec{\gamma})\ \exp(-i\vec{\gamma}\cdot\vec{r})d^3\gamma \qquad \text{---- (2d)}$$

Eq. (1), after Fourier transformation and rearrangement, can be written in vector notation as:

$$\rho_o \omega^2 \, \vec{\bar{U}}^{(1)}(\vec{r}) - (\mu - i\omega\eta) \, \nabla \times \nabla \times \vec{\bar{U}}^{(1)}(\vec{r})$$

$$+ [(\lambda + 2\mu) - i\omega(\zeta + \eta/3)] \, \nabla(\nabla \cdot \vec{\bar{U}}^{(1)}) = 0 \qquad \text{--------- (3)}$$

By letting the displacement vector be represented by the sum of a longitudinal displacement $\vec{\bar{U}}_L(\vec{r})$ and a transverse displacement $\vec{\bar{U}}_T(\vec{r})$, and by noting the properties of these displacements as listed in the last section, the equation can be separated into two vector Helmholtz equations:

$$(\nabla \cdot \nabla + \vec{K}_L \cdot \vec{K}_L) \, \vec{\bar{U}}_L(\vec{r}) = 0 \qquad \text{------------ (4)}$$

$$(\nabla \cdot \nabla + \vec{K}_T \cdot \vec{K}_T) \, \vec{\bar{U}}_T(\vec{r}) = 0 \qquad \text{------------ (5)}$$

where the magnitude of the propagation vectors (K) can be separated into real and imaginary parts:

$$|\vec{K}_L| = |\vec{K}_L'| + i|K_L''| \qquad \text{--------------- (6)}$$

$$|\vec{K}_T| = |\vec{K}_T'| + |\vec{K}_T''| \qquad \text{--------------- (7)}$$

$$|\vec{K}_L'| = \frac{\omega}{C_{Lo}} \left[\frac{(1+\omega^2\bar{A}^2)^{1/2}+1}{2(1+\omega^2\bar{A}^2)} \right]^{1/2} \qquad \text{---------- (8)}$$

$$|\vec{K}_T'| = \frac{\omega}{C_{To}} \left[\frac{(1+\omega^2\bar{B}^2)^{1/2}+1}{2(1+\omega^2\bar{B}^2)} \right]^{1/2} \qquad \text{---------- (9)}$$

$$|K_L''| = \frac{\omega}{C_{Lo}} \left[\frac{(1+\bar{A}^2\omega^2)^{1/2}-1}{2(1+\bar{A}^2\omega^2)} \right]^{1/2} \qquad \text{-------- (10)}$$

$$|K_T''| = \frac{\omega}{C_{To}} \left[\frac{(1+\bar{B}^2\omega^2)^{1/2}-1}{(1+\bar{B}^2\omega^2)} \right]^{1/2} \qquad \text{-------- (11)}$$

$$C_{Lo} = \left(\frac{\lambda + 2\mu}{\rho_o}\right)^{1/2} \qquad \text{--------------- (12)}$$

$$C_{To} = (\mu/\rho_o)^{1/2} \qquad \text{--------------- (13)}$$

$$\bar{A} = \frac{\zeta + \eta/3}{\lambda + 2\mu} \qquad \text{--------------- (14)}$$

$$\bar{B} = \eta/\mu \qquad \text{--------------- (15)}$$

The primary waves can be represented as the inverse time transform of the real part of the s-lution to the linear equation, or for this problem, the primary displacement field will be the sum of two waves:

$$U_i^{(1)}(\vec{r},t) = A_{i(\alpha)}\exp(-\vec{K}''_{(\alpha)}\cdot\vec{r})\cos(\omega_\alpha t-\vec{K}'_{(\alpha)}\cdot\vec{r})$$

$$+ A_{i(\beta)}\exp(-\vec{K}''_{(\beta)}\cdot\vec{r})(\cos\omega_\beta t-\vec{K}'_{(\beta)}\cdot\vec{r}) \quad ------ (16)$$

The primary displacement field is written in this manner because we are interested in the interaction of two distinct waves. In the above equation $A_{i(\alpha)}$, $A_{i(\beta)}$ are the polarization vectors and the subscripts α and β are intended to only identify the waves and they are in no way related to the $\bar{A}$ and $\bar{B}$ of Eqs. (14) and (15).

Substitution of the primary wave into the source term given by Eq. (42) of Section III, results in the following expression if the self interaction terms are disregarded.

$$S_i^{(2)}(\vec{r},t) = \sum_{\alpha\pm\beta}\{V^{(2)}_{i(\alpha\pm\beta)}\sin[(\omega_\alpha\pm\omega_\beta)t - (\vec{K}'_\alpha \pm K'_\beta)\cdot\vec{r}]$$

$$+ W^{(2)}_{i(\alpha\pm\beta)}\cos[(\omega_\alpha\pm\omega_\beta)t-(\vec{K}'_\alpha \pm \vec{K}'_\beta)\cdot\vec{r}]\}\exp[-(K''_\alpha+K''_\beta)\cdot\vec{r}] \quad --- (17)$$

Where the vector nature of the source term is given by:

$$V_i^{(2)} = -\frac{1}{2}(\mu+A/4)\ [({}_1\gamma_i \pm {}_2\gamma_i) + ({}_5\gamma_i \pm {}_6\gamma_i) + 2\ ({}_9\gamma_i \pm {}_{10}\gamma_i)]$$

$$+ (\lambda+\mu+A/4+B)\ [({}_{13}\gamma_i \pm {}_{15}\gamma_i) + ({}_{17}\gamma_i \pm {}_{18}\gamma_i)]$$

$$+ (\lambda+B)\ [({}_{21}\gamma_i \pm {}_{22}\gamma_i)] + (B+2C)\ [{}_{33}\gamma_i \pm {}_{34}\gamma_i]$$

$$+ (A/4+B)\ [({}_{25}\gamma_i \pm {}_{26}\gamma_i) + ({}_{29}\gamma_i \pm {}_{30}\gamma_i)]\} \quad ------- (18a)$$

$$W_i^{(2)} = -\frac{1}{2}\{(\mu+A/4)\ [({}_3\gamma_i \pm {}_4\gamma_i) + ({}_7\gamma_i \pm {}_8\gamma_i) + 2\ ({}_{11}\gamma_i \pm {}_{12}\gamma_i)]$$

$$+ (\lambda+\mu+A/4+B)\ [({}_{15}\gamma_i \pm {}_{16}\gamma_i) + ({}_{19}\gamma_i \pm {}_{20}\gamma_i)]$$

$$+ (\lambda+B)\ [({}_{23}\gamma_i \pm {}_{29}\gamma_i)] + (B+2C)\ [{}_{35}\gamma_i \pm {}_{36}\gamma_i]$$

$$+ (A/4+B)\ [({}_{27}\gamma_i \pm {}_{28}\gamma_i) + ({}_{31}\gamma_i \pm {}_{32}\gamma_i)]\} \quad ------- (18b)$$

and the γ_i vectors are as follows:

$$_1\gamma_i = (\vec{A}_\beta \cdot \vec{A}_\alpha)(\vec{K}'_\beta \cdot \vec{K}'_\beta)\, K'_{i(\alpha)} - (\vec{A}_\beta \cdot \vec{A}_\alpha)(\vec{K}''_\beta \cdot \vec{K}''_\alpha)\, K_{I(\alpha)} - 2\,(\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}''_\alpha \cdot \vec{K}'_\alpha)\, K''_{i(\beta)}$$

$$_2\gamma_i = (\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}'_\alpha \cdot \vec{K}'_\alpha)\, K'_\beta - (\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}''_\alpha \cdot \vec{K}''_\beta)\, K'_{i(\beta)} - 2\,(\vec{A}_\beta \cdot \vec{A}_\alpha)(\vec{K}''_\beta \cdot \vec{K}'_\beta)\, K''_{\alpha(\alpha)}$$

$$_3\gamma_i = (\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}''_\alpha \cdot \vec{K}''_\alpha)\, K''_{i(\beta)} - (\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}'_\alpha \cdot \vec{K}'_\beta)\, K_{i(\beta)} + (\vec{A}_\beta \cdot \vec{A}_\alpha)(\vec{K}''_\beta \cdot \vec{K}''_\beta)\, K''_{\alpha(\alpha)}\ (\vec{A}_\beta \cdot \vec{A}_\alpha)(\vec{K}_\beta \cdot \vec{K}_\alpha)\, K''_{i(\alpha)}$$

$$_4\gamma_i = 2\,[(\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}''_\alpha \cdot \vec{K}'_\alpha)\, K_{i(\beta)} + (\vec{A}_\beta \cdot \vec{A}_\alpha)(\vec{K}''_\beta \cdot \vec{K}'_\beta)\, K'_{i(\alpha)}]$$

$$_5\gamma_i = (\vec{A}_\beta \cdot \vec{K}'_\alpha)(\vec{K}'_\beta \cdot \vec{K}'_\beta)\, A_{i(\alpha)} - (\vec{A}_\beta \cdot \vec{K}'_\alpha)(\vec{K}''_\beta \cdot \vec{K}''_{(\beta)})\, A_{i(\alpha)} + 2\,(\vec{A}_\alpha \cdot \vec{K}''_\beta)\ (\vec{K}''_\alpha \cdot \vec{K}'_\alpha)\, A_{i(\beta)}$$

$$_6\gamma_i = (\vec{A}_\alpha \cdot \vec{K}'_\beta)(\vec{K}_\alpha \cdot \vec{K}_\alpha)\, A_{i(\beta)} - (\vec{A}_\alpha \cdot \vec{K}_\beta)(\vec{K}''_\alpha \cdot \vec{K}''_\alpha)\, A_{i(\beta)} + 2\,(\vec{A}_\beta \cdot \vec{K}''_\alpha)\ (\vec{K}''_\beta \cdot \vec{K}'_\beta)\, A_{i(\alpha)}$$

$$_7\gamma_i = (\vec{A}_\alpha \cdot \vec{K}''_\beta)(\vec{K}''_\alpha \cdot \vec{K}''_\alpha)\, A_{i(\beta)} - (\vec{A}_\alpha \cdot \vec{K}''_{(\beta)})\ (\vec{K}'_\alpha \cdot \vec{K}'_\alpha)\, A_{i(\beta)} + (\vec{A}_\beta \cdot \vec{K}''_\alpha)(\vec{K}''_\beta \cdot \vec{K}''_\beta)\, A_{i(\alpha)} - (\vec{A}_\beta \cdot \vec{K}''_\alpha)(\vec{K}'_\beta \cdot \vec{K}'_\beta)\, A_{i(\alpha)}$$

$$_8\gamma_i = 2\,(\vec{A}_\alpha \cdot \vec{K}'_\beta)(\vec{K}''_\alpha \cdot \vec{K}'_\alpha)\, A_{i(\beta)} + 2\,(\vec{A}_\beta \cdot \vec{K}'_\alpha)\ (\vec{K}''_\alpha \cdot \vec{K}'_\alpha)\, A_{i(\alpha)}$$

$$_9\gamma_i = (\vec{K}'_\beta \cdot \vec{K}'_\alpha)(\vec{K}'_\beta \cdot \vec{A}'_\alpha)\, A_{i(\beta)} - (\vec{K}''_\beta \cdot \vec{K}'_\alpha)(\vec{K}''_\beta \cdot \vec{A}_\alpha)\, A_{i(\beta)} - (\vec{K}''_\alpha \cdot \vec{K}''_\beta)(\vec{K}'_\alpha \cdot \vec{A}_\beta)\, A_{i(\alpha)} + (\vec{K}'_\alpha \cdot \vec{K}''_\beta)(\vec{K}''_\alpha \cdot \vec{A}_\beta)\, A_{i(\alpha)}$$

$$_{10}\gamma_i = (\vec{K}'_\alpha \cdot \vec{K}'_\beta)(\vec{K}'_\alpha \cdot \vec{A}_\beta)\, A_{i(\alpha)} - (\vec{K}''_\alpha \cdot \vec{K}'_\beta)(\vec{K}''_\alpha \cdot \vec{A}_\beta)\, A_{i(\alpha)} - [(\vec{K}''_\beta \cdot \vec{K}''_\alpha)(\vec{K}'_\beta \cdot \vec{A}_\alpha)\, A_{i(\beta)} + (\vec{K}'_\beta \cdot \vec{K}''_\alpha)(\vec{K}''_\beta \cdot \vec{A}_\alpha)\, A_{i(\beta)}]$$

$$_{11}\gamma_i = (\vec{K}''_\alpha \cdot \vec{K}''_\beta)(\vec{K}''_\alpha \cdot \vec{A}_\beta)\, A_{i(\alpha)} - (\vec{K}'_\alpha \cdot \vec{K}''_\beta)(\vec{K}'_\alpha \cdot \vec{A}_\beta)\, A_{i(\alpha)} + (\vec{K}''_\beta \cdot \vec{K}''_\alpha)(\vec{K}''_\beta \cdot \vec{A}_\alpha)\, A_{i(\beta)} - (\vec{K}'_\beta \cdot \vec{K}''_\alpha)(\vec{K}'_\beta \cdot \vec{A}_\alpha)\, A_{i(\beta)}$$

$$_{12}\gamma_i = (\vec{K}''_\alpha \cdot \vec{K}'_\beta)(\vec{K}'_\alpha \cdot \vec{A}_\beta)\ A_{\alpha i} + (\vec{K}'_\beta \cdot \vec{K}'_\alpha)(\vec{K}''_\alpha \cdot \vec{A}_\beta)\ A_{i(\alpha)}$$
$$+ (\vec{K}''_\beta \cdot \vec{K}'_\alpha)(\vec{K}'_\beta \cdot \vec{A}_\alpha)\ A_{i(\beta)} + (\vec{K}'_\alpha \cdot \vec{K}'_\beta)(\vec{K}''_\beta \cdot \vec{A}_\alpha)\ A_{i(\beta)}$$

$$_{13}\gamma_i = (\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}'_\beta \cdot \vec{K}'_\alpha)\ K'_{i(\beta)} - (\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}''_\beta \cdot \vec{K}'_\alpha)\ K''_{i(\beta)}$$
$$+ (\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}''_\alpha \cdot \vec{K}''_\beta)\ K'_{i(\alpha)} - (\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}'_\alpha \cdot \vec{K}''_\beta)\ K''_{i(\alpha)}$$

$$_{14}\gamma_i = (\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}'_\alpha \cdot \vec{K}'_\beta)\ K'_{i(\alpha)} - (\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}''_\alpha \cdot \vec{K}'_\beta)\ K''_{i(\alpha)}$$
$$- (\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}''_\beta \cdot \vec{K}''_\alpha)\ K'_{i(\beta)} - (\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}'_\beta \cdot \vec{K}''_\alpha)\ K''_{i(\beta)}$$

$$_{15}\gamma_i = [(K''_\alpha \cdot K''_\beta)\ K''_{i\alpha} - (\vec{K}'_\alpha \cdot \vec{K}''_\beta)\ K'_{i(\beta)} + (\vec{K}''_\beta \cdot \vec{K}''_\alpha)\ K''_{i(\beta)}$$
$$- (\vec{K}'_\beta \cdot \vec{K}''_\alpha)\ K'_{i(\beta)}](\vec{A}_\alpha \cdot \vec{A}_\beta)$$

$$_{16}\gamma_i = (\vec{K}'_\alpha \cdot \vec{K}'_\beta)\ K''_{i(\alpha)} + (\vec{K}''_\alpha \cdot \vec{K}'_\beta)\ K'_{i(\alpha)} + (\vec{K}_\beta \cdot \vec{K}_\alpha)\ K''_{i(\beta)}$$
$$+ (\vec{K}''_\beta \cdot \vec{K}'_\alpha)\ K_{i(\beta)}\ \ (\vec{A}_\alpha \cdot \vec{A}_\beta)$$

$$_{17}\gamma_i = (\vec{K}'_\beta \cdot \vec{A}_\beta)(\vec{K}'_\alpha \cdot \vec{K}'_\beta)\ A_{i(\alpha)} - (\vec{A}_\beta \cdot \vec{K}''_\beta)(\vec{K}''_\beta \cdot \vec{K}'_\alpha)\ A_{i(\alpha)}$$
$$- (\vec{A}_\alpha \cdot \vec{K}''_\alpha)(\vec{K}'_\alpha \cdot \vec{K}''_\beta)\ A_{i(\beta)} + (\vec{A}_\alpha \cdot \vec{K}'_\alpha)(\vec{K}''_\alpha \cdot \vec{K}''_\beta)\ A_{i(\beta)}$$

$$_{18}\gamma_i = (\vec{K}'_\alpha \cdot \vec{A}_\alpha)(\vec{K}'_\alpha \cdot \vec{K}'_\beta)\ A_{i(\beta)} - (\vec{A}_\alpha \cdot \vec{K}''_\alpha)(\vec{K}''_\alpha \cdot \vec{K}'_\beta)\ A_{i(\beta)}$$
$$- (\vec{A}_\beta \cdot \vec{K}''_\beta)(\vec{K}'_\beta \cdot \vec{K}''_\alpha)\ A_{i(\alpha)} + (\vec{A}_\beta \cdot \vec{K}_\beta)(\vec{K}''_\beta \cdot \vec{K}''_\alpha)\ A_{i(\alpha)}$$

$$_{19}\gamma_i = (\vec{A}_\alpha \cdot \vec{K}''_\alpha)(\vec{K}''_\alpha \cdot \vec{K}''_\beta)\ A_{i(\beta)} - (\vec{A}_\alpha \cdot \vec{K}'_\alpha)(\vec{K}'_\alpha \cdot \vec{K}''_\beta)\ A_{i(\beta)}$$
$$+ (\vec{A}_\beta \cdot \vec{K}''_{(\beta)})(\vec{K}''_\beta \cdot \vec{K}''_\alpha)\ A_{i(\alpha)} - (\vec{A}_\beta \cdot \vec{K}'_\beta)(\vec{K}'_\beta \cdot \vec{K}'_\alpha)\ A_{i(\alpha)}$$

$$_{20}\gamma_i = (\vec{A}_\alpha \cdot \vec{K}'_\alpha)(\vec{K}''_\alpha \cdot \vec{K}'_\beta)\ A_{i(\beta)} + (\vec{A}_\alpha \cdot \vec{K}''_\alpha)(\vec{K}'_\alpha \cdot \vec{K}'_\beta)\ A_{i(\beta)}$$
$$+ (\vec{A}_\alpha \cdot \vec{K}'_\alpha)(\vec{K}''_\beta \cdot \vec{K}'_\alpha)\ A_{i(\alpha)} + (\vec{A}_\beta \cdot K''_{(\beta)})(\vec{K}'_\beta \cdot \vec{K}'_\alpha)\ A_{i(\alpha)}$$

$$_{21}\gamma_i = (\vec{K}'_\beta \cdot \vec{K}'_\beta)(\vec{A}_\alpha \cdot \vec{K}'_\alpha)\ A_{i(\beta)} - (\vec{K}''_\beta \cdot \vec{K}''_\beta)(\vec{A}_\alpha \cdot \vec{K}'_\alpha)\ A_{i(\beta)}$$

$$- 2(\vec{K}''_\alpha \cdot \vec{K}'_\alpha)(\vec{A}_\beta \cdot \vec{K}''_\beta)\ A_{i(\alpha)}$$

$$_{22}\gamma_i = (\vec{K}'_\alpha \cdot \vec{K}'_\alpha)(\vec{A}_\beta \cdot \vec{K}'_\beta)\ A_{i(\alpha)} - (\vec{K}''_\alpha \cdot \vec{K}''_\alpha)(\vec{A}_\beta \cdot \vec{K}'_\beta)\ A_{i(\alpha)}$$

$$- 2\ (\vec{K}''_\beta \cdot \vec{K}'_\beta)(\vec{A}_\alpha \cdot \vec{K}''_\alpha)\ A_{i(\beta)}$$

$$_{23}\gamma_i = (\vec{K}''_\alpha \cdot \vec{K}''_\alpha)(\vec{A}_\beta \cdot \vec{K}''_\beta)\ A_{i(\alpha)} - (\vec{K}'_\alpha \cdot \vec{K}'_\alpha)(\vec{A}_\beta \cdot \vec{K}''_\beta)\ A_{i(\alpha)}$$

$$+ (\vec{K}''_\beta \cdot \vec{K}''_\beta)(\vec{A}_\alpha \cdot \vec{K}''_\alpha)\ A_{i(\beta)} - (\vec{K}'_\beta \cdot \vec{K}'_\beta)(\vec{A}_\alpha \cdot \vec{K}''_\alpha)\ A_{i(\beta)}$$

$$_{24}\gamma_i = 2\ (\vec{K}''_\alpha \cdot \vec{K}'_\alpha)(\vec{A}_\beta \cdot \vec{K}'_\beta)\ A_{i(\alpha)} + 2\ (\vec{K}''_\beta \cdot K'_\beta)(A_\alpha \cdot K'_{(\alpha)})\ A_{i(\beta)}$$

$$_{25}\gamma_i = (\vec{A}_\beta \cdot \vec{K}'_\beta)(\vec{A}_\alpha \cdot \vec{K}'_\alpha)\ \vec{K}'_{i(\alpha)} - (\vec{A}_\beta \cdot \vec{K}''_\beta)(\vec{A}_\alpha \cdot \vec{K}''_\alpha)\ \vec{K}'_{i(\alpha)}$$

$$- (\vec{K}'_\alpha \cdot \vec{K}'_\alpha)(\vec{A}_\beta \cdot \vec{K}''_{(\alpha)})\ \vec{K}''_{i(\beta)} - (\vec{A}_\alpha \cdot \vec{K}''_\alpha)(\vec{A}_\beta \cdot \vec{K}'_\alpha)\ K''_{i(\beta)}$$

$$_{26}\gamma_i = (\vec{A}_\alpha \cdot \vec{K}'_\alpha)(\vec{A}_\beta \cdot \vec{K}'_\beta)\ \vec{K}'_\beta - (\vec{A}_\alpha \cdot \vec{K}''_\alpha)(\vec{A}_\beta \cdot \vec{K}''_\beta)\ K'_{i(\beta)}$$

$$- (\vec{K}'_\beta \cdot \vec{K}'_\beta)(\vec{A}_\alpha \cdot \vec{K}''_\beta)\ K''_{i(\alpha)} - (\vec{A}_\beta \cdot \vec{K}''_{(\beta)})(\vec{A}_\alpha \cdot \vec{K}'_\beta)\ K''_{i(\alpha)}$$

$$_{27}\gamma_i = (\vec{A}_\alpha \cdot \vec{K}''_\alpha)(\vec{A}_\beta \cdot \vec{K}''_\beta)\ K'_{i(\beta)} - (\vec{A}_\alpha \cdot \vec{K}'_\alpha)(\vec{A}_\beta \cdot \vec{K}'_\beta)\ K''_{i(\beta)}$$

$$+ (\vec{A}_\beta \cdot \vec{K}''_\beta)(\vec{A}_\alpha \cdot \vec{K}''_\alpha)\ K''_{i(\alpha)} - (\vec{A}_\beta \cdot \vec{K}'_\beta)(\vec{A}_\alpha \cdot \vec{K}'_\alpha)\ K''_{i(\alpha)}$$

$$_{28}\gamma_i = (\vec{A}_\alpha \cdot \vec{K}'_\alpha)(\vec{A}_\beta \cdot \vec{K}''_\alpha)\ K'_{i(\beta)} + (\vec{A}_\alpha \cdot \vec{K}''_\alpha)(\vec{A}_\beta \cdot \vec{K}'_\alpha)\ K_{i(\beta)}$$

$$+ (\vec{A}_\beta \cdot \vec{K}'_\beta)(\vec{A}_\alpha \cdot \vec{K}''_\beta)\ K_{i(\alpha)} + (\vec{A}_\beta \cdot \vec{K}''_\beta)(\vec{A}_\alpha \cdot \vec{K}'_\beta)\ K'_{i(\alpha)}$$

$$_{29}\gamma_i = (\vec{A}_\beta \cdot \vec{K}'_\alpha)(\vec{A}_\alpha \cdot \vec{K}'_\beta)\ K'_{i(\beta)} - (\vec{A}_\beta \cdot \vec{K}'_\alpha)(\vec{A}_\alpha \cdot \vec{K}''_\beta)\ K''_{i(\beta)}$$

$$- (\vec{A}_\beta \cdot \vec{K}''_\alpha)(\vec{A}_\alpha \cdot \vec{K}''_\beta)\ K_{i(\alpha)} + (\vec{A}_\alpha \cdot \vec{K}''_\beta)(\vec{A}_\beta \cdot \vec{K}'_\alpha)\ K''_{i(\alpha)}$$

$$_{30}\gamma_i = (\vec{A}_\alpha \cdot \vec{K}'_\beta)(\vec{A}_\beta \cdot \vec{K}'_\alpha)\ K'_{i(\alpha)} - (\vec{A}_\alpha \cdot \vec{K}'_\beta)(\vec{A}_\beta \cdot \vec{K}''_\alpha)\ K''_{i(\alpha)}$$

$$- (\vec{A}_\alpha \cdot \vec{K}''_\beta)(\vec{A}_\beta \cdot \vec{K}''_\alpha)\ K'_{i(\beta)} + (\vec{A}_\beta \cdot \vec{K}''_\alpha)(\vec{A}_\alpha \cdot \vec{K}'_\beta)\ K''_{i(\beta)}$$

$$_{31}\gamma_i = (\vec{A}_\alpha \cdot \vec{K}''_\beta)(\vec{A}_\beta \cdot \vec{K}''_\alpha)\, K''_{i(\alpha)} - (\vec{A}_\alpha \cdot \vec{K}''_\beta)(\vec{A}_\beta \cdot \vec{K}'_\alpha)\, K'_{i(\alpha)}$$

$$+ (\vec{A}_\beta \cdot \vec{K}''_\alpha)(\vec{A}_\alpha \cdot \vec{K}''_\beta)\, K''_{i(\beta)} - (\vec{A}_\beta \cdot \vec{K}''_\alpha)(\vec{A}_\alpha \cdot \vec{K}'_\beta)\, K'_{i(\beta)}$$

$$_{32}\gamma_i = (\vec{A}_\alpha \cdot \vec{K}'_\beta)(\vec{A}_\beta \cdot \vec{K}''_\alpha)\, K_{i(\alpha)} + (\vec{A}_\alpha \cdot \vec{K}'_\alpha)(\vec{A}_\beta \cdot \vec{K}'_\alpha)\, K''_{i(\alpha)}$$

$$+ (\vec{A}_\beta \cdot \vec{K}'_\alpha)(\vec{A}_\alpha \cdot \vec{K}''_\beta)\, K'_{i(\beta)} + (\vec{A}_\beta \cdot \vec{K}'_\alpha)(\vec{A}_\alpha \cdot \vec{K}'_\beta)\, K''_{i(\beta)}$$

$$_{33}\gamma_i = (\vec{A}_\beta \cdot \vec{K}'_\beta)(\vec{A}_\alpha \cdot \vec{K}'_\alpha)\, K'_{i(\beta)} - (\vec{A}_\beta \cdot \vec{K}''_\beta)(\vec{A}_\alpha \cdot \vec{K}'_\alpha)\, K''_{i(\beta)}$$

$$- (\vec{A}_\alpha \cdot \vec{K}'_\alpha)(\vec{A}_\beta \cdot \vec{K}''_\beta)\, K''_{i(\alpha)} + (\vec{A}_\alpha \cdot \vec{K}''_\alpha)(\vec{A}_\beta \cdot \vec{K}''_\beta)\, K'_{i(\alpha)}$$

$$_{34}\gamma_i = (\vec{A}_\beta \cdot \vec{K}'_\beta)(\vec{A}_\alpha \cdot \vec{K}'_\alpha)\, K'_{i(\beta)} - (\vec{A}_\beta \cdot \vec{K}''_\beta)(\vec{A}_\alpha \cdot \vec{K}'_\alpha)\, K''_{i(\beta)}$$

$$- (\vec{A}_\alpha \cdot \vec{K}'_\alpha)(\vec{A}_\beta \cdot \vec{K}''_\beta)\, K''_{i(\alpha)} + (\vec{A}_\alpha \cdot \vec{K}''_\beta)(\vec{A}_\alpha \cdot \vec{K}''_\alpha)\, K_{i(\beta)}$$

$$_{35}\gamma_i = (\vec{A}_\alpha \cdot \vec{K}''_\alpha)(\vec{A}_\beta \cdot \vec{K}''_\beta)\, K''_{i(\alpha)} - (\vec{A}_\alpha \cdot \vec{K}'_\alpha)(\vec{A}_\beta \cdot \vec{K}''_\beta)\, K'_{i(\alpha)}$$

$$+ (\vec{A}_\beta \cdot \vec{K}''_\beta)(\vec{A}_\alpha \cdot \vec{K}''_\alpha)\, K''_{i(\beta)} - (\vec{A}_\beta \cdot \vec{K}'_\beta)(\vec{A}_\alpha \cdot \vec{K}''_\alpha)\, K'_{(\beta)}$$

$$_{36}\gamma_i = (\vec{A}_\alpha \cdot \vec{K}'_\alpha)(\vec{A}_\beta \cdot \vec{K}'_\beta)\, K''_{i(\alpha)} + (\vec{A}_\alpha \cdot \vec{K}''_\alpha)(\vec{A}_\beta \cdot \vec{K}'_\beta)\, K'_{i(\alpha)}$$

$$+ (\vec{A}_\beta \cdot \vec{K}'_\beta)(\vec{A}_\alpha \cdot \vec{K}'_\alpha)\, K''_{i(\beta)} + (\vec{A}_\beta \cdot \vec{K}''_\beta)(\vec{A}_\alpha \cdot \vec{K}'_\alpha)\, K'_{i(\beta)}$$

2) The Second Order Wave

The second order displacement field can be calculated by using the appropriate Green's function. The equation to be solved can be Fourier transformed and rearranged as follows:

$$[(\partial^2_k + K_T^2)\delta_{ik} - (1 - K_T^2/K_L^2)\partial_i \partial_k]\ \bar{U}_k^{(2)}(\vec{r}) = -\frac{1}{\rho_0 C_T^{\,2}}\ S_i^{(2)}(\vec{r}) \quad ---- \quad (19)$$

Where K_T, K_L and C_T have both real and imaginary parts. If a tensor valued Green's function is defined in such a way that it satisfies the following equation:

$$[(\partial^2{}_k + K_T^2)\ \delta_{ik} - (1 - K_T^2/K_L^2)\partial_i\partial_k]G_{kj}(\vec{r};\vec{r}';\omega) = -\frac{1}{\rho_o C_T{}^2}\delta_{ik}\delta(\vec{r}-\vec{r}') \quad --\ (20)$$

then the displacement field will be given by:

$$U_k^{(2)}(\vec{r}) = \int_V G_{ik}(\vec{r};\vec{r}';\omega)\ S_i^{(2)}(\vec{r}')d\vec{r}' \quad ----------------- (21)$$

The tensor valued Green's function is derived in the appendix and is:

$$G_{ik}(\vec{r};\vec{r}';\omega) + \frac{1}{4\pi\rho_o C_T^2}\quad (\delta_{ik} - \frac{\partial_i\partial_k'}{K_T^2})\ \frac{\exp\ (iK_T'|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}\ \exp\ (-K_T''|\vec{r}-\vec{r}'|)$$

$$+ \frac{1}{4\pi\rho_o C_T^2}\ (\frac{\partial_i\partial_k'}{K_T^2})\ \frac{\exp(iK_L'|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}\exp\ (-K_L''|\vec{r}-\vec{r}'|) \quad ----\ (22)$$

The use of the Green's function as expressed above would be prohibitively difficult because of the amount of algebra necessary to complete the problem; however, the far field approximation of this Green's function is relatively easy to use. This Green's function is then valid for the radiation zone and thus for the remainder of this section, we will be investigating the radiated waves due to interaction.

The far field Green's function is:

$$G_{ik}(\vec{r};\vec{r}';\omega) = \frac{1}{4\pi\rho_o C_T^2 r}\ (\delta_{ik} - \hat{r}_i\hat{r}_k)\exp[iK_T'(r-\hat{r}\cdot\vec{r}')]\exp[-K_T''(r-\hat{r}\cdot\vec{r}')]$$

$$+ \frac{1}{4\pi\rho_o C_L^2 r}\ (\hat{r}_i\hat{r}_k)\exp[iK_L'(r-\hat{r}\cdot\vec{r}')]\exp[-K_L''(r-\hat{r}\cdot\vec{r}')] \quad ----\ (23)$$

After substitution of the Fourier transformed source term into Eq. (21), and performing the operations indicated by the Green's function of Eq. (23) and inverse time transformation of the result, the solution to the inhomogeneous equation can be found to be:

$$U_k^{(2)}(\vec{r},t) = \sum_{\alpha\pm\beta} \frac{V_{k\ (\alpha\pm\beta)}^{(2)} - (V_{i\ (\alpha\pm\beta)}^{(2)}\hat{r}_i)\hat{r}_k}{8\pi^2\ \rho_o\ C_{T(\alpha\pm\beta)}^2 r} \{$$

$$\int_{V^1} \sin(\vec{M}_T\cdot\vec{r}')d^3r'\ \cos[(\omega_\alpha\pm\omega_\beta)t-(K'_{T(\alpha\pm\beta)} + iK''_{T(\alpha\pm\beta)})r]$$

$$+ \int_{V^1} \cos(\vec{M}_T\cdot\vec{r}')d^3r'\ \sin[(\omega_\alpha\pm\omega_\beta)t - (K'_{T(\alpha\pm\beta)} + iK''_{T(\alpha\pm\beta)})r]\}$$

$$+ \sum_{\alpha\pm\beta} \frac{(V_{i(\alpha\pm\beta)}\hat{r}_i)\ \hat{r}_k}{8\pi^2\rho_o\ C_{L(\alpha\pm\beta)}^2 r} \{$$

$$\int_{V^1} \sin(\vec{M}_L\cdot\vec{r}')d^3r'\ \cos[(\omega_\alpha\pm\omega_\beta)t - (K'_{L(\alpha\pm\beta)} + iK''_{L(\alpha\pm\beta)})r]$$

$$+ \int_{V^1} \cos(\vec{M}_L\cdot\vec{r}')d^3r'\ \sin[(\omega_\alpha\pm\omega_\beta)t - (K'_{L(\alpha\pm\beta)} + iK''_{L(\alpha\pm\beta)})\ r]\}$$

$$+ \sum_{\alpha\pm\beta} \frac{W_{k(\alpha\pm\beta)}^{(2)} - (W_{i(\alpha\pm\beta)}\cdot r_i)\hat{r}_k}{8\pi^2\rho_o\ \ C_{T(\alpha\pm\beta)}^2 r} \{$$

$$\int_{V^1} \cos(\vec{N}_r\cdot\vec{r}')d^3r'\ \cos[(\omega_\alpha\pm\omega_\beta)t - (K'_{T(\alpha\pm\beta)} + iK''_{T(\alpha\pm\beta)})\ r]$$

$$- \int_{V^1} \sin(\vec{N}_T\cdot\vec{r}')d^3r'\ \sin\ [(\omega_\alpha\pm\omega_\beta)t - (K'_{T(\alpha\pm\beta)} + iK''_{T(\alpha\pm\beta)})\ r]\}$$

$$+ \sum_{\alpha\pm\beta} \frac{(W_{i(\alpha\pm\beta)}^{(2)}\hat{r}_i)\hat{r}_k}{8\pi^2\rho_o C_{L(\alpha\pm\beta)}^2 r} \{$$

$$\int_{V^1} \cos(\vec{N}_L\cdot\vec{r}')d^3r'\ \cos[(\omega_\alpha\pm\omega_\beta)t - (K'_{L(\alpha\pm\beta)} + iK''_{L(\alpha\pm\beta)})\ r]$$

$$- \int_{V^1} \sin(\vec{N}_L\cdot\vec{r}')d^3r'\ \sin[(\omega_\alpha\pm\omega_\beta)t - (K'_{L(\alpha\pm\beta)} + iK''_{L(\alpha\pm\beta)})\ r]\} \quad ---- \quad (24)$$

where:

$$\vec{M}_T = (K'_{T(\alpha\pm\beta)} + iK''_{T(\alpha\pm\beta)})\ \hat{r} - [(\vec{K}'_\alpha \pm \vec{K}'_\beta) + i\ (\vec{K}''_\alpha \pm \vec{K}''_\beta)]$$

$$\vec{M}_L = (K'_{L(\alpha\pm\beta)} + i\ K''_{L(\alpha\pm\beta)})\ \hat{r} - [(\vec{K}_{(\alpha)} \pm\ \vec{K}'_{(\beta)}) + i\ (\vec{K}''_\alpha \pm \vec{K}''_\beta)]$$

$$\vec{N}_T = (K'_{T(\alpha\pm\beta)} + i\, K''_{T(\alpha\pm\beta)})\ \hat{r} - [(\vec{K}'_{(\alpha)} \pm \vec{K}'_{(\beta)}) + i\ (\vec{K}''_{\alpha} \pm \vec{K}''_{\beta})]$$

$$\vec{N}_L = (K'_{L(\alpha\pm\beta)} + i\, K''_{L(\alpha\pm\beta)})\ \hat{r} - [(\vec{K}'_{(\alpha)} \pm \vec{K}'_{(\beta)}) + i\ (\vec{K}''_{\alpha} \pm \vec{K}''_{\beta})]$$

The second order displacement can be thought of as the real part of the above expression. Although Eq. (24) does not explicitly express the second order displacement, investigation of this expression will show several of the interesting characteristics of the displacement. First the functional dependence of the displacement of the wave is of the form:

$$\frac{{}^{\sin}_{\cos}[(\omega_\alpha \pm \omega_\beta)t - (K'_{(\alpha\pm\beta)})r]\}\{{}^{i\ \sinh}_{\cosh}[-K''_{(\alpha\pm\beta)}r]\}}{r}$$

This is a radiated wave of frequency $(\omega_\alpha \pm \omega_\beta)$ propagating with the speed $C_{(\alpha\pm\beta)} = \frac{\omega_\alpha \pm \omega_\beta}{K'_{(\alpha\pm\beta)}}$ and attenuating at the rate of $K''_{(\alpha\pm\beta)}$. The first and third term in Eq. (24) are transverse waves and the second and fourth terms are longitudinal waves. Also notice that for a given interaction mode, the first two terms will increase (or decrease) the spectrum and the last two terms will decrease (or increase) the spectrum respectively. In general, all of the terms will have significant results for both normal mode interactions $(\omega_\alpha \pm \omega_\beta)$ and flipped mode interactions $(\omega_\alpha - \omega_\beta)$. The polarization of the second order wave is determined by the projection operators. These vectors do not have unit magnitude and therefore will contribute to the amplitude of the wave for a given mode. The amplitude of the wave is also very much dependent on the volume of the integration. For instance consider the following.

$$\int_{V'} {}^{\sin}_{\cos}[\vec{M},\vec{N}\cdot\vec{r}\,']\ d^3r'$$

This integration is the most physical part of the calculation and it is also the most difficult. The volume V' is the interaction volume of the primary waves, hence, this section is appropriately named "Volume interaction of Viscoelastic Waves". Because it is extremely difficult to integrate this term, the entire problem is avoided by considering resonant interaction of the waves. For resonance, we have

$$|\vec{M}| = |\vec{N}| = 0$$

for some specific case and the entire integral expression reduces to V'. Although the analysis is simplified in cases of resonance, the solution is still not very physical because one is left with the problem of specifying the interaction volume V'. It is because the Green's function was formulated in spherical geometry that we say V' must be a sphere but the question remains as to the size of that sphere.

3) Resonant Interaction of Elastic Waves

The unphysicalness of this problem and the amount of algebra involved precludes the solution of volume interaction of viscoelastic waves at this time; however, the volume interaction of purely elastic waves will be carried to its conclusion. This will give an indication of the type of analysis that needs to be done and the results that can be expected.

The above mentioned equations can be simplified to the elastic case by letting $K''_{T,L} = 0$. In this case the solution to the inhomogeneous equation becomes the displacement because no imaginary terms remain. This second order displacement takes the following form:

$$U_k^{(2)}(\vec{r},t) = \sum_{\alpha\pm\beta} \frac{S^{(2)}_{i(\alpha\pm\beta)}\hat{r}_i)\hat{r}_k}{8\pi^2\rho_o C_L^2 r} \int_V \sin[(\frac{\omega_\alpha\pm\omega_\beta}{\;}\hat{r} - (\vec{K}_\alpha\pm\vec{K}_\beta))\cdot\vec{r}\,' + (\omega_\alpha\pm\omega_\beta)(t-r/C_L)]\,d^3r$$

$$+ \frac{S^{(2)}_{k(\alpha\pm\beta)} - (S^{(2)}_{i(\alpha\pm\beta)} r_i) r_k}{8\pi^2\,\rho_o\,C_T^2\,r} \int_V \sin\,[(\frac{\omega_\alpha\pm\omega_\beta}{C_T}\hat{r} - (K_\alpha\pm K_\beta))\cdot\vec{r}\,' + (\omega_\alpha\pm\omega_\beta)(t-r/C_T)]\,d^3r$$

where:

$$\begin{aligned}
S^{(2)}_{i(\alpha\pm\beta)} = &-\tfrac{1}{2}(\mu + A/4)\,[(\vec{A}_\alpha\cdot\vec{A}_\beta)(\vec{K}_\beta\cdot\vec{K}_\beta)K_{i(\alpha)}\;\;(\vec{A}_\alpha\cdot\vec{A}_\beta)(\vec{K}_\alpha\cdot\vec{K}_\alpha)\;K_{i(\beta)} \\
&+ (\vec{A}_\beta\cdot\vec{K}_\alpha)(\vec{K}_\beta\cdot\vec{K}_\beta)\;A_{i(\alpha)} \pm (\vec{A}_\alpha\cdot\vec{K}_\beta)(\vec{K}_\alpha\cdot\vec{K}_\alpha)\;A_{i(\beta)} \\
&+ 2\,(\vec{A}_\alpha\cdot\vec{K}_\beta)(\vec{K}_\alpha\cdot\vec{K}_\beta)\;A_{i(\beta)} \pm 2\,(\vec{A}_\beta\cdot\vec{K}_\alpha)\;(\vec{K}_\alpha\cdot\vec{K}_\beta)\;A_{i(\alpha)}] \\
&- \tfrac{1}{2}\,(\lambda + \mu + A/4 + B)\,[(\vec{A}_\alpha\cdot\vec{A}_\beta)(\vec{K}_\alpha\cdot\vec{K}_\beta)\;K_{i(\beta)} \pm (\vec{A}_\alpha\cdot\vec{A}_\beta)(\vec{K}_\alpha\cdot\vec{K}_\beta)\;K_{i(\alpha)} \\
&\qquad + (\vec{A}_\beta\cdot\vec{K}_\beta)(\vec{K}_\alpha\cdot\vec{K}_\beta)\;A_{i(\alpha)} \pm (\vec{A}_\alpha\cdot\vec{K}_\alpha)(\vec{K}_\alpha\cdot\vec{K}_\beta)\;A_{i(\beta)}] \\
&- \tfrac{1}{2}(A/4 + B)\,[(\vec{A}_\alpha\cdot\vec{K}_\beta)(\vec{A}_\beta\cdot\vec{K}_\beta)\;K_{i(\alpha)} \pm (\vec{A}_\alpha\cdot\vec{K}_\alpha)(\vec{A}_\beta\cdot\vec{K}_\alpha)\;K_{i(\beta)} \\
&\qquad + (\vec{A}_\alpha\cdot\vec{K}_\beta)(\vec{A}_\beta\cdot\vec{K}_\alpha)\;K_{i(\beta)} \pm (\vec{A}_\alpha\cdot\vec{K}_\beta)(\vec{A}_\beta\cdot\vec{K}_\alpha)\;K_{i(\alpha)}] \\
&- \tfrac{1}{2}\,(B + 2C)\,[(\vec{A}_\alpha\cdot\vec{K}_\alpha)(\vec{A}_\beta\cdot\vec{K}_\beta)\;K_{i(\beta)} \pm (\vec{A}_\alpha\cdot\vec{K}_\alpha)(\vec{A}_\beta\cdot\vec{K}_\beta)\;K_{i(\alpha)}] \\
&- \tfrac{1}{2}\,(\lambda + B)\;\;[(\vec{K}_\beta\cdot\vec{K}_\beta)(\vec{A}_\alpha\cdot\vec{K}_\alpha)\;A_{i(\beta)} \pm (\vec{K}_\alpha\cdot\vec{K}_\alpha)(\vec{A}_\beta\cdot\vec{K}_\beta)\;A_{i(\alpha)}]
\end{aligned}$$

As mentioned before, if $\vec{r}$ is such that the coefficient of $\vec{r}\,'$ is zero, the volume integration of Eq. (25a) would reduce to the volume of interaction multiplied by a harmonic term. If $\vec{r} = \vec{r}_s$ meets this condition and the wave generated is due to resonant interaction of the primary waves. Here the conditions for resonance will be investigated and the radiated wave for three types of interaction will be studied. The three types of interaction to be studied are:

1. interaction of two distinct transverse waves (T - T)

2. interaction of two distinct longitudinal waves (L - L)

3. interaction of a longitudinal and a transverse wave (L - T).

For distinct waves the resonance conditions for the two expressions in the displacement equation are:

$$(\alpha\pm\beta)\ \text{LONGITUDINAL} \qquad \frac{\omega_\alpha\pm\omega_\beta}{C_L}\,\hat{r}_S - (\vec{K}_\alpha \pm \vec{K}_\beta) = 0$$

$$(\alpha\pm\beta)\ \text{TRANSVERSE} \qquad \frac{\omega_\alpha\pm\omega_\beta}{C_L}\,\hat{r}_S - (\vec{K}_\alpha \pm \vec{K}_\beta) = 0$$

By squaring the above conditions and by denoting the angle between $\vec{K}_\alpha$ and $\vec{K}_\beta$ as ψ, one can obtain the constraints on ψ (or $\cos\psi$) for resonance to occur. Because, $-1 \geq \cos\psi \geq 1$, limits can be established for the frequencies of the primary waves. The frequency limits, cos , and direction of radiation caused by resonance are tabulated for (T - T), (L - L) and (L - T) interaction in Tables 1, 2, and 3.

The individual types of interaction will next be considered

a) Transverse-Transverse Interaction

In this case, the primary waves will be two distinct transverse waves with frequencies that satisfy the resonance conditions. First we will consider colinearly propagating, orthogonally polarized waves. The following characterizes this relationship: $\vec{K}_\alpha \cdot \vec{A}_\alpha = 0$, $\vec{K}_\beta \cdot \vec{A}_\beta = 0$, $\vec{K}_\beta \cdot \vec{A}_\alpha = 0$, $\vec{K}_\alpha \cdot \vec{A}_\beta = 0$ and $\vec{A}_\alpha \cdot \vec{A}_\beta = 0$. When one examines the $S_i^{(2)}$ vector expressed in Eq. (25b) one observes that it is equal to zero, and we therefore conclude that no interaction takes place. For colinearly propagating, non-orthogonally polarized transverse waves, the following characterizes the relationship: $\vec{K}_\alpha \cdot \vec{A}_\alpha = 0$, $\vec{K}_\beta \cdot \vec{A}_\beta = 0$, $\vec{K}_\alpha \cdot \vec{A}_\alpha = 0$, $\vec{K}_\beta \cdot \vec{A}_\alpha = 0$ then the source vector becomes:

TABLE 1: (T-T) interaction

wave	frequency limit	cos ψ	f_s
$(\alpha+\beta)L$	$2\frac{C_L^2+C_T^2}{C_L^2-C_T^2} \le (\frac{\omega_\alpha}{\omega_\beta}+\frac{\omega_\beta}{\omega_\alpha}) \le -2$	$\frac{1}{2}(\frac{C_T^2}{C_L^2}-1)(\frac{\omega_\alpha}{\omega_\beta}+\frac{\omega_\beta}{\omega_\alpha})+\frac{C_T^2}{C_L^2}$	$\frac{\omega_\alpha\hat{K}_\alpha+\omega_\beta\hat{K}_\beta}{(\omega_\alpha^2+\omega_\beta^2)^{1/2}}$
$(\alpha-\beta)L$	$-2\frac{C_L^2+C_T^2}{C_L^2-C_T^2} \le (\frac{\omega_\alpha}{\omega_\beta}+\frac{\omega_\beta}{\omega_\alpha}) \le 2$	$\frac{1}{2}(1-C_T^2/C_L^2)(\frac{\omega_\alpha}{\omega_\beta}+\frac{\omega_\beta}{\omega_\alpha})+\frac{C_T^2}{C_L^2}$	$\frac{\omega_\alpha\hat{K}_\alpha+\omega_\beta\hat{K}_\beta}{(\omega_\alpha^2+\omega_\beta^2)^{1/2}}$
$(\alpha\pm\beta)T$	$-\infty < \frac{\omega_\alpha}{\omega_\beta} < \infty$	1	$\frac{\omega_\alpha\hat{K}_\alpha\pm\omega_\beta\hat{K}_\beta}{(\omega_\alpha^2+\omega_\beta^2)^{1/2}}$

TABLE 2: (L-L) interaction

wave	frequency limit	cos ψ	f_s
$(\alpha\pm\beta)L$	$-\infty < \frac{\omega_\alpha}{\omega_\beta} < \infty$	1	$\frac{\omega_\alpha\hat{K}_\alpha\pm\omega_\beta\hat{K}_\beta}{(\omega_\alpha^2+\omega_\beta^2)^{1/2}}$
$(\alpha+\beta)T$	$-2\frac{C_T^2+C_L^2}{C_L^2-C_T^2} \le (\frac{\omega_\alpha}{\omega_\beta}+\frac{\omega_\beta}{\omega_\alpha}) \le 2$	$\frac{1}{2}(\frac{\omega_\alpha}{\omega_\beta}+\frac{\omega_\beta}{\omega_\alpha})(\frac{C_L^2}{C_T^2}-1)+\frac{C_L^2}{C_T^2}$	$\frac{\omega_\alpha\hat{K}_\alpha+\omega_\beta\hat{K}_\beta}{(\omega_\alpha^2+\omega_\beta^2)^{1/2}}$
$(\alpha-\beta)T$	$2\frac{C_T^2+C_L^2}{C_L^2-C_T^2} \le (\frac{\omega_\alpha}{\omega_\beta}+\frac{\omega_\beta}{\omega_\alpha}) \le 2$	$\frac{1}{2}(\frac{\omega_\alpha}{\omega_\beta}+\frac{\omega_\beta}{\omega_\alpha})(1-C_L^2/C_T^2)+\frac{C_L^2}{C_T^2}$	$\frac{\omega_\alpha\hat{K}_\alpha-\omega_\beta\hat{K}_\beta}{(\omega_\alpha^2+\omega_\beta^2)^{1/2}}$

TABLE 3: (L-T) interaction

wave	frequency limit	cos ψ	f_s
$(\alpha\pm\beta)L$	$-2\frac{C_T^2+C_TC_L}{\mp C_L^2\pm C_T^2} \le \frac{\omega_\beta}{\omega_\alpha} \le 2\frac{C_TC_L-C_T^2}{\mp C_L^2\pm C_T^2}$	$\frac{1}{2}(\mp\frac{C_L}{C_T}\pm\frac{C_T}{C_L})\frac{\omega_\beta}{\omega_\alpha}+\frac{C_T}{C_L}$	$\frac{\omega_\alpha\hat{K}_\alpha\pm\frac{C_L}{C_T}\omega_\beta\hat{K}_\beta}{[\omega_\alpha^2+(\frac{C_L}{C_T}\omega_\beta)^2]^{1/2}}$
$(\alpha\pm\beta)T$	$-2\frac{C_TC_L\mp C_L^2}{\pm C_L^2\mp C_T^2} \le \frac{\omega_\alpha}{\omega_\beta} \le 2\frac{C_LC_T-C_L^2}{\pm C_L^2\mp C_T^2}$	$\frac{1}{2}(\pm\frac{C_L}{C_T}\mp\frac{C_T}{C_L})\frac{\omega_\alpha}{\omega_\beta}+\frac{C_L}{C_T}$	$\frac{\frac{C_T}{C_L}\omega_\alpha\hat{K}_\alpha\pm\omega_\beta\hat{K}_\beta}{[(\frac{C_T}{C_L}\omega_\alpha)^2+\omega_\beta^2]^{1/2}}$

$$S^{(2)}_{i(\alpha\pm\beta)} = -1/2(\mu + A/4)(\vec{A}_\alpha \cdot \vec{A}_\beta)[|\vec{K}_\beta|^2 K_{i(\alpha)} \pm |\vec{K}_\alpha|^2 K_{i(\beta)}]$$
$$- 1/2(\lambda + \mu + A/4 + B)(\vec{A}_\alpha \cdot \vec{A}_\beta)(\vec{K}_\alpha \cdot \vec{K}_\beta)[K_{i(\beta)} \pm K_{i(\alpha)}] \quad --- (21)$$

By examining the table, we can see that there is no frequency limit for the transverse wave, and that the direction of radiation is colinear with the propagation vectors. Now let us investigate the projection vector

$$S^{(2)}_{k(\alpha\pm\beta)} - (S^{(2)}_{i(\alpha\pm\beta)}\, \hat{r}_i)\, \hat{r}_k$$

Because $\hat{r}_i$ and $\hat{r}_k$ are components of unit vectors, the above expression is zero. Because of this we conclude that a transverse wave does not radiate due to resonant interaction of colinearly propagating transverse waves. This result is not physical and will be considered again. We can also conclude that a longitudinal wave will not radiate when the interaction takes place in a medium where $C_L > C_T$, because there is no frequency ratio that will satisfy the resonance condition.

Noncolinear, orthogonally polarized transverse-transverse interaction is next investigated. The relationship between the primary waves is characterized by: $\vec{K}_\alpha \cdot \vec{A}_\alpha = 0$, $\vec{A}_\alpha \cdot \vec{A}_\beta = 0$, $\vec{K}_\beta \cdot \vec{A}_\beta = 0$. Then:

$$S^{(2)}_{i(\alpha\pm\beta)} = -1/2(\mu + A/4)\{[(\vec{A}_\beta \cdot \vec{K}_\alpha)|\vec{K}_\beta|^2 A_{i(\alpha)} \pm (\vec{A}_\alpha \cdot \vec{K}_\beta)|\vec{K}_\alpha|^2 A_{i(\beta)}]$$
$$+ 2(\vec{A}_\alpha \cdot \vec{K}_\beta)(\vec{K}_\alpha \cdot \vec{K}_\beta)\, A_{i(\beta)} \pm 2(\vec{A}_\beta \cdot \vec{K}_\alpha)(\vec{K}_\alpha \cdot \vec{K}_\beta)\, A_{i(\alpha)}\}$$
$$- \frac{1}{2}(\lambda + \mu + A/4 + B)[(\vec{A}_\beta \cdot \vec{K}_\beta)(\vec{K}_\alpha \cdot \vec{K}_\beta)\, A_{i(\alpha)} + (\vec{A}_\alpha \cdot \vec{K}_\alpha)(\vec{K}_\alpha \cdot \vec{K}_\beta)\, A_{i(\alpha)}]$$
$$- \frac{1}{2}(A/4 + B)[(\vec{A}_\alpha \cdot \vec{K}_\beta)(\vec{A}_\beta \cdot \vec{K}_\alpha)\, K_{i(\beta)} \pm (\vec{A}_\alpha \cdot \vec{K}_\beta)(\vec{A}_\beta \cdot \vec{K}_\alpha)\, K_{i(\alpha)}]$$
$$---- (28)$$

For noncolinear propagation, the transverse waves cannot excite a second order transverse wave and for a material where $C_L > C_T$, the $(\alpha+\beta)$ second order longitudinal wave cannot exist. Therefore the displacement due to resonant interaction in this case is the following longitudinal wave.

$$U^{(2)}_{k(\alpha-\beta)}(\vec{r},t) = \frac{(S^{(2)}_{i(\alpha-\beta)}\hat{r}_i)\hat{r}_k\ V'}{8\pi^2 \rho_o C_L^2 r} \sin[(\omega_\alpha - \omega_\beta)(t - r/C_L)] \quad \text{- - - - (29)}$$

If the primary waves had the same frequency, then there would be no second order displacement. For nonorthogonal polarizations, the above expression is valid, that is, the $(\alpha-\beta)$ longitudinal wave will be radiated; however, the $S^{(2)}_{k(\alpha-\beta)}$ expression will be as expressed in Eq. (19) with $\vec{K}_\alpha \cdot \vec{A}_\alpha = 0$, $\vec{K}_\beta \cdot \vec{A}_\beta = 0$.

b) Longitudinal-Longitudinal interaction.

First the colinear propagation of primary waves will be considered. In this case, the $S^{(2)}_{k(\alpha\pm\beta)}$ vector remains the same as expressed in Eq. (19), with the exception of some of the vector products being easily written as scalars. Note that this vector is colinear with the propagation (and polarization) vectors of the primary waves. By examination of Table 2, one observes that a longitudinal wave will radiate for any frequency ratio, and that the direction of resonance is colinear with the direction of propagation. The displacement is:

$$U^{(2)}_{k(\alpha\pm\beta)}(\vec{r},t) = \frac{(S^{(2)}_{i(\alpha\pm\beta)}\hat{r}_i)\hat{r}_k\ V'}{8\pi^2 \rho_o C_L^2\ r} \sin[(\omega_\alpha \pm \omega_\beta)(t - r/C_L)] \quad \text{- - - - (30)}$$

For a media where $C_L > C_T$, there is no possible frequency ratio for resonance of the $(\alpha-\beta)$ transverse wave and the frequency ratio for the $(\alpha+\beta)$ transverse wave is -1. This is not physical so we conclude that no transverse wave can be generated by colinear longitudinal-longitudinal wave interaction. For oblique interaction of the primary waves, one observes that there is no longitudinal wave radiated due to resonant interaction; however, if the frequency limits of the $(\alpha+\beta)$ transverse wave are met, the radiated displacement will be:

$$U^{(2)}_{k(\alpha+\beta)}(r,t) = \frac{S^{(2)}_{k(\alpha+\beta)} - (S^{(2)}_{i(\alpha+\beta)}\hat{r}_i)\hat{r}_k \; V'}{8\pi^2 \rho_o C_T^2 \, r} \sin[(\omega_\alpha + \omega_\beta)(t-r/C_T)] \; - - - \; (31)$$

This can be shown to be equal to zero for orthogonal propagation by letting $\cos \phi = 0$ in the resonance conditions. The vectors $\hat{r}_k$ and $S^{(2)}_{k(\alpha+\beta)}$ are coplaner with the propagation vectors; therefore, the displacement and direction of propagation of the $(\alpha+\beta)$ transverse wave will also be coplaner with this plane of interaction.

c) Longitudinal-Transverse Interaction

Colinear propagation of the primary wave ($\vec{A}_\beta \cdot \vec{K}_\beta = 0$, $\vec{A}_\beta \cdot \vec{K}_\alpha = 0$, $\vec{A}_\alpha \cdot \vec{A}_\beta = 0$) will be considered first. Here the α wave is longitudinal and the β wave is transverse. In this case the source vector becomes:

$$\begin{aligned} S^{(2)}_{i(\alpha\pm\beta)} = -\,1/2\; \{ &(\mu + A/4)\,[2(\vec{A}_\alpha \cdot \vec{K}_\alpha)(\vec{K}_\alpha \cdot \vec{K}_\beta) \pm (\vec{A}_\alpha \cdot \vec{K}_\beta)(\vec{K}_\beta \cdot \vec{K}_\alpha)] \\ &+ (\lambda + \mu + A/4 + B)(\vec{A}_\alpha \cdot \vec{K}_\beta)(\vec{K}_\alpha \cdot \vec{K}_\beta) \\ &+ (\lambda + B)(\vec{A}_\alpha \cdot \vec{K}_\alpha)|\vec{K}_\beta|^2 \}\, A_{i(\beta)} \; - - - - - - - - - - \; (32) \end{aligned}$$

Upon examination of Table 3, one sees that there is no frequency ratio that can satisfy the resonance condition for the $(\alpha+\beta)$ longitudinal and the $(\alpha-\beta)$ transverse second order waves. The $(\alpha-\beta)$ longitudinal and the $(\alpha+\beta)$ transverse waves have possible frequency ratios, and therefore, resonant radiation of these waves are possible. The direction of radiation for each wave is colinear with the axis of the propagation of the primary waves. The source vector is colinear with the direction of the polarization vector $A_{i(\beta)}$. Then

$$S^{(2)}_{i(\alpha\pm\beta)}\hat{r}_i = 0$$

and we conclude that the only second order displacement due to longitudinal-transverse interaction is the transverse wave:

$$U^{(2)}_{k(\alpha+\beta)}(\vec{r},t) = \frac{S^{(2)}_{k(\alpha+\beta)}\ V'}{8\pi^2 \rho_o C_T^2\ r} \sin[(\omega_\alpha + \omega_\beta)(t - r/C_T)] \qquad (33)$$

For oblique interaction of the primary waves, the vector has components both coplaner with the plane of the propagation vectors and perpendicular to this plane. Therefore, the $(\alpha-\beta)$ longitudinal wave will be excited as well as the $(\alpha+\beta)$ transverse wave mentioned above. Thus it can be concluded that a normal mode transverse wave and a flipped mode longitudinal wave will be generated when a longitudinal and a transverse wave interact obliquely.

Before discussing the results found above, interaction of visco-elastic plane waves will be considered. Once this is done, the certain inadequacies of volume interaction will become evident.

CHAPTER V

INTERACTION OF VISCOELASTIC PLANE WAVES

In this chapter, the interaction of viscoelastic waves that are propagating colinearly will be considered. Also for convenience these waves will be assumed to be propagating in the same direction and have the same phase at the origin ($x = t = o$). Nothing is lost by assuming these two conditions and the analysis can easily be redone to include either or both of the excluded conditions. Also it is important to note that these interaction cases cannot be derived from the previous volume interaction cases for reasons that will become evident later. There are several distinct types of interaction to be considered when studying the interaction of viscoelastic plane waves. They are:

1. Interaction of transverse and longitudinal waves
2. Interaction of two transverse waves
3. Interaction of two longitudinal waves
4. Self interaction of a transverse wave
5. Self interaction of a longitudinal wave

Let us consider plane waves propagating in the x direction. In this case u_x is a longitudinal wave displacement field and u_y and u_z are two orthogonally polarized transverse waves. If $u_i(r,t) = u_{x,y,z}(x,t)$ is substituted into Eq. (I-37), the resultant equations will be:

$$\rho_o\partial_t^2u_x-(\zeta + \eta/3)\partial_t\partial_x^2u_x - (\lambda + 2\mu)\ \partial_x^2u_x =$$

$$[3(\lambda + 2\mu) + 2A + 6B + 2C]\ \partial_x^2u_x\partial_xu_x$$

$$+ [(\lambda + 2\mu) + A/2 + B][\partial_x^2u_y\partial_xu_y + \partial_x^2u_z\partial_xu_z] \quad --- \quad (1)$$

$$\rho_o \partial_t^2 u_y - \eta \partial_t \partial_x^2 u_y - \mu\, \partial_x^2 u_y =$$

$$[(\lambda + 2\mu) + A/2 + B][\partial_x^2 u_y \cdot \partial_x u_x + \partial_x^2 u_x \cdot \partial_x u_y] \quad ------ \quad (2)$$

$$\rho_o \partial_t^2 u_z - \eta \partial_t \partial_x^2 u_z - \eta\, \partial_x^2 u_z =$$

$$[(\lambda + 2\mu) + A/2 + B][\partial_x^2 u_z \cdot \partial_x u_x + \partial_x^2 u_x \cdot \partial_x u_z] \quad ------ \quad (3)$$

The above three equations could have been written as one equation as was done in the preceding chapter, however, with the equations written in this manner, it is easy to see what primary waves will interact to form the various types of second order waves. Also the scalar Green's function can be used to solve these equations whereas the tensor valued Green's function was used to solve the vector equation of the last chapter.

1) Longitudinal and Transverse wave interaction

In this section, the interaction of a longitudinal and a transverse wave will be considered. The equations to be solved are equations (1) and (3), where the polarization of the transverse wave was arbitrarily chosen to be in the z direction. The same equation governs the transverse wave polarized along the y axis and by choosing either polarization, the problem is completely solved as equations 2 and 3 show, orthogonally polarized transverse waves propagate independently of each other. Investigation of the equations will show that perhaps two second order waves will result when longitudinal and transverse waves interact. They are a transverse wave and a longitudinal wave; however, the interaction that causes these waves is distinctly different. The transverse wave is generated by the interaction of the primary longitudinal and transverse waves. Later it will be shown that this interaction does not occur. The second order wave is generated by the sum of the self interaction of the two primary waves. That is the transverse primary wave interacts with itself to form a second order longitudinal wave and the primary longitudinal wave does the same.

Self interaction of waves will be considered in a later section; however, the second order displacement field will be calculated here in a slightly different manner.

As in the last section, the primary waves will be represented by the real parts of the solution to the linear equations. The linear equations in this case are the scalar wave equations:

$$\rho_o \partial_t^2 u_x^{(1)} - (\zeta + \eta/3)\, \partial_t \partial_x^2 u_x^{(1)} - (\lambda + 2\mu)\, \partial_x^2 u_x^{(1)} = 0 \quad \text{------} \quad (4)$$

$$\rho_o \partial_t^2 u_z^{(1)} - \eta\, \partial_t \partial_x^2 u_z^{(1)} - \mu\, \partial_x^2 u_z^{(1)} = 0 \quad \text{-------------} \quad (5)$$

The real part of the solution to these equations are:

$$u_x^{(1)} = |\vec{A}_\alpha| \cos[\omega_\alpha t - K'_{L(\alpha)} x] \exp[-K''_{L(\alpha)}\, x] \quad \text{-----------} \quad (6)$$

$$u_z^{(1)} = |\vec{A}_\beta| \cos[\omega_\beta t - K'_{T(\beta)} x] \exp[-K''_{T(\beta)}\, x] \quad \text{-----------} \quad (7)$$

After substitution of the primary waves into the source terms and after Fourier transformation, the inhomogeneous equations governing the second order displacements can be found to be:

$$(\partial_x^2 + K_L^2)\, \bar{U}_k^{(2)} = \bar{S}_x^{(2)} \quad \text{---------} \quad (8)$$

$$(\partial_x^2 + K_T^2)\, \bar{U}_z^{(2)} = \bar{S}_z^{(2)} \quad \text{---------} \quad (9)$$

where in $\bar{S}_x^{(2)}$ both waves are longitudinal.

$$\bar{S}_x^{(2)} = -\frac{3(\lambda + 2\mu) + 2(A + 3B + C)}{4\pi\ \rho_o\ \omega^2} K_L^2\ A_\alpha A_\alpha\ (K'^{\,3}_{L(\alpha)} + 3\ K''^{\,2}_{L(\alpha)}\ K'_{L(\alpha)})$$

$$\frac{1}{2_i}[\exp[-2K'_{L(\alpha)}x]\ \ \delta(\omega + 2\omega_\alpha) - \exp[2K'_{L(\alpha)}x]\ \ \delta(\omega - 2\omega_\alpha)]\exp[-2K''_{L(\alpha)}x]$$

$$+ \frac{2\ K'^{\,2}_{L(\alpha)}\ K''_{L(\alpha)}}{2}[\exp[-2K'_{L(\alpha)}x]\ \ \delta(\omega+2\omega_\alpha) + \exp[2\ K'_{L(\alpha)}x]\ \ \delta(\omega-2\omega_\alpha)]\exp[-2K''_{L(\alpha)}]$$

$$- \frac{(\lambda + 2\mu) + A/2 + B}{4\pi\,\rho_o\ \omega^2} K_L^2 A_\beta A_\beta\ \ (K'^{\,3}_{T(\beta)} + 3K''^{\,2}_{T(\beta)}\ K'_{T(\beta)})$$

$$\frac{1}{2_i}[\exp[-2K'_{T(\beta)}x]\ \ \delta(\omega + 2\omega_\beta) - \exp[2K'_{T(\beta)}x]\ \ \delta(\omega - 2\omega_\beta)]\exp[-2K''_{T(\beta)}x]$$

$$+ \frac{2\ K'^{\,2}_{T(\beta)}\ K''_{T(\beta)}}{2}[\exp[-2K'_{T(\beta)}x]\ \ \delta(\omega + 2\omega_\beta) + \exp\ [2K'_{T(\beta)}x]\ \ \delta(\omega - 2\omega_\beta)\exp[-2K''_{T(\beta)}]$$

For the z equation a longitudinal and a transverse wave that are propagating colinearly are the components of the source term. These two waves will always have a zero resultant for the source term because $A_\alpha \cdot A_\beta = 0$. Thus the only second order wave generated will be the longitudinal wave, and this wave will be due to the self interaction of the primary waves. The solution to the inhomogeneous equation can be found by using the scalar Green's function:

$$\bar{G}_L(x,x') = \begin{cases} -\ \dfrac{i}{2}\ \dfrac{\exp[iK_L(x - x')]}{K_L} & x>x' \\ -\ \dfrac{i}{2}\ \dfrac{\exp[-iK_L(x - x')]}{K_L} & x<x' \end{cases} \qquad - - - - (11)$$

Then the second order displacement will be given by the real part of the solution of the inhomogeneous equation

$$U_x^{(2)}(x,t) = \mathrm{RE}\int_0^{\ell}\int_{-\infty}^{\infty} \exp(-i\omega t)\ \bar{G}_L(x,x')\ \bar{S}_x^{(2)}(x')\ d\omega dx' \qquad ------ (12)$$

where ℓ is the interaction length or the distance through which the wave interacted. This length would in general be the field variable x if the waves were propagating in a homogeneously nonlinear media and also if the primary waves were generated at the origin. There is one interesting and physical case where the field variable is not equal to the interaction length ℓ and that is as follows. If the primary waves (or wave) were generated at the origin with sufficient amplitude ($A_{\alpha,\beta}$) and that there was nonlinear interaction, either self interaction or mixed interaction, a second order wave would be generated up to a point $x = \ell$, at which time the primary wave or waves have been damped to such an extent that the argument of second order smallness becomes valid and the second order wave generated after the point $x = \ell$ becomes insignificant. In this case the second order wave would be "radiated" along the axis of propagation of the primary waves until it also damps to an insignificant amplitude. The case where the field variable x is equal to ℓ is derivable from the results obtained by the integration of Eq. (13) and will be discussed further for the cases of self interaction of waves.

The cases involving second order wave propagation in nonlinearly inhomogeneous media will not be solved here; however, the methods for solving these problems are similar to those considered and these problems are at least conceptually no more difficult to solve. For instance, if the medium of propagation was characterized by a strip of nonlinear material confined between the end points (a,b) the second order displacement could be solved for by changing the limits of integration of the variable x' in Eq. (12) to (a,b) and by letting the Lamé parameters λ and μ be zero where they appear explicitly in the source term.

As mentioned before, the second order displacement is due in the case of longitudinal-transverse interaction only to the self interaction of the primary waves. Because self interaction will be considered later, the result in this case will be written down with only the following explanation.

The second order displacement is the real part of the solution to the inhomogeneous equation. If Equations (10), (11), and (12) are combined, the real part of the solution can be found to be:

$$U_x^{(2)}(x,t) = \frac{3(\lambda + 2\mu) + 2(A + 3B + C)}{16\pi\ \rho_o\ (2\omega_\alpha)^2} A_\alpha A_\alpha \{$$

$$- \frac{\nabla_1}{\nabla_3} \{[\cos M_1 \cos P_1 - \sin M_1 \sin P_1] \exp(N_1) \exp(Q_1) - \cos M_1 \exp(N_1)\}$$

$$+ \frac{\nabla_2}{\nabla_3} \{[\sin M_1 \cos P_1 + \cos M_1 \sin P_1] \exp(N_1) \exp(Q_1) - \sin(M_1) \exp(N_1)\}$$

$$- \frac{\nabla_4}{\nabla_6} \{[\cos M_1 \cos P_1 - \sin M_1 \sin P_1] \exp(-N_1) \exp(-Q_1') - \cos M_1 \exp(N_1)\}$$

$$- \frac{\nabla_5}{\nabla_6} \{[\sin M_1 \cos P_1 + \cos M_1 \sin P_1] \exp(-N_1) \exp(-Q_1') + \sin M_1 \exp(-N_1)\}\}$$

$$+ \frac{(\lambda + 2\mu) + A/2 + B}{16\pi\ \rho_o\ (2\omega_\beta)^2} A_\beta A_\beta \{$$

$$- \frac{\nabla_7}{\nabla_9} \{[\cos M_2 \cos P_2 - \sin M_2 \sin P_2] \exp(N_2) \exp(Q_2) - \cos M_2 \exp(N_2)\}$$

$$+ \frac{\nabla_8}{\nabla_9} \{[\sin M_2 \cos P_2 + \cos M_2 \sin P_2] \exp(N_2) \exp(Q_2) - \sin M_2 \exp(N_2)\}$$

$$\frac{\nabla_{10}}{\nabla_{12}} \{[\cos M_2 \cos P_2 + \sin M_2 \sin P_2] \exp(-N_2) \exp(-Q_2') + \cos M_2 \exp(-N_2)\}$$

$$- \frac{\nabla_{11}}{\nabla_{12}} \{[\sin M_2 \cos P_2 + \cos M_2 \sin P_2] \exp(-N_2) \exp(-Q_2') + \sin M_2 \exp(-N_2]\}\}$$

$$\text{- - - - - - - - - - - (13)}$$

$$M_1 = 2\omega_\alpha t - 2\gamma' K'_{L(\alpha)} x$$

$$N_1 = + 2\delta' \, K''_{L(\alpha)} x$$

$$P_1 = 2 \, (\gamma' - 1) K'_{L(\alpha)} \, \ell$$

$$Q_1 = + 2 \, (\delta' - 1) K''_{L(\alpha)} \ell$$

$$Q'_1 = + 2 \, (\delta' + 1) K''_{L(\alpha)} \ell$$

$$M_2 = 2\omega_\beta t - 2 \, \gamma'' K'_{L(\beta)} x$$

$$N_2 = 2\delta'' \, K''_{L(\beta)} x$$

$$P_2 = 2 \, (\gamma'' K'_{L(\beta)} - K'_{T(\beta)}) \ell$$

$$Q_2 = 2 \, (\delta K''_{L(\beta)} - K''_{T(\beta)}) \ell$$

$$Q'_2 = 2 \, (\delta K''_{L(\beta)} + K''_{T(\beta)}) \ell$$

$$\begin{matrix} \gamma' \\ \delta' \end{matrix} = \frac{(1 + \omega_\alpha^2 \, A^2)^{1/2} \, [(1 + 4\omega_\alpha^2 A^2)^{1/2} [\pm] \, 1]}{(1 + 4\omega_\alpha^2 A \,)^{1/2} \, [(1 + \omega_\alpha^2 A^2)^{1/2} [\pm] \, 1]}$$

γ'' and δ'' are as above with ω_α replaced by ω_β.

$$\Delta_1 = [\delta(\gamma-1)\ K''_{L(\alpha)}\ K'_{L(\alpha)} + \gamma(\delta+1)\ K'_{L(\alpha)}\ K''_{L(\alpha)}]\Gamma_1$$

$$- [\delta(\delta+1)\ K''^{2}_{L(\alpha)} - \gamma(\delta-1)\ K'_{L(\alpha)}\ K''_{L(\alpha)}]\Gamma_2$$

$$\Delta_2 = [\delta(\delta+1)\ K''^{2}_{L(\alpha)} - \gamma(\gamma-1)\ K'^{2}_{L(\alpha)}]\Gamma_1$$

$$- [\delta(\gamma-1)\ K''_{L(\alpha)}K'_{L(\alpha)} + \gamma(\delta+1)\ K'_{L(\alpha)}K''_{L(\alpha)}]\Gamma_2$$

$$\Delta_3 = [(\gamma-1)\ K'_{L(\alpha)}]^2 + [(\delta+1)\ K''_{L(\alpha)}]^2$$

$$\Delta_4 = [\delta(\gamma-1)\ K''_{L(\alpha)}K'_{L(\alpha)} + \gamma(\delta-1)\ K'_{L(\alpha)}K''_{L(\alpha)}]\Gamma_1$$

$$- [\delta(\gamma-1)\ K''_{L(\alpha)}K'_{L(\alpha)} - \gamma(\gamma-1)\ K'^{2}_{L}]\Gamma_2$$

$$\Delta_5 = [\delta(\gamma-1)\ K''_{L(\alpha)}K'_{L(\alpha)} - \gamma(\gamma-1)\ K'_{L(\alpha)}\ K'_{L(\alpha)}]\Gamma_1$$

$$- [\delta(\gamma-1)\ K''_{L(\alpha)}K'_{L(\alpha)} + \gamma(\delta-1)\ K'_{L(\alpha)}K''_{L(\alpha)}]\Gamma_2$$

$$\Delta_6 = [(\gamma-1)\ K'_{L(\alpha)}]^2 + [(\delta-1)\ K''_{L(\alpha)}]^2$$

$$\Delta_7 = [\delta''K''_{L(\beta)}\ (\gamma''K'_{L(\beta)} - K'_{T(\beta)}) + \gamma''K'_{L(\beta)}(\delta''K''_{L(\beta)} + K''_{T(\beta)})]\Gamma_3$$

$$- [\delta''K''_{L(\beta)}(\delta''K''_{L(\beta)} + K''_{T(\beta)}) - \gamma''K''_{L(\beta)}(\gamma''K'_{L(\beta)} - K''_{T(\beta)})]\Gamma_4$$

$$\Delta_8 = [\delta''K''_{L(\beta)}\ (\delta''K''_{L(\beta)} + K''_{T(\beta)}) - \gamma''K'_{L(\beta)}\ (\gamma''K'_{L(\beta)} - K'_{T(\beta)})]\Gamma_3$$

$$- [\delta''K''_{L(\beta)}\ (\gamma K'_{L(\beta)} - K'_{T(\beta)}) + \gamma''K'_{L(\beta)}\ (\delta''K''_{L(\beta)} + K'_{T(\beta)}]\Gamma_4$$

$$\Delta_9 = [\gamma''K'_{L(\beta)} - K'_{T(\beta)}]^2 + [\delta K''_{L(\beta)} + K'_{T(\beta)}]^2$$

$$\Delta_{10} = [\delta''K''_{L(\beta)}\ (\gamma''K'_{L(\beta)} - K'_{T(\beta)}) + \gamma''K'_{L(\beta)}\ (\delta''K''_{L(\beta)} - K'_{T(\beta)})]\Gamma_3$$

$$- [\delta''K''_{L(\beta)}\ (\delta''K''_{L(\beta)} - K'_{T(\beta)}) - \gamma''K'_{L(\beta)}\ (\gamma''K'_{L(\beta)} - K'_{T(\beta)})]\Gamma_4$$

$$\Delta_{11} = [\gamma''K'_{L(\beta)}\ (\delta''K''_{L(\beta)} - K''_{T(\beta)}) - \gamma''K'_{L(\beta)}\ (\gamma''K'_{L(\beta)} - K'_{T(\ )})]\Gamma_3$$

$$- [\delta''K''_{L(\beta)}\ (\gamma''K'_{L(\beta)} - K'_{T(\beta)}) + \gamma K'_{L(\beta)}\ (\delta K''_{L(\beta)} - K'_{T(\beta)})]\Gamma_4$$

$$\Delta_{12} = [\delta''K'_{L(\beta)} - K'_{T(\beta)}]^2 + [\delta K''_{L(\beta)} - K''_{T(\beta)}]^2$$

and where:

$$\Gamma_1 = K'^{\,3}_{L(\alpha)} + 3\ K''^{\,2}_{L(\alpha)}\ K'_{L(\alpha)}$$

$$\Gamma_2 = K'^{\,2}_{L(\alpha)}\ K''_{L(\alpha)}$$

$$\Gamma_3 = K'^{\,3}_{T(\beta)} + 3\ K''^{\,2}_{T(\beta)}\ K'_{T(\beta)}$$

$$\Gamma_4 = K'^{\,2}_{T(\beta)}\ K''_{T(\beta)}$$

Now the nature of the second order wave can be investigated. The terms whose argument is M show that the wave oscillates with frequency $2\omega_\alpha$ or $2\omega_\beta$ and propagates with a speed that a viscoelastic wave of this frequency would. Also as the terms with N as the argument shows, the wave dissipates at a rate indicative of a wave of this frequency. The terms with P and Q as arguments show the functional dependence of the wave interaction. It is interesting to note that the wave will increase in amplitude in an oscillatory manner, then decrease in the same manner. In this case, the P terms modulate the wave and the Q terms dissipate the wave. Investigation of the P and Q terms of the contribution for the transverse wave will show that the amplitude of the second order wave is dependent on the difference between the magnitudes of the propagation (and dissipation) vectors of the longitudinal and transverse waves. This is a very interesting and physical phenomenon and will be discussed in the next section.

2) Interaction of Longitudinal Waves

The interaction of two distinct longitudinal waves will next be considered. As seen in the last section, a second order longitudinal wave will be formed when these waves interact. The equation to be solved is:

$$\rho_o \partial_t^2 U_x - (\zeta + \eta/3)\ \partial_t \partial_x^2 U_x - (\lambda + 2\mu)\ \partial_x^2 U_x =$$

$$[3(\lambda + 2\mu) + 2(A + 3B + C)]\ \partial_x^2 U_x \partial_x U_x \quad \text{--------- (14)}$$

The primary waves can be written in the form:

$$U_x^{(1)} = A_\alpha \cos[\omega_\alpha t - K'_{L(\alpha)} x] \exp[-K''_{L(\alpha)} x]$$

$$+ A_\beta \cos[\omega_\beta t - K'_{L(\beta)} x] \exp[-K''_{L(\beta)} x] \quad \text{-------- (15)}$$

After substitution of the primary waves into the perturbed equation of motion, and after Forier transformation, the equation to be solved for the second order displacement can be shown to be:

$$(\partial_x^2 + K_L^2)\ \bar{U}_x^{(2)} = \bar{S}_x^{(2)} \qquad \text{- - - - - - - - - - - - - - - (16)}$$

where:

$$\bar{S}_x^{(2)} = \sum_{\alpha\pm\beta} - \frac{3(\lambda+2\mu) + 2(A+3B+C)}{4\pi\rho_o\ \omega^2}\ K_L^2\ A_\alpha A_\beta \{$$

$$- \frac{\Gamma_5 \pm \Gamma_6}{2i}\ [\exp[-(K'_{L(\alpha)} \pm K'_{L(\beta)})x]\,\delta(\omega + (\omega_\alpha \pm \omega_\beta))$$

$$- \exp[-(K'_{L(\alpha)} \pm K'_{L(\beta)})x]\,\delta(\omega - (\omega_\alpha \pm \omega_\beta))]$$

$$+ \frac{\Gamma_7 \pm \Gamma_8}{2}\ [\exp[-(K'_{L(\alpha)} \pm K'_{L(\beta)})x]\,\delta(\omega + (\omega_\alpha \pm \omega_\beta))$$

$$- \exp[-(K'_{L(\alpha)} \pm K'_{L(\beta)})x]\,\delta(\omega - (\omega_\alpha \pm \omega_\beta))] \cdot$$

$$\cdot \exp[-(K''_{L(\alpha)} + K''_{L(\beta)})x]\} \qquad \text{- - - - (17)}$$

where:

$$\Gamma_5 = K'^{\,2}_{L(\alpha)}\ K'_{L(\beta)} + K''^{\,2}_{L(\alpha)}\ K'_{L(\beta)} + 2\ K'_{L(\beta)}\ K''_{L(\beta)}\ K''_{L(\alpha)}$$

$$\Gamma_6 = K'^{\,2}_{L(\beta)}\ K'_{L(\alpha)} + K''^{\,2}_{L(\beta)}\ K'_{L(\alpha)} + 2\ K'_{L(\alpha)}\ K''_{L(\alpha)}\ K''_{L(\beta)}$$

$$\Gamma_7 = K'^{\,2}_{L(\beta)}\ K''_{L(\alpha)} + K''^{\,2}_{L(\beta)}\ K''_{L(\alpha)} - (K'^{\,2}_{L(\alpha)}\ K''_{L(\beta)} + K''^{\,2}_{L(\alpha)}\ K''_{L(\beta)})$$

$$\Gamma_8 = 2\ K'_{L(\alpha)}\ K''_{L(\alpha)}\ K'_{L(\beta)} + 2\ K'_{L(\beta)}\ K''_{L(\beta)}\ K'_{L(\alpha)}$$

As before the second order displacement is found by using the Green's function expressed in Eq. (11). Then:

$$U_x^{(2)}(x,t) = \text{RE} \int_0^{\ell}\int_{-\infty}^{\infty} \exp(-i\omega t)\ \bar{G}_L(x,x')\ \bar{S}_x^{(2)}(x')\,d\omega dx' \qquad \text{- - - - - - - (18)}$$

By performing the indicated operations, the second order displacement will be found to be:

$$U_x^{(2)}(x,t) = \sum_{\alpha\pm\beta} \frac{3(\lambda + 2\mu) + 2(A + 3B + C)}{16\pi\rho_0\ (\omega_\alpha \pm \omega_\beta)^2}\ A_\alpha A_\beta \{$$

$$- \frac{\Delta_{13}}{\Delta_{15}} \{[\cos M_3 \cos P_3 - \sin M_3 \sin P_3] \exp(N_3) \exp(Q_3) - \cos M_3 \exp(N_3)\}$$

$$+ \frac{\Delta_{14}}{\Delta_{15}} \{[\sin M_3 \cos P_3 + \cos M_3 \sin P_3] \exp(N_3) \exp(Q_3) - \sin M_3 \exp(N_3)\}$$

$$\frac{\Delta_{16}}{\Delta_{18}} \{[\cos M_3 \cos P_3 - \sin M_3 \sin P_3] \exp(-N_3) \exp(-Q_3') - \cos M_3 \exp(-N_3)\}$$

$$\frac{\Delta_{17}}{\Delta_{18}} \{[\sin M_3 \cos P_3 + \cos M_3 \sin P_3] \exp(-N_3) \exp(-Q_3') - \sin M_3 \exp(-N_3)\}\}$$

where:

$$\Delta_{13} = [K''_{L(\alpha\pm\beta)} (K'_{L(\alpha\pm\beta)} - (K'_{L(\alpha)} \pm K'_{L(\beta)})) + K'_{L(\alpha\pm\beta)} (K''_{L(\alpha\pm\beta)} + (K''_{L(\alpha)} + K''_{L(\beta)}] (\Gamma_5 \pm \Gamma_6)$$

$$- [K''_{L(\alpha\pm\beta)} (K''_{L(\alpha\pm\beta)} + (K''_{L(\alpha)} + K''_{L(\beta)})) - K'_{L(\alpha\pm\beta)} (K''_{L(\alpha\pm\beta)} + (K'_{L(\alpha)}\ K'_{L(\beta)}))] (\Gamma_7 \pm \Gamma_8)$$

$$\Delta_{14} = [K''_{L(\alpha\pm\beta)} (K''_{L(\alpha\pm\beta)} + (K''_{L(\alpha)} + K''_{L(\beta)})) - K'_{L(\alpha\pm\beta)} (K'_{L(\alpha\pm\beta)} - (K'_{L(\alpha)} \pm K'_{L(\beta)}))] (\Gamma_5 \pm \Gamma_6)$$

$$- [K''_{L(\alpha\pm\beta)} (K'_{L(\alpha\pm\beta)} - (K'_{L(\alpha)} \pm K'_{L(\beta)})) + K'_{L(\alpha\pm\beta)} (K''_{L(\alpha\pm\beta)} + (K''_{L(\alpha)} + K''_{L(\beta)}))] (\Gamma_7 \pm \Gamma_8)$$

$$\Delta_{15} = [K'_{L(\alpha\pm\beta)} - (K'_{L(\alpha)} \pm K'_{L(\beta)})]^2 + [K''_{L(\alpha\pm\beta)} + (K''_{L(\alpha)} + K''_{L(\beta)})]^2$$

$$\Delta_{16} = [K''_{L(\alpha\pm\beta)} (K'_{L(\alpha\pm\beta)} - (K'_{L(\alpha)} \pm K'_{L(\beta)})) + K'_{L(\alpha\pm\beta)} (K''_{L(\alpha\pm\beta)} - (K''_{L(\alpha)} + K''_{L(\beta)}))] (\Gamma_5 \pm \Gamma_6)$$

$$- [K''_{L(\alpha\pm\beta)} (K'_{L(\alpha\pm\beta)} - (K'_{L(\alpha)} \pm K'_{L(\beta)}) - K'_{L(\alpha\pm\beta)} (K'_{L(\alpha\pm\beta)} - (K'_{L(\alpha)} \pm K'_{L(\beta)}))] (\Gamma_7 \pm \Gamma_8)$$

$$\Delta_{17} = [K''_{L(\alpha\pm\beta)}(K'_{L(\alpha\pm\beta)}-(K'_{L(\alpha)}\pm K'_{L(\beta)}))-K'_{L(\alpha\pm\beta)}(K'_{L(\alpha\pm\beta)}-(K'_{L(\alpha)}\;K'_{L(\beta)}))](\Gamma_5\pm\Gamma_6)$$

$$-\ [K''_{L(\alpha\pm\beta)}(K'_{L(\alpha\pm\beta)}-(K'_{L(\alpha)}\pm K'_{L(\beta)}))+K'_{L(\alpha\pm\beta)}(K''_{L(\alpha\pm\beta)}-(K''_{L(\alpha)}+K''_{L(\beta)}))](\Gamma_7\pm\Gamma_8)$$

$$\Delta_{18} = [K'_{L(\alpha\pm\beta)}-(K'_{L(\alpha)}\pm K'_{L(\beta)})]^2 + [K''_{L(\alpha\pm\beta)}-(K''_{L(\alpha)}+K''_{L(\beta)})]^2$$

and where:

$$M_3 = (\omega_\alpha\pm\omega_\beta)t - K'_{L(\alpha\pm\beta)}x$$

$$N_3 = K''_{L(\alpha\pm\beta)}x$$

$$P_3 = [K'_{L(\alpha\pm\beta)} - (K'_{L(\alpha)} \pm K'_{L(\beta)})]\ell$$

$$Q_3 = [-K''_{L(\alpha\pm\beta)} - (K''_{L(\alpha)} + K''_{L(\beta)})]\ell$$

$$Q_3' = [-K''_{L(\alpha\pm\beta)} + (K''_{L(\alpha)} + K''_{L(\beta)})]\ell$$

Thus we see that two second order longitudinal waves are generated when two primary longitudinal waves interact. The second order waves have the frequencies of $(\omega_\alpha + \omega_\beta)$ and $(\omega_\alpha - \omega_\beta)$. Because of the frequency dependence of dispersion and dissipation, the wave of frequency $(\omega_\alpha - \omega_\beta)$ is generally regarded as the important product of this type of interaction. These two second order waves, when added to the primary waves will tend to alter the shape of the primary waves. Under some conditions, the shape of the total displacement field approaches that of a shock wave.

3) Interaction of Two Transverse Waves

Previously it was shown that when two primary longitudinal waves interact, a second order longitudinal wave will be generated. In this paragraph, it will be shown that a similar longitudinal wave will be generated when two transverse waves interact.

The equations to be solved are:

$$\rho_o \partial_t^2 U_x^{(2)} - (\zeta + \eta/3)\ \partial_t \partial_x^2 U_x^{(2)} - (\lambda + 2\mu)\ \partial_x^2 U_x^{(2)} =$$

$$[(\lambda + 2\mu) + A/2 + B][\partial_x^2 U_z^{(1)} \partial_x U_z^{(1)} + \partial_x^2 U_y^{(1)} \partial_x U_y^{(1)}] \quad (20)$$

and

$$\rho_o \partial_t^2\ U_{y,z}^{(1)} - \eta\ \partial_t\ \partial_x^2\ U_{y,z}^{(1)} - \mu\ \partial_x^2\ U_{y,z}^{(1)} = 0 \quad (21)$$

The primary waves could be written in the form:

$$U_{y,z}^{(1)}(x,t) = (A_{y(\alpha)}, A_{z(\alpha)})\ \cos\ (\omega_\alpha t - K'_{T(\alpha)}x)\ \exp\ (-K''_{T(\alpha)}x)$$

$$+ (A_{y(\beta)}, A_{z(\beta)})\ \cos\ (\omega_\beta t - K'_{T(\beta)}x)\ \exp(-K''_{T(\beta)}x) \quad (22)$$

and substituted into the source term of the inhomogeneous equation; however, it is easier and more instructive to write the source term of Eq. (20) as:

$$[(\lambda + 2\mu) + A/2 + B]\ \partial_x^2 \vec{U}_T^{(1)} \cdot \partial_x \vec{U}_T^{(1)} \quad (23)$$

where $\vec{U}_T^{(1)}$ is the primary transverse displacement:

$$\vec{U}_T^{(1)} = \vec{A}_{(\alpha)}\ \cos[\omega_\alpha t - K'_{T(\alpha)}x]\exp[-K''_{T(\alpha)}x] + \vec{A}_{(\beta)}\ \cos[\omega_\beta t - K'_{T(\beta)}x]\exp[-K''_{T(\beta)}x] \quad (24)$$

After the source term of Eq. (21) is replaced by Eq. (23) and the primary displacement substituted into it, the Fourier transform of the equation to be solved will take the form:

$$(\partial_x^2 + K_L^2)\ \tilde{U}_x^{(2)}(x) = \tilde{S}_x^{(2)}(x) \quad (25)$$

where:

$$\bar{S}_x^{(2)}\ (x) = \sum_{\alpha\pm\beta} - \frac{(\lambda + 2\mu) + A/2 + B}{4\pi\rho_o\ \omega^2}\ \vec{A}_\alpha \cdot \vec{A}_\beta\ \ K_L^2\ \{$$

$$(\Gamma_9 \pm \Gamma_{10})\ \frac{1}{2_i}\ \{\ \exp[-(K'_{T(\alpha)} \pm\ \ K'_{T(\beta)})]\delta[\omega + (\omega_\alpha \pm \omega_\beta)]$$

$$- \exp[+(K'_{T(\alpha)}\ \ K'_{T(\beta)})x]\delta(\omega-(\omega_\alpha \pm \omega_\beta))]\exp[-(K''_{T(\alpha)} + K''_{T(\beta)})x$$

$$+ (\Gamma_{11} \pm \Gamma_{12})\ \frac{1}{2}\ [\exp[-K'_{T(\alpha)}\ \pm K'_{T(\beta)})x]\delta[\omega + (\omega_\alpha \pm \omega_\beta)]$$

$$+ \exp[(K'_{T(\alpha)} \pm K'_{T(\beta)})x]\delta(\omega-(\omega_\alpha \pm \omega_\beta))]\ \ \exp[-(K''_{T(\alpha)} + K''_{T(\beta)})x \qquad (25)$$

where:

$$\Gamma_9 = K'^2_{T(\alpha)}\ K'_{T(\beta)} + K''^2_{T(\alpha)}\ K'_{T(\beta)} + 2K'_{T(\beta)}\ K''_{T(\beta)}\ K''_{T(\alpha)}$$

$$\Gamma_{10} = K'^2_{T(\beta)}\ K'_{T(\alpha)} + K''^2_{T(\beta)}\ K'_{T(\alpha)} + 2K'_{T(\alpha)}\ K''_{T(\alpha)}\ K''_{T(\beta)}$$

$$\Gamma_{11} = K'^2_{T(\beta)}\ K''_{T(\alpha)} + K''^2_{T(\beta)}\ K''_{T(\alpha)} - (K'^2_{T(\alpha)}\ K''_{T(\beta)} + K''^2_{T(\alpha)}\ K''_{T(\beta)})$$

$$\Gamma_{12} = 2\ K'_{T(\alpha)}\ K''_{T(\alpha)}\ K'_{T(\beta)} + 2K'_{T(\beta)}\ K''_{T(\beta)}\ K'_{T(\alpha)}$$

The second order wave will be given by:

$$U_x^{(2)}(x,t) = RE \int_{-\infty}^{\infty}\int_0^{\ell} \exp(-i\omega t)\ \bar{G}_L(x,x')\ \bar{S}_x^{(2)}(x')\ dx'd\omega \qquad (26)$$

After performing the indicated operations the displacement is:

$$U_x^{(2)}(x,t) = \sum_{\alpha\pm\beta} \frac{\lambda + 2\mu + A/2 + B}{16\pi\ \rho_o(\omega_\alpha \pm \omega_\beta)}\ \ \vec{A}_\alpha \cdot \vec{A}_\beta\ \{$$

$$- \frac{\Delta_{19}}{\Delta_{21}} \{[\cos M_4 \cos P_4 - \sin M_4 \sin P_4] \exp(N_4) \exp(Q_4) - \cos M_4 \exp(N_4)\}$$

$$+ \frac{\Delta_{20}}{\Delta_{21}} \{[\sin M_4 \cos P_4 + \cos M_4 \sin P_4] \exp(N_4) \exp(Q_4) - \sin M_4 \exp N_4\}$$

$$- \frac{\Delta_{22}}{\Delta_{24}} \{[\cos M_4 \cos P_4 - \sin M_4 \sin P_4] \exp(-N_4) \exp(-Q_4') - \cos M_4 \exp(-N_4)\}$$

$$- \frac{\Delta_{23}}{\Delta_{24}} \{[\sin M_4 \cos P_4 + \cos M_4 \sin P_4] \exp(-N_4) \exp(-Q_4') - \sin M_4 \exp(-N_4)\}$$

where:

$$M_4 = (\omega_\alpha \pm \omega_\beta)t - K'_{L(\alpha\pm\beta)}x$$

$$N_4 = K''_{L(\alpha\pm\beta)}x$$

$$P_4 = [K'_{L(\alpha\pm\beta)} - (K'_{T(\alpha)} \pm K'_{T(\beta)})]\ell$$

$$Q_4 = [-K''_{L(\alpha\pm\beta)} - (K''_{T(\alpha)} + K''_{T(\beta)})]\ell$$

$$Q_4' = [-K''_{L(\alpha\pm\beta)} + (K''_{T(\alpha)} + K''_{T(\beta)}]\ \ell$$

$$\Delta_{19} = [K''_{L(\alpha\pm\beta)}(K'_{L(\alpha\pm\beta)} - (K'_{T(\alpha)}\ K'_{T(\beta)})) + K'_{L(\alpha\pm\beta)}(K''_{L(\alpha\pm\beta)} + (K''_{T(\alpha)} + K''_{T(\beta)}))](\Gamma_9 \pm \Gamma_{10})$$

$$-[K''_{L(\alpha\pm\beta)}(K''_{L(\alpha\pm\beta)} + (K''_{T(\alpha)} + K''_{T(\beta)})) - K''_{L(\alpha\pm\beta)}(K'_{L(\alpha\pm\beta)} - (K'_{T(\alpha)}\ K'_{T(\beta)}))](\Gamma_{11} \pm \Gamma_{12})$$

$$\Delta_{20} = [K''_{L(\alpha\pm\beta)}(K''_{L(\alpha\pm\beta)} + (K''_{T(\alpha)} + (K''_{T(\beta)})) - K'_{T(\alpha\pm\beta)}(K'_{L(\alpha\pm\beta)} - (K'_{T(\alpha)}\ K'_{T(\beta)}))](\Gamma_9 \pm \Gamma_{10})$$

$$-[K''_{L(\alpha\pm\beta)}(K'_{L(\alpha\pm\beta)} - (K'_{T(\alpha)}\ K'_{T(\beta)})) + K'_{L(\alpha\pm\beta)}(K''_{L(\alpha\pm\beta)} + (K''_{T(\alpha)} + K''_{T(\beta)}))](\Gamma_{11} \pm \Gamma_{12})$$

$$\Delta_{21} = [K'_{L(\alpha\pm\beta)} - (K'_{T(\alpha)} \pm K'_{T(\beta)})]^2 + [K''_{L(\alpha\pm\beta)} + (K''_{T(\alpha)} + K''_{T(\beta)})]^2$$

$$\Delta_{22} = [K''_{L(\alpha\pm\beta)}(K'_{L(\alpha\pm\beta)}-(K'_{T(\alpha)}\ K'_{T(\beta)})+K'_{L(\alpha\pm\beta)}(K''_{L(\alpha\pm\beta)}-(K_{T(\alpha)}+K_{T(\beta)}))](\Gamma_9\pm\Gamma_{10})$$

$$-[K''_{L(\alpha\pm\beta)}(K''_{L(\alpha\pm\beta)}-(K''_{T(\alpha)}+K''_{T(\beta)})-K'_{L(\alpha\pm\beta)}(K'_{L(\alpha\pm\beta)}-(K'_{T(\alpha)}\pm K'_{T(\beta)}))](\Gamma_{11}\pm\Gamma_{12})$$

$$\Delta_{23} = [K'_{L(\alpha\pm\beta)}(K''_{L(\alpha\pm\beta)}-(K''_{T(\alpha)}-K''_{T(\beta)}))-K'_{L(\alpha\pm\beta)}(K'_{L(\alpha\pm\beta)}-(K'_{T(\alpha)}\pm K'_{T(\beta)}))](\Gamma_9\pm\Gamma_{10})$$

$$-\ [K''_{L(\alpha\pm\beta)}(K'_{L(\alpha\pm\beta)}-(K'_{T(\alpha)}\pm K'_{T(\beta)}))+K'_{L(\alpha\pm\beta)}(K''_{L(\alpha\pm\beta)}-(K''_{T(\alpha)}+K''_{T(\beta)}))](\Gamma_{11}\pm\Gamma_{12})$$

$$\Delta_{24} = [K'_{L(\alpha\pm\beta)}-(K'_{T(\alpha)}\pm K'_{T(\beta)})]^2+\ [K''_{L(\alpha\pm\beta)}-(K''_{T(\alpha)}+K''_{T(\beta)})]^2$$

4) Self Interaction of Viscoelastic Transverse Waves

Self interaction of a transverse wave will be considered next. The result of a transverse-transverse wave interaction is the generation of a longitudinal second order wave. Also, the second order wave will be a normal mode wave with the flipped mode wave being zero in the case of self interaction. Unlike the previous analysis, the interaction length will be specified as the entire length of the field variable x. That is the second order displacement will be given by:

$$U_x^{(2)}(x,t) = RE \int_0^X\int_{-\infty}^{\infty} \exp(-i\omega t)\ \bar{G}_L(x,x')\ \bar{S}_x^{(2)}(x')\ d\omega dx' \text{ - - - - - - } (27)$$

The equation to be solved for the primary wave is:

$$\rho_o\, \partial_t^2 U_z^{(1)} - \eta\ \partial_t \partial_x^2 U_z^{(1)}\ -\ \mu\ \partial_x^2\ U_z^{(1)} = 0 \text{ - - - - - - - - - - - - - - - - } (28)$$

The real part of the solution of this equation is:

$$U_z^{(1)}(x,t) = A_\beta \cos(\omega_\beta t - K'_{T(\beta)}x)\ \exp(-K''_{T(\beta)}x) \text{ - - - - - - - - - - - - } (29)$$

Here the polarization was arbitrarily chosen to be in the z direction. The inhomogeneous equation whose solution will yield the second order displacement is:

$$\rho_o \partial_t^2 U_x^{(2)} - (\zeta + \eta/3)\, \partial_t \partial_x^2 U_x^{(2)} - (\lambda + 2\mu)\, \partial_x^2 U_x^{(2)} =$$

$$[(\lambda + 2\mu) + \Lambda/2 + B]\, \partial_x^2 U_z^{(1)} \partial_x U_z^{(1)} \quad \text{- - - - - - - - - - -} \quad (30)$$

After the primary wave is substituted into the inhomogeneous equation and after Fourier transformation, the equation takes the familiar form:

$$\partial_x^2 \bar{U}_x^{(2)} + K_L^2 \bar{U}_x^{(2)} = \bar{S}_x^{(2)} \quad \text{- - - - - - - - - - - - - -} \quad (31)$$

where the effective source term is given by:

$$\bar{S}_x^{(2)}(x) = - \frac{(\lambda + 2\mu) + \Lambda/2 + B}{4\pi\ \rho_o \omega^2} \Lambda_\alpha A_\beta K_L^2 \{$$

$$\frac{\Gamma_3}{2i} \left[\exp[-2iK'_{T(\beta)}x]\,\delta(\omega+2\omega_\beta) - \exp[2K'_{T(\beta)}x]\,\delta(\omega-2\omega_\beta)\right]\exp[-2K''_{T(\beta)}x]$$

$$+ \frac{\Gamma_4}{2} \left[\exp[-2iK'_{T(\beta)}x]\,\delta(\omega+2\omega_\beta) + \exp[2K'_{T(\beta)}x]\,\delta(\omega-2\omega_\beta)\right]\exp[-2K''_{T(\beta)}x]\} \quad \text{- -} \quad (32)$$

where:

$$\Gamma_3 = 2\ (K'^3_{T(\beta)} + 3\ K''_{T(\beta)}\ K'_{T(\beta)})$$

$$\Gamma_4 = 4\ K'^2_{T(\beta)}\ K''_{T(\beta)}$$

By using Eq. (27) and the Green's function expressed in Eq. (11), the real part of the solution and hence the second order displacement can be found to be:

$$U_x^{(2)}(x,t) = + \frac{\lambda + 2\mu + \Lambda/2 + B}{16\pi\rho_o\ (2\omega_\beta)^2} \Lambda_\beta A_\beta\ \{$$

$$- \frac{\Delta_{25}}{\Delta_{27}} \{\cos[2(\omega_\beta t - K'_{T(\beta)}x)] \exp[-2 K''_{T(\beta)}x]$$

$$-\cos[2(\omega_\beta t - \gamma'' K'_{L(\beta)}x)]\exp[2\delta'' K''_{L(\beta)}x]\}$$

$$+ \frac{\Delta_{26}}{\Delta_{27}} \{\sin[2(\omega_\beta t - K'_{T(\beta)}x)] \exp[-2 K''_{T(\beta)}x]$$

$$- \sin[2(\omega_\beta t - \gamma'' K'_{L(\beta)}x)] \exp[2\delta'' K''_{L(\beta)}x]\}$$

$$- \frac{\Delta_{28}}{\Delta_{30}} \{\cos[2(\omega_\beta t - K'_{T(\beta)}x)] \exp[2 K''_{T(\beta)}x]$$

$$- \cos[2(\omega_\beta t - \gamma'' K'_{L(\beta)}x)] \exp[-2\delta'' K''_{L(\beta)}\lambda]\}$$

$$+ \frac{\Delta_{29}}{\Delta_{30}} \{-\sin[2(\omega_\beta t - K'_{T(\beta)}x)] \exp[2K''_{T(\beta)}x]$$

$$+ \sin[2(\omega_\beta t - \gamma'' K'_{L(\beta)}x)] \exp[-2\delta'' K''_{L(\beta)}x]\}$$

$$\text{- - - - - - - - - -} \quad (33)$$

where the coefficients are:

$$\Delta_{25} = [\Gamma_3 \gamma'' K'_{L(\beta)} - \Gamma_4 \delta'' K''_{L(\beta)}][\delta'' K''_{L(\beta)} + K''_{T(\beta)}]$$

$$-[\Gamma_4 \gamma'' K'_{L(\beta)} + \Gamma_3 \delta'' K''_{L(\beta)}][\gamma'' K'_{L(\beta)} - K'_{T(\beta)}]$$

$$\Delta_{26} = [\Gamma_3 \gamma'' K'_{L(\beta)} - \Gamma_4 \delta'' K''_{L(\beta)}][\gamma'' K'_{L(\beta)} - K'_{T(\beta)}]$$

$$+ [\Gamma_4 \gamma'' K'_{L(\beta)} + \Gamma_3 \delta'' K''_{L(\beta)}][\delta'' K''_{L(\beta)} + K''_{T(\beta)}]$$

$$\Delta_{27} = [\delta'' K''_{L(\beta)} + K''_{T(\beta)}]^2 + [\gamma'' K'_{L(\beta)} - K'_{T(\beta)}]^2$$

$$\Delta_{28} = [\Gamma_3\ \gamma''K'_{L(\beta)} + \Gamma_4\ \delta''K''_{L(\beta)}]\,[\delta''K''_{L(\beta)} - K''_{T(\beta)}]$$

$$- [\Gamma_4\ \gamma''K'_{L(\beta)} - \Gamma_3\ \delta''K''_{L(\beta)}]\,[\delta''K'_{L(\beta)} - K'_{T(\beta)}]$$

$$\Delta_{29} = -\ [\Gamma_3\ \gamma''K'_{L(\beta)} + \Gamma_4\ \delta''K''_{L(\beta)}]\,[\gamma''K'_{L(\beta)} - K'_{T(\beta)}]$$

$$+ [\Gamma_4\ \gamma''K'_{L(\beta)} - \Gamma_3\ \delta''K''_{L(\beta)}]\,[\delta''K''_{L(\beta)} - K''_{T(\beta)}]$$

$$\Delta_{30} = [\delta''K''_{L(\beta)} - K''_{T(\beta)}]^2 + [\gamma''K'_{L(\beta)} - K'_{T(\beta)}]^2$$

Investigation of Eq. (33) seems to indicate that the second order longitudinal wave is due to a sum of two waves of frequencies $2\omega_\beta$. One of these waves propagates with the speed of a transverse wave and the other propagates with the speed of a longitudinal wave. This is an incorrect interpretation and is due to the fact that the second order displacement is written in the most compact manner and not in a form which is instructive.

The expression from which Eq. (33) was extracted is:

$$U_x'^{(2)}(x,t) = \frac{(\lambda + 2\mu) + A/2 + B \quad A_\beta A_\beta}{16\pi\ \rho_o\ \ (2\omega_\beta)^2} \{$$

$$\frac{\Delta_{25}+i\Delta_{26}}{\Delta_{27}}\ \{\exp[2i(\omega_\beta t - \gamma''K'_{L(\beta)}x)]\ \exp[2\delta''K''_{L(\beta)}x]$$

$$[1-\exp[2i(\gamma''K'_{L(\beta)} - K'_{T(\beta)})x]\ \exp[-2(\delta''K''_{L(\beta)} + K''_{T(\beta)})x]\}$$

$$+ \frac{\Delta_{28}+i\Delta_{29}}{\Delta_{30}}\ \{\exp[-2i(\omega_\beta t - \gamma''K'_{L(\beta)}x)]\ \exp[-2\delta''K''_{L(\beta)}x]$$

$$[1-\exp[-2i(\gamma''K'_{L(\beta)} - K'_{T(\beta)})x]\ \exp[2(\delta''K''_{L(\beta)} + K''_{T(\beta)}x]\}\ \}$$

$$\text{-------------} \quad (34)$$

This expression, although it is not the second order displacement (its real part is) shows the nature of transverse wave self interaction.

In each term there is an expression for a second order longitudinal wave with a coefficient which is of the form:

$$[1-\exp[2i(\gamma''K'_{L(\beta)} - K'_{T(\beta)}x] \exp[-2(\delta''K''_{L(\beta)} + K''_{T(\beta)})x]$$

This is in effect a modulation function and it is due to the interaction nature of the wave. The harmonic term can be thought of as a longitudinal wave propagating away from a transverse wave. This is exactly the case of self interaction of a transverse wave.

5) Self Interaction of Longitudinal Waves

From the results of longitudinal-longitudinal interaction, we can conclude that a second order longitudinal wave will be produced when a primary longitudinal wave self interacts. The equation to be solved for the second order displacement is:

$$\rho_0 \partial_t^2 U_x^{(2)} - (\zeta + \eta/3)\ \partial_t\ \partial_x^2\ U_x^{(2)} - (\lambda + 2\mu)\ \partial_x^2 U_x^{(2)} =$$

$$[3(\lambda + 2\mu) + 2(A + 3B + C)]\ \partial_x^2 U_x^{(1)}\ \ \partial_x U_x^{(1)} \quad ------ \quad (35)$$

and the primary wave is:

$$U_x^{(1)}(x,t) = A_\alpha \cos[\omega_\alpha t - K'_{L(\alpha)}x]\ \exp(-K''_{L(\alpha)}x) \quad ------------ \quad (36)$$

Substitution of Eq. (36) into Eq. (35) and Fourier Transformation results in:

$$\partial_x^2 \bar{U}_x^{(2)} + K_L^2 \bar{U}_x^{(2)} = \bar{S}_x^{(2)} \quad ----------- \quad (37)$$

Where the effective source term is given by:

$$\bar{S}_x^{(2)}(x) = -\frac{3(\lambda + 2\mu) + 2(A + 3B + C)}{4\pi \rho_o \quad (\omega)^2} A_\alpha A_\beta K_L^2 \{$$

$$\frac{\Gamma_1}{2} [\exp[-2i K'_{L(\alpha)} x]\delta(\omega+2\omega_\alpha) - \exp[2i K'_{L(\alpha)} x]\delta(\omega-2\omega_\alpha)] \exp[-2K''_{L(\alpha)} x]]$$

$$+ \frac{\Gamma_2}{2} [\exp[-2i K'_{L(\alpha)} x]\delta(\omega+2\omega_\alpha) + \exp[2K'_{L(\alpha)} x]\delta(\omega-2\omega_\alpha)] \exp[-2K''_{L(\alpha)} x]]\}$$

$$\text{- - - - - - - - - - - - (38)}$$

where:

$$\Gamma_1 = 2[K'^3_{L(\alpha)} + 3K''^2_{L(\alpha)} K'_{L(\alpha)}]$$

$$\Gamma_2 = 4[K'^2_{L(\alpha)} K''_{L(\alpha)}]$$

By using Eq. (27) and the Green's function expressed in Eq. (11), the solution to the inhomogeneous equation can be found to be:

$$U'^{(2)}_x(x,t) = \frac{3(\lambda + 2\mu) + 2(A + 3B + C)}{16\pi \rho_o (2\omega_\alpha)^2} A_\alpha A_\alpha \{$$

$$\frac{\Delta_{31} + i\Delta_{32}}{\Delta 33} \{\exp[2i(\omega_\alpha t - \gamma'' K'_{L(\alpha)} x)] \exp[2\delta' K''_{L(\alpha)} x] \cdot$$

$$\cdot [1 - \exp[2i(\gamma' - 1) K'_{L(\beta)} x] \exp[-2(\delta' + 1) K''_{L(\alpha)}]]$$

$$+ \frac{\Delta_{34} + i\Delta_{35}}{\Delta_{36}} \{\exp[-2i(\omega_\alpha t - \gamma' K'_{L(\alpha)} x)] \exp[-2\delta' K''_{L(\alpha)} x] \cdot$$

$$\cdot [1 - \exp[-2i(\gamma' - 1) K'_{L(\alpha)} x] \exp[2(\delta + 1) K''_{L(\alpha)}]]\} \text{ - - - (39)}$$

The real part of the above yields the displacement:

$$U_x^{(2)}(x,t) = \frac{3(\lambda + 2\mu) + 2(A + 3B + C)}{16\pi\ \rho_0\ (2\omega_\alpha)^2}\ A_\alpha A_\alpha \{$$

$$- \frac{\Delta_{31}}{\Delta_{33}} \quad \{\cos[2(\omega_\alpha t - K'_{L(\alpha)}x)]\ \exp(-2K''_{L(\alpha)}x)$$

$$- \cos[2(\omega_\alpha t - \gamma' K'_{L(\alpha)}x)]\ \exp(2\delta' K''_{L(\alpha)}x)\}$$

$$+ \frac{\Delta_{32}}{\Delta_{33}} \quad \{\sin[2(\omega_\alpha t - K'_{L(\alpha)}x)]\ \exp(-2K''_{L(\alpha)}x)$$

$$- \sin\ [2(\omega_\alpha t - \gamma' K'_{L(\alpha)}x)]\ \exp(2\delta' K''_{L(\alpha)}x)$$

$$- \frac{\Delta_{34}}{\Delta_{35}} \quad \{\sin[2(\omega_\alpha t - K'_{L(\alpha)}x)]\ \exp(2K''_{L(\alpha)}x)$$

$$- \cos[2(\omega_\alpha t - \gamma' K'_{L(\alpha)}x)]\ \exp(-2\delta' K''_{L(\alpha)}x)$$

$$+ \frac{\Delta_{35}}{\Delta_{36}} \quad \{-\sin[2(\omega_\alpha t - K'_{L(\alpha)}x)]\ \exp(2K''_{L(\alpha)}x)$$

$$+ \sin[2(\omega_\alpha t - \gamma' K'_{L(\alpha)}x)]\ \exp(-2\delta' K''_{L(\alpha)}x)\}\} \quad --- \quad (40)$$

where:

$$\Delta_{31} = [\Gamma_1 \gamma' K'_{L(\alpha)} - \Gamma_2 \delta' K''_{L(\alpha)}](1 + \delta')\ K''_{L(\alpha)}$$

$$-[\Gamma_2 \gamma' K'_{L(\alpha)} + \Gamma_1 \delta' K''_{L(\alpha)}](\gamma' - 1)\ K'_{L(\alpha)}$$

$$\Delta_{32} = [\Gamma_1 \gamma' K'_{L(\alpha)} - \Gamma_2 \delta' K''_{L(\alpha)}](\gamma' - 1)\ K'_{L(\alpha)}$$

$$+ [\Gamma_2 \gamma' K'_{L(\alpha)} + \Gamma_1 \delta' K''_{L(\alpha)}](\delta' + 1)\ K''_{L(\alpha)}$$

$$\Delta_{33} = [(\delta' + 1)\ K''_{L(\alpha)}]^2 + [(\gamma' - 1)\ K'_{L(\alpha)}]^2$$

$$\Delta_{34} = -\ [\Gamma_1\gamma' K'_{L(\alpha)} + \Gamma_2\delta' K''_{L(\alpha)}](\delta' - 1)\ K''_{L(\alpha)}$$

$$-[\Gamma_2\gamma' K'_{L(\alpha)} - \Gamma_1\delta' K''_{L(\alpha)}](\gamma' - 1)\ K'_{L(\alpha)}$$

$$\Delta_{35} = -\ [\Gamma_1\gamma' K'_{L(\alpha)} + \Gamma_2\delta' K''_{L(\alpha)}](\gamma' - 1)\ K'_{L(\alpha)}$$

$$+\ [\Gamma_2\gamma' K'_{L(\alpha)} - \Gamma_1\delta' K''_{L(\alpha)}](\delta' - 1)\ K''_{L(\alpha)}$$

$$\Delta_{36} = [(\delta' - 1)\ K''_{L(\alpha)}]^2 + [(\gamma' - 1)\ K'_{L(\alpha)}]^2$$

The significance of this result will now be discussed.

CHAPTER VI

SUMMARY AND DISCUSSION

Two types of interaction of waves in a solid have been considered: volume interaction of viscoelastic waves and interaction of viscoelastic plane waves. Volume interaction of viscoelastic waves was investigated only to the extent of calculating the far field displacement. Furthermore, only the purely elastic wave was considered when resonant interaction was studied. The results are summarized below.

1) Transverse-transverse wave interaction.

   a) Colinearly propagating, orthogonally polarized transverse waves not to interact.

   b) Colinearly propagating, non-orthogonally polarized waves will not interact resonantly.

   c) Obliquely propagating, orthogonally polarized transverse waves interact resonantly to form a flipped(α-β) mode longitudinal wave.

   d) A flipped mode longitudinal wave is also produced when obliquely propagating, nonorthogonally polarized waves interact.

2) Longitudinal-longitudinal interactions.

   a) Colinearly propagating longitudinal waves will interact to form both a normal mode (α+β) and a flipped (α-β) mode longitudinal wave.

   b) Non-colinearly propagating longitudinal waves will interact to form a normal mode transverse wave. The polarization and direction of propagation of this wave is coplaner with the propagation plane of the primary waves.

c) Orthogonally propagating longitudinal waves do not interact.

3) Longitudinal-Transverse wave interaction.

a) Colinearly propagating longitudinal and transverse waves interact to form a normal mode transverse wave.

b) Obliquely propagating longitudinal and transverse waves produce a flipped mode longitudinal wave and a normal mode transverse wave when they interact resonantly.

It must be pointed out that these waves are not the only waves present in a physical interaction situation. The above mentioned generated waves are due to "strong" or resonant interaction and they are the waves that are radiated away from the primary deformations. The far-field approximation of the Green's function was used to calculate these second order waves. Had the entire Green's function as expressed in Eq. (IV-22) been used rather than this approximation, the resulting second order displacement would include all of the nonlinear waves. For instance, in the resonant interaction of colinearly propagating transverse and longitudinal waves, the flipped mode longitudinal wave was found to be zero by using the far-field Green's function; however, this wave will have a nonzero value if the entire Green's function was used. Also by using the entire Green's function, it can be seen that both the normal and flipped mode transverse waves will be generated when two transverse primary waves propagate colinearly.

From the above statements one might conclude that the analysis should be done by using the entire Green's function rather than the approximation. This conclusion would be valid if it were not for the following points.

The volume of interaction must be spherical for the Green's function in question to be valid, and the wave radiated must be a radial wave. If one is to calculate the near field effect of interaction by using the entire Green's function, then the interaction volume must be specified more precisely and the appropriateness of the Green's function reestablished. One need only consider the interaction of colinearly propagating waves to see the point.

One very physical way of handling the interaction of colinearly propagating waves is by the methods of interaction of plane waves. By using these methods, we found that either a normal or flipped mode longitudinal wave was formed when any of the following interactions took place:

Longitudinal - Longitudinal

Transverse - Transverse

Longitudinal - Transverse

Transverse self interaction

Longitudinal self interaction

The resultant nonlinear wave due to Longitudinal-Transverse interaction was actually a sum of two longitudinal waves that were caused by self interaction of the primary waves. Therefore we conclude that colinearly propagating longitudinal and transverse waves do not interact with each other.

The form that the second order longitudinal wave takes when two longitudinal waves interact is given by Eq. (19). Here we see that both normal and flipped mode waves are generated. We can also see that no interaction will occur when

$$[K'_{L(\alpha\pm\beta)} - (K'_{L(\alpha)} \pm K'_{L(\beta)})]\ell = N\pi \qquad N = 1,2,3 \ldots$$

and that the second order wave is maximized when:

$$[K'_{L(\alpha\pm\beta)} - (K'_{L(\alpha)} \pm K'_{L(\beta)})]\ell = \frac{N\pi}{2} \qquad N = 0,1,2,\ldots$$

In spite of the term $(\omega_\alpha \pm \omega_\beta)^2$ in the denominator, the flipped mode wave due to primary waves of nearly the same frequency will not have an amplitude of infinity. In fact the entire contribution of the coefficients will tend to make a flipped mode wave of low frequency have an amplitude of zero. Also the normal mode wave with high frequency will not be unstable because of the familiar effects of dissipation. One then concludes that there is an ideal frequency for the second order wave that maximizes its amplitude. This wave of ideal frequency can either be a normal or flipped mode wave; however, the mode with nonideal frequency will probably be much smaller than the ideal wave. Such a wave would then correspond to the wave due to resonant interaction of the volume interaction cases. No attempt was made to find this "ideal" frequency because it is believed that considerable numerical analysis will be necessary to find this frequency analytically.

A similar second order longitudinal wave will be generated when two transverse waves interact. In this case, the conditions of no interaction are:

$$[K'_{L(\alpha\pm\beta)} - (K'_{T(\alpha)} \pm K'_{T(\beta)})]\ell = N\pi \qquad N=0,1,2,\ldots$$

and the conditions for maximum interaction are:

$$[K'_{L(\alpha\pm\beta)} - (K'_{T(\alpha)} \pm K'_{T(\beta)})]\ell = \frac{N\pi}{2} \qquad N=0,1,2,\ldots$$

The same frequency dependence of the amplitude of the wave in the longitudinal interaction is present here. This strong frequency dependence seems to indicate that the second order wave is highly dispersive. That is, a spectra of nonlinear waves may tend to filter itself to the "ideal" frequency as it propagates. A spectra of nonlinear waves may be generated by a similar spectra of linear waves or by higher order interaction of the primary and nonlinear waves. This filtering effect of nonlinear waves may be responsible for the characteristic shape that nonlinear waves which propagate long distances have (24).

The magnitude that the second order wave will have is very much dependent on the interaction length. This can be seen by not writing the displacement but rather writing the solution to the inhomogeneous equation.

For Longitudinal-Longitudinal interaction this is:

$$U_x'^{(2)}(x,t) = \sum_{\alpha\pm\beta} \frac{3(\lambda + 2\mu) + 2(A + 3B + C)}{16\pi\ \rho_o\ (\omega_\alpha \pm \omega_\beta)^2}\ A_\alpha A_\beta \{$$

$$\frac{\Delta_{13}+i\Delta_{14}}{\Delta_{15}}\ \exp[i((\omega_\alpha \pm \omega_\beta)t - K'_{L(\alpha\pm\beta)}x)]\ \exp[K''_{L(\alpha\pm\beta)}x]\ \cdot$$

$$\cdot\ [1-\exp[i((K'_{L(\alpha\pm\beta)} - (K'_{L(\alpha)} \pm K'_{L(\beta)})\ell)]\ \exp[-(K''_{L(\alpha\pm\beta)} - (K''_{L(\alpha)} + K''_{L(\beta)})\ell]]$$

$$+ \frac{\Delta_{16}+i\Delta_{17}}{\Delta_{18}}\ \exp[-i((\omega_\alpha \pm \omega_\beta)t - K'_{L(\alpha\pm\beta)}x)]\ \exp[-K''_{L(\alpha\pm\beta)}x]\ \cdot$$

$$\cdot\ [1-\exp[-i(K'_{L(\alpha\pm\beta)} - (K'_{L(\alpha)} \pm K'_{L(\beta)})\ell)]\exp[K''_{L(\alpha\pm\beta)} + (K''_{L(\alpha)} + K''_{L(\beta)})\ell]] \qquad (1)$$

and for Transverse-Transverse interaction, a similar term exists.

$$U_x'^{2}(x,t) = \sum_{\alpha\pm\beta} \frac{(\lambda + 2\mu) + A/2 + B}{16\pi\ \rho_o\ (\omega_\alpha \pm \omega_\beta)^2}\ A_\alpha A_\beta \{$$

$$\frac{\Delta_{19}+i\Delta_{20}}{\Delta_{21}}\ \exp[i((\omega_\alpha \pm \omega_\beta)t - K'_{L(\alpha\pm\beta)}x)]\ \exp[K''_{L(\alpha\pm\beta)}x]$$

$$[1-\exp[i(K'_{L(\alpha\pm\beta)} - (K'_{T(\alpha)} \pm K'_{T(\beta)}))\ell]\ \exp[-(K''_{L(\alpha\pm\beta)} - (K''_{T(\alpha)} + K''_{T(\beta)}))\ell]$$

$$+ \frac{\Delta_{22}+i\Delta_{23}}{\Delta_{24}} \exp[i((\omega_\alpha \pm \omega_\beta)t - K'_{L(\alpha\pm\beta)}x)] \exp[-K''_{L(\alpha\pm\beta)}x]$$

$$[1-\exp[-i(K'_{L(\alpha\pm\beta)}-(K'_{T(\alpha)}\pm K'_{T(\beta)}))\ell] \exp[(K''_{L(\alpha\pm\beta)}-(K''_{T(\alpha)}+K''_{T(\beta)}))\ell]] \quad \text{- - - - - - - - - (2)}$$

Both equations are very similar, each term consisting of a harmonic expression that represents the second order wave multiplied by an expression that represents the interaction of the waves. One can see that the amplitude of the second order wave will be dependent on the interaction length ℓ, and that there will be definite values for ℓ that will maximize or minimize the magnitude of the second order wave.

If the primary waves interact for the distance up to or beyond the field variable, then the above equations are modified by letting $\ell = x$. In this case the coefficients of the second order waves become modulation functions and it can be seen that the amplitude of the second order wave will fluctuate along the x axis. Also the amplitude will grow or decay in this fluctuating manner depending on the values of $K_{(\alpha)}$ and $K_{(\beta)}$.

A second order longitudinal wave is formed when either a longitudinal or a transverse wave interacts with itself. In the cases of self interaction, the interaction length was assumed to be the entire length of propagation of the primary wave. Again the nonlinear wave will be modulated in time and space, and hence there is a specific length for the field variable that will maximize or minimize the second order effects of the wave in a specific region.

With all of the types of interaction having been investigated, an interesting phenomenon can be observed. Because a second order transverse wave is not generated when a transverse primary wave self-interacts, the form of the primary transverse wave will be independent of nonlinear effects, whereas the second order longitudinal wave produced when a primary longitudinal wave self interacts will add to the primary wave and alter its form. Why anharmonic effects alter the form of a longitudinal wave and not that of a transverse wave can be explained as follows.

The speed of any point on the profile of a longitudinal wave is given by:

$$\vec{v} = \vec{C}_L + \partial_t \vec{U}(t)$$

The change in the waveform of the wave is due to the particle velocity v being different for the various points on the wave form. This can be caused by two mechanisms.

1) C_L is dependent on the compressed or extended state of the media and therefore it varies over the range of the period of the waveform.

2) The particle velocity coincides in direction to C_L (and v) for a longitudinal wave and decreases or increases v accordingly.

This is not the case with a transverse wave though. First compressions and extensions do not occur when a transverse wave propagates and secondly the particle velocity is perpendicular to the propagation velocity for transverse waves. Therefore the particle velocity will remain constant at all points on the profile for a transverse wave and the wave will retain its form. The longitudinal wave will distort due to the change in particle velocity on the profile as we have found in the analysis. This distortion (due to anharmonic effects) is often referred to as intermodulation distortion.

A final note on how the nonlinear elastic constants can be found is now in order. We have found that there are two types of interactions that occur when the primary waves are propagating colinearly. They are transverse-transverse and longitudinal-longitudinal interactions and they both result in a longitudinal second order wave. Although there are three nonlinear constants they only appear as two algebraic combinations.

$$A + 3B + C$$

$$\text{and } A/4 + B$$

The first appears in the coefficient of the second order wave when longitudinal waves interact and the second appears in that when transverse waves interact. This interaction can be either distinct or self. Thus these combinations of constants can be found by investigating, say, the self interaction of longitudinal and transverse waves. In this case equations IV-35 and IV-38 would give the nonlinear coefficients if the linear parameters $(\lambda, \mu, \zeta, \eta)$, frequency of the wave, distance of propagation, amplitude of the primary wave, and amplitude of the second order wave were known. The linear (isothermal) parameters can also be determined by a wave propagation experiment; however, inclusion of that experiment in this discussion is not appropriate. A convenient way of determining the amplitudes of the waves is described below and the necessary apparatus is shown on Plate 1.

The apparatus has provisions for monitoring the input spectra and output spectra of waves of two polarizations, longitudinal and one transverse direction. From the output spectra, both the amplitude of the second order wave and that of the linear wave can be obtained. This is accomplished by knowing the coupling coefficients of the crystals and the setting on the amplifier. In order to perform the necessary calculations, the primary wave amplitude must be known. If the output amplitude is known, then the input amplitude can be found by using the following:

$$A_{out} = A_{in} \exp(-K''\ell) \quad .$$

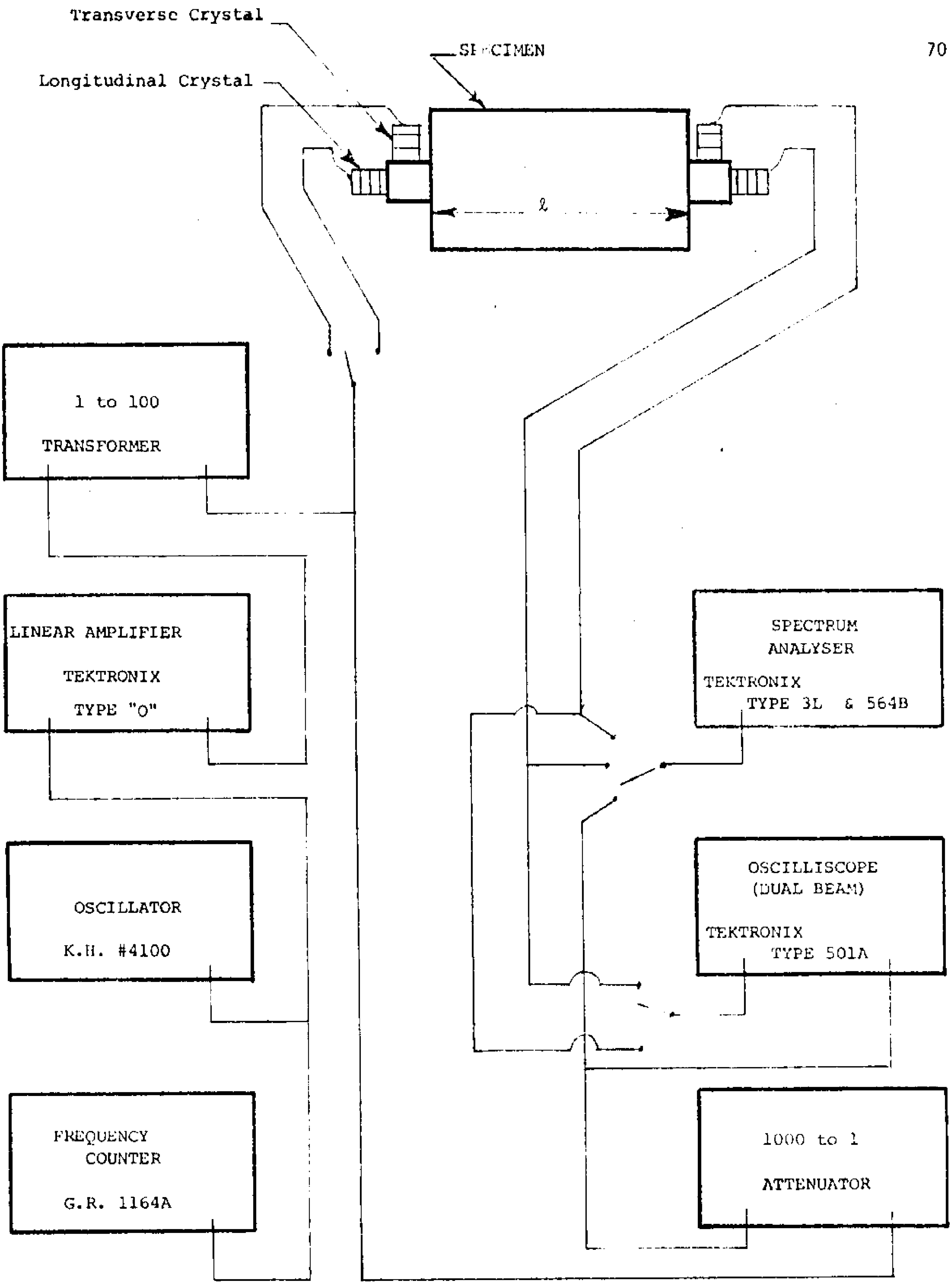


PLATE I

Apparatus for finding the second order coefficients.

The frequency of the oscillator is precisely determined by the frequency counter and the input spectra is monitored to be sure that the primary wave is monochromatic. The photographs on Plate II show the input and output spectra for a 5 K Hz longitudinal wave propagating through a path length of 20 cm of an acrylic resin plastic. The input spectra shows only a 5 K Hz primary wave and the output spectra shows the primary wave and the nonlinear wave. This will be the type of data needed to obtain the nonlinear constant combinations.

In conclusion, we must state that in addition to being interesting, nonlinear viscoelastic waves can be significant and that this anharmonic phenomenon deserves more theoretical, numerical and empirical study.

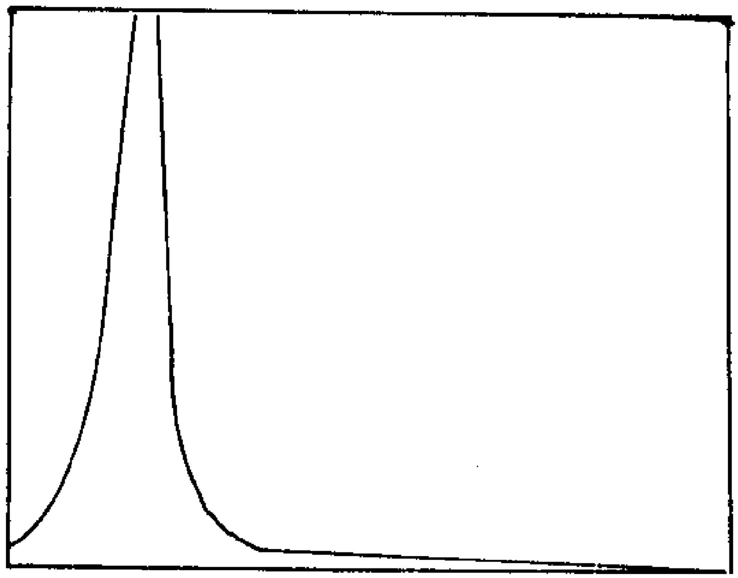


Figure (1a)

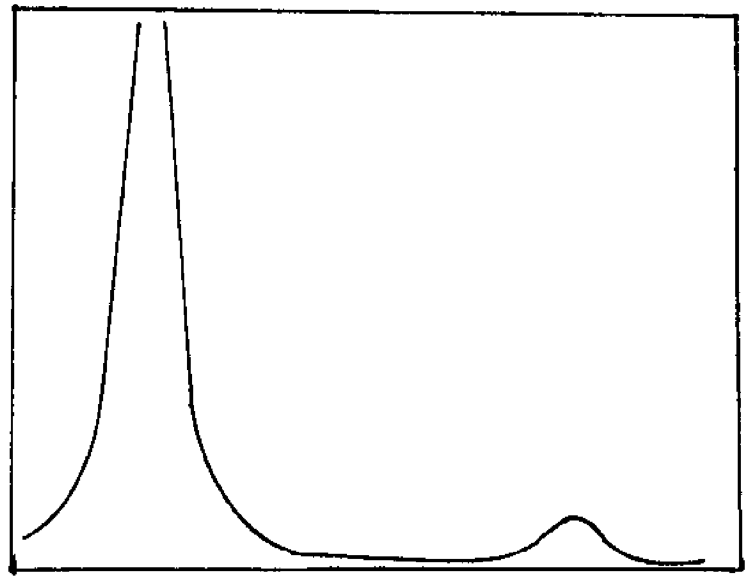


Figure (1b)

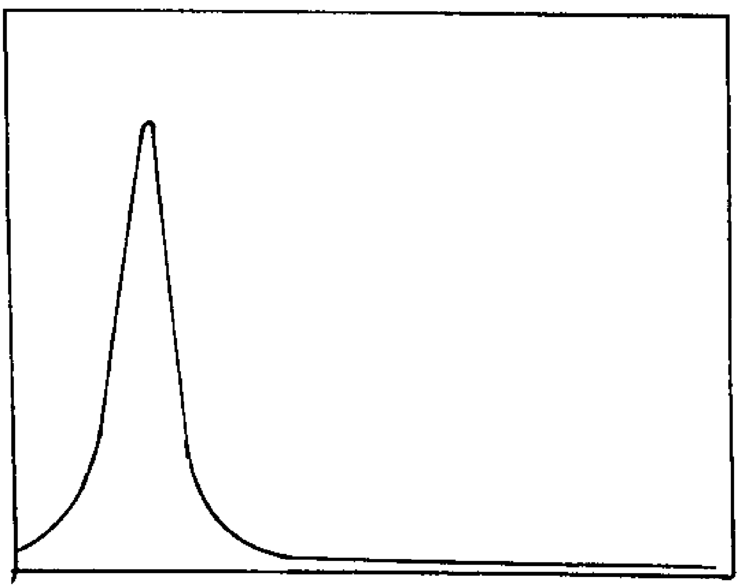


Figure (2a)

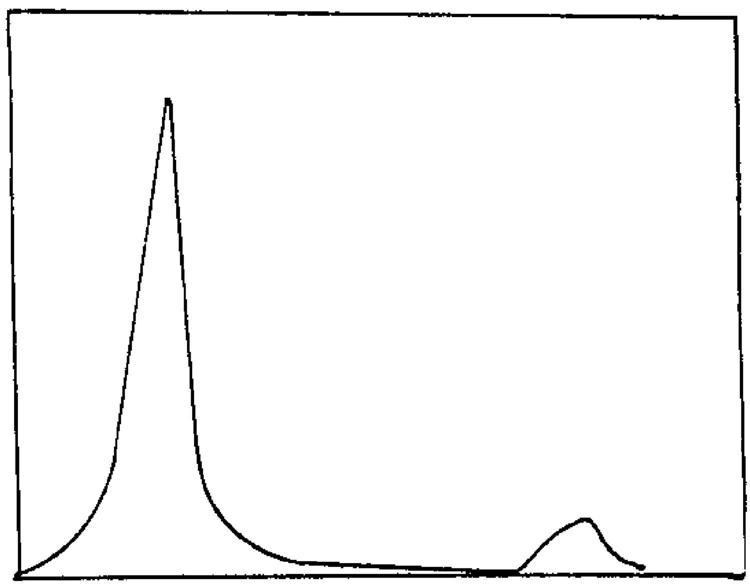


Figure (2b)

Input (Figs.(1a) and(2a)) and output (Figs.(1b) and(2b)) spectra for a 5K Hz wave propagating in acrylic resin plastic.

Plate II

REFERENCES

(1) A. L. Thuras, R. T. Jenkins, and H. T. O'Neil, "Extraneous Frequencies Generated in Air Carrying Intense Sound Waves," J.A.S.A. 6, 173 (1934).

(2) M. J. Lighthill, "On Sound Generated Aerodynamically; I General Theory," Proceedings of the Royal Society A211, pp. 564-578 (1952).

(3) G. S. Gorelih, V. A. Zverev, "Problem of Mutual Interaction between Sound Waves," Ahust. Zh. 1, 4, 339-342 (1955) [Soviet Physics-Acoustics, Vol. 1, p. 353].

(4) S. I. Solvyan and R. V. Khokhlov, "Propagation of Finite-Amplitude Acoustic Waves in a Dissipative Medium," Vestn. M.G.U., 3, 52-61 (1961).

(5) N. N. Andrew, "Concerning Certain Second-Order Quantities in Acoustics," Ahust. Zh. 1, 1, 3-11 (1955) [Sov. Phys.-Acoust., 1,2, (1957)].

(6) L. K. Zarembo and V. V. Shklovshaya-Kordi, "Propagation Velocity of Finite Amplitude Ultrasonic Waves in a Liquid," Ahust. Zh. 6, 1, 47-51 (1960) [Soviet Physics-Acoustics, 6, pp. 42, (1961)].

(7) I. G. Mikhailov, V. A. Shutilov, "Distortion of the Finite Amplitude Ultrasonic Waveform in Various Liquids," Ahust. Zh., Vol. 6, No. 3, pp. 340-346 (1960) [Soviet Physics-Acoustics, Vol. 6, pp. 340 (1961)].

(8) K. A. Naugolnykh, S. I. Solvyan, R. V. Khakhov, "Nonlinear Interaction of Sound Waves in an Absorbing Medium," Ahust. Zh. Vol. 9, No. 2, pp. 192-197 (1963) [Soviet Physics-Acoustics , Vol.9, No. 2, pp. 155, (1963)].

(9) L. A. Ostroshii, "Second-order Terms in a Traveling Sound Wave," Ahust. Zh. Vol. 14, No. 1, pp. 82-89 (1968) [Soviet Physics-Acoustics, Vol. 14, No. 1, pp. 61 (1968)].

(10) Z. A. Goldberg, "Plane Acoustic Waves of Finite Amplitude in a Viscous Heat-conducting Medium," (Disseratation) [Acoustics Institute, AN SSSR,(1958)].

(11) V. R. Larevstad, "Nonlinear Interaction of Two Monochromatic Sound-Waves," Acoustica, Vol. 16, No. 4, pp. 191 (1965).

(12) N. Romilly, "One-dimensional Nonlinear Waves in a Dissipative Gas," Acoustica, Vol. 25, No. 5, pp. 248 (1971).

(13) H. O. Berktay, "Possible Exploitation of Nonlinear Acoustics in Underwater Transmission Applications," J. of Sound and Vib. Vol. 2, No. 9, pp. 435-461 (1965).

(14) D. G. Tucher, "The Exploitation of Non-linearity in Underwater Acoustics," J. of Sound and Vib., Vol. 2, No. 4, pp. 429-434 (1965).

(15) D. T. Blackstoch "Propagation of Plane Sound Waves of Finite Amplitude in Nondissipative Fluids," J.A.S.A. Vol. 34, No. 1, pp. 9, (1962).

(16) R. Y. Chiao, E. Garmire, and G. H. Townes, " Self Trapping of Optical Beams" , Phys. Rev. Letters 13, 479 (1964).

(17) A. Yildiz, "Light and Sound Emissions from non-linear Plasma Fluxuations," Physical Review Vol. 136, No. 2A, pp. A393-A409, (1964).

(18) Landau and Lifshitz, "Theory of Elasticity," Second Edition, Reading, Mass.: Addison Wesley.

(19) Z. A. Goldberg, "Interaction of a Plane Longitudinal and Transverse waves," Ahust. Zh., Vol. 6, No. 3, pp. 307-310 (1960) [Soviet Physics-Acoustics, Vol. 6, No. 3, pp. 306 (1961)].

(20) G. L. Jones and D. R. Kobett, "Interaction of Elastic Waves in an Isotropic Solid," J.A.S.A., Vol. 35, No. 1, pp. 5, (1963).

(21) J. D. Childress, C. G. Hanbrick, "Interactions between Elastic waves in an Isotropic Solid," Physical Review, Vol. 136, No. 2A, pp. A411, (1964).

(22) L. H. Taylor, F. R. Rollins, "Ultrasonic Study of Three-Phonon Interactions," Physical Review, Vol. 136, No. 319, pp. A591 (1964).

(23) A. A. Gedroits and V. A. Krasilnikov, "Finite-Amplitude Elastic Waves in Solids and Deviations from Holli's Law," J. of Explt. Theoret. Phys. 43, pp. 1592-1599 (1962) [Soviet Physics JETP, Vol. 16, No. 5, (1963)].

(24) I. A. Viktorov, "Effects of a Second Approximation in the Propagation of Waves Through Solids." Ahust. Zh., Vol. 9, No. 3, pp. 296-300 (1963) [Soviet Physics-Acoustics, Vol. 9, No. 3, pp. 242 (1964)].

(25) L. A. Pospelov, "Propagation of Finite Amplitude Elastic Waves," Ahust. Zh. Vol. 11, No. 3, pp. 359-362 (1965) [Soviet Physics-Acoustics, Vol. 11, No. 3, pp. 302, (1966)].

(26) N. S. Stepanov, "Interaction of Longitudinal and Transverse Elastic Waves," Ahust. Zh., Vol. 13, No. 2, pp. 270-275 (1967) [Soviet Physics-Acoustics, Vol. 13, No. 2, pp. 230 (1967)].

(27) A. A. Gedroits, L. K. Zarembo, and V. A. Krasilnilov, "Shear waves of finite Amplitude in Polycrystals and Single Crystals of Metals," Doklady Ohadimic Nauh. SSSR., Vol. 150, No. 3, pp. 515-518 (1963) [Soviet Physics-Doklady, Vol. 8, No. 5, pp. 478 (1964)].

(28) Kung Hsiu-jen, L. K. Zarembo and V. A. Krasilnilov, "Nonlinear Interaction of Elastic Waves in Solids," Ahust. Zh., Vol. 11, No. 1, pp. 112-115 (1965) [Soviet Physics-Acoustics, Vol. 11, No. 1, pp. 89-92 (1965)].

(29) F. R. Rollins, L. H. Taylor, and P. H. Todd, "Ultrasonic Study of Three Phonon Interaction," Physical Review, Vol. 136, No. 3A, pp. A597 (1964).

(30) Y. M. Chen, "Interaction of Longitudinal Waves with Transverse Waves in Dispersive Nonlinear Elastic Media." Quarterly J. of Applied Math., April, 1971, pp. 125.

(31) F. D. Murnaghan, "Finite Deformations of an Elastic Solid", John Wiley & Sons, New York (1951).

(32) Hughes and Kelly, "Second-Order elastic deformations of solids," Phy. Rev. Vol. 92 , pp. 1145 (1953).

APPENDIX

APPENDIX A

GREEN'S FUNCTION FOR VOLUME INTERACTION

The Fourier Transformed inhomogeneous equation is:

$$\bar{\mathcal{L}}_{ik}(\vec{r})\ \bar{U}_k^{(2)}(\vec{r}) = -\frac{1}{\rho_o C_T^2}\ \bar{S}_i^{(2)}(\vec{r}) \text{ - - - - - - - - } (A1)$$

where:

$$\bar{\mathcal{L}}_{ik}(\vec{r}) = (\partial_k^2 + K_T^2)\delta_{ik} - (1 - K_T^2/K_L^2)\partial_i\partial_k \text{ - - - - - - - } (A2)$$

This has the solution:

$$\bar{U}_k^{(2)}(\vec{r}) = -\frac{1}{\rho_o C_T^2}\ \bar{S}_i^{(2)}(\vec{r})\ \bar{\mathcal{L}}_{ik}^{(-1)}(\vec{r}) \text{ - - - - - - } (A3)$$

However, if a Green's function is defined as a tensor of rank two such that:

$$\bar{\mathcal{L}}(\vec{r})\ \bar{G}_{kj}(\vec{r},\vec{r}\,') = -\frac{1}{\rho_o C_T^2}\ \delta_{ik}\partial(\vec{r}-\vec{r}\,') \text{ - - - - - - } (A4)$$

then:

$$\bar{G}_{kj}(\vec{r},\vec{r}\,') = -\frac{1}{\rho_o C_T^2}\ \delta_{ik}\ \partial(\vec{r}-\vec{r}\,')\ \bar{\mathcal{L}}_{ij}^{-(1)}(\vec{r}) \text{ - - - } (A5)$$

from which it can be seen that:

$$\bar{U}_k^{(2)}(\vec{r}) = \int_{V'} \bar{G}_{kj}(\vec{r},\vec{r}\,')\ \bar{S}_j^{(2)}(\vec{r}\,')\ d^3r' \text{ - - - - - - } (A6)$$

and the problem reduces to one of finding the Green's function that satisfies Eq. (A4). For simplicity, the term $(-\ ^1/_{\rho_o C_T^2})$ will be ignored for the present, then, Eq. (A4) becomes:

$$[(\partial_k^2 + K_T^2)\ \partial_{ik} - (1 - K_T^2/K_L^2)\ \partial_i\partial_k]\ \bar{G}_{kj}\ (\vec{r},\vec{r}\,') = \delta_{ik}\ \partial(\vec{r}-\vec{r}\,') \text{ - - - - } (A7)$$

The spatial Fourier transform pair is:

$$\tilde{f}(\gamma) = \int_V f(\vec{r}) \exp(i\vec{\gamma}\cdot\vec{R})d^3r \quad \text{------------- (A8)}$$

and:

$$f(r) = 1/(2\pi)^3 \int_\gamma \tilde{f}(\gamma) \exp(-i\vec{\gamma}\cdot\vec{R})d^3\gamma i \quad \text{--------- (A9)}$$

If the divergence of Eq. (A7) is taken and the resulting equation Fourier transformed in space coordinates, the result will be:

$$[(-\gamma^2{}_k + K_T^2)\ \gamma_i\ \delta_{ik} + (1 - K_T^2/K_L^2)\ \gamma_i\gamma_i\gamma_k]\ \tilde{\tilde{G}}_{kj}(\vec{\gamma}) = \delta_{ik}\gamma_i \quad \text{--- (A10)}$$

if

$$\lim_{r\to\pm\infty} \bar{G}_{kj}(\vec{r},\vec{r}') = 0$$

Eq. (A10) can be written as:

$$(K_T{}^2 - K_T^2/K_L^2\ \gamma_i{}^2)\ \gamma_k\ \tilde{\tilde{G}}_{kj}(\gamma) = \gamma_j \quad \text{------- (A11)}$$

from which we conclude:

$$\gamma_k\ \tilde{\tilde{G}}_{kj}(\vec{\gamma}) = -\frac{K_L^2}{K_T^2}\ \frac{\gamma_j}{\gamma_i{}^2 - K_L{}^2} \quad \text{------- (A12)}$$

Rearrangement of Eq. (A10) will result in:

$$(\gamma_k{}^2 + K_T{}^2)\ \gamma_i\ \tilde{\tilde{G}}_{ij}\ (\vec{\gamma}) = \delta_{ij}\gamma_i - (1 - K_T^2/K_L^2)\gamma_i{}^2\ \gamma_k\ \tilde{\tilde{G}}_{kj}(\vec{\gamma}) \quad \text{--- (A13)}$$

Substitution of Eq. (A12) into the above and multiplication of the equation by $^{-1}/\gamma_K^2 - K_T^2$ will result in:

$$\tilde{\tilde{G}}_{ij}(\vec{\gamma}) = \frac{-\delta_{ij}}{\gamma_k^2 - K_T^2} + \left(\frac{K_T^2}{K_L^2} - 1\right)\left(\frac{K_L^2}{K_T^2}\right)\ \frac{\gamma_i\gamma_j}{(\gamma_k^2 - K_L^2)\ (\gamma_k^2 - K_T^2)} \quad \text{-------- (A14)}$$

or

$$\tilde{\tilde{G}}_{ij}(\vec{\gamma}) = \frac{-\delta_{ij}}{\gamma_k^2 - K_T^2} + \left(1 - \frac{K_L^2}{K_T^2}\right)\ \frac{\gamma_i\gamma_j}{(\gamma_k^2 - K_L^2)\ (\gamma_k^2 - K_T^2)} \quad \text{-------- (A14b)}$$

Furthermore

$$(1 - \frac{K_L^2}{K_T^2}) \frac{1}{(\gamma_k^2 - K_L^2)(\gamma_k^2 - K_T^2)} = \frac{K_T^2 - K_L^2}{K_T^2(\gamma_k^2 - K_L^2)(\gamma_k^2 - K_T^2)}$$

and

$$\frac{K_T^2 - K_L^2}{(\gamma_i^2 - K_L^2)} = \frac{K_T^2 - K_L^2}{\gamma_i^2 - K_L^2} -1 +1 = \frac{-(\gamma_i^2 - K_T^2)}{(\gamma_i^2 - K_L^2)} + 1$$

Substitution of these results into Eq. (A14b) results in:

$$\tilde{\tilde{G}}_{ij}(\vec{\gamma}) = \frac{-\delta_{ij}}{\gamma_i^2 - K_T^2} + \frac{\gamma_i \gamma_k}{K_T^2} [\frac{1}{\gamma_i^2 - K_T^2} - \frac{1}{\gamma_i^2 - K_L^2}] \quad \text{(A15a)}$$

or:

$$\tilde{\tilde{G}}_{ij}(\vec{\gamma}) = -[\delta_{ij} - \frac{\gamma_i \gamma_j}{K_T^2}] \frac{1}{\gamma_i^2 - K_T^2} - \frac{\gamma_i \gamma_j}{K_T^2} \frac{1}{\gamma_i^2 - K_L^2} \quad \text{(A15b)}$$

This can be inverse Fourier transformed in γ space to yield:

$$\tilde{G}_{ij}(\vec{r},\vec{r}\,') = -[\delta_{ij} + \frac{\partial_i \partial_j}{K_T^2}] \frac{1}{(2\pi)^3} \int_\gamma \frac{\exp[-i\vec{\gamma}\cdot(\vec{r}-\vec{r}\,')]}{\gamma_i^2 - K_T^2} (d\gamma_i)^3$$

$$+ \frac{\partial_i \partial_k}{K_T^2} \frac{1}{(2\pi)^3} \int_\gamma \frac{\exp[-i\vec{\gamma}\cdot(\vec{r}-\vec{r}\,')]}{\gamma_i^2 - K_L^2} (d\gamma_i)^3 \quad \text{(A16)}$$

In spherical γ space, the integral

$$\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\exp[-i\vec{\gamma}\cdot\vec{R}]}{\gamma_i^2 - K_{L,T}{}^2} (d\gamma_i)^3$$

becomes:

$$\frac{1}{(2\pi)^3}\int_0^{2\pi} d\psi \int_0^{\pi} d\theta \int_0^{\infty} d\gamma \, \frac{\exp[-i\gamma R\cos\theta]\,\sin\theta\,\gamma^2}{\gamma_i^2 - K_{L,T}{}^2} \quad \text{--------} \quad (A17)$$

The tesserial part can be integrated by use of real variable techniques.

$$\int_0^{2\pi} d\psi \int_0^{\pi} d\theta \, \exp[-i\gamma R\cos\theta]\,\sin\theta\,d\theta$$

$$= 2\pi \int_0^{\pi} \frac{1}{i\gamma R}\exp[-i\gamma R\cos\theta]\; i\gamma R\sin\theta\,d\theta$$

$$= \frac{4\pi}{\gamma R}\;\frac{\exp(i\gamma R) - \exp(-i\gamma R)}{2_i} = 4\pi\,\frac{\sin(\gamma R)}{\gamma R}$$

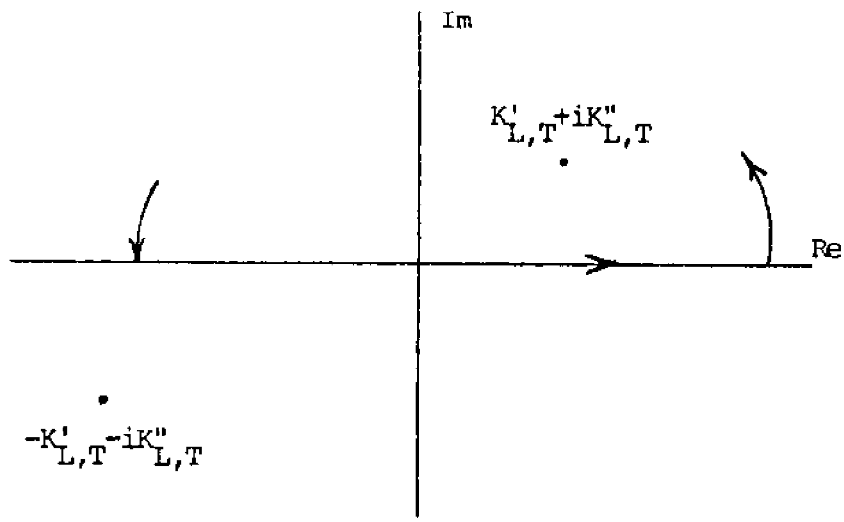
The integral (A17) becomes:

$$\frac{1}{2\pi^2}\int_0^{\infty} \frac{\sin(\gamma R)}{\gamma R}\;\frac{\gamma^2}{(\gamma^2 - K_{L,T}{}^2)}\,d\gamma \quad \text{---------------} \quad (A18)$$

The integrand is an even function, therefore the above is one half the integral from $-\infty$ to ∞. Then the integral becomes:

$$-\frac{1}{i8\pi^2 R}\int_{-\infty}^{\infty} \frac{\exp(i\gamma|\vec{R}|) - \exp(-i\gamma|\vec{R}|)}{(\gamma - K_{L,T})(\gamma + K_{L,T})}\,\gamma d\gamma \quad \text{-------------} \quad (A19)$$

The above is to be integrated by using Cauchy's theorem and therefore the contour must be chosen so that a physical Green's function results. Because we are interested in finding the waves that radiate away from the interaction volume, the terms in the Green's function must be of the type $\exp(iK|\vec{R}|)$. The first term in Eq. (A19) is integrated by using the contour shown in the sketch,

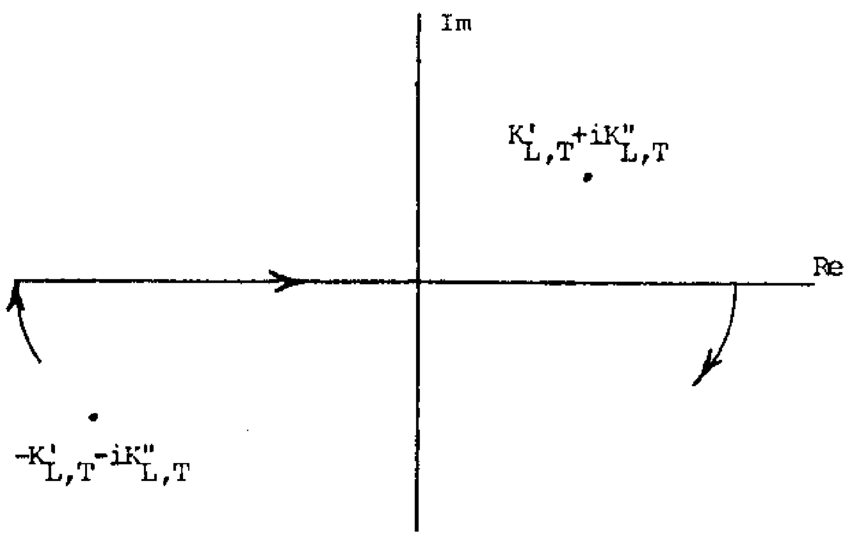


and the result is:

$$i\pi \exp(iK_{L,T}|\vec{R}|)$$

The contour for the second term is shown below and the result is:

$$-i\pi \exp(iK_{L,T}|\vec{R}|)$$



Combining the results yields:

$$-\frac{\exp(i\, K_{L,T}|\vec{R}|)}{4\pi R}$$

or

$$-\frac{\exp(i\, K_{L,T}|\vec{r}-\vec{r}\,'|)}{4\pi\, |\vec{r}-\vec{r}\,'|}$$

Thus the resulting Green's function is:

$$\bar{G}_{ij}(\vec{r},\vec{r}\,') = [\delta_{ij} - \frac{\partial_i \partial_j}{K_T^2}] \frac{\exp[iK_T|\vec{r}-\vec{r}\,'|]}{4\pi|\vec{r}-\vec{r}\,'|} + \frac{\partial_i \partial_j}{K_T^2} \frac{\exp[i\, K_L|\vec{r}-\vec{r}\,'|]}{4\pi|\vec{r}-\vec{r}\,'|}$$

The above can be expanded to yield

$$\bar{G}_{ij}(\vec{r},\vec{r}\,') = [\delta_{ij} - \hat{r}_i\hat{r}_j + \hat{r}_i\hat{r}_j \frac{2_i}{|\vec{r}-\vec{r}\,'|K_T} + \hat{r}_i\hat{r}_j \frac{2}{|\vec{r}-\vec{r}\,'|^2 K_T^2}] \frac{\exp[iK_T|\vec{r}-\vec{r}\,'|]}{4\pi|\vec{r}-\vec{r}\,'|}$$

$$+ [\hat{r}_i\hat{r}_j \frac{K_L^2}{K_T^2} - \hat{r}_i\hat{r}_j \frac{2_i K_L}{|\vec{r}-\vec{r}\,'|K_T^2} - \hat{r}_i\hat{r}_j \frac{2}{|\vec{r}-\vec{r}\,'|\, K_T^2}] \frac{\exp[iK_L|\vec{r}-\vec{r}\,'|]}{4\pi|\vec{r}-\vec{r}\,'|} \quad \text{- - -(A21)}$$

In order to obtain a far field approximation of the Green's function, all quantities involving r will be expanded.

$$|\vec{r}-\vec{r}\,'| = [(\vec{r}-\vec{r}\,') \cdot (\vec{r}-\vec{r}\,')]^{1/2} = [\vec{r} \cdot \vec{r} - 2\vec{r} \cdot \vec{r}\,' + \vec{r}\,' \cdot \vec{r}\,']^{1/2}$$

$$= r[I - 2\frac{\hat{r} \cdot \vec{r}\,'}{r} + (\vec{r}\,'/\vec{r})^2]^{1/2} = r(1 - \hat{r} \cdot \vec{r}\,' + - - - -)$$

Similarly:

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r(1 - \frac{\hat{r}\cdot\vec{r}\,'}{r} + - - -)} = \frac{1}{r} + \frac{1}{r^2}\ \hat{r} \cdot \vec{r}\,' \quad - - - - - - -$$

If we consider only the lowest terms in the expansions, the far field Green's function will be given by:

$$G_{ij}(\vec{r},\vec{r}'_j\omega) = \frac{1}{4\pi\rho_o C_T{}^2 r}(\delta_{ij}-\hat{r}_i\hat{r}_j)\exp[iK_T'(r-\hat{r}\cdot\vec{r}')]\ \exp[-K_T''(r-\hat{r}\cdot\vec{r}')$$

$$+ \frac{1}{4\pi\rho_o C_L{}^2 r}\hat{r}_i\hat{r}_j\ \exp[iK_L'(r-\hat{r}\cdot\vec{r})]\ \exp[-K_L''(r-\hat{r}\cdot\vec{r}')] \text{ ---- (A22)}$$

where the Green's function of Eq. (A21) was multiplied by and the real and imaginary parts of the wave numbers were explicitly written as:

$$K_L = K_L' + i\ K_L'' \text{ ---------------------------- (A23)}$$

$$K_T = K_T' + i\ K_T'' \text{ ---------------------------- (A24)}$$

where:

$$K_T' = \frac{\omega}{C_{To}}\left[\frac{2(1+\bar{B}^2\omega^2)}{(1+\bar{B}^2\omega^2)^{\frac{1}{2}}+1}\right]^{\frac{1}{2}}$$

$$K_T'' = \frac{\omega}{C_{To}}\left[\frac{(1+\bar{B}^2\omega^2)^{\frac{1}{2}}-1}{2(1+\bar{B}^2\omega^2)}\right]^{\frac{1}{2}}$$

$$K_L' = \frac{}{C_{Lo}}\left[\frac{2(1+\bar{A}^2\omega^2)}{(1+\bar{A}^2\omega^2)^{\frac{1}{2}}+1}\right]^{\frac{1}{2}}$$

$$K_L'' = \frac{\omega}{C_{Lo}}\left[\frac{(1+\bar{A}^2\omega^2)^{\frac{1}{2}}-1}{2(1+\bar{A}^2\omega^2)}\right]^{\frac{1}{2}}$$

$$\bar{A} = \frac{\zeta + \eta/3}{\lambda + 2\mu}$$

$$\bar{B} = \eta/\mu$$

$$C_{To} = (\mu/\rho_o)^{\frac{1}{2}}$$

$$C_{Lo} = (^{\lambda+\ 2\mu}/\rho_o)^{\frac{1}{2}}$$

These expressions are derived in Appendix C.

APPENDIX B

GREEN'S FUNCTION FOR PLANE WAVE INTERACTION

The equation to be solved is:

$$\bar{\mathcal{L}}\,(x)\ \bar{U}^{(2)}(x) = \bar{S}^{(2)}(x) \qquad \text{(B1)}$$

This equation has the solution:

$$\bar{U}^{(2)}(x) = \bar{\mathcal{L}}^{(-1)}(x)\ \bar{S}^{(2)}(x) \qquad \text{(B2)}$$

However if a Green's function is defined in such a way that

$$\bar{\mathcal{L}}\,(x)\ \bar{G}\,(x,x') = \delta(x-x') \qquad \text{(B3)}$$

or

$$\bar{G}(x,x') = \bar{\mathcal{L}}^{-1}(x)\,\delta(x-x') \qquad \text{(B4)}$$

then it can be seen that

$$\bar{U}^{(2)}(x) = \int_{x'} G(x,x')S(x')dx' \qquad \text{(B5)}$$

In this case the linear operator is:

$$\mathcal{L}(x) = \partial^2 x + K_{L,T}^{\ 2} \qquad \text{(B6)}$$

where $K_{L,T}$ is complex.

The spatial Fourier transform pair is:

$$\tilde{\bar{f}}\,(\gamma) = \int_{\ell} \bar{f}\,(x)\ \exp(i\gamma\bar{x})\ dx \qquad \text{(B7)}$$

$$\bar{f}\,(x) = 1/2\pi \int_{\gamma} \tilde{\bar{f}}\,(\gamma)\ \exp(-i\gamma\bar{x})\ d\gamma \qquad \text{(B8)}$$

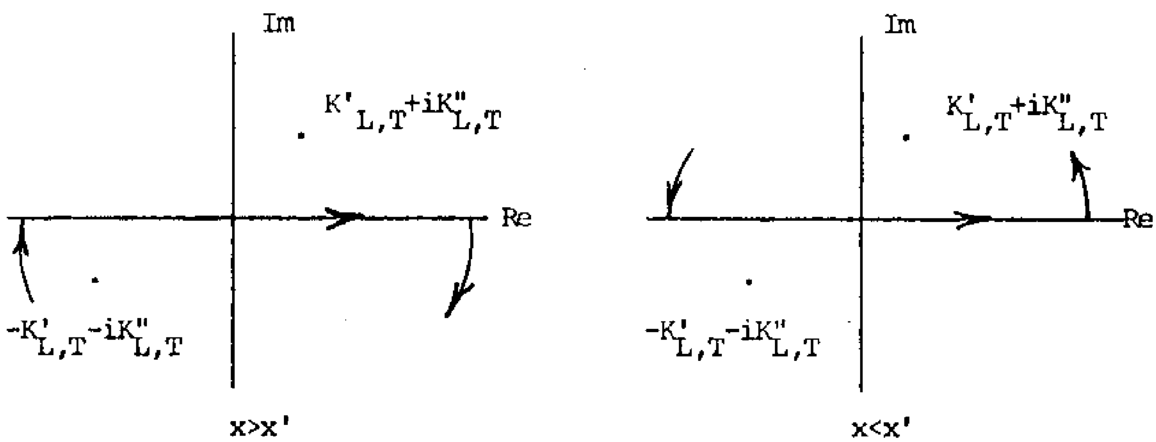
Transformation of Eq. (B3) results in:

$$(-\gamma^2 + K_{L,T}^2)\ \tilde{G}(\gamma) = \exp(i\gamma x') \quad \text{(B9)}$$

Multiplication of Eq. (B9) by $^{-1}/(-\gamma^2 + K_{L,T}^2)$ and applying inverse transformation results in:

$$G(x,x') = 1/2\pi \int_\gamma - \frac{\exp[-i\gamma(x-x')]}{(\gamma-K_{L,T})(\gamma+K_{L,T})}\, d\gamma \quad \text{(B10)}$$

This will be integrated by using Cauchy's residue theorem and the contours shown below.



The result is:

$$G(x,x') = \begin{cases} -\dfrac{i}{2K_{L,T}} \exp[i\,K'_{L,T}(x-x')]\ \exp[-K''_{L,T}(x-x')] & x>x' \\ -\dfrac{i}{2K_{L,T}} \exp[-i\,K'_{L,T}(x-x')]\ \exp[-K''_{L,T}(x-x')] & x<x' \end{cases} \quad \text{(B11)}$$

APPENDIX C

DERIVATION OF THE COMPLEX PROPAGATION VECTORS

The equation to be solved is:

$$\rho_o \partial^2{}_t U_i - \partial_t [\eta \partial^2{}_k U_i + (\zeta - 2/3\eta) \partial_i \partial_k U_k] - \mu \partial^2{}_k U_i + (\lambda+\mu) \partial_i \partial_k U_k = S_i \quad \text{- - - - (C1)}$$

The Fourier Transformation Pair is:

$$\bar{f}(\omega) = 1/2\pi \int_{-\infty}^{\infty} f(t) \exp(i\omega t) dt \quad \text{- - - - - - - - - - - - (C2)}$$

$$f(t) = \int_{-\infty}^{\infty} \bar{f}(\omega) \exp(-i\omega t) d\omega \quad \text{- - - - - - - - - - - - - (C3)}$$

If $f(t)$ is zero at $t = +\infty$ and $-\infty$, then the above equation can be transformed to:

$$-\rho_o \omega^2 \bar{U}_i + i\omega [\eta \partial^2{}_k \bar{U}_i + (\zeta - 2/3\eta) \partial_i \partial_k \bar{U}_k] - \mu \partial^2{}_k \bar{U}_i - (\lambda+\mu) \partial_i \partial_k \bar{U}_k = \bar{S}_i \quad \text{- - - (C4)}$$

Rearrangements yields:

$$\rho_o \omega^2 \bar{U}_i - [\mu - i\omega\eta] \varepsilon_{ijk} \varepsilon_{k\ell m} \partial_i \partial_\ell \bar{U}_m + [\lambda+2\mu) - i\omega(\zeta - \eta/3)] \partial_i \partial_k \bar{U}_k = \bar{S}_i \quad \text{- - - - - (C5)}$$

After taking the curl of the above equation the following results:

$$(\partial^2{}_k + K_L^2)\, \bar{U}_{i(L)} = \frac{1}{[(\lambda+2\mu) - i\omega(\zeta+\eta/3)]}\, \bar{S}_{i(L)} \quad \text{- - - - - - - - - - - - (C6)}$$

and after taking the divergence of the equation:

$$(\partial^2{}_k + K_T^2)\, \bar{U}_{i(T)} = \frac{-1}{\mu + i\omega\eta}\, \bar{S}_{i(T)} \quad \text{- - - - - - - - - - - - - - - - - - - (C7)}$$

where:

$$\partial_i U_{i(T)} = 0$$

and

$$e_{ijk}\partial_j U_{k(L)} = 0$$

and

$$K_L^2 = \frac{\rho_o \omega^2}{[(\lambda+2\mu) - i\omega(\zeta + \eta/3)]}$$

and

$$K_T^2 = \frac{\rho_o \omega^2}{\mu - i\omega\eta}$$

The remainder of the analysis will be devoted to finding the real and imaginary parts of K_L and K_T. Eq. (C8) and (C9) can be arranged to yield:

$$K_L^2 = \frac{\rho_o \omega^2}{\lambda+2\mu} \frac{(1+i\omega\bar{\Lambda})}{(1+\omega^2\bar{\Lambda}^2)} \quad \text{-------------- (C10)}$$

and

$$K_T^2 = \frac{\rho_o \omega^2}{\mu} \frac{(1+i\omega\bar{B})}{(1+\omega^2\bar{B}^2)} \quad \text{-------------- (C11)}$$

where:

$$\bar{\Lambda} = \frac{\zeta + \eta/3}{\lambda + 2\mu} \quad , \quad \bar{B} = \frac{\eta}{\mu}$$

Then:

$$K_L = \frac{\omega}{C_{Lo}} \frac{(1+i\omega\bar{\Lambda})^{\frac{1}{2}}}{(1+\omega^2\bar{\Lambda}^2)^{\frac{1}{2}}} \quad \text{-------------- (C12)}$$

$$K_T = \frac{\omega}{C_{To}} \frac{(1+i\omega\bar{B})^{\frac{1}{2}}}{(1+\omega^2\bar{B}^2)^{\frac{1}{2}}} \quad \text{-------------- (C13)}$$

where

$$C_{Lo} = \left(\frac{\lambda+2\mu}{\rho_o}\right)^{\frac{1}{2}}$$

and

$$C_{To} = \left(\frac{\mu}{\rho_o}\right)^{\frac{1}{2}}$$

The bracketed term in the numerator of each expression for k can be represented by:

$$Z = R\exp(i\theta)$$

where:

$$\exp(i\theta) = \cos\theta + i\sin\theta$$

and where

$$R_L = (1 + \omega^2\bar{A}^2)^{\frac{1}{2}}$$

$$R_T = (1 + \omega^2\bar{B}^2)^{\frac{1}{2}}$$

and

$$\theta_L = \tan^{-1}\bar{A}$$

$$\Theta_T = \tan^{-1}\bar{B}$$

also

$$\exp(\tfrac{i\Theta}{2}) = [\frac{1+\cos\Theta}{2}]^{\frac{1}{2}} + i\,[\frac{1-\cos\Theta}{2}]^{\frac{1}{2}}$$

$$\cos\Theta_L = {}^{1}/(1 + (\omega\bar{A})^2)^{\frac{1}{2}}$$

$$\cos\Theta_T = {}^{1}/(1 + (\omega\bar{B})^2)^{\frac{1}{2}}$$

Thus:

$$K_L = \frac{\omega}{C_L}\ \frac{1}{(1+\omega^2\bar{A}^2)^{\frac{1}{2}}}\ (1+\omega^2A^2)^{\frac{1}{4}}\ \left\{\left[\frac{1+\frac{1}{(1+\omega^2\bar{A}^2)^{\frac{1}{2}}}}{2}\right]^{\frac{1}{2}} + i\left[\frac{\frac{1}{(1+\omega^2\bar{A}^2)^{\frac{1}{2}}}}{2}\right]^{\frac{1}{2}}\right\}$$

and

$$K_T = \frac{\omega}{C_T}\ \frac{1}{(1+\omega^2\bar{B}^2)^{\frac{1}{2}}}\ (1+\omega^2\bar{B}^2)^{\frac{1}{4}}\left\{\left[\frac{1+\frac{1}{(1+\omega^2\bar{B}^2)^{\frac{1}{2}}}}{2}\right]^{\frac{1}{2}} + i\left[\frac{1-\frac{1}{(1+\omega^2B^2)^{\frac{1}{2}}}}{2}\right]^{\frac{1}{2}}\right\}$$

From which we conclude:

$$K_L = K_L' + iK_L'' \qquad \text{and} \qquad K_T = K_T' + iK_T''$$

where:

$$K'_L = \frac{\omega}{C_{Lo}} \left[\frac{(1+\omega^2\bar{A}^2)^{1/2} + 1}{2(1+\omega^2\bar{A}^2)} \right]^{1/2} \qquad K''_L = \frac{\omega}{C_{Lo}} \left[\frac{(1+\omega^2\bar{A}^2)^{1/2} - 1}{2(1+\omega^2\bar{A}^2)} \right]^{1/2}$$

$$K'_T = \frac{\omega}{C_{To}} \left[\frac{(1+\omega^2\bar{B}^2)^{1/2} + 1}{2(1+\omega^2\bar{B}^2)} \right]^{1/2} \qquad K''_T = \frac{\omega}{C_{To}} \left[\frac{(1+\omega^2\bar{B}^2)^{1/2} - 1}{2(1+\omega^2\bar{B}^2)} \right]^{1/2}$$