The Center for Marine and Coastal Studies — North Carolina State University

STATISTICAL PROPERTIES OF FLUID MOTION AND FLUID FORCE IN A RANDOM WAVE FIELD

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K. PAJOUHI
and
C. C. TUNG

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NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NORTH CAROLINA

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LIST OF SYMBOLS

A, $A(z,t)$, A_1	horizontal component of fluid particle acceleration at (o,z,t)
A_2	horizontal component of fluid particle acceleration at (o,z,t+τ)
A_j	polynomial function of $r_{\eta U}(o)$ and λ_1
AA	symbol used in Eq. (4.3.4)
$AR_{\overline{U}\overline{U}}(\tau)$	approximate covariance function of $\overline{U}$
$AR_{\overline{Y}\overline{Y}}(\tau)$	approximate covariance function of $\overline{Y}$
$AS_{\overline{U}\overline{U}}(n)$	approximate spectrum function of $\overline{U}$
$AS_{\overline{Y}\overline{Y}}(n)$	approximate spectrum function of $\overline{Y}$
$\overline{A}$, $\overline{A}(z,t)$	horizontal component of fluid particle acceleration at (0,z,t) considering the free surface fluctuation phenomenon
a_1, a_2	coefficients used for conditional statistical moment (Eq. (3.3.4))
BB	symbol used in Eq. (4.3.4)
B_j	functions of η
b	$= z/\sigma_\eta$, dimensionless variable
b_1, b_2	coefficients used for conditional statistical moment Eq. (4.3.5)
$C_{\cdot\mid\cdot}$	conditional covariance function
C_D	$= K_D \frac{\rho}{2} (D \times 1)$
C_M	$= K_M \rho (\frac{\pi D^2}{4} \times 1)$
$Cov(\cdot,\cdot)$	covariance function
D	diameter of pile
dA(n)	complex random function
dB(n)	complex random function
dB*(n)	complex conjugate of dB(n)

$E\{M\}$	first statistical moment (mean) of random variable M
$E\{M^2\}$	second statistical moment of random variable M
$E\{M^3\}$	third statistical moment of random variable M
$E\{\cdot \mid \cdot\}$	conditional expectation
$f(\cdot)$	probability density function
$f_{\cdot \mid \cdot}(\cdot)$	conditional probability density function
$f_{\cdot,\cdot}(\cdot,\cdot)$, $f_{\cdot,\cdot,\cdot}(\cdot,\cdot,\cdot)$	joint probability density function
$G(\cdot,\cdot,\cdot)$	a function of correlation coefficients
G_0, G_1, G_2	symbols used in Eq. (4.3.8)
g	gravitational acceleration
$H(\cdot)$	Heaviside unit function
I	symbol used in Eqs. (3.1.7) and (4.1.7)
i	$= \sqrt{-1}$, imaginary unit
J	Jacobian of transformation
j	dummy index
K_D	drag coefficient
K_M	inertia coefficient
k, k_1, k_2	wave - number
$L(b, b, r)$	$= \int_b^\infty Z(\lambda) Q\left(\frac{b - r\lambda}{\sqrt{1-r^2}}\right) d\lambda$
M, M_1	symbols used to denote events or random variables
$M_{\overline{U}}(s)$	moment generating function of $\overline{U}$
$M_{\overline{Y}}(s)$	moment generating function of $\overline{Y}$
m	an integer
$m_{\cdot \mid \cdot}$	conditional mean

n, n_1, n_2	frequency in rad./sec.
$P_r\{M\}$	Probability of event M
P, $P(z,t)$	pressure at $(0,z,t)$
$\overline{P}$, $\overline{P}(z,t)$	pressure at $(0,z,t)$ considering the free surface fluctuation
$Q(b)$	$= \int_b^\infty Z(\lambda)d\lambda$
$q(x,z,t)$	total fluid particle velocity at (x,z,t)
$R_{MM}(\cdot)$	covariance function of random variable M
$R_{MM_1}(\cdot)$	cross covariance function of random variables M, M_1
$r(\cdot)$	conditional correlation coefficient used in Eq. (4.3.4)
$r_{MM}(\cdot)$	correlation coefficient of M
$r_{MM_1}(\cdot)$	cross correlation of M, M_1
$S_{MM}(\cdot)$	spectrum of M
$S_{MM_1}(\cdot)$	cross spectrum of M, M_1
s	dummy variable
t	time or dummy variable
$t_1, t_2, \cdots, t_m$	time
U, $U(z,t)$, U_1	a random process at $(0,z,t)$
U_2	$= U(z,t+\tau)$
$\overline{U}$, $\overline{U}(z,t)$, $\overline{U}_1$	$= U(z,t)\ H(\eta(t)-z)$
V, $V(z,t)$, V_1	horizontal component of water particle velocity at $(0,z,t)$
V_2	$= V(z,t+\tau)$

$\overline{V}$, $\overline{V}(z,t)$, $\overline{V}_1$	$= V(z,t)\ H(\eta(t)-z)$, horizontal component of water particle velocity at (0,z,t) considering the free surface fluctuation phenomenon
v	dummy variable
W	mean wind speed
x	horizontal coordinate
$\hat{x}$	unit vector in x direction
Y(z,t)	horizontal component of wave force at (0,z,t)
Y_0, Y_1, Y_2	auxiliary random variables
$\overline{Y}$, $\overline{Y}(z,t)$, $\overline{Y}_1$	horizontal component of wave force at (0,z,t), considering the free surface fluctuation phenomenon
y, y_0, y_1, y_2	dummy variables
z	vertical coordinate
$\hat{z}$	unit vector in z direction
Z(y)	$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right)$
α	$= C_M \sigma_A / 2\ C_D \sigma_V^{\ 2}$
β_1, β_2	constant coefficients in spectrum of sea surface
$\eta(t)$	sea surface elevation at (0,t)
η_j, $\eta(t_j)$	sea surface elevation at $(0,t_j)$
λ, λ_0, λ_1	dummy variables
$\delta(\cdot)$	Dirac delta function
τ	time lag or dummy variable
ρ	density of water
σ_M	standard deviation of random variable M
ϕ (x,z,t)	velocity potential at (x,z,t)

Ω	dummy variable
Δ	symbol used in Eq. (3.3.4)
Δ_1	symbol used in Eq. (4.3.5)
∇	$= (\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial z} \hat{z})$, gradient operator
∇_h	$= (\frac{\partial}{\partial x})$, horizontal gradient operator
∇^2	$= (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2})$, the Laplace operator

1. INTRODUCTION

1.1. MOTIVATION OF RESEARCH

The design of marine structures requires knowledge of the forces to which they are subjected. Of primary importance are the wave forces (American Petroleum Institute, 1971). The evaluation of wave force on structures has been the subject of research for many years. For slender members, the Morison formula has been widely used.

Let Y denote the wave force per unit length of a vertical cylinder. According to the Morison formula

$$Y = C_D V|V| + C_M A$$

in which V and A are respectively the horizontal components of fluid particle velocity and acceleration at a specific point under consideration at time t. C_D and C_M are respectively the drag and inertia coefficients which are determined experimentally.

To evaluate the fluid particle velocity and acceleration analytically, the potential theory of fluid flow has been shown to be generally statisfactory (Kinsman, 1965). That is, the fluid is assumed to be incompressible, inviscid, and its motion irrotational. Under these assumptions, the fluid particle velocity is the gradient of a potential function which is governed by the Laplace equation. The fluid motion everywhere below the free surface can be determined from the solution of the Laplace equation using the boundary conditions at the free surface and at the bottom of the fluid.

In an open sea, most of the energy of the waves comes from wind. Under extreme design conditions, the waves are generated by strong winds such as hurricanes. These waves are random in nature and therefore require probabilistic descriptions.

It has been shown (Kinsman, 1965) that if the duration of the storm is long, the sea surface elvation can be reasonably represented by a Gaussian stationary process.

Utilizing these assumptions on wave characteristics, statistical properties of the random wave field and wave force were studied analytically by Borgman (1967, 1972). There are, however, some important considerations that have been overlooked. These are explained briefly in the following:

1. It is noted that the potential theory for fluid motion and hence the Morison formula are applicable everywhere below the free surface. Failure to recognize this has led past researchers (Borgman, 1967, 1972, Pierson and Holmes, 1965) to refrain from evaluating statistical properties of the wave field and wave force above the mean water level.

2. Due to fluctuations of the free surface, a fixed point on a cylinder in the vicinity of the mean water level may rise above or fall below the water. At instants when the point is above the free surface, the velocity and acceleration of fluid particles at the point are zero and the element of the cylinder experiences no wave force. The Morison formula given above, which has been the basis of derivation of wave force statistics in the past, does not reflect this phenomenon.

Preliminary studies of the effects of the free surface fluctuations on the statistical properties of wave field kinematics, pressure, and wave force were carried out recently (Tung, 1975 a, b). For the restricted statistical properties examined therein, and by comparisons made with past results, it was shown that appreciable differences were observed especially at and above the mean water level.

1.2. OBJECTIVE AND SCOPE OF RESEARCH

Due to the demonstrated importance of the effects of the free surface fluctuations phenomenon and the obvious implications on the analysis and design of marine structures, it is the purpose of this study to extend the idea developed by Tung (1975 a, b) to further derive the probability density function, mean, variance, skewness, covariance function, and spectrum of the horizontal components of fluid particle velocity, acceleration, pressure, and wave force.

Numerical results are obtained for mean wind velocity W = 40 miles per hour (mph), presented graphically and compared with those obtained previously in which the free surface fluctuation phenomenon was ignored.

In this study, the potential theory for fluid motion, carried to the first order, is used. The sea surface is assumed to be Gaussian and stationary in time (and homogeneous in space). For simplicity, waves are considered to be one-dimensional. However, the ideas underlying the derivation are general and can be extended to the two-dimensional case. Also, the waves are assumed to be in deep water, and wave force computation is based on the Morison formula.

For convenience of presentation, the description of random surface waves and those quantities associated with wave field kinematics and pressure which will be repeatedly used in the text is first recapitulated briefly.

2. DESCRIPTION OF RANDOM SEA

The waves formed on the surface of the sea are almost always random. This is especially true for wind-driven sea waves.

When the storm duration is long compared with typical wave periods as is the case in most circumstances, the surface wave elevation at a specific point can be adequately regarded as a stationary random process in time. That is, the statistical properties of the surface wave elevation at the point are independent of time (Kinsman, 1965).

Considering the sea surface elevation as a stationary random process, the statistical properties of the sea surface elevation and the associated wave field kinematics and pressure are discussed in the following sections. Materials in sections 2.1 and 2.2 are extracted mainly from Phillips (1969).

2.1. DESCRIPTION OF SEA SURFACE ELEVATION

To describe the statistical properties of the random sea surface, consider, for brevity, only one-dimensional waves. Let x be the horizontal axis in the direction of wave propagation. The z-axis is considered positive upwards with origin at the mean water level. Denote the random sea surface elevation at $x = 0$ by $z = \eta(t)$ in which t is time.

The fundamental measure of the random process $\eta(t)$ is the joint probability density function $f(\eta_1,\eta_2,\ldots,\eta_m)$ of $\eta_1 = \eta(t_1)$, $\eta_2 = \eta(t_2),\ldots,$ $\eta_m = \eta(t_m)$. That is, $f(\eta_1,\eta_2,\ldots,\eta_m)\, d\eta_1 d\eta_2 \ldots d\eta_m$ represents the probability that the surface wave elevation at a specified point $x = 0$ and at all the times t_1, t_2, ..., t_m lies within assigned limits $\eta_1,\eta_1 + d\eta_1,\eta_2,$ $\eta_2 + d\eta_2$, ..., $\eta_m,\eta_m + d\eta_m$. The joint probability density function, however, is difficult to use without further assumptions and simplifications of the random process $\eta(t)$.

If the distribution of the sea surface elevation is considered to consist of contributions arising from relatively unrelated forces originating at different times, then the sea surface, considered as the sum of the statistically independent contributions of these elements, may be assumed to be Gaussian. Gross observation of the sea appears to confirm this assumption (Kinsman, 1965). The Gaussian assumption of the sea surface elevation is therefore adopted throughout this research.

Under the Gaussian assumption, the probability law for the process is completely determined by the mean and covariance function of η (t).

Since the origin of z is selected at the mean water level, $\eta(t)$ is a zero mean process and knowledge of its covariance function suffices to determine the probability law of the process completely.

The covariance function, $R_{\eta\eta}(\tau)$, of the stationary process, $\eta(t)$, is

$$R_{\eta\eta}(\tau) = E\{(\eta(t) - E\{\eta(t)\})\ (\eta(t+\tau) - E\{\eta(t+\tau)\})\}$$

in which $E\{\cdot\}$ is the expected value of the quantity enclosed in the bracket. Due to the zero mean property of $\eta(t)$, the covariance function is simply

$$R_{\eta\eta}(\tau) = E\{\eta(t)\eta(t+\tau)\}. \qquad (2.1.1)$$

It is noted that

$$R_{\eta\eta}(o) = E\{\eta^2(t)\} = \sigma_\eta^2 \qquad (2.1.2)$$

is the mean square surface elevation or variance σ_η^2 of $\eta(t)$ since $\eta(t)$ has zero mean. The quantity σ_η which is the square root of the variance, is the standard deviation of $\eta(t)$.

Associated with the covariance function $R_{\eta\eta}(\tau)$ is the frequency spectrum $S_{\eta\eta}(n)$ of $\eta(t)$ defined as

$$S_{\eta\eta}(n) = \frac{1}{2\Pi} \int_\tau R_{\eta\eta}(\tau)\ e^{in\tau} d\tau \qquad (2.1.3)$$

in which $i = \sqrt{-1}$ is the imaginary unit, n, ranging between $-\infty$ to $+\infty$, is the frequency and the integration is over all values of τ from $-\infty$ to $+\infty$.

The inverse relation is

$$R_{\eta\eta}(\tau) = \int_n S_{\eta\eta}(n)\ e^{-in\tau} dn. \qquad (2.1.4)$$

In particular,

$$R_{\eta\eta}(o) = \int_n S_{\eta\eta}(n)\, dn \tag{2.1.5}$$
$$= \sigma_\eta^2$$

so that $S_{\eta\eta}(n)$ can be interpreted as the density of contributions of energy among all the frequencies present.

The correlation coefficient $r_{\eta\eta}(\tau)$ of $\eta(t)$ which appears in subsequent derivations, is defined as

$$r_{\eta\eta}(\tau) = R_{\eta\eta}(\tau)/\sigma_\eta^2 \tag{2.1.6}$$

with

$$r_{\eta\eta}(o) = 1.$$

Much effort on the part of oceanographers and engineers has been spent on determining the characterization of wind generated wave spectrum. The underlying theories of wind wave generation and statistical analysis of wave records are well summarized in Phillips (1969), Kinsman (1965), and Pierson and Moskowitz (1964) and are therefore not repeated here.

For engineering applications, the Pierson-Moskowitz-Kitaigorodskii spectrum is commonly used. That is, for a fully aroused sea, the one-sided frequency spectrum of the sea surface elevation, $S_{\eta\eta}(n)$, is (Pierson and Moskowitz, 1964)

$$S_{\eta\eta}(n) = \frac{\beta_1 g^2}{n^5} \exp\{-\beta_2 (\frac{g}{Wn})^4\} \quad , n>o \tag{2.1.7}$$

in which $\beta_1 = 0.81 \times 10^{-2}$, $\beta_2 = 0.74$, W is the mean wind speed and g is gravitational acceleration. This spectrum, with a cut-off frequency n = 0.8 rad/sec is used for subsequent numerical computation throughout this study.

2.2. FOURIER-STIELTJES REPRESENTATION OF SEA SURFACE ELEVATION

In dealing with random process, it is often convenient to decompose the process into Fourier components. This is particularly true when relations between the statistical properties of wave field and those of $\eta(t)$ are required. That is, the sea surface elevation may be conceived as consisting of the sum of the infinite numbers of infinitesimal harmonic waves.

The Fourier-Stieltjes representation of $\eta(t)$ is given by (Phillips, 1969)

$$\eta(t) = \int_n dB(n) e^{-int} \tag{2.2.1}$$

in which $dB(n)$ is a Gaussian, zero mean, complex random function of frequency n of component waves and the integration is over all the frequencies present.

The function $dB(n)$ has the property that

$$dB(n) = dB^*(-n)$$

in which "*" denotes the complex conjugate. This is due to the fact that $\eta(t)$ is real, so that

$$\eta(t) = \eta^*(t).$$

That is

$$\begin{aligned} \eta(t) &= \int_n dB(n) e^{-int} \\ &= \int_n dB^*(n) e^{int} \\ &= \int_n dB^*(-n) e^{-int} . \end{aligned}$$

The frequency spectrum $S_{\eta\eta}(n)$ of $\eta(t)$ may also be represented in terms of the Fourier-Stieltjes coefficient $dB(n)$ of $\eta(t)$. That is, from Eq. (2.2.1)

$$R_{\eta\eta}(\tau) = E\{\eta^*(t)\eta(t+\tau)\}$$

$$= \int_{n_1}\int_{n_2} E\{dB^*(n_1)dB(n_2)\}\, e^{i(n_1-n_2)t}\, e^{-in_2\tau}.$$

Since the process $\eta(t)$ is stationary, the covariance function $R_{\eta\eta}(\tau)$ is a function of τ only, so that the condition $n_1 = n_2$ must be satisfied, giving

$$E\{dB^*(n_1)dB(n_2)\} = \begin{cases} 0 & \text{if} \quad n_1 \neq n_2 \\ S_{\eta\eta}(n)dn & \text{if} \quad n_1 = n_2 = n \end{cases} \qquad (2.2.2.)$$

which follows by reference to Eq. (2.1.4).

It is noted that since $S_{\eta\eta}(n)$ is the expected value of the product of the complex random function $dB(n)$ and its complex conjugate, it must be real and positive for all n. Also, from Eq. (2.2.2), the different Fourier components are uncorrelated and are therefore statistically independent due to the Gaussian assumption on $\eta(t)$ (Papoulis, 1965).

2.3. FOURIER-STIELTJES REPRESENTATION OF WAVE FIELD KINEMATICS AND PRESSURE

Let the fluid be assumed to be incompressible, inviscid, and the motion irrotational. Following the exposition of Phillips (1960), there exists a velocity potential $\phi(x,z,t)$ that satisfies the Laplace equation

$$\nabla^2 \phi(x,z,t) = 0$$

everywhere below the free surface in which $\nabla^2 \equiv (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2})$ is the Laplace operator.

To relate the velocity potential and the surface elevation in a random sea, it is convenient to represent the former also in terms of a Fourier-Stieltjes integral. In deep water, it may be verified that the Fourier-Stieltjes representation of $\phi(\cdot)$ is

$$\phi(x,z,t) = \int_n dA(n)\, e^{i(kx - nt)}\, e^{|k|z} \qquad (2.3.1)$$

which satisfies the Laplace equation and the kinematic boundary condition that

$$\lim_{z \to -\infty} \bar{\phi}(x,z,t) = 0.$$

The quantity dA(n) in Eq. (2.3.1) is another Gaussian, zero mean, complex random function and k is the wave-number.

To relate dA(n) to dB(n), the nonlinear kinematic boundary condition of the free surface

$$\frac{\partial \eta}{\partial t} = \left(\frac{\partial \phi}{\partial z}\right)_\eta - (\nabla_h \phi)_\eta\, \nabla_h \eta$$

is used in which $\nabla_h \equiv (\frac{\partial}{\partial x})$ is the horizontal gradient operator. The arguments of $\phi(x,z,t)$ and $\eta(t)$ are dropped here for convenience. Considering infinitesimal waves, it can be shown that (Phillips, 1960), to the first order of approximation,

$$dA(n) = -\frac{in}{|k|}\, dB(n).$$

Thus

$$\phi(x,z,t) = -i \int_n \frac{n}{|k|}\, dB(n)\, e^{i(kx-nt)} e^{|k|z}. \qquad (2.3.2)$$

The dispersive relationship relating frequency n with wave-number k, of component waves is, to the first order,

$$n^2 = gk$$

which is obtained from the dynamic boundary condition

$$g\eta = -\left(\frac{\partial \phi}{\partial t}\right)_\eta - \frac{1}{2}(\nabla \phi)^2_\eta$$

at the free surface in which $\nabla \equiv (\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial z}\hat{z})$ is the gradient operator, $\hat{x}$, $\hat{z}$ being unit vectors in the x and z directions.

Having the velocity potential $\phi(x,z,t)$, the horizontal fluid particle velocity at point (0,z) is

$$V(z,t) = \frac{\partial \phi(x,z,t)}{\partial x}\Big|_{x=o} = \int_n |n| \, dB(n) \, e^{|k|z} e^{-int} \tag{2.3.3}$$

and the horizontal fluid particle acceleration at point (0,z), to the first order, is

$$A(z,t) = \frac{\partial V(z,t)}{\partial t} = -i \int_n n|n| dB(n) \, e^{|k|z} e^{-int}. \tag{2,3,4}$$

The pressure, p(z,t), at point (0,z), is given by the Bernoulli's equation

$$P(z,t) = -\rho\left(\frac{\partial\phi(x,z,t)}{\partial t} + \frac{1}{2} q^2(x,z,t) + gz\right)\Big|_{x=o} \tag{2.3.5}$$

in which ρ is density of water and q(x,z,t) is the total fluid particle velocity at point (0,z). To the first order of approximation, the non-linear term $q^2(x,z,t)$ may be neglected and by using Eqs. (2.3.2) and (2.3.5), the Fourier-Stieltjes representation of p(z,t) is, disregarding hydrostatic pressure,

$$P(z,t) = -\rho g \int_n dB(n) \, e^{|k|z} e^{-int}. \tag{2.3.6}$$

It is noted here that Eqs. (2.3.3), (2.3.4) and (2.3.6) hold everywhere below the free surface.

In subsequent derivations of the statistical properties of wave field kinematics, pressure and wave force, several of the statistical properties of $V(z,t)$, $A(z,t)$, $P(z,t)$, and $\eta(t)$ are required. These properties are discussed in the following.

From Eqs. (2.3.3), (2.3.4), and (2.3.6) it is seen that, to the first order of approximation, $V(z,t)$, $A(z,t)$, and $p(z,t)$ are individually Gaussian process since $dB(n)$ is Gaussian and statistically independent of each other for different n. Furthermore the quantities $V(z,t)$, $A(z,t)$, and $P(z,t)$ are jointly Gaussian with themselves as well as with $\eta(t)$ since, to the first order, they are linearly related to each other and to $\eta(t)$.

Also, from Eqs. (2.3.3), (2,3,4), and (2.3.6), since $dB(n)$ is zero mean, $V(z,t)$, $A(z,t)$ and $p(z,t)$ are zero mean.

The covariance function of $V(z,t)$ and $V(z,t+\tau)$, in anticipation that it is independent of time t, is denoted by $R_{VV}(\tau)$ and is

$$R_{VV}(\tau) = E\{V^*(z,t)V(z,t+\tau)\}.$$

Thus, from Eq. (2.3.3)

$$R_{VV}(\tau) = \int_{n_1}\int_{n_2} |n_1 n_2| E\{dB^*(n_1)dB(n_2)\} e^{(|k_1|+|k_2|)z} e^{in_1 t} e^{-in_2(t+\tau)}.$$

In view of Eq. (2.2.2), the above equation can be written in terms of the frequency spectrum $S_{\eta\eta}(n)$ of $\eta(t)$. That is

$$R_{VV}(\tau) = \int_n n^2 S_{\eta\eta}(n)\, e^{2|k|z} e^{-in\tau} dn. \tag{2.3.7}$$

It is noted that the covariance function of $V(z,t)$ and $V(z,t+\tau)$ is indeed independent of t and is an even function of τ. Also, when $\tau = 0$

$$R_{VV}(0) = E\{V^2(z,t)\} \tag{2.3.8}$$

$$= \sigma_V^2$$

in which σ_V^2 is the variance of V(z,t) and σ_V is its standard deviation.

Denoting $S_{VV}(n)$ the frequency spectrum of V(z,t),

$$S_{VV}(n) = \frac{1}{2\Pi} \int_\tau R_{VV}(\tau)e^{in\tau}d\tau \tag{2.3.9}$$

with inverse relation

$$R_{VV}(\tau) = \int_n S_{VV}(n)e^{-in\tau}dn \tag{2.3.10}$$

it is seen from Eqs. (2.3.7) and (2.3.10) that

$$S_{VV}(n) = n^2 \, S_{\eta\eta}(n)e^{2|k|z} . \tag{2.3.11}$$

The correlation coefficient $r_{VV}(\tau)$ of V(z,t) and V(z,t+τ) is

$$r_{VV}(\tau) = R_{VV}(\tau) \, /\sigma_V^2 \tag{2.3.12}$$

with

$$r_{VV}(0) = 1.$$

In the same manner, it can be verified that the covariance function, standard deviation, spectrum, and correlation coefficient for A(z,t) and p(z,t) are respectively

$$R_{AA}(\tau) = \int_n n^4 \, S_{\eta\eta}(n) \; e^{2|k|z}e^{-in\tau}dn \tag{2.3.13}$$

$$\sigma_A^2 = R_{AA}(0)$$

$$S_{AA}(n) = n^4 \; S_{\eta\eta}(n) e^{2|k|z}$$

$$r_{AA}(\tau) = R_{AA}(\tau) \; / \; \sigma_A^2$$

and

$$R_{pp}(\tau) = \rho^2 g^2 \int_n S_{\eta\eta}(n) e^{2|k|z} e^{-in\tau} dn \tag{2.3.14}$$

$$\sigma_p^2 = R_{pp}(0)$$

$$S_{pp}(n) = \rho^2 g^2 \; S_{\eta\eta}(n) \; e^{2|k|z}$$

$$r_{pp}(\tau) = R_{pp}(\tau) \; / \; \sigma_p^2 \; .$$

In addition to these quantities, the cross-correlation coefficients $r_{\eta A}(\tau)$, $r_{\eta V}(\tau)$, and $r_{\eta P}(\tau)$ of $\eta(t)$ and $A(z,t)$, $\eta(t)$ and $V(z,t)$, and $\eta(t)$ and $P(z,t)$ of a given point $(0,z)$ and at time instants t and $t + \tau$, are required. These quantities may be obtained in much the same way that $r_{AA}(\tau)$, $r_{VV}(\tau)$, and $r_{pp}(\tau)$ are determined and are therefore merely given below without further derivations.

$$r_{\eta A}(\tau) = i \int_n n|n| S_{\eta\eta}(n) e^{|k|z} e^{-in\tau} dn / \sigma_\eta \sigma_A \tag{2.3.15}$$

with

$$r_{\eta A}(0) = 0$$

and

$$r_{\eta A}(\tau) = -r_{\eta A}(-\tau)$$

$$r_{\eta V}(\tau) = \int_n |n| S_{\eta\eta}(n) e^{|k|z} e^{-in\tau} dn / \sigma_\eta \sigma_V \tag{2.3.16}$$

with

$$r_{\eta V}(0) = \int_{n} |n| \; S_{\eta\eta}(n)e^{|k|z}dn/\sigma_{\eta}\sigma_{V}$$

and

$$r_{\eta V}(\tau) = r_{\eta V}(-\tau)$$

$$r_{\eta P}(\tau) = -\rho g \int_{n} S_{\eta\eta}(n)e^{|k|z}e^{-in\tau}dn/\sigma_{\eta}\sigma_{P} \qquad (2.3.17)$$

with

$$r_{\eta P}(0) = -\rho g \int_{n} S_{\eta\eta}(n)e^{|k|z}dn/\sigma_{\eta}\sigma_{P}$$

and

$$r_{\eta P}(\tau) = r_{\eta P}(-\tau).$$

3. STATISTICAL PROPERTIES OF VELOCITY, ACCELERATION, AND PRESSURE

In this chapter, expressions of the probability density function, mean, variance, skewness, covariance function, and spectrum of the horizontal component of fluid particle velocity, acceleration, and pressure, taking into consideration the free surface fluctuation effects, are derived.

The horizontal velocity, acceleration, and pressure at point (0,z) at time t given by Eqs. (2.3.3), (2.3.4), and (2.3.6) are valid everywhere below the free surface. It is recognized that due to fluctuations of the free surface, a point under consideration, especially when it is around and above the mean water level, may rise above the free surface, in which case, the quantities under consideration are all equal to zero and the above equations are no longer valid.

To take into account the free surface fluctuations phenomenon, the horizontal component of velocity, acceleration, and pressure at point (0,z) at time t should be written as

$$\bar{V}(z,t) = V(z,t)\ H(\eta(t)-z) \tag{3.1}$$

$$\bar{A}(z,t) = A(z,t)\ H(\eta(t)-z) \tag{3.2}$$

and

$$\bar{P}(z,t) = P(z,t)\ H(\eta(t)-z) \tag{3.3}$$

in which $H(\cdot)$ is the Heaviside unit function. That is

$$H(y) = \begin{cases} 0 & \text{if} \quad y < 0 \\ 1 & \text{if} \quad y > 0 \,. \end{cases}$$

Examination of Eqs. (3.1), (3.2), and (3.3) shows that while $V(z,t)$, $A(z,t)$, and $P(z,t)$ are Gaussian, $\bar{V}(z,t)$, $\bar{A}(z,t)$, and $\bar{P}(z,t)$, being nonlinear functions of these quantities and $\eta(t)$, are obviously non-Gaussian. Furthermore, although $V(z,t)$, $A(z,t)$, and $P(z,t)$ are zero mean, $\bar{V}(z,t)$, and $\bar{P}(z,t)$ are not as will be shown subsequently.

It is the lack of consideration of the free surface fluctuation phenomenon that is responsible for the errors committed in the past and the present study seeks to rectify.

Before proceeding to derive the statistical properties of $\bar{V}(z,t)$, $\bar{A}(z,t)$, and $\bar{P}(z,t)$, it is noted that these three quantities are similar in form as shown in Eqs. (3.1), (3.2), and (3.3). This observation suggests that their respective statistical properties can be deduced from those of a random process $\bar{U}(z,t)$ such that

$$\bar{U}(z,t) = U(z,t)\ H(\eta(t)-z) \tag{3.4}$$

in which $U(z,t)$ is stationary in time with zero mean and standard deviation σ_U and jointly Gaussian with $\eta(t)$. That is, the joint probability density function of $U(z,t)$ and $\eta(t)$ is

$$f_{U\eta}(y,\S) = \frac{1}{2\Pi\sigma_U\sigma_\eta\sqrt{1-r_{\eta U}^2(0)}} \exp\{-\frac{1}{2(1-r_{\eta U}^2(0))}[(\frac{y}{\sigma_U})^2-2r_{\eta U}(0)\frac{y\S}{\sigma_U\sigma_\eta}+(\frac{\S}{\sigma_\eta})^2]\} \quad (3.5)$$

in which $r_{\eta U}(0)$ is the cross-correlation coefficient of $\eta(t)$ and $U(z,t)$.

3.1. PROBABILITY LAW OF VELOCITY, ACCELERATION, AND PRESSURE

To derive the probability density function of $\bar{U}(z,t)$, the theorem of total probability is used (Papoulis, 1965). That is

$$f_{\bar{U}}(y) = Pr[\eta \leq z]\, f_{\bar{U}|\eta \leq z}(y) + Pr[\eta > z]\, f_{\bar{U}|\eta > z}(y) \quad (3.1.1)$$

in which $Pr[\cdot]$ is the probability of the event enclosed in the bracket, $f_{\bar{U}}(.)$ is the probability density function of $\bar{U}$ and $f_{\bar{U}|M}(\cdot)$ is the conditional probability density of $\bar{U}$ given the event M. The arguments t and z of $\eta(t)$ and $U(z,t)$, $\bar{U}(z,t)$ and subsequently the same for $V(z,t)$, $\bar{V}(z,t)$, $A(z,t)$, $\bar{A}(z,t)$, $P(z,t)$, and $\bar{P}(z,t)$ are dropped for convenience.

Noting that η is a zero mean, Gaussian process

$$Pr[\eta > z] = Q(b) \quad (3.1.2)$$

$$Pr[\eta \leq z] = 1-Q(b) \quad (3.1.3)$$

in which

$$Q(b) = \int_b^\infty Z(\lambda)d\lambda$$

$$Z(\lambda) = \frac{1}{\sqrt{2\Pi}} \exp(-\frac{1}{2}\lambda^2)$$

and

$$b = z/\sigma_\eta .$$

Since, given the event $[\eta<z]$, the point under consideration is above the free surface, it is obvious that

$$f_{\bar{U}|\eta\leq z}(y) = \delta(y) \quad (3.1.4)$$

in which $\delta(\cdot)$ is the Dirac delta function.

The term $f_{\bar{U}|\eta>z}(\cdot)$ is given by,

$$f_{\bar{U}|\eta>z}(y) = \int_{\S = z}^{\infty} f_{U\eta}(y,\S)d\S/Q(b) \tag{3.1.5}$$

by definition of conditional probability.

Upon substituting Eq. (3.5) into Eq. (3.1.5), performing the integration and using Eqs. (3.1.1) to (3.1.5), the probability density function $f_{\bar{U}}(\cdot)$ of $\bar{U}$ is

$$f_{\bar{U}}(y) = [1 - Q(b)]\delta(y) + \frac{1}{\sigma_U} Z(y/\sigma_U)Q\left(\frac{b-r_{\eta U}(0)y/\sigma_U}{\sqrt{1-r^2_{\eta U}(0)}}\right) . \tag{3.1.6}$$

To show

$$I = \int_{\infty}^{\infty} f_{\bar{U}}(y)dy = 1 \tag{3.1.7}$$

it suffices to substitute Eq.(3.1.6) into the above equation, giving,

$$I = \int_{-\infty}^{\infty} [1-Q(b)]\delta(y)dy + \int_{-\infty}^{\infty} \frac{1}{\sigma_U} Z\left(\frac{y}{\sigma_U}\right) Q\left(\frac{b-r_{\eta U}(0)y/\sigma_U}{\sqrt{1-r^2_{\eta}(0)}}\right)dy$$

$$= [1-Q(b)] + \int_{-\infty}^{\infty} Z\left(\frac{\S}{\sigma_\eta}\right) \int_{-\infty}^{\infty} Z\left(\frac{\frac{y}{\sigma_U} - r_{\eta U}(0)\frac{\S}{\sigma_\eta}}{\sqrt{1-r^2_{\eta U}(0)}}\right) \frac{dy}{\sigma_U\sqrt{1-r^2_{\eta U}(0)}} \frac{d\S}{\sigma_\eta} .$$

Noting that

$$\int_{-\infty}^{\infty} Z\left(\frac{\frac{y}{\sigma_U} - r_{\eta U}(0)\frac{\S}{\sigma_\eta}}{\sqrt{1 - r^2_{\eta U}(0)}}\right) \frac{dy}{\sigma_U\sqrt{1 - r^2_{\eta U}(0)}} = 1,$$

$$I = 1 - Q(b) + \int_{z}^{\infty} Z\left(\frac{\S}{\sigma_\eta}\right) \frac{d\S}{\sigma_\eta}$$

the required result is obtained.

The probability density function $f_{\bar{V}}(y)$ of the horizontal fluid particle velocity $\bar{V}$ may be obtained from Eq. (3.1.6) by replacing σ_U and $r_{\eta U}(0)$ by σ_V and $r_{\eta V}(0)$ respectively, giving

$$f_{\bar{V}}(y) = [1-Q(b)]\delta(y) + Z(\frac{y}{\sigma_V})\ Q(\frac{b-r_{\eta V}(0)\ \frac{y}{\sigma_V}}{\sqrt{1 - r^2_{\eta V}(0)}})\ /\sigma_V \tag{3.1.8}$$

in which σ_V and $r_{\eta V}(0)$ are given by Eqs. (2.3.8) and (2.3.16), respectively.

Similarly, noting that $r_{\eta A}(0) = 0$,

$$f_{\bar{A}}(y) = [1 - Q(b)]\delta(y) + Z(\frac{y}{\sigma_A})\ Q(b)\ /\sigma_A \tag{3.1.9}$$

σ_A being given by Eq. (2.3.13).

Also,

$$f_{\bar{P}}(y) = [1 - Q(b)]\delta(y) + Z\ (\frac{y}{\sigma_P})\ Q(\frac{b - r_{\eta P}(0)y/\sigma_P}{\sqrt{1 - r^2_{\eta P}(0)}})\ /\sigma_P \tag{3.1.10}$$

in which σ_P and $r_{\eta P}(0)$ are given by Eqs. (2.3.14) and (2.3.17), respectively.

From Eqs. (3.1.8), (3.1.9), and (3.1.10) it is seen that when the point under consideration is far below the mean water level $(z \to -\infty)$, the influence of the free surface fluctuation on the statistical properties of wave kinematics and pressure diminishes $(r_{\eta V}(0) \to 0,\ r_{\eta P}(0) \to 0)$ and the probability density functions of $\bar{V}(z,t)$, $\bar{A}(z,t)$, and $\bar{P}(z,t)$ approach the Gaussian distributions $f_V(y) = \frac{1}{\sigma_V} Z(\frac{y}{\sigma_V})$, $f_A(y) = \frac{1}{\sigma_A} Z(\frac{y}{\sigma_A})$, and $f_P(y) = \frac{1}{\sigma_P} Z(\frac{y}{\sigma_P})$ of V, A and P, respectively.

Also, far above the mean water level $(z \to +\infty)$, $f_{\bar{V}}(y)$, $f_{\bar{A}}(y)$, and $f_{\bar{P}}(y)$

all approach the Dirac delta function.

Numerical results of $f_{\bar{V}}(y)$, $f_{\bar{A}}(y)$, and $f_{\bar{P}}(y)$ are obtained and compared with $f_V(y)$, $f_A(y)$, and $f_P(y)$, respectively. These are shown in Figures (3.1) to (3.9) for a mean wind speed W = 40 miles per hour at three locations $z = +\sigma_\eta$, $z = 0$ and $z = -\sigma_\eta$ in which σ_η is computed to be 5.59 feet. The abscissa in these figures are non-dimensionalized.

The discrepancies between $f_{\bar{V}}(y)$ and $f_V(y)$, $f_{\bar{A}}(y)$ and $f_A(y)$, and $f_{\bar{P}}(y)$, $f_P(y)$ are clearly seen; the differences being larger for points above the mean water level ($z > o$) than below it. As the point under consideration moves below the mean water level, it is seen that the probability density functions of $\bar{V}$, $\bar{A}$, and $\bar{P}$ approach the probability density functions of V, A, and P, respectively. That $\bar{V}$ and $\bar{P}$ are skewed with non-zero mean, are seen in these figures. Also, $\bar{A}$ is unskewed with zero mean, albeit non-Gaussian.

3.2. MEAN, VARIANCE, AND SKEWNESS OF VELOCITY, ACCELERATION, AND PRESSURE

The mean, variance, and skewness of $\bar{V}$, $\bar{A}$, and $\bar{P}$ can be deduced from the corresponding quantities of $\bar{U}$. To find the mean, variance, and skewness of $\bar{U}$, it suffices to determine its first three statistical moments.

The first statistical moment $E\{\bar{U}\}$is in fact the mean, and the variance $\sigma^2_{\bar{U}}$ is given by

$$\sigma^2_{\bar{U}} = E\{\bar{U}^2\} - E^2\{\bar{U}\} \qquad (3.2.1)$$

in which $E\{\bar{U}^2\}$ is the second statistical moment of $\bar{U}$. Also, the skewness $\gamma_{\bar{U}}$ of $\bar{U}$ is

$$\gamma_{\bar{U}} = E\{(\bar{U} - E\{\bar{U}\})^3\}/\sigma^3_{\bar{U}} \qquad (3.2.2)$$

in which $E\{\bar{U}^3\}$ is the third statistical moment of $\bar{U}$.

In this study, the statistical moments of $\bar{U}$ are obtained by three

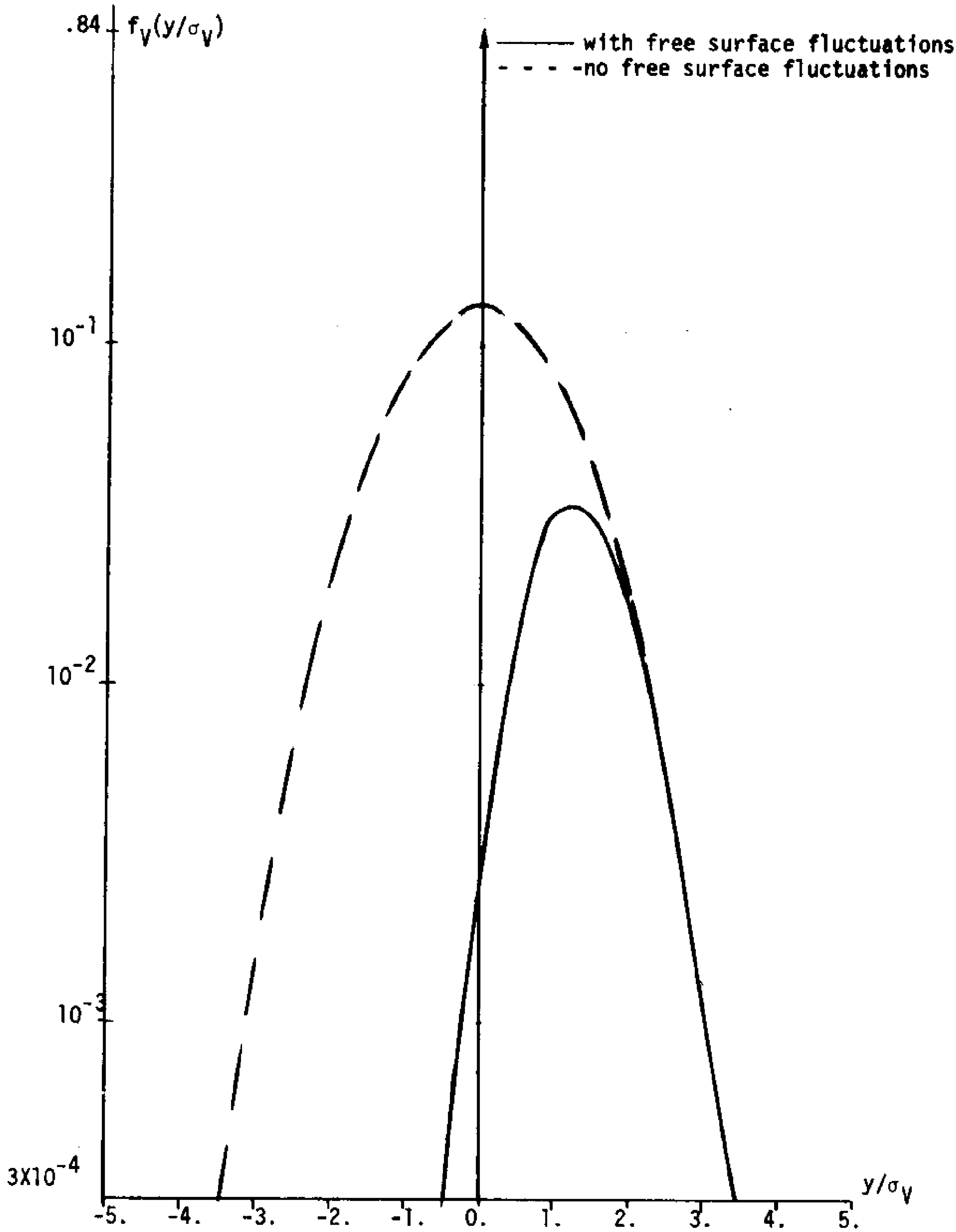

Fig. 3.1 Probability Density Function of Horizontal Component of Velocity at $z=+\sigma_\eta$ = +5.59 ft., Mean Wind Speed = 40 mph

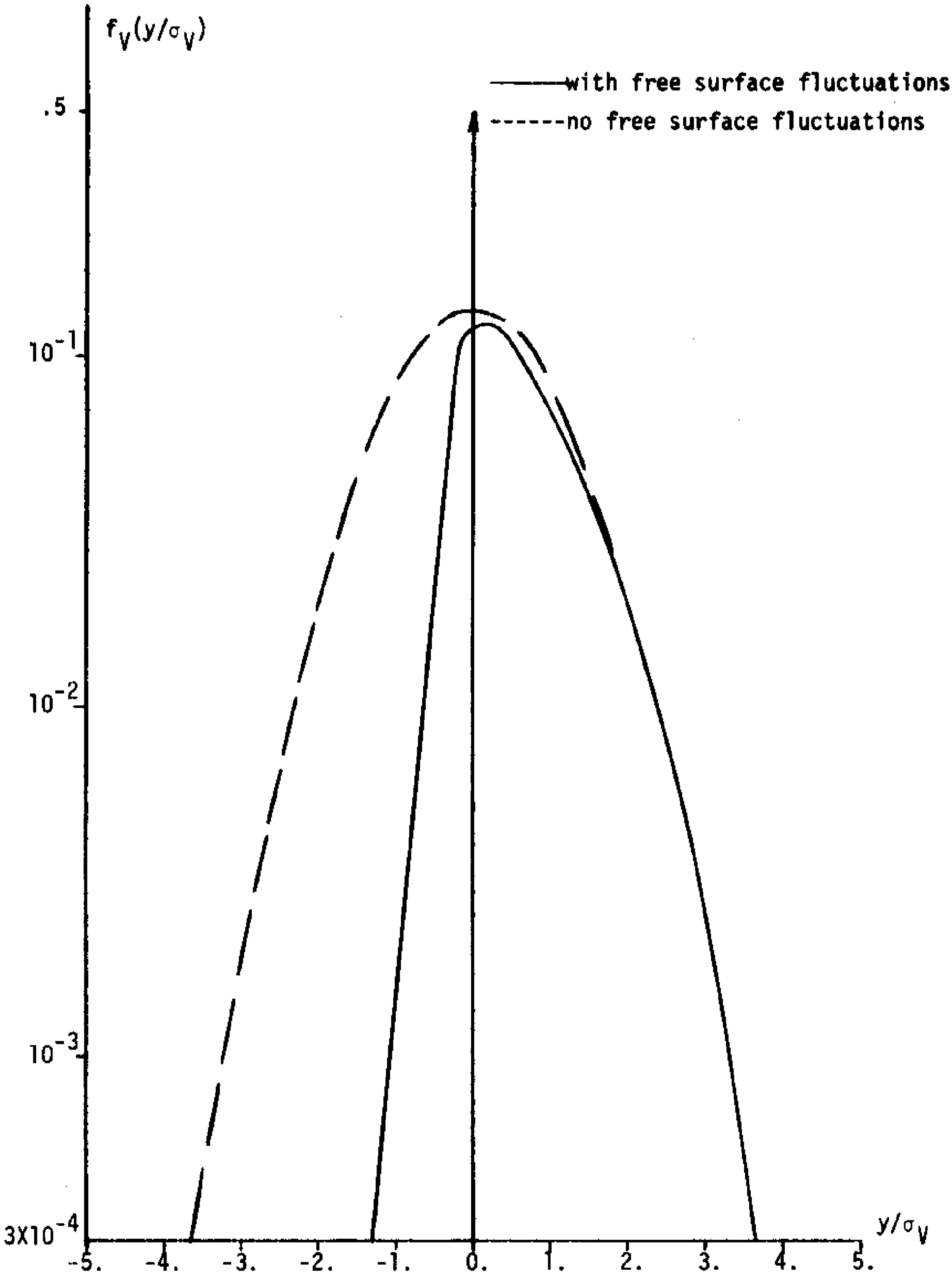

Fig. 3.2. Probability Density Function of Horizontal Component of Velocity at z = 0 ft., Mean Wind Speed = 40 mph

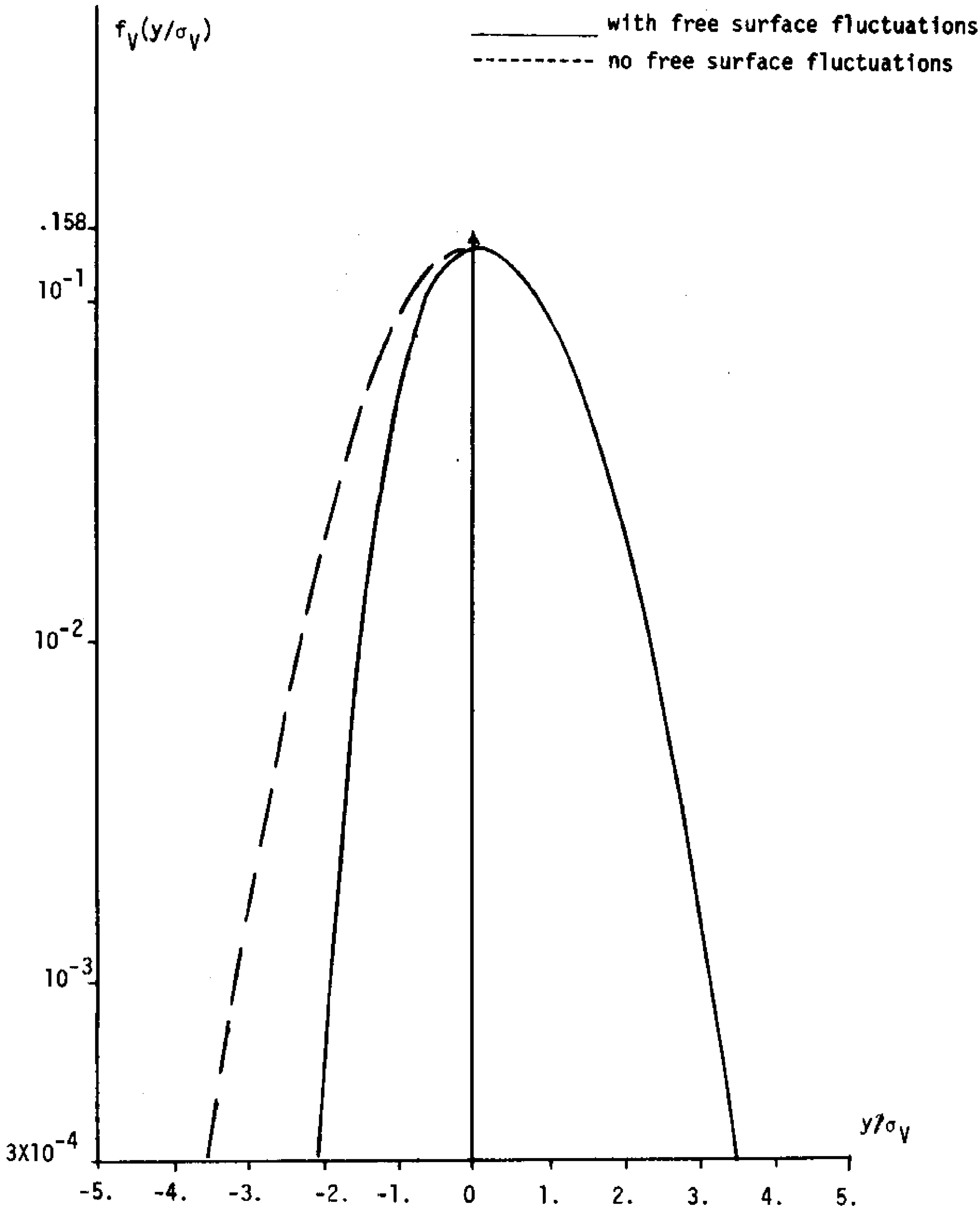

Fig. 3.3. Probability Density Function of Horizontal Component of Velocity at $z = -\sigma_\eta = -5.59$ ft., Mean Wind Speed = 40 mph

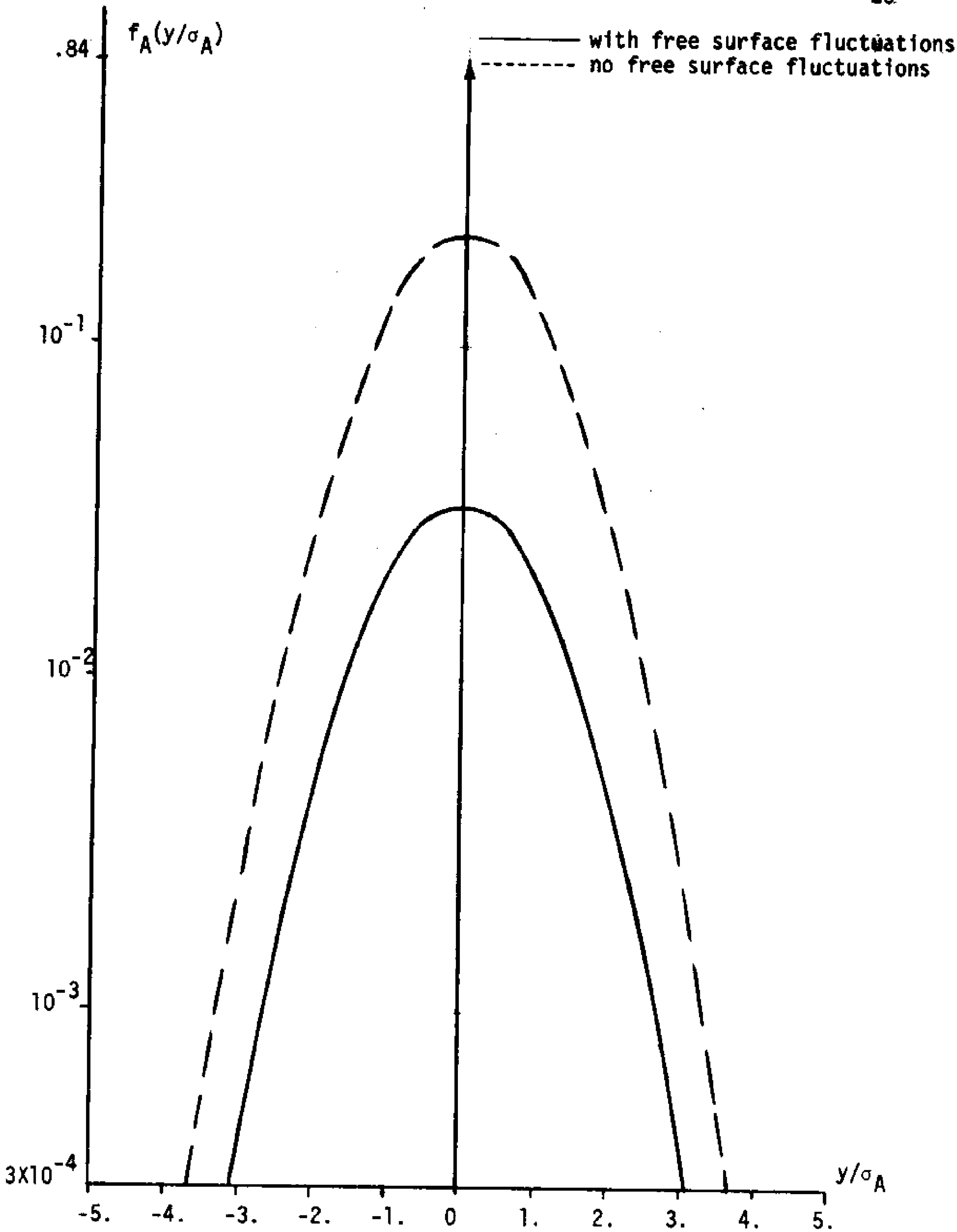

Fig. 3.4 Probability Density Function of Horizontal Component of Acceleration at $z = +\sigma_\eta = +5.59$ ft., Mean Wind Speed = 40 mph

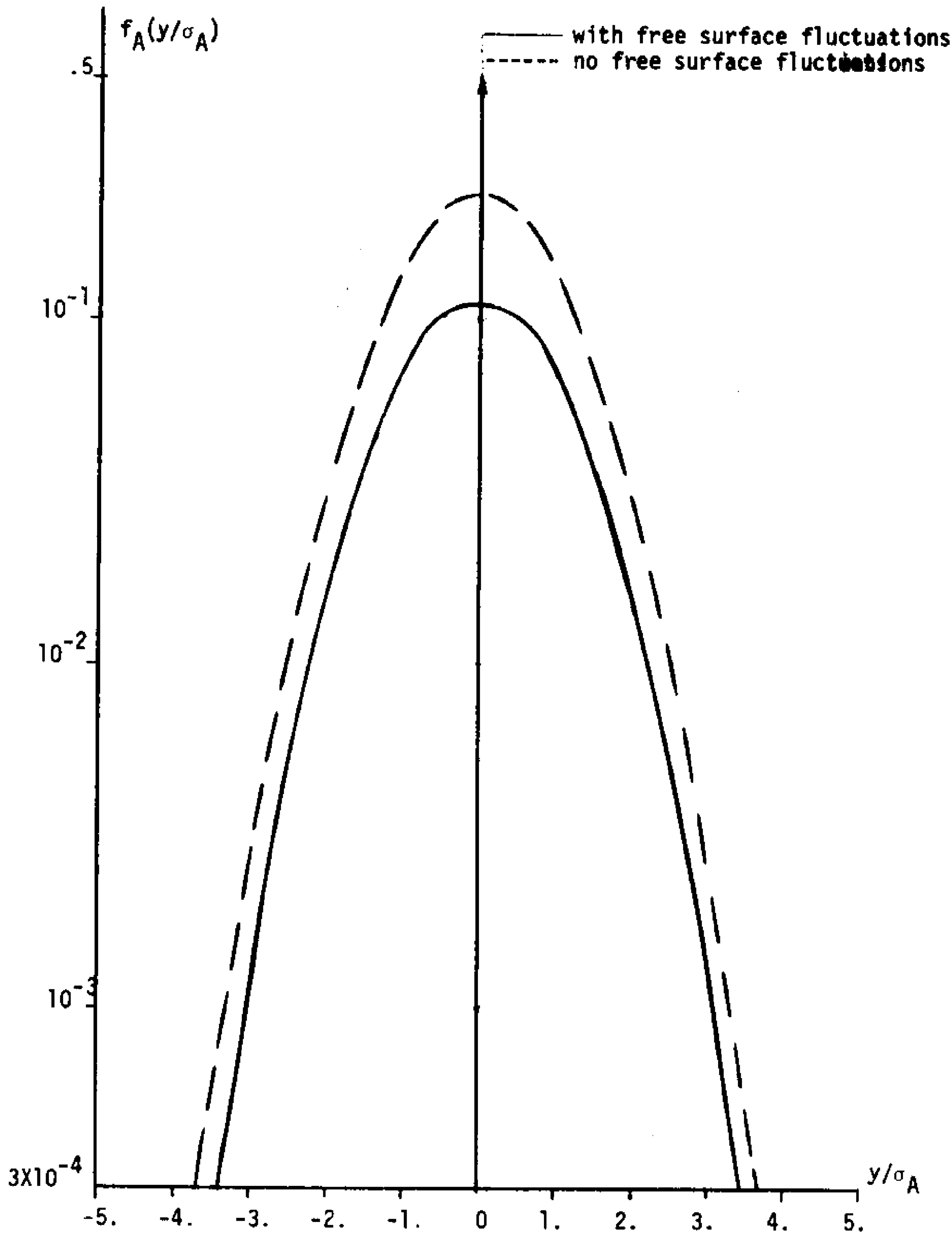

Fig. 3.5 Probability Density Function of Horizontal Component of Acceleration at z = 0 ft., Mean Wind Speed = 40 mph

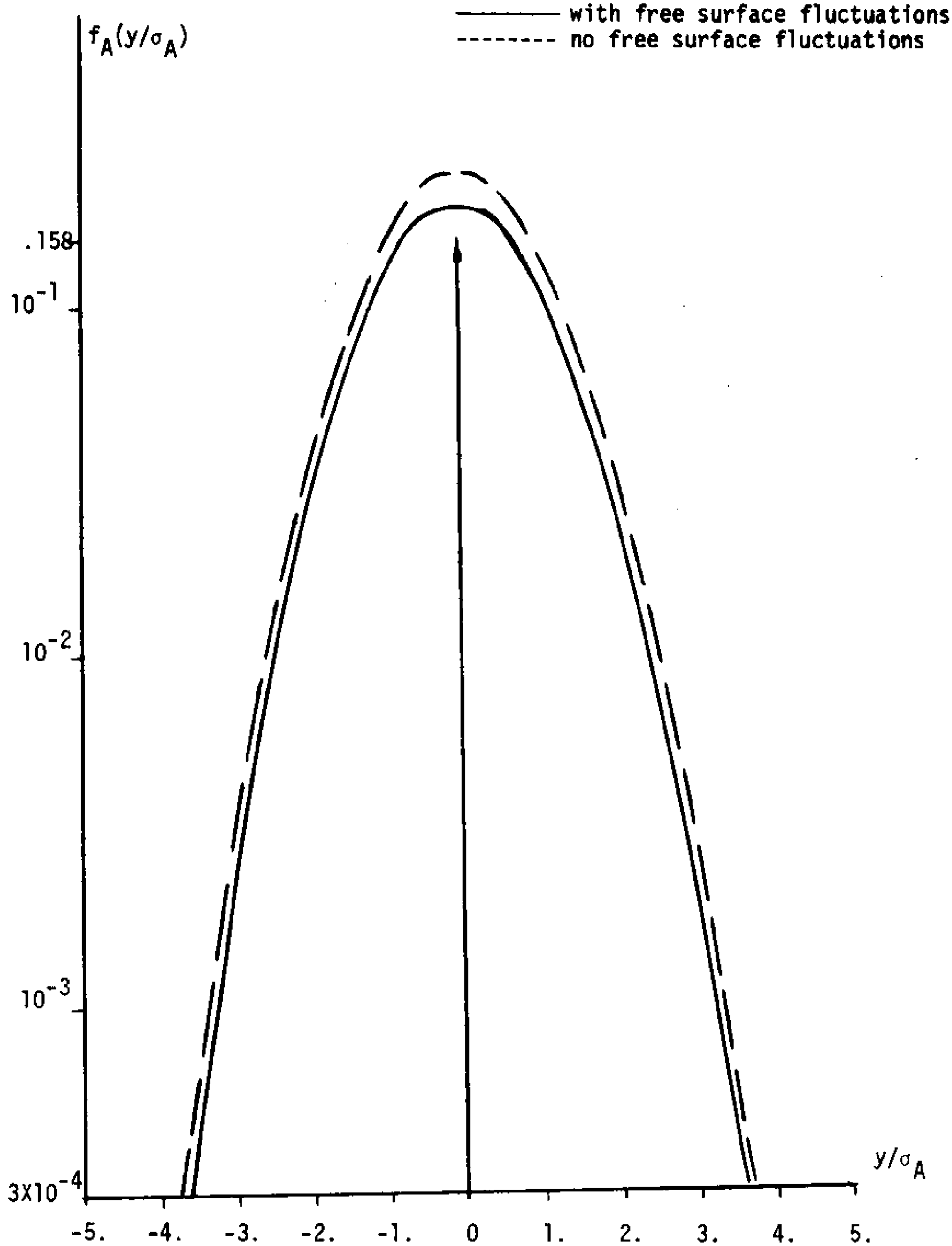

Fig. 3.6 Probability Density Function of Horizontal Component of Acceleration at $z = -\sigma_{\eta} = -5.59$ ft., Mean Wind Speed = 40 mph

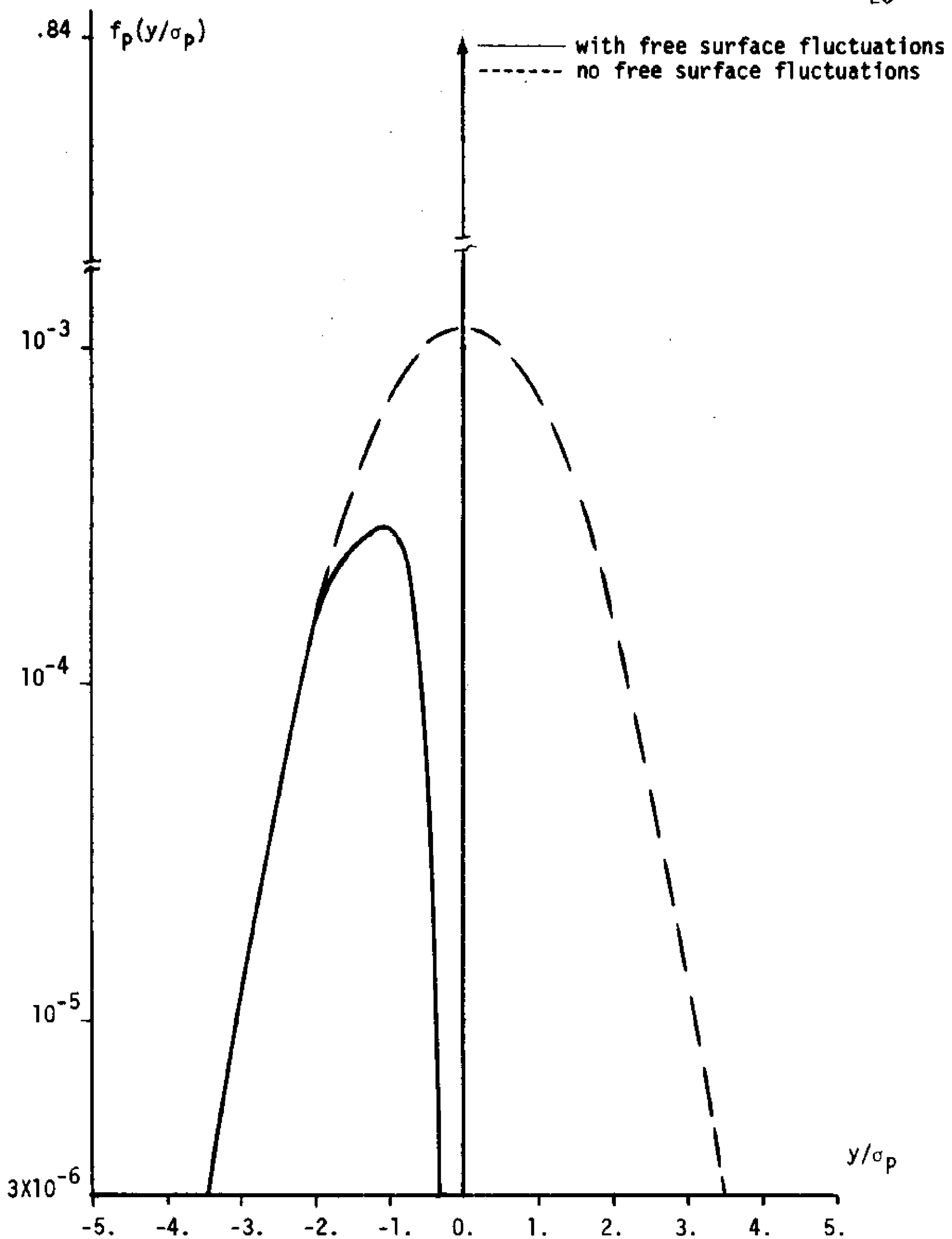

Fig. 3.7 Probability Density Function of Pressure at $z = +\sigma_\eta =$ +5.59 ft., Mean Wind Speed = 40 mph

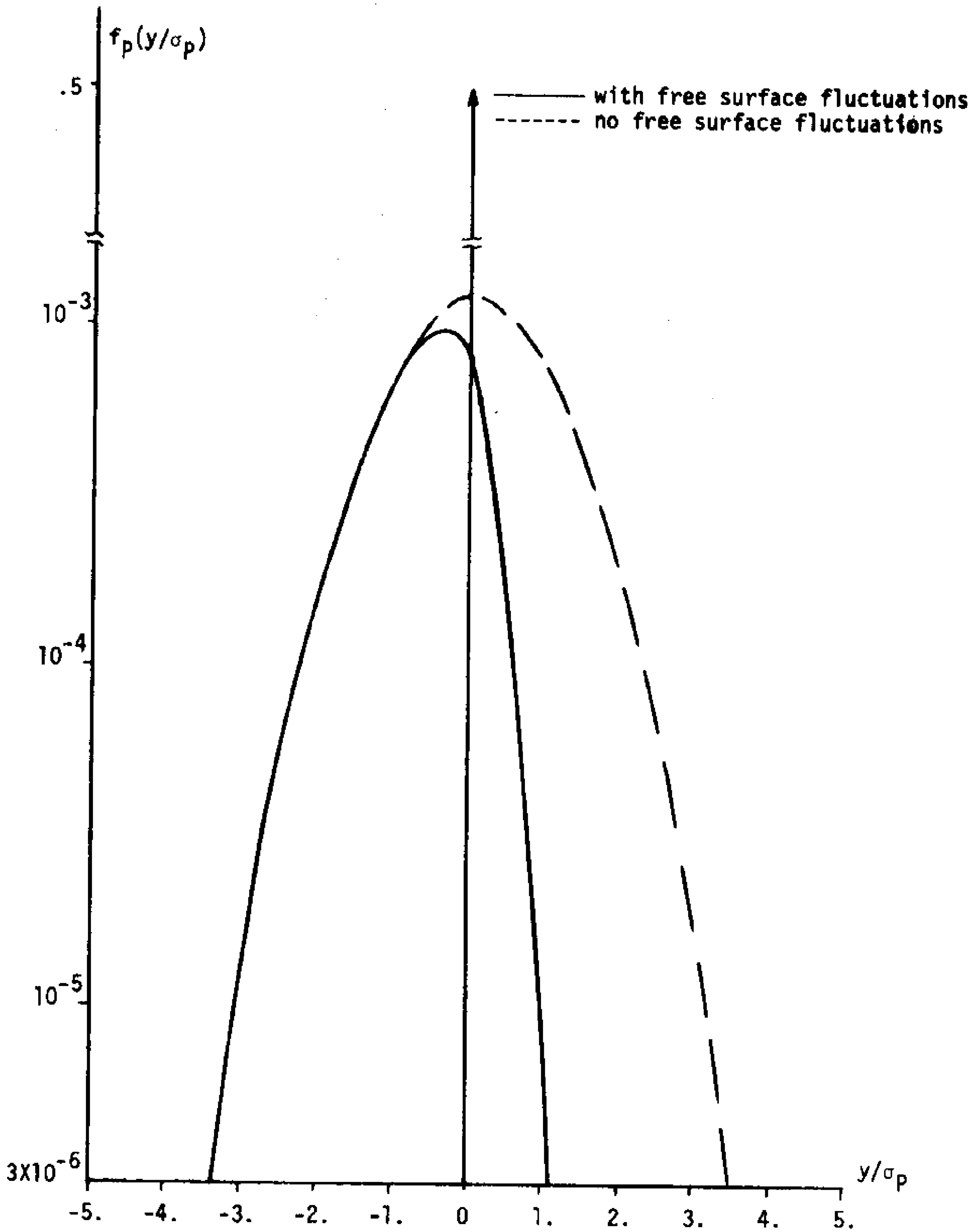

Fig. 3.8 Probability Density Function of Pressure at z = 0 ft., Mean Wind Speed = 40 mph

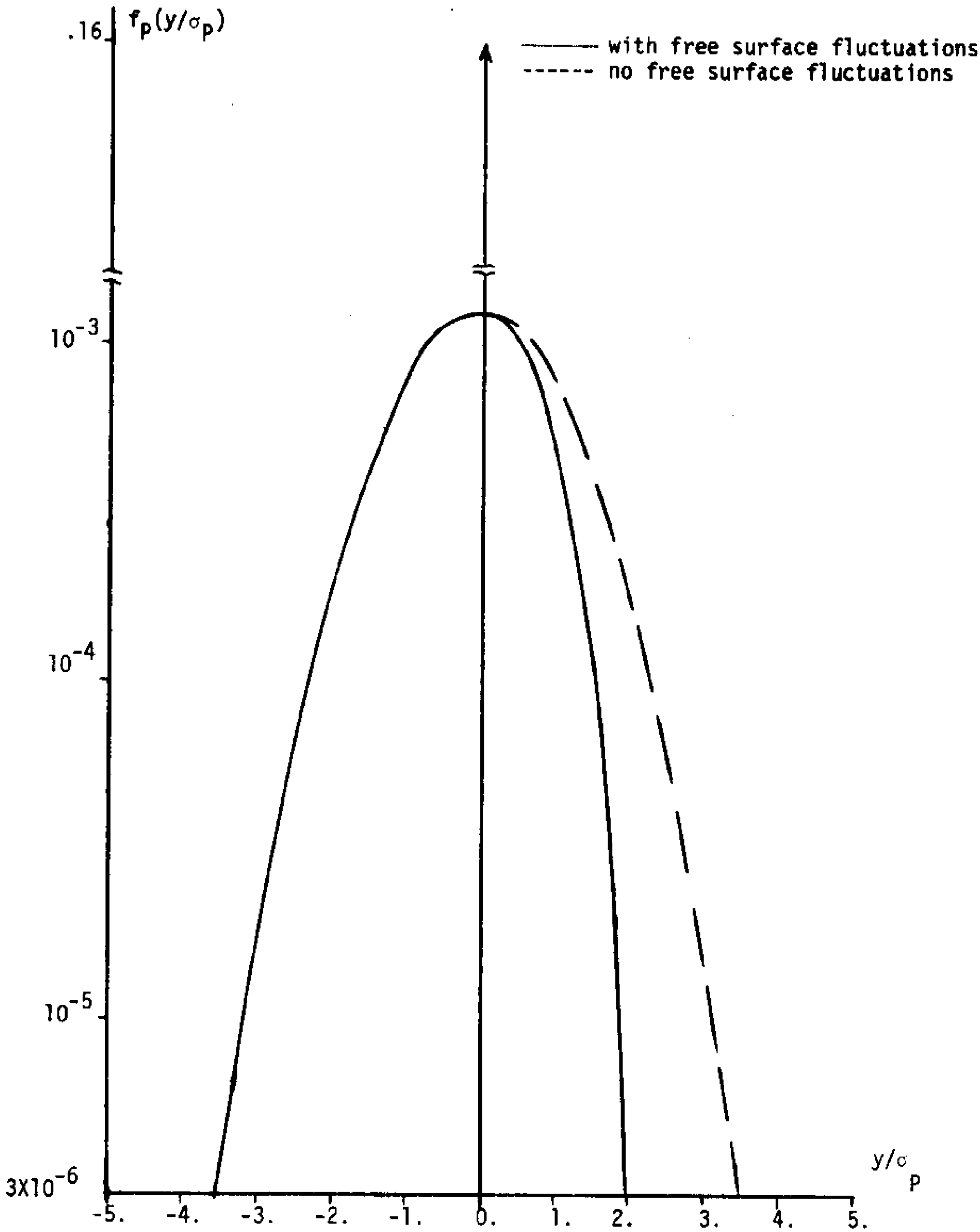

Fig. 3.9 Probability Density Function of Pressure at $z = -\sigma_\eta =$ -5.59 ft., Mean Wind Speed = 40 mph

different methods. That is, they are obtained

1. by means of the probability density function of $\bar{U}$,
2. by method of moment generating function of $\bar{U}$, and
3. directly from Eq. (3.4).

METHOD 1.

To obtain the first three statistical moments of $\bar{U}$ from its probability density function, note that by definition of statistical moments, they are

$$E\{\bar{U}^j\} = \int_{-\infty}^{\infty} y^j f_{\bar{U}}(y)dy \ , \quad j = 1, 2, 3.$$

Using Eq. (3.1.6)

$$E\{\bar{U}^j\} = \int_{-\infty}^{\infty} y^j Z(\frac{y}{\sigma_U}) \int_{z}^{\infty} Z(\frac{\frac{\S}{\sigma_\eta} - r_{\eta U}(0)\frac{y}{\sigma_U}}{\sqrt{1 - r^2_{\eta U}(0)}}) \frac{d\S}{\sigma_\eta \sqrt{1 - r^2_{\eta U}(0)}} \frac{dy}{\sigma_U}$$

$$= \int_{b}^{\infty} Z(\lambda) \int_{-\infty}^{\infty} y^j Z((\frac{y}{\sigma_U} - r_{\eta U}(0)\lambda) / \sqrt{1 - r^2_{\eta U}(0)}) dy \, d\lambda/\sigma_U\sqrt{1-r^2_{\eta U}(0)}$$

in which $\lambda = \frac{\S}{\sigma_\eta}$. Letting

$$\lambda_1 = (\frac{y}{\sigma_U} - r_{\eta U}(0)\lambda) / \sqrt{1-r^2_{\eta U}(0)}$$

the above equation becomes

$$E\{\bar{U}^j\} = \int_{b}^{\infty} Z(\lambda) \int_{-\infty}^{\infty} A_j Z(\lambda_1) \, d\lambda_1 \, d\lambda, \quad J = 1,2,3$$

in which A_j, $j = 1,2,3$, are polynomial functions of $r_{\eta U}(0)$ and λ_1. The expressions of A_j are rather lengthy and are therefore not given; the resulting integrations, however, are straightforward and can be performed easily by means of integration by parts giving

$$E\{\bar{U}\} = \sigma_U \, r_{\eta U}(0) \, Z(b) \tag{3.2.3}$$

$$E\{\bar{U}^2\} = \sigma_U^2 \, (Q(b) + r_{\eta U}^2(0) \, b \, Z(b)) \tag{3.2.4}$$

and

$$E\{\bar{U}^3\} = \sigma_U^3 \, (3 + r_{\eta U}^2(0) \, (b^2 - 1)) \, r_{\eta U}(0) \, Z(b). \tag{3.2.5}$$

METHOD 2.

The moment generating function $M_{\bar{U}}(s)$ of $\bar{U}$ is,

$$M_{\bar{U}}(s) = E\{e^{s\bar{U}}\}$$

$$= \int_{-\infty}^{\infty} e^{sy} f_{\bar{U}}(y) dy$$

by definition. Substituting Eq. (3.1.6) into the above equation

$$M_{\bar{U}}(s) = 1 - Q(b) + \int_{-\infty}^{\infty} e^{sy} \, Z\left(\frac{y}{\sigma_U}\right) Q\left(\frac{b - r_{\eta U}(0) y/\sigma_U}{\sqrt{1 - r_{\eta U}^2(0)}}\right) \frac{dy}{\sigma_U}$$

$$= 1 - Q(b) + e^{\frac{1}{2}\sigma_U^2 s^2} \int_{b - r_{\eta U}(0) s \sigma_U}^{\infty} Z(\lambda_o) d\lambda_o$$

in which $\lambda_o = \frac{\xi}{\sigma_\eta} - r_{\eta U}(0)\sigma_U s$. Therefore

$$M_{\bar{U}}(s) = 1 - Q(b) + e^{\frac{1}{2}\sigma_U^2 s^2} \, Q(b - r_{\eta U}(0) \, \sigma_U \, s).$$

The jth statistical moment of $\bar{U}$ is, by definition,

$$E\{\bar{U}^j\} = \frac{d^j}{d s^j} M_{\bar{U}}(s)\Big|_{s=o}. \tag{3.2.6}$$

which can be performed in a straightforward manner giving the same results as those obtained by Method 1.

METHOD 3.

The first three statistical moments of $\bar{U}$ may also be obtained directly from Eq. (3.4) without resort to the probability density function of $\bar{U}$ or its moment generating function. That is, take the expected value of both sides of Eq. (3.4)

$$E\{\bar{U}\} = E\{UH(\eta - z)\}.$$

The above equation can be written as (Papoulis, 1965)

$$E\{\bar{U}\} = E\{H(\eta-z)E\{U|\eta\}\} \tag{3.2.7}$$

in which $E\{U|\eta\}$ is the conditional expectation of the random variable U given the random variable η. Noting that U and η are jointly Gaussian, it follows that (Papoulis, 1965)

$$E\{U|\eta\} = r_{\eta U}(o)\ \sigma_U\ (\frac{\eta}{\sigma_\eta}).$$

Eq. (3.2.7) can then be written as

$$E\{\bar{U}\} = r_{\eta U}(o)\ \sigma_U\ E\{\frac{\eta}{\sigma_\eta} H(\eta-z)\}.$$

Thequantity $E\{\frac{\eta}{\sigma_\eta} H\ (\eta-z)\}$ is easily obtained by integration by parts, giving

$$E\{\bar{U}\} = r_{\eta U}(o)\ \sigma_U Z(b).$$

The second statistical moment of $\bar{U}$ can also be found by squaring and taking the expectation of both sides of Eq. (3.4), giving

$$E\{\bar{U}^2\} = E\{U^2H(\eta-z)\} \quad = E\{H(\eta-z)E\{U^2|\eta\}\} \tag{3.2.8}$$

in which (Papoulis, 1965)

$$E\{U^2|\eta\} = r_{\eta U}^2(0)\ \sigma_U^2(\frac{\eta}{\sigma_\eta})^2 + \sigma_U^2\ (1-r_{\eta U}^2(0)).$$

Using this result, Eq. (3.2.8) becomes

$$E\{\bar{U}^2\} = \sigma_U^2\{(1-r_{\eta U}^2(o))Q(b) + r_{\eta U}^2(o)E\{(\frac{\eta}{\sigma_\eta})^2\ H(\eta-z)\}\}$$

in which $E\{(\frac{\eta}{\sigma_\eta})^2 H(\eta-z)\}$ can again be obtained by integration by parts. After rearranging, Eq. (3.2.4) is obtained.

In much the same manner, the third statistical moment of $\bar{U}$ can be derived. Thus

$$E\{\bar{U}^3\} = E\{U^3H(\eta-z)\} \tag{3.2.9}$$

$$= E\{H(\eta-z)\ E\{U^3|\eta\}\}$$

in which (Papoulis, 1965)

$$E\{U^3|\eta\} = \sigma_U^3\ \{3(1 - r_{\eta U}^2(0))r_{\eta U}(0)\ \frac{\eta}{\sigma_\eta} + r_{\eta U}^3(0)\ (\frac{\eta}{\sigma_\eta})^3\}.$$

Substituting the above result into Eq. (3.2.9) and integrating by parts, Eq. (3.2.5) is recovered.

The statistical moments of $\bar{V}$, $\bar{A}$, and $\bar{P}$ are easily deduced from those of $\bar{U}$. They are

$$E\{\bar{V}\} = \sigma_V\ r_{\eta V}(0)\ Z(b) \tag{3.2.10}$$

$$E\{\bar{V}^2\} = \sigma_V^2\ (Q(b) + r_{\eta V}^2(0)\ bZ(b)) \tag{3.2.11}$$

and

$$E\{\bar{V}^3\} = \sigma_V^3\ (3 + r_{\eta V}^2(0)\ (b^2 - 1))\ r_{\eta V}(0)\ Z(b). \tag{3.2.12}$$

Similarly

$$E\{\bar{A}\} = E\{\bar{A}^3\} = o$$

and

$$E\{\bar{A}^2\} = \sigma_A^2 \; Q(b). \tag{3.2.13}$$

Also

$$E\{\bar{P}\} = \sigma_P \; r_{\eta P}(0) \; Z(b) \tag{3.2.14}$$

$$E\{\bar{P}^2\} = \sigma_P^2 \; (Q(b) + r_{\eta P}^2(0) \; bZ(b)) \tag{3.2.15}$$

and

$$E\{\bar{P}^3\} = \sigma_P^3 \; (3 + r_{\eta P}(0) \; (b^2 - 1)) \, r_{\eta P}(0) \; Z(b). \tag{3.2.16}$$

From Eqs. (3.2.10), (3.2.12), (3.2.14), and (3.2.16) it is seen that $\bar{V}$ and $\bar{P}$ have mean value and are skewed, as observed earlier in section 3.1, contrary to opinions commonly held in the past.

From Eqs. (3.2.10) to (3.2.16) it can be shown that as the point under consideration is far removed from the mean water level, all these quantities approach zero. In this connection, it should be mentioned that below the mean water level, the quantities $E\{\bar{V}^2\}$, $E\{\bar{A}^2\}$, and $E\{\bar{P}^2\}$ approach ${\sigma_V}^2$, ${\sigma_A}^2$, and ${\sigma_P}^2$, respectively. Also, above the mean water level, while $\sigma_{\bar{V}}$, $\sigma_{\bar{A}}$, and $\sigma_{\bar{P}}$ converge to zero as z becomes large, σ_V, σ_A, and σ_P grow indefinitely with z. It is for this reason that no attempt was made in the past to evaluate these quantities above the mean water level.

Numerical results of the mean, variance, and skewness are given in Figures (3.10) to (3.13) together with those of σ_V, σ_A, and σ_P, as function of z. The observations made earlier in regard to these quantities are vividly seen from these figures.

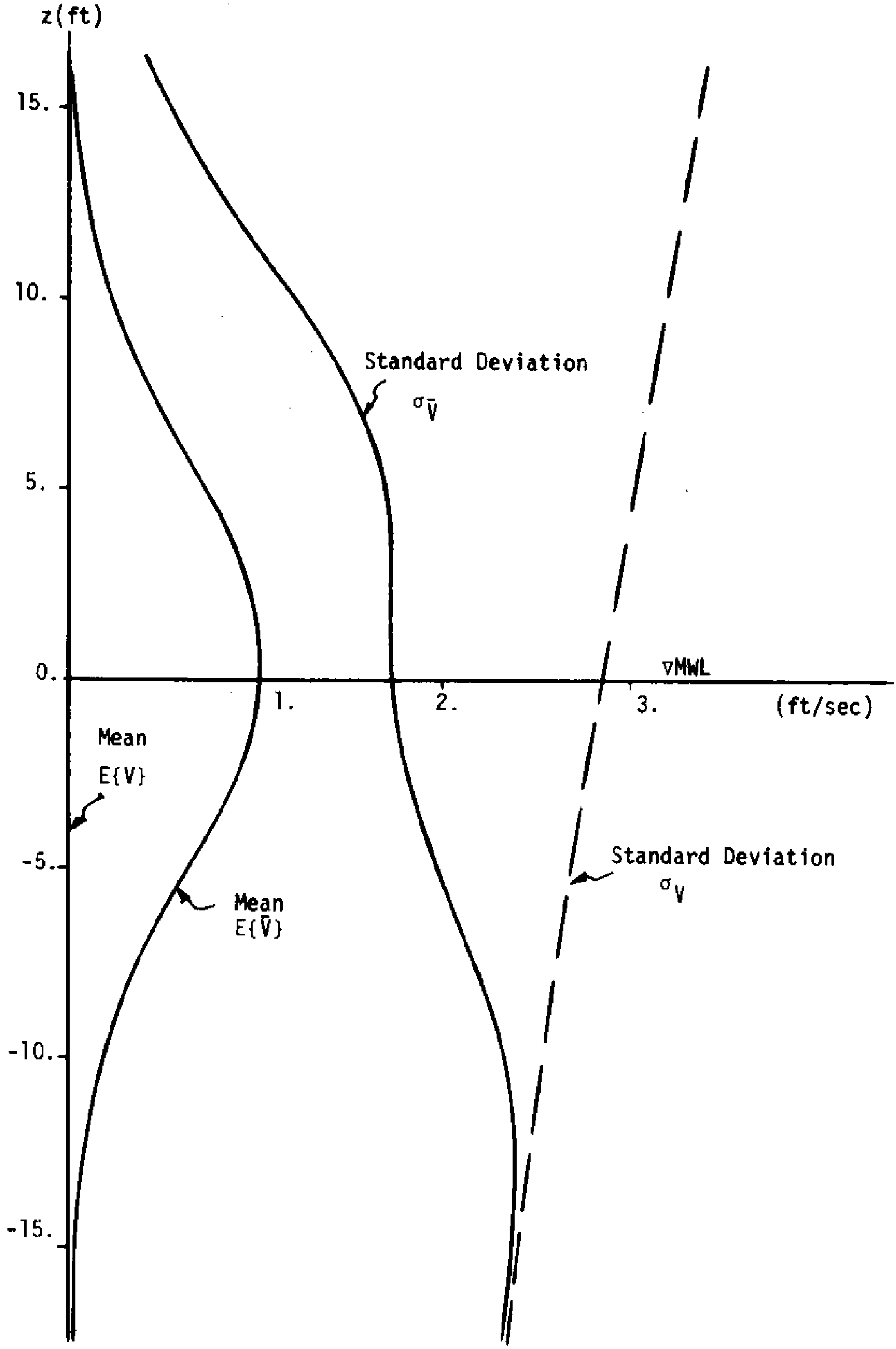

Fig. 3.10 Mean and Standard Deviation of Horizontal Component of Velocity, Mean Wind Speed = 40 mph

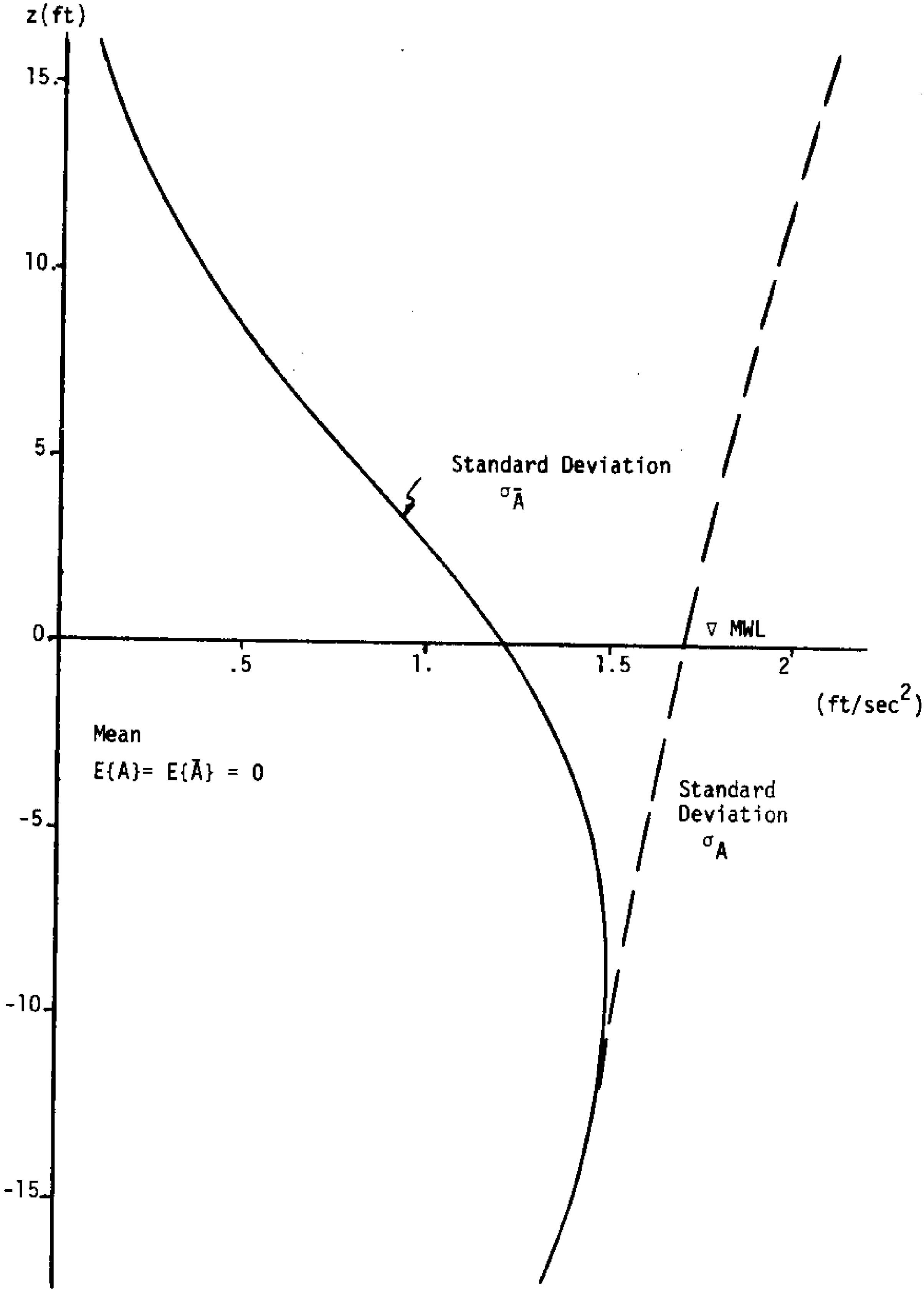

Fig. 3.11 Mean and Standard Deviation of Horizontal Component of Acceleration, Mean Wind Speed = 40 mph

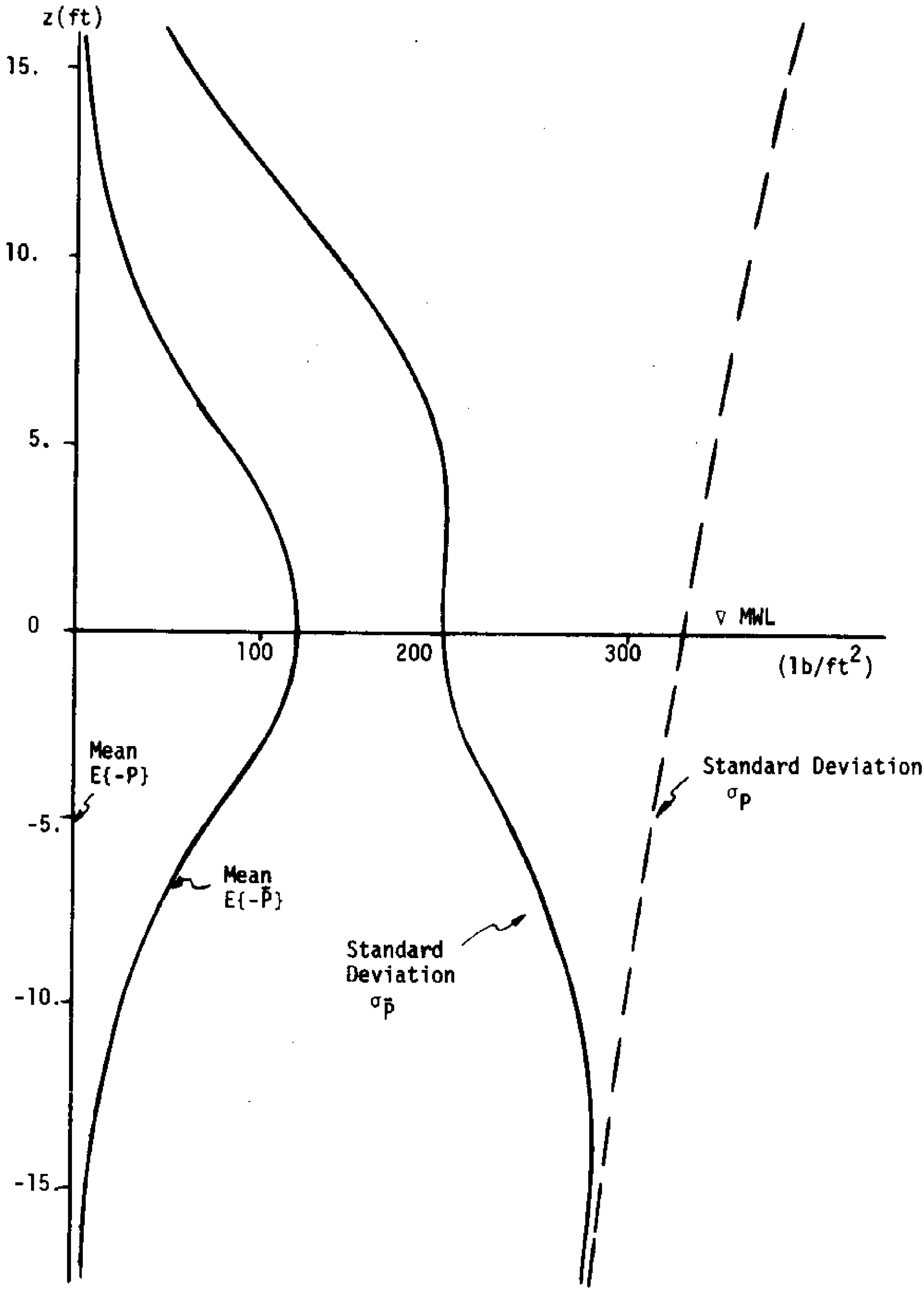

Fig. 3.12 Mean and Standard Deviation of Pressure, Mean Wind Speed = 40 mph

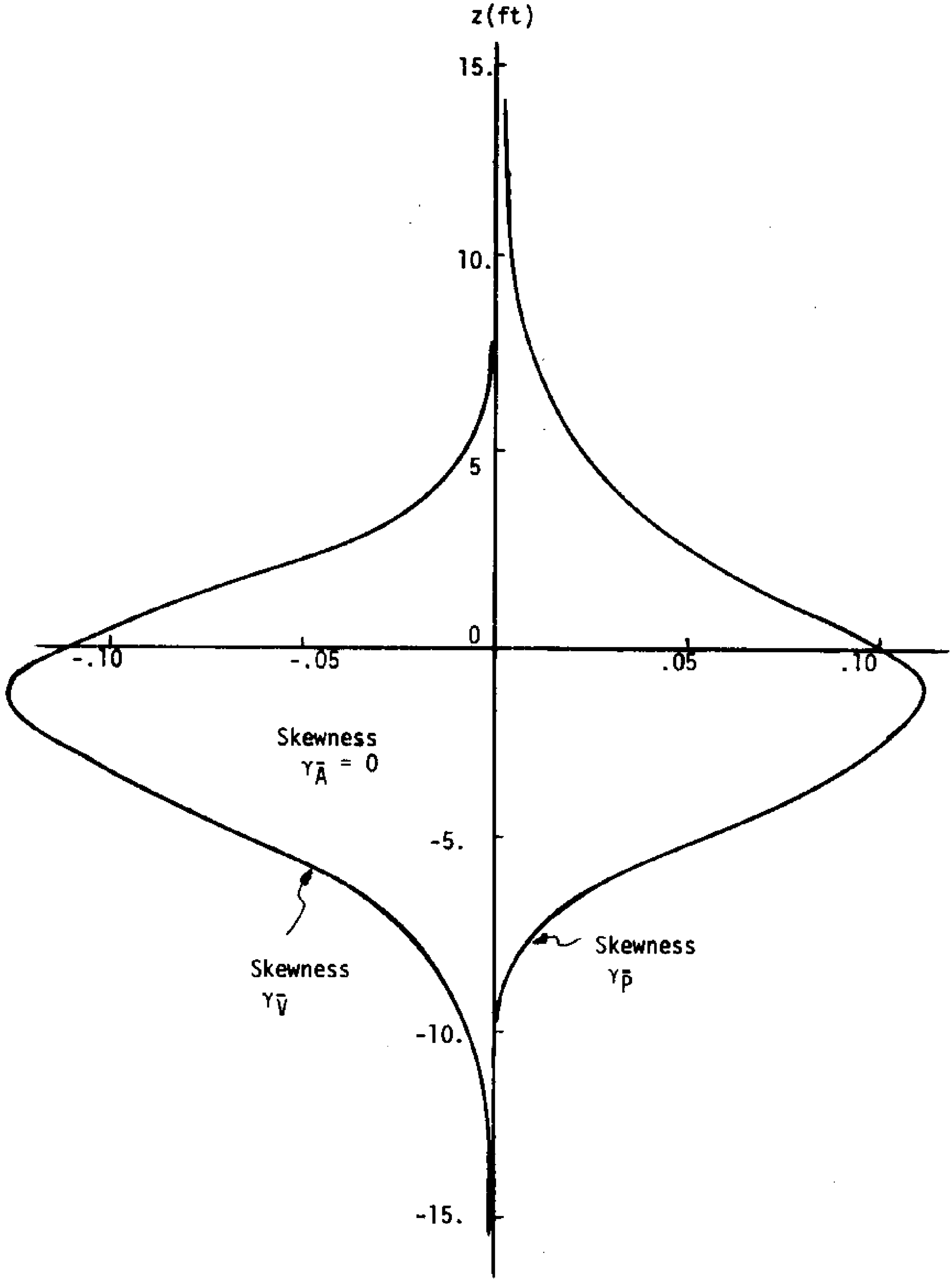

Fig. 3.13 Skewness of Horizontal Component of Velocity, Acceleration, and pressure, Mean Wind Speed = 40 mph

3.3. COVARIANCE AND SPECTRUM OF VELOCITY, ACCELERATION, AND PRESSURE

To derive the title quantities, those of $\bar{U}$ are obtained first. The covariance function of $\bar{U}(z,t)$ and $\bar{U}(z,t+\tau)$ is, by definition

$$\mathrm{COV}(\bar{U}(z,t),\ \bar{U}(z,t+\tau)) = E\{(\bar{U}(z,t) - E\{\bar{U}(z,t)\})(\bar{U}(z,t+\tau)$$

$$-E\{\bar{U}(z,t+\tau)\})\} = E\{\bar{U}(z,t)\bar{U}(z,t+\tau)\} - E^2\{\bar{U}(z,t)\} \qquad (3.3.1)$$

in which the property that $E\{\bar{U}(z,t)\} = E\{\bar{U}(z,t+\tau)\}$ is used.

For convenience, let η_1, U_1, and $\bar{U}_1$ denote $\eta(t)$, $U(z,t)$, and $\bar{U}(z,t)$, respectively and η_2, U_2, and $\bar{U}_2$ denote the corresponding quantities at $(z,t+\tau)$.

Thus

$$\bar{U}_1 = U_1\ H(\eta_1 - z)$$

$$\bar{U}_2 = U_2\ H(\eta_2 - z).$$

Multiplying these two equations and taking the expected value of both sides of the resulting product gives

$$E\{\bar{U}_1\bar{U}_2\} = E\{U_1U_2\ H(\eta_1 - z)\ H(\eta_2 - z)\} \qquad (3.3.2)$$

$$= E\{H(\eta_1 - z)\ H(\eta_2 - z)\ E\{U_1U_2|\eta_1\eta_2\}\}$$

in which $E\{U_1U_2|\eta_1\eta_2\}$ is the conditional expectation of the quantity U_1U_2 given the values of η_1 and η_2. Since U_1, U_2, η_1 and η_2 are zero mean random variables having joint Gaussian distribution, it can be shown that (Papoulis, 1965, p. 257)

$$E\{U_1U_2|\eta_1\eta_2\} = m_{U_1|\eta_1\eta_2}m_{U_2|\eta_1\eta_2} + C_{U_1U_2|\eta_1\eta_2} \qquad (3.3.3)$$

in which $m_{U_1|\eta_1\eta_2}$ is the conditional mean of U_1 given η_1 and η_2 and may be written as

$$m_{U_1|\eta_1\eta_2} = a_1\,\eta_1 + a_2\eta_2$$

a linear function of η_1 and η_2 in which

$$a_1 = \sigma_U\,\sigma_\eta^{\ 3}\,(r_{\eta U}(0) - r_{\eta\eta}(\tau)r_{\eta U}(\tau))\ /\Delta \qquad (3.3.4)$$

$$a_2 = \sigma_U\,\sigma_\eta^{\ 3}\,(r_{\eta U}(\tau) - r_{\eta\eta}(\tau)r_{\eta U}(0))\ /\Delta$$

and

$$\Delta = \sigma_\eta^{\ 4}\,(1 - r_{\eta U}^2(\tau)) \qquad (3.3.5)$$

the quantity $r_{\eta U}(\tau)$ being the cross-correlation coefficient of $\eta(t)$ and $U(z,t+\tau)$. Similarly, it can be shown that the conditional mean $m_{U_2|\eta_1\eta_2}$ of U_2 given η_1 and η_2 is $m_{U_2|\eta_1\eta_2} = a_2\eta_1 + a_1\eta_2$. The conditional covariance function $C_{U_1U_2|\eta_1\eta_2}$ of U_1U_2 given η_1 and η_2 is

$$C_{U_1U_2|\eta_1\eta_2} = \sigma_U^{\ 2}\{r_{UU}(\tau) - \sigma_\eta^{\ 4}[2r_{\eta U}(0)r_{\eta U}(\tau) - r_{\eta\eta}(\tau)(r_{\eta U}^2(0) + r_{\eta U}^2(\tau))]\ /\Delta\}$$

and is independent of η_1 and η_2; $r_{UU}(\tau)$ is the correlation coefficient of $U(z,t)$ and $U(z,t+\tau)$.

Substituting these quantities into Eq. (3.3.3) and subsequently into Eq. (3.3.2) yields

$$E\{U_1U_2H(\eta_1-z)H(\eta_2-z)\} = C_{U_1U_2|\eta_1\eta_2}E\{H(\eta_1-z)H(\eta_2-z)\} + 2a_1a_2E\{\eta_1^2H(\eta_1 - z)H(\eta_2-z)\} + (a_1^2 + a_2^2)E\{\eta_1\eta_2H(\eta_1-z)H(\eta_2-z)\}.$$

The above equation can be written as

$$E\{U_1U_2H(\eta_1-z)H(\eta_2-z)\} = \sigma_U^2\{r_{UU}(\tau)L(b,b,r_{\eta\eta}(\tau)) + 2r_{\eta U}(0)r_{\eta U}(\tau)$$

$$bZ(b)Q(\frac{b(1-r_{\eta\eta}(\tau))}{\sqrt{1-r_{\eta\eta}^2(\tau)}}) + \frac{r_{\eta U}^2(0) + r_{\eta U}^2(\tau) - 2r_{\eta U}(0)r_{\eta\eta}(\tau)r_{\eta U}(\tau)}{\sqrt{2\pi}\ \sqrt{1-r_{\eta\eta}^2(\tau)}}$$

$$Z(\frac{\sqrt{2}\ b}{\sqrt{1+r_{\eta\eta}(\tau)}})\} \qquad (3.3.6)$$

in which

$$L(b,b,r_{\eta\eta}(\tau)) = \int_b^\infty Z(\lambda)\ Q(\frac{b - r_{\eta\eta}(\tau)\lambda}{\sqrt{1-r_{\eta\eta}^2(\tau)}})d\lambda.$$

The covariance function is then obtained from Eq. (3.3.1). That it is a function of τ only and is even is clearly seen, indicating that $\bar{U}$ is covariance stationary. Let it be denoted by $R_{\bar{U}\bar{U}}(\tau)$.

The frequency spectrum $S_{\bar{U}\bar{U}}(n)$ of $\bar{U}$ can be obtained by taking the Fourier transform of $R_{\bar{U}\bar{U}}(\tau)$. That is

$$S_{\bar{U}\bar{U}}\ (n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{\bar{U}\bar{U}}(\tau)\ e^{in\tau}d\tau\,. \qquad (3.3.7)$$

The above integral can not be carried out in closed form and must be performed numerically.

In Eq. (3.3.6), it can be shown that since

$$\lim_{\tau \to 0} L(b,b,r_{\eta\eta}(\tau)) = Q(b)$$

$$\lim_{\tau \to 0} Q(\frac{b(1-r_{\eta\eta}(\tau))}{\sqrt{1-r_{\eta\eta}^2(\tau)}}) = \frac{1}{2}$$

and

$$\lim_{\tau\to 0} (r^2_{\eta U}(0) + r^2_{\eta U}(\tau) - 2r_{\eta U}(0) r_{\eta\eta}(\tau) r_{\eta U}(\tau)) / (1 - r^2_{\eta\eta}(\tau))^{1/2} = 0$$

the covariance function $R_{\bar{U}\bar{U}}(\tau)$ approaches the variance $\sigma^2_{\bar{U}}$ of the process $\bar{U}$ as τ approaches zero. That is

$$\lim_{\tau \to 0} R_{\bar{U}\bar{U}}(\tau) = \sigma^2_U \{Q(b) + r^2_{\eta U}(0) bZ(b)\}$$

$$= \sigma^2_{\bar{U}} \ .$$

When the point under consideration is far below the free surface, $r_{\eta U}(0) = r_{\eta U}(\tau) = 0$ and

$$\lim_{r_{\eta U}(\tau) \to 0} R_{\bar{U}\bar{U}}(\tau) = \sigma^2_U \, r_{UU}(\tau)$$

whereas far above the free surface

$$\lim_{z \to +\infty} R_{\bar{U}\bar{U}}(\tau) = 0.$$

The covariance function and spectrum of $\bar{V}$, $\bar{A}$, and $\bar{P}$ are obtained by simply replacing the quantities $\bar{U}$, U, by the respective appropriate counterparts and will not be repeated here.

AN APPROXIMATE REPRESENTATION

To facilitate computation of $S_{\bar{U}\bar{U}}(n)$, it is desirable that the expression for $R_{\bar{U}\bar{U}}(\tau)$ be simplified so that the Fourier transorm in Eq. (3.3.7) may be carried out in closed form.

Let $G(r_{\eta\eta}(\tau), r_{\eta U}(\tau), r_{UU}(\tau))$ be used to denote $R_{\bar{U}\bar{U}}(\tau)$ as a function of the three correlation coefficients indicated. Following Borgman (1967), the series representation of $G(\cdot)$ around $r_{\eta\eta}(\tau) = r_{\eta U}(\tau) = r_{UU}(\tau) = 0$ may be obtained by using the Taylor series expansion,

$$G(r_{\eta\eta}(\tau), r_{\eta U}(\tau), r_{UU}(\tau)) = G(0,0,0)$$

$$+ \sum_j \frac{1}{j} [(r_{\eta\eta}(\tau) \frac{\partial}{\partial r_{\eta\eta}(\tau)} + r_{\eta U}(\tau) \frac{\partial}{\partial r_{\eta U}(\tau)} + r_{UU}(\tau) \frac{\partial}{\partial r_{UU}(\tau)})^j$$

$$G(r_{\eta\eta}(\tau), r_{\eta U}(\tau), r_{UU}(\tau))]_{r_{\eta\eta}(\tau) = r_{\eta U}(\tau) = r_{UU}(\tau)=0.}$$

After some algebra, it may be verified that

$$G(0,0,0) = \sigma_U^2 \, r_{\eta U}^2(0) \, Z^2(b) - E^2\{\bar{U}\} = 0$$

$$\frac{\partial}{\partial r_{\eta\eta}(\tau)} G(\cdot) \, |_{0,0,0} = \sigma_U^2 \, r_{\eta U}^2(0) \, b^2 \, Z^2(b)$$

$$\frac{\partial}{\partial r_{\eta U}(\tau)} \, G(\cdot)|_{0,0,0} = 2\sigma_U^2 \, r_{\eta U}(0) \, bZ(b) \, Q(b)$$

$$\frac{\partial}{\partial r_{UU}(\tau)} G(\cdot)|_{0,0,0} = \sigma_U^2 \, Q^2(b).$$

By taking only the first two terms of the series (j = 1), the approximate covariance function $AR_{\bar{U}\bar{U}}(\tau)$ of $\bar{U}_1$ and $\bar{U}_2$ is,

$$AR_{\bar{U}\bar{U}}(\tau) = \sigma_U^2\{r_{\eta\eta}(\tau) r_{\eta U}^2(0) b^2 Z^2(b) + 2r_{\eta U}(\tau) r_{\eta U}(0) bZ(b) \, Q(b)$$

$$+ r_{UU}(\tau) \, Q^2(b)\}. \tag{3.3.8}$$

The corresponding approximate spectrum, $AS_{\bar{U}\bar{U}}(n)$, is, by taking the Fourier transform of Eq. (3.3.8),

$$AS_{\bar{U}\bar{U}}(n) = (\frac{\sigma_U}{\sigma_\eta})^2 \ r_{\eta U}^2(0) b^2 Z^2(b) \ S_{\eta\eta}(n) + (\frac{\sigma_U}{\sigma_\eta}) r_{\eta U}(0) b Z(b) \ Q(b)$$

$$S_{\eta U}(n) + Q^2(b) \ S_{UU}(n) \qquad (3.3.9)$$

in which $S_{\eta\eta}(n)$ is given by Eq. (2.1.3) and

$$S_{\eta U}(n) = \frac{1}{2\pi} \ \sigma_\eta \ \sigma_U \int_{-\infty}^{\infty} \ r_{\eta U}(\tau) e^{in\tau} d\tau$$

and

$$S_{UU}(n) = \frac{1}{2\pi} \ \sigma_U^2 \int_{-\infty}^{\infty} r_{UU}(\tau) e^{in\tau} d\tau. \qquad (3.3.10)$$

Again, by replacing of the quantities corresponding to $\bar{U}$ and U by their appropriate counterparts, the approximate covariance function and spectrum of $\bar{V}$, $\bar{A}$, and $\bar{P}$ are obtained.

Numerical results of the exact and approximate covariance functions and spectrum of $\bar{V}$, $\bar{A}$, and $\bar{P}$ are computed and shown in Figures (3.14) to (3.31) together with the exact covariance functions and spectra of V, A and P as obtained by Borgman (1967). The covariance functions are even functions of τ and only the portion for which $\tau \geq 0$ are shown. Also, the spectra shown are the one-sided spectra. It is seen that there is good agreement between the exact and approximate covariance functions and spectra. Appreciable differences are noted, however, between the present results and those of Borgman (1967) at z = 0 and $z = \sigma_\eta$; below the mean water level, as the influence of the free surface fluctuation lessens, so do these differences.

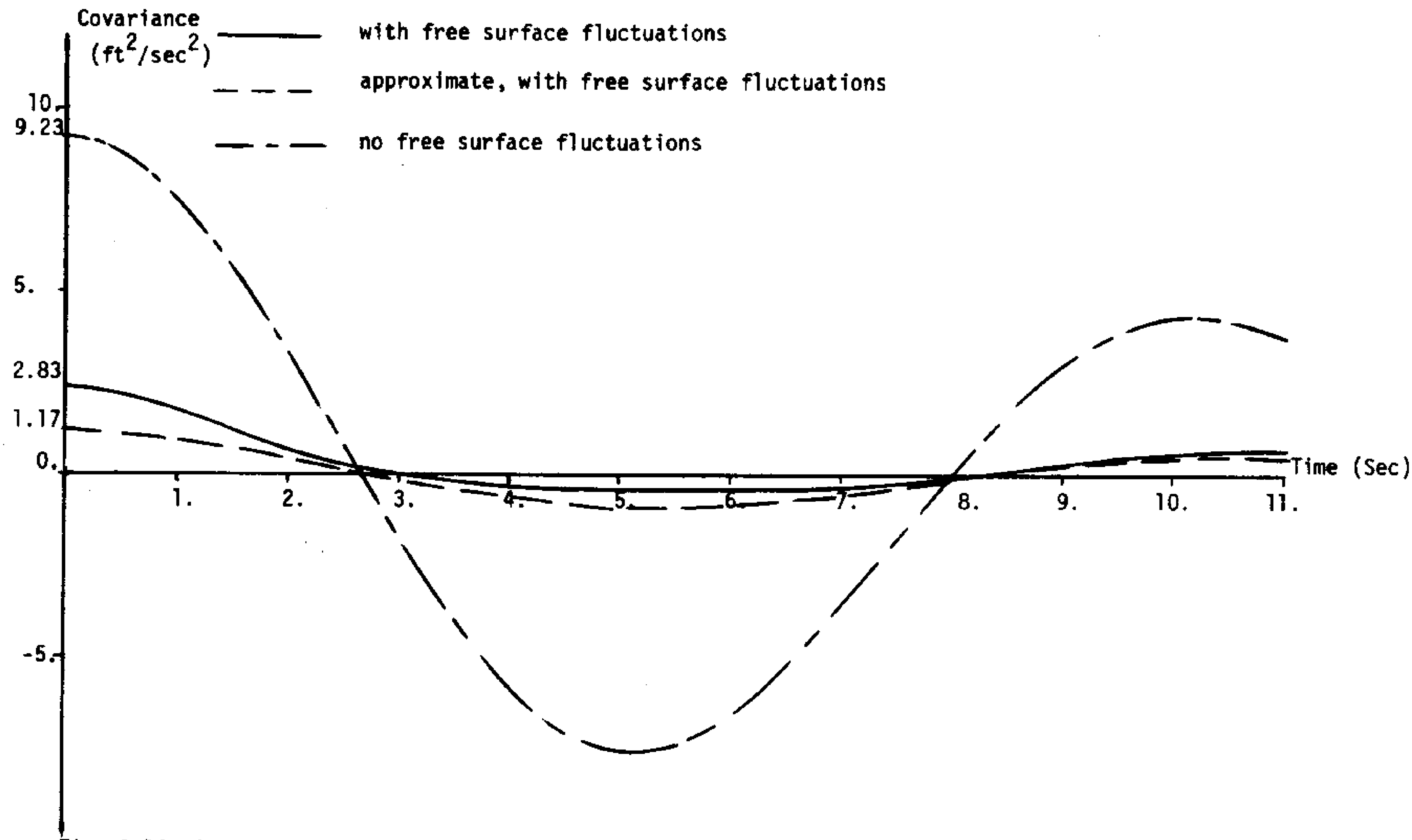

Fig. 3.14 Covariance Function of Horizontal Component of Velocity at $z = +\sigma_\eta = +5.59$ ft., Mean Wind Speed = 40 mph

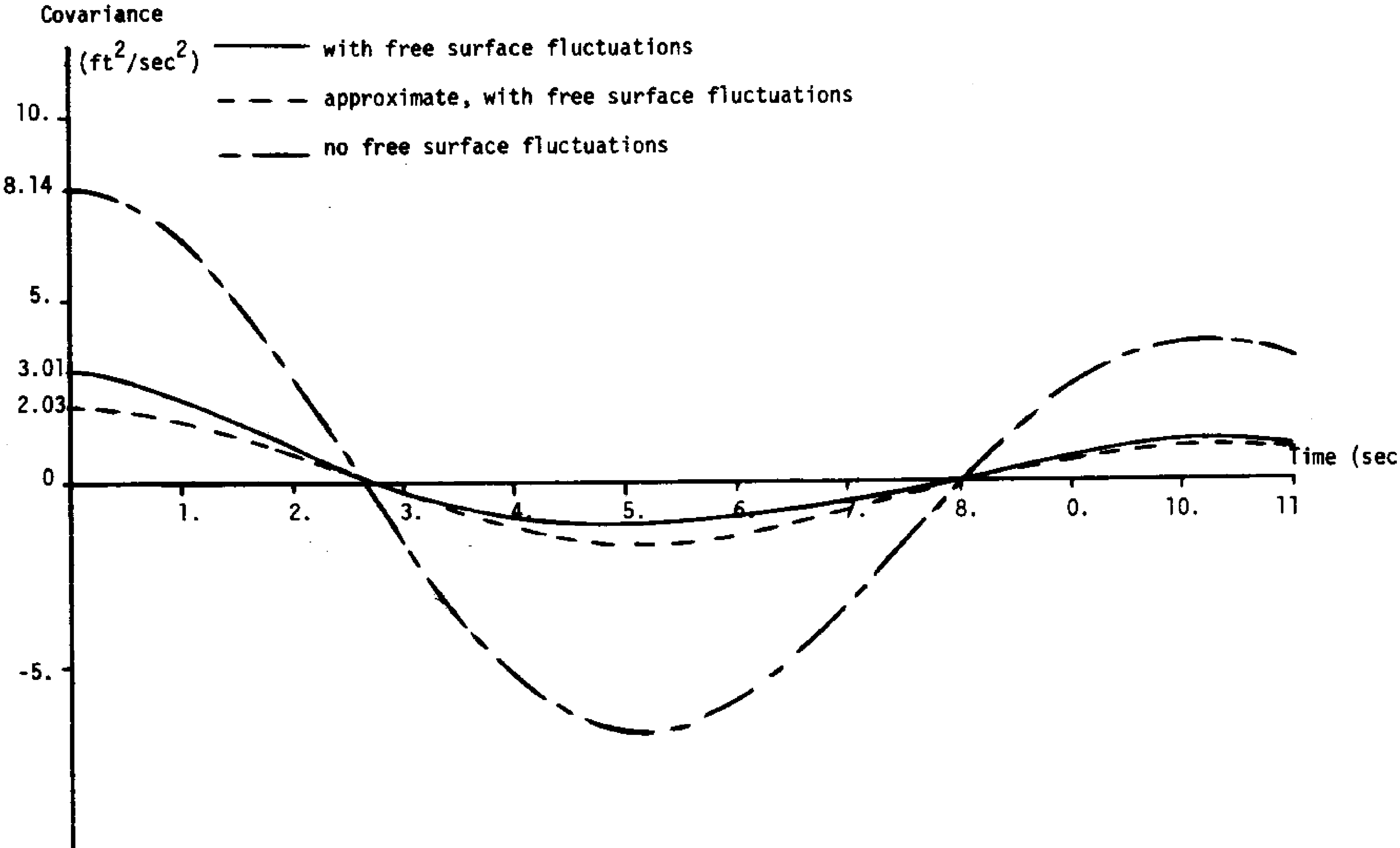

Fig. 3.15 Covariance Function of Horizontal Component of Velocity at z = 0 ft., Mean Wind Speed = 40 mph

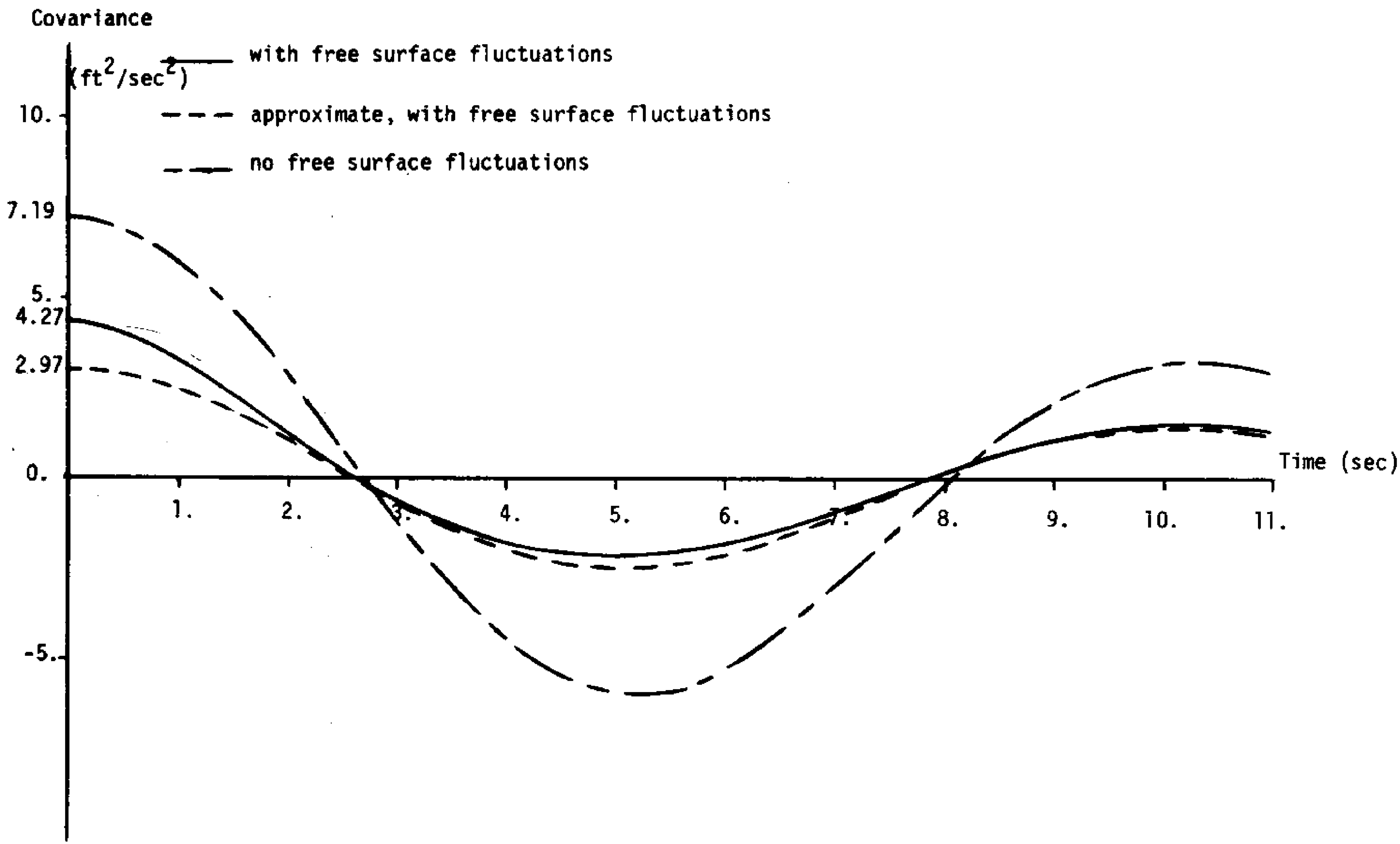

Fig. 3.16 Covariance Function of Horizontal Component of Velocity at $z = -\sigma_\eta = -5.59$ ft., Mean Wind Speed = 40 mph

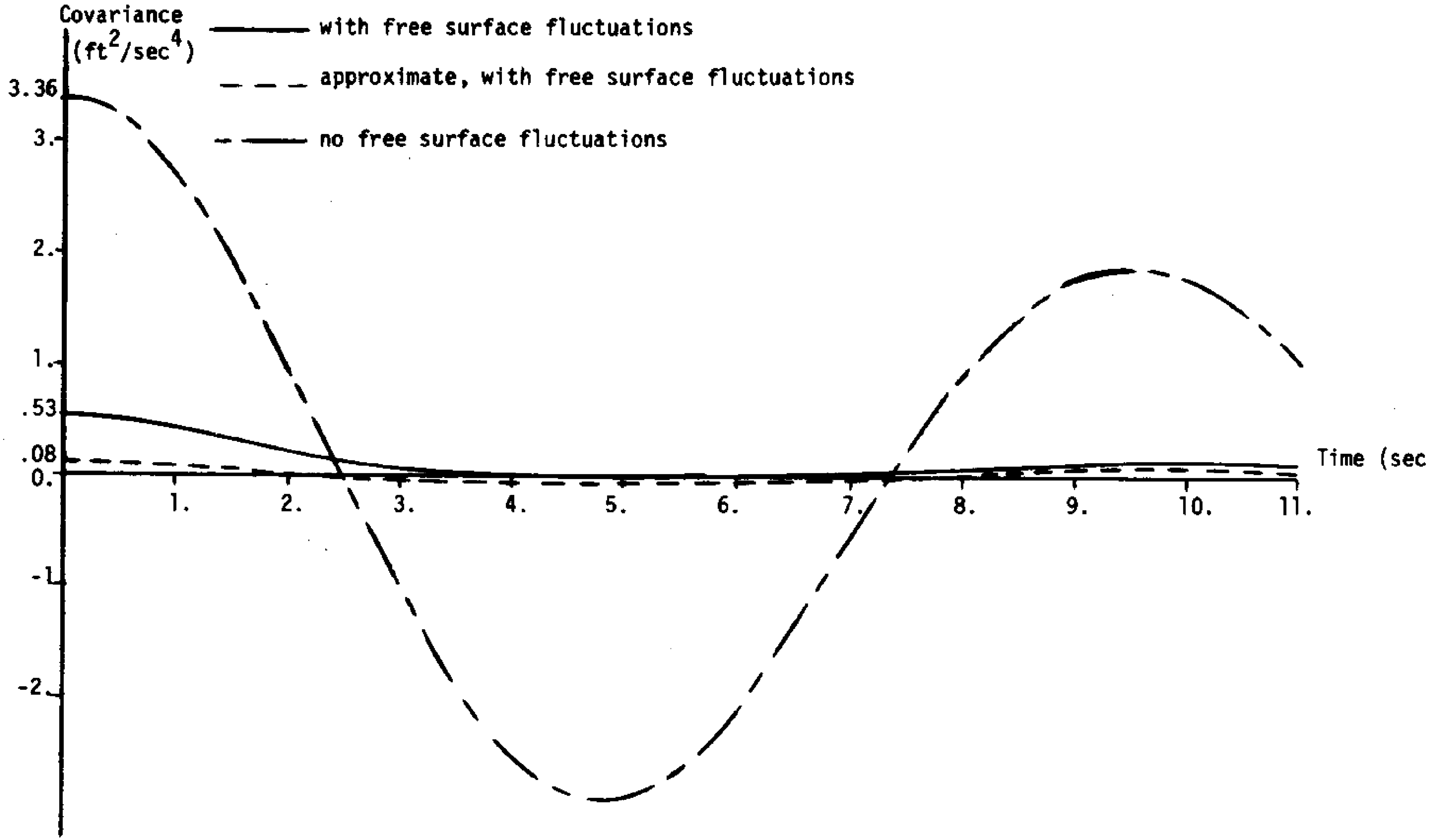

Fig. 3.17 Covariance Function of Horizontal Component of Acceleration at $z = +\sigma_\eta = +5.59$ ft., Mean Wind Speed = 40 mph

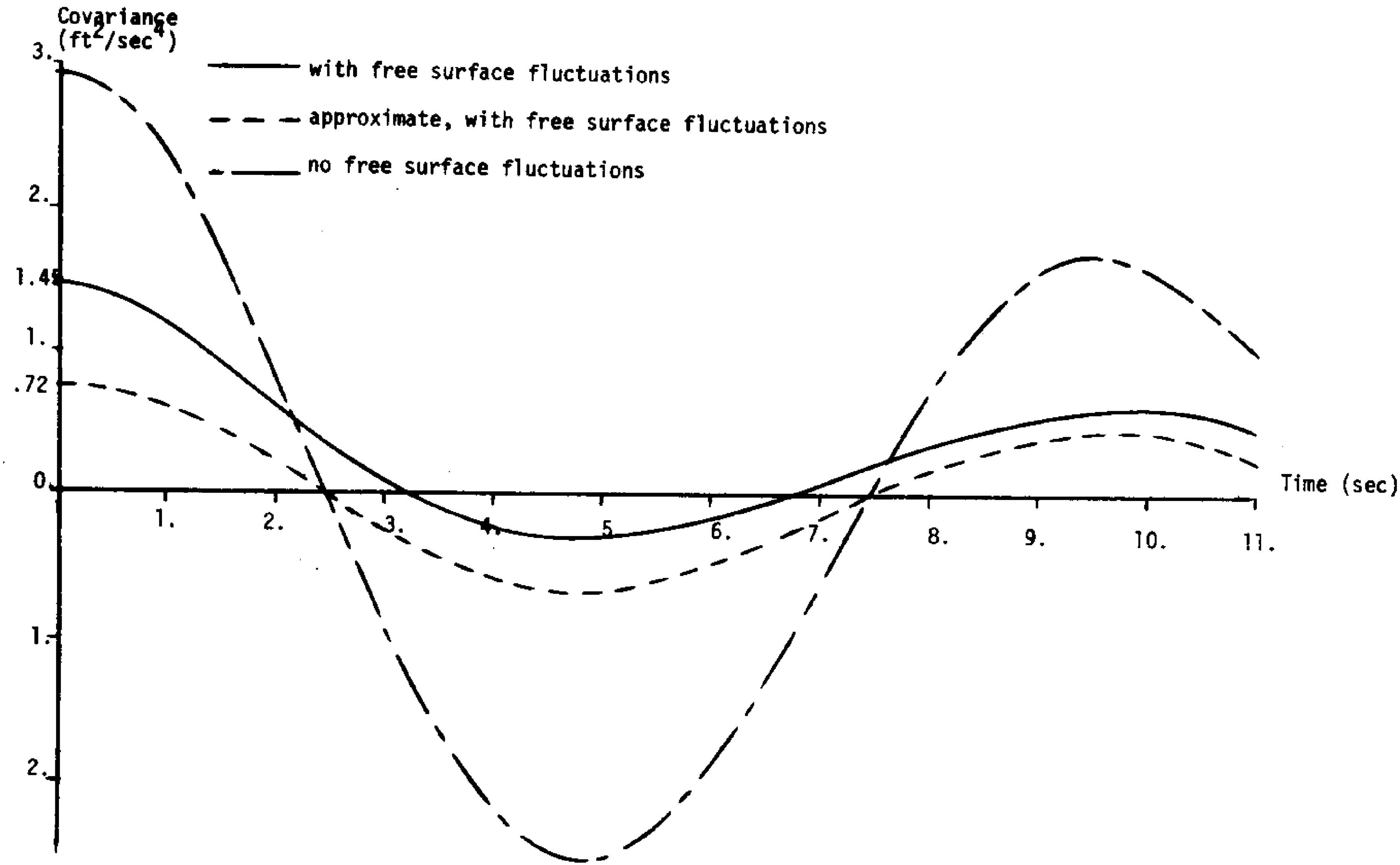

Fig. 3.18 Covariance Function of Horizontal Component of Acceleration at $z = 0$ ft., Mean Wind Speed = 40 mph

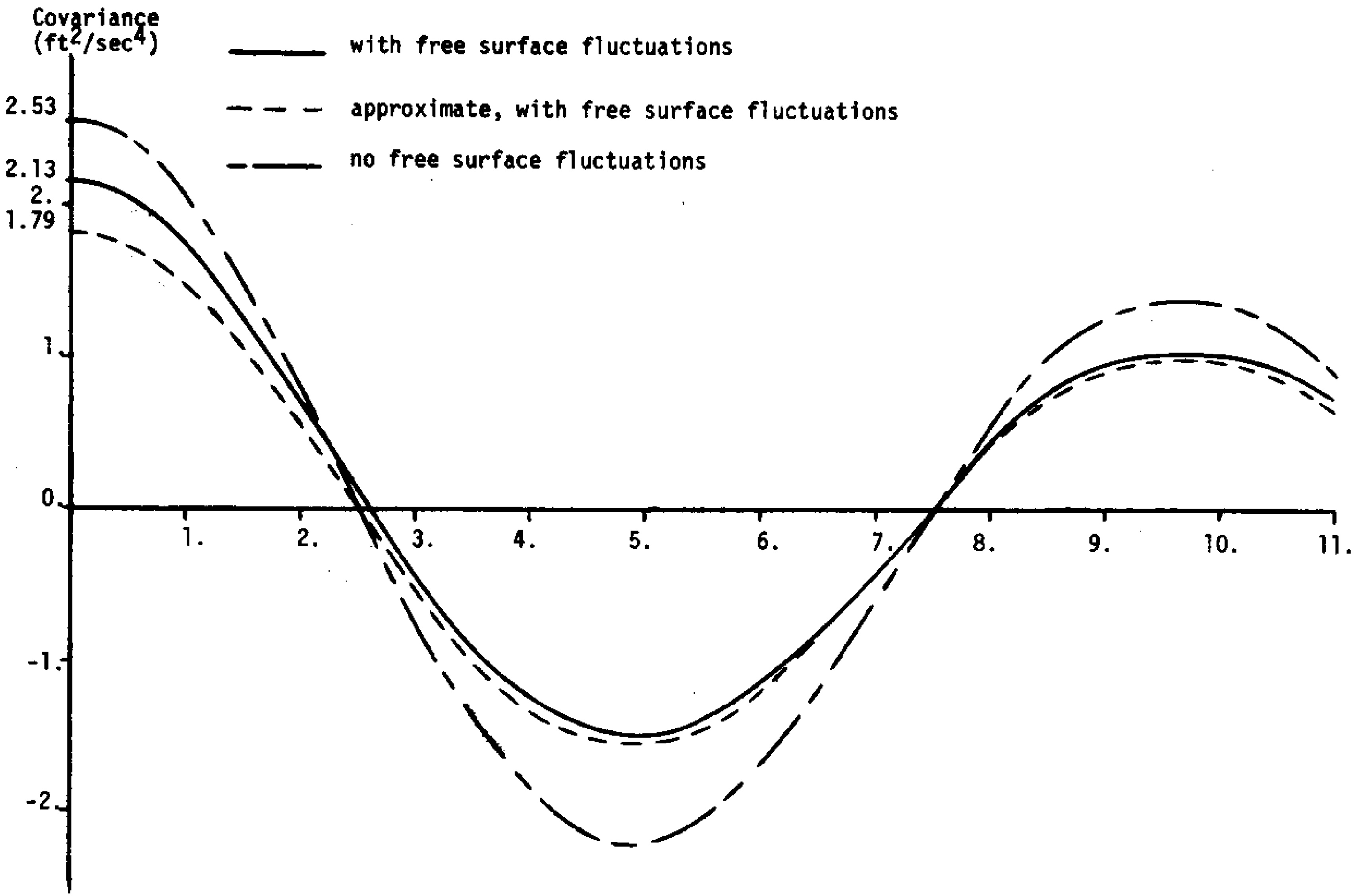

Fig. 3.19 Covariance Function of Horizontal Component of Acceleration at $z = -\sigma_\eta = -5.59$ ft., Mean Wind Speed = 40 mph

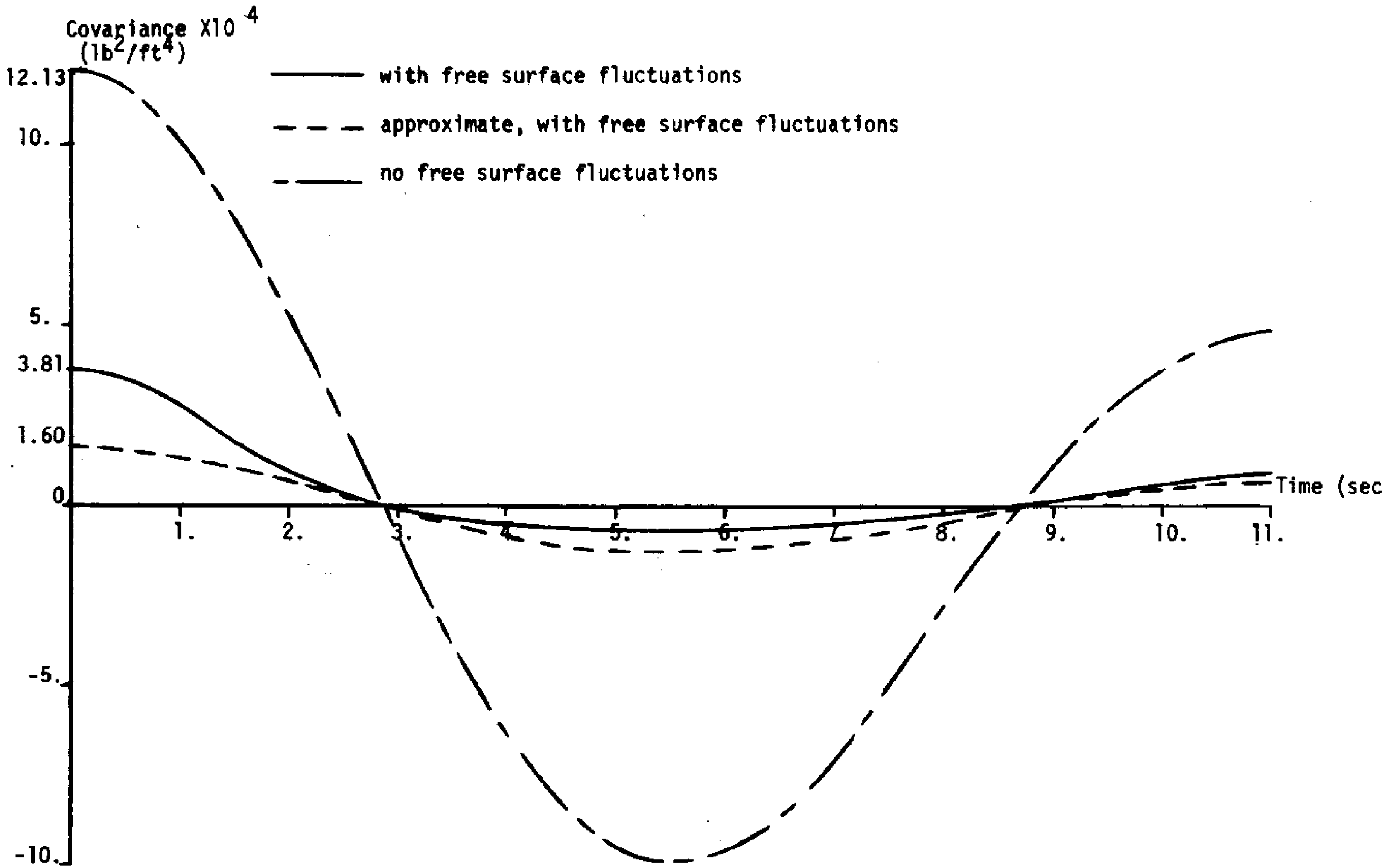

Fig. 3.20 Covariance Function of pressure at $z = +\sigma_\eta = +5.59$ ft., Mean Wind Speed = 40 mph

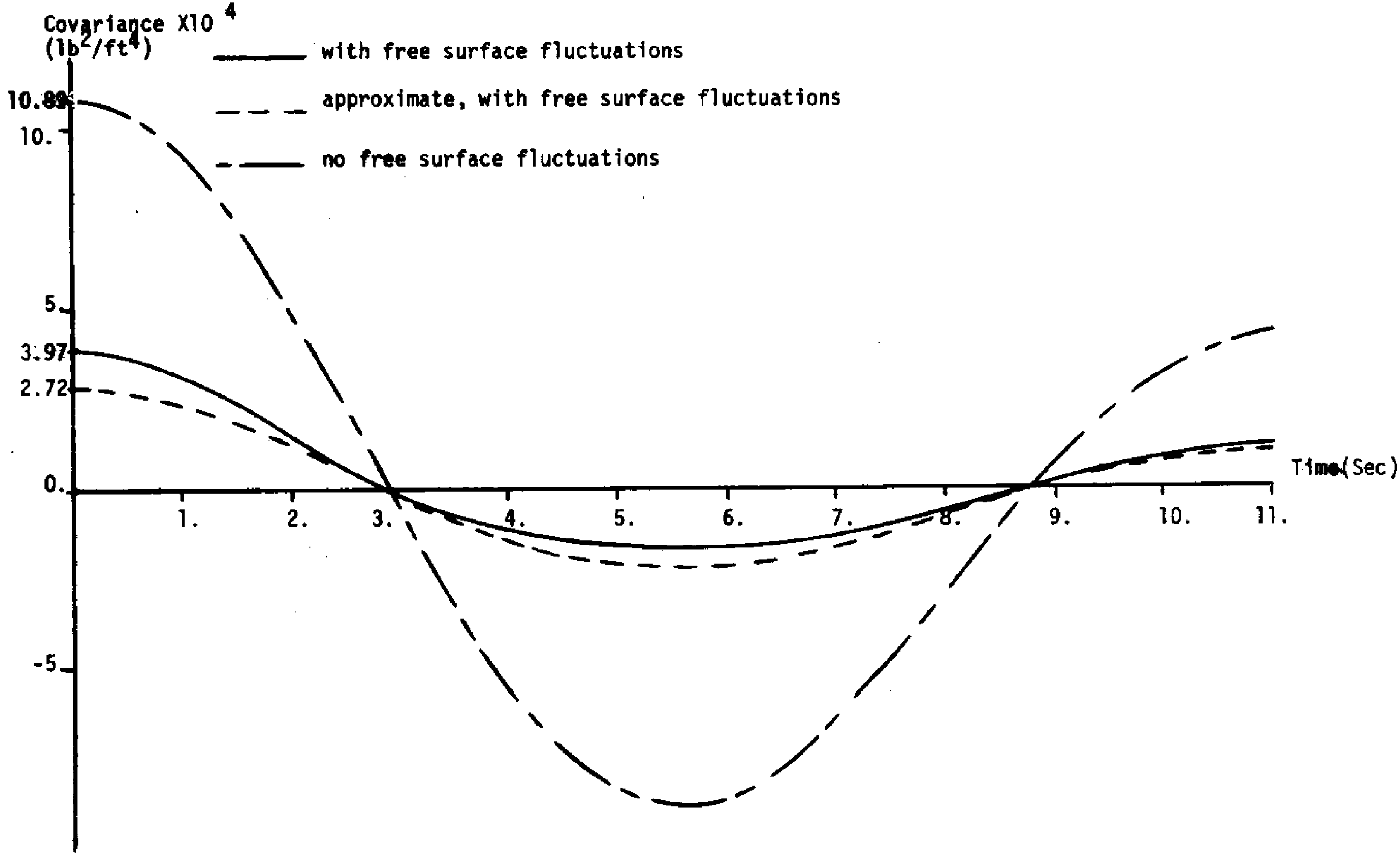

Fig. 3.21 Covariance Function of Pressure at z = 0 ft., Mean Wind Speed = 40 mph

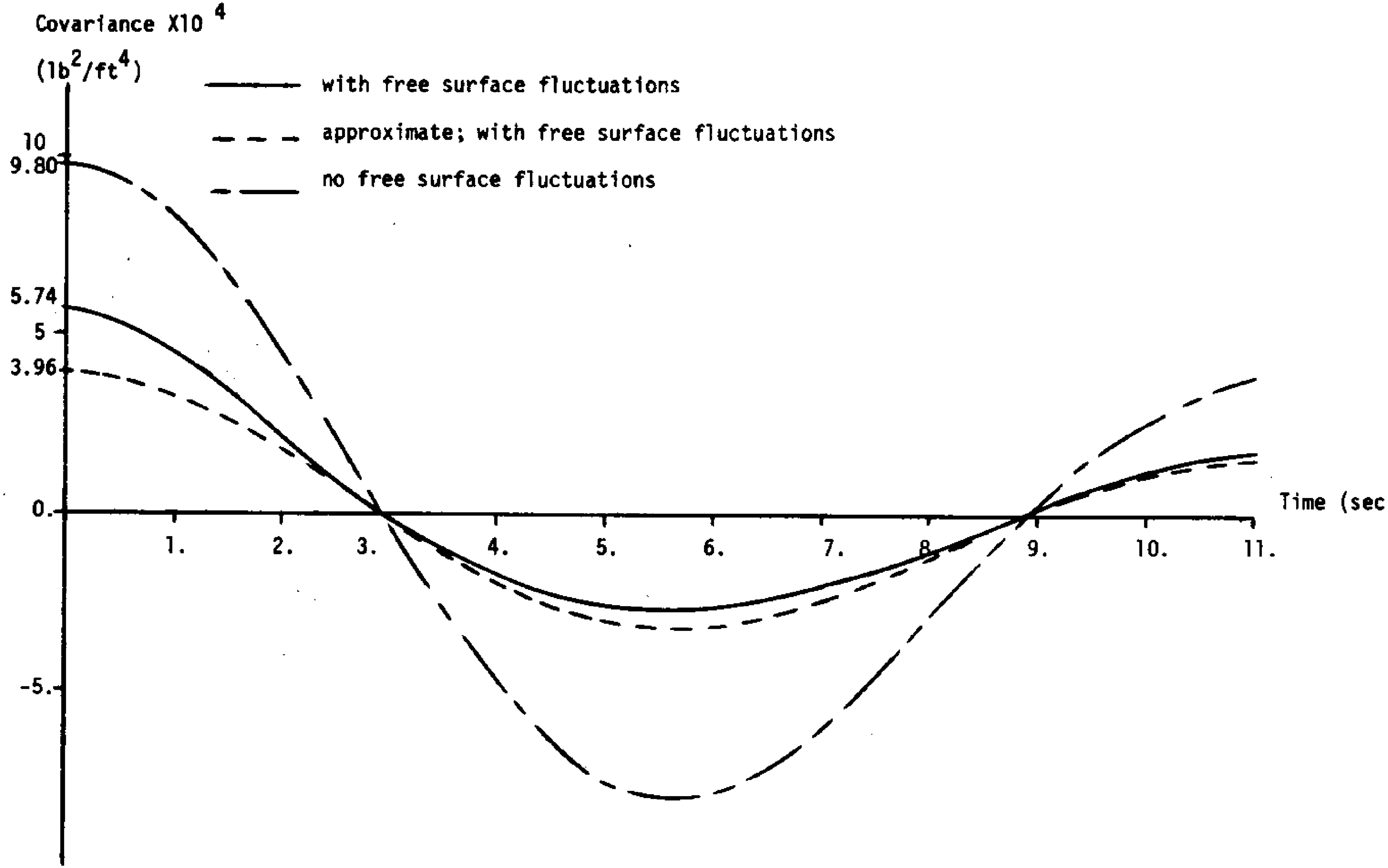

Fig. 3.22 Covariance Function of Pressure at $z = -\sigma_\eta = -5.59$ ft., Mean Wind Speed = 40 mph

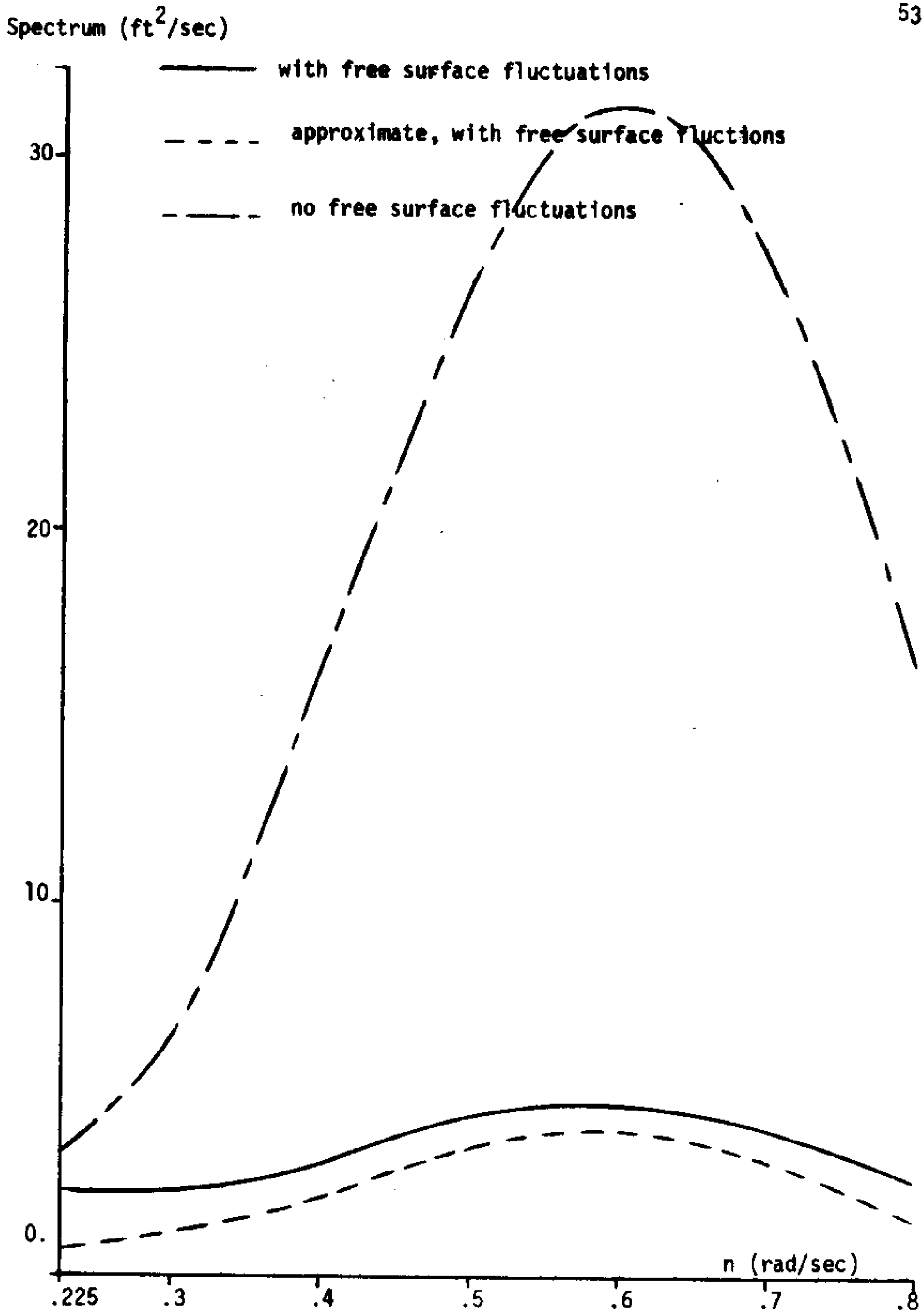

Fig. 3.23 Spectrum of Horizontal Component of Velocity at $z = +\sigma_\eta =$ +5.59 ft., Mean Wind Speed = 40 mph

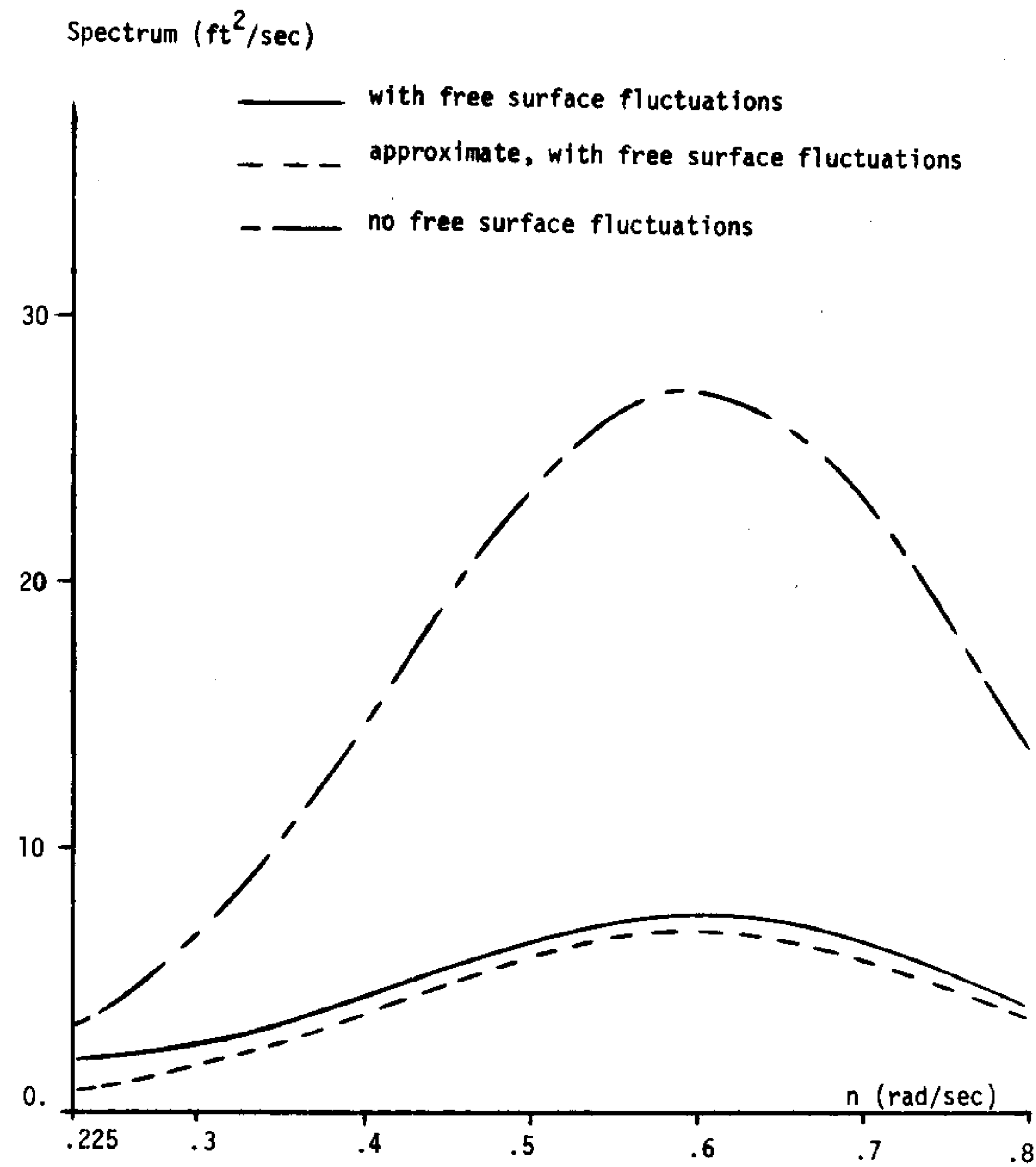

Fig. 3.24 Spectrum of Horizontal Component of Velocity at z = 0
Mean Wind Speed = 40 mph

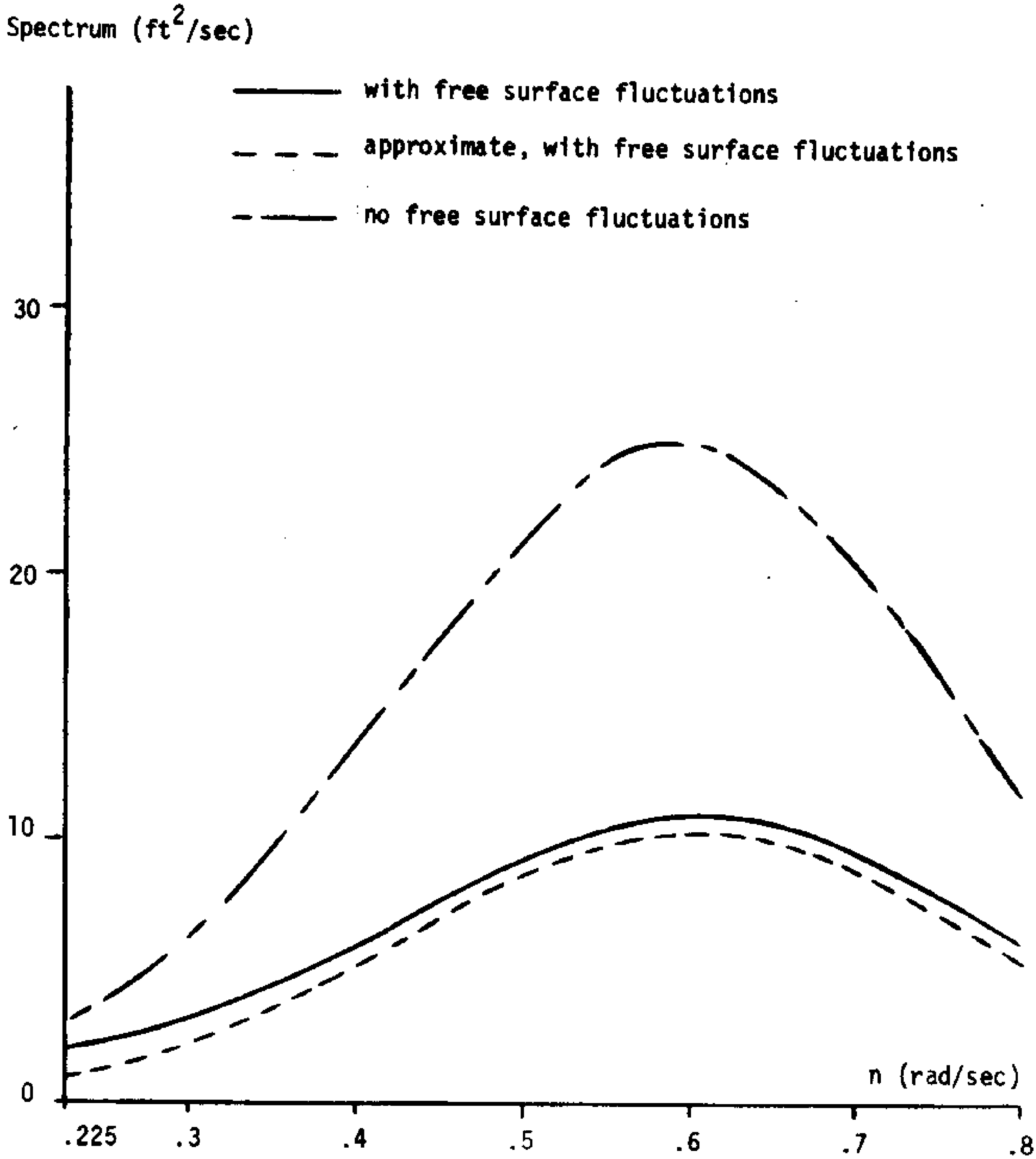

Fig. 3.25 Spectrum of Horizontal Component of Velocity at $z = -\sigma_\eta = -5.59$ ft., Mean Wind Speed = 40 mph

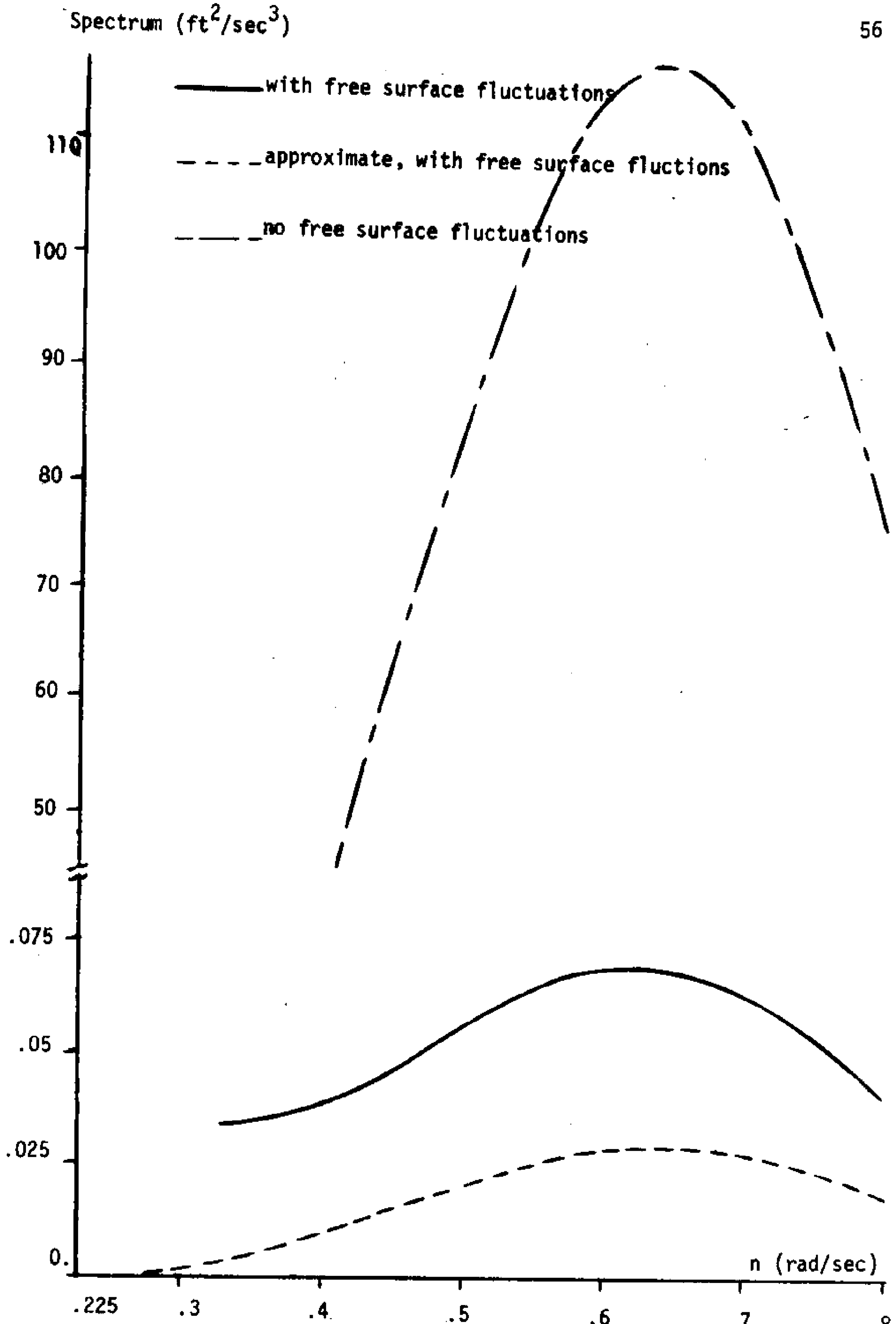

Fig. 3.26 Spectrum of Horizontal Component of Acceleration at $z = +\sigma_n =$ +5.59 ft., Mean Wind Speed = 40 mph

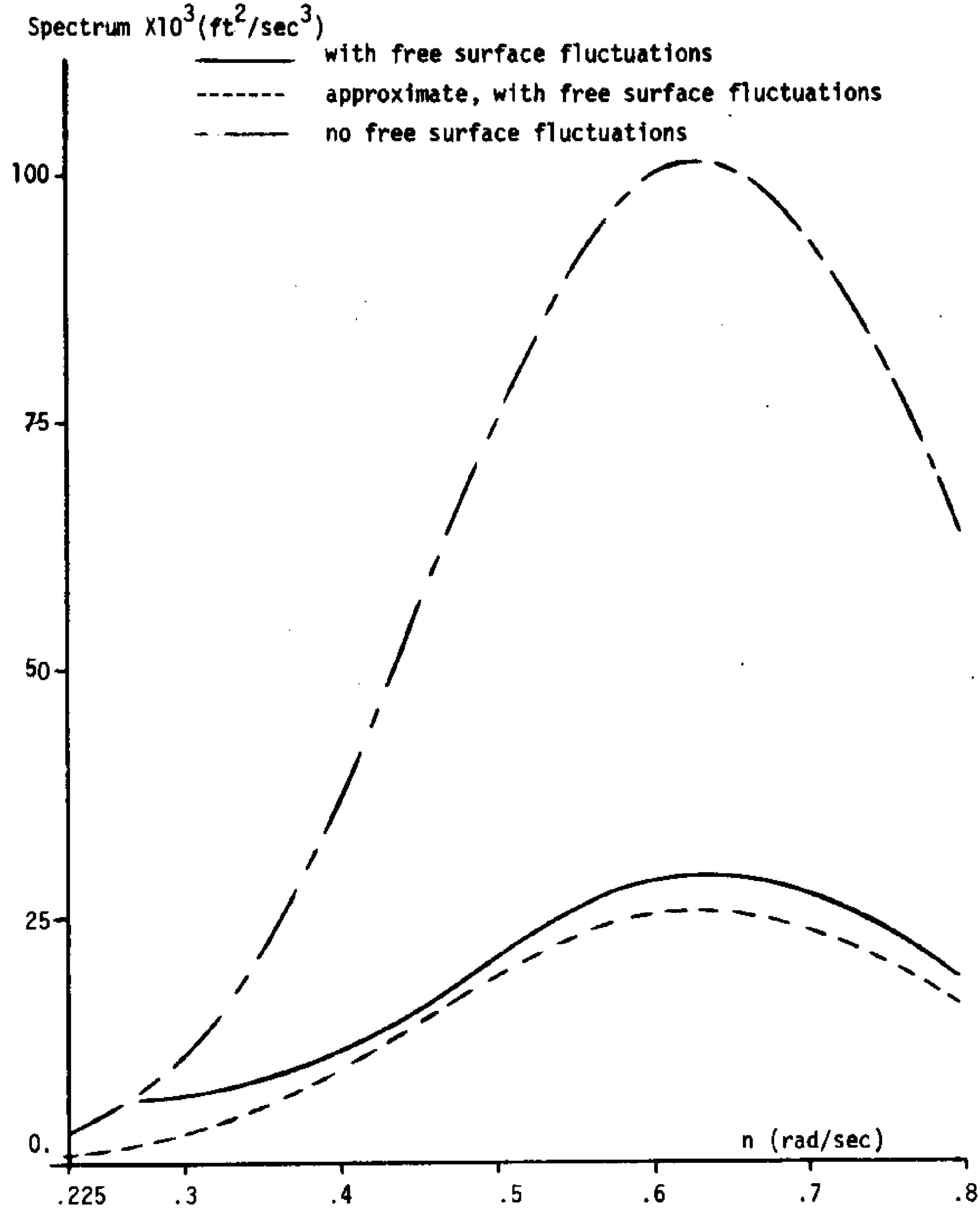

Fig. 3.27 Spectrum of Horizontal Component of Acceleration at z = 0 ft, Mean Wind Speed = 40 mph

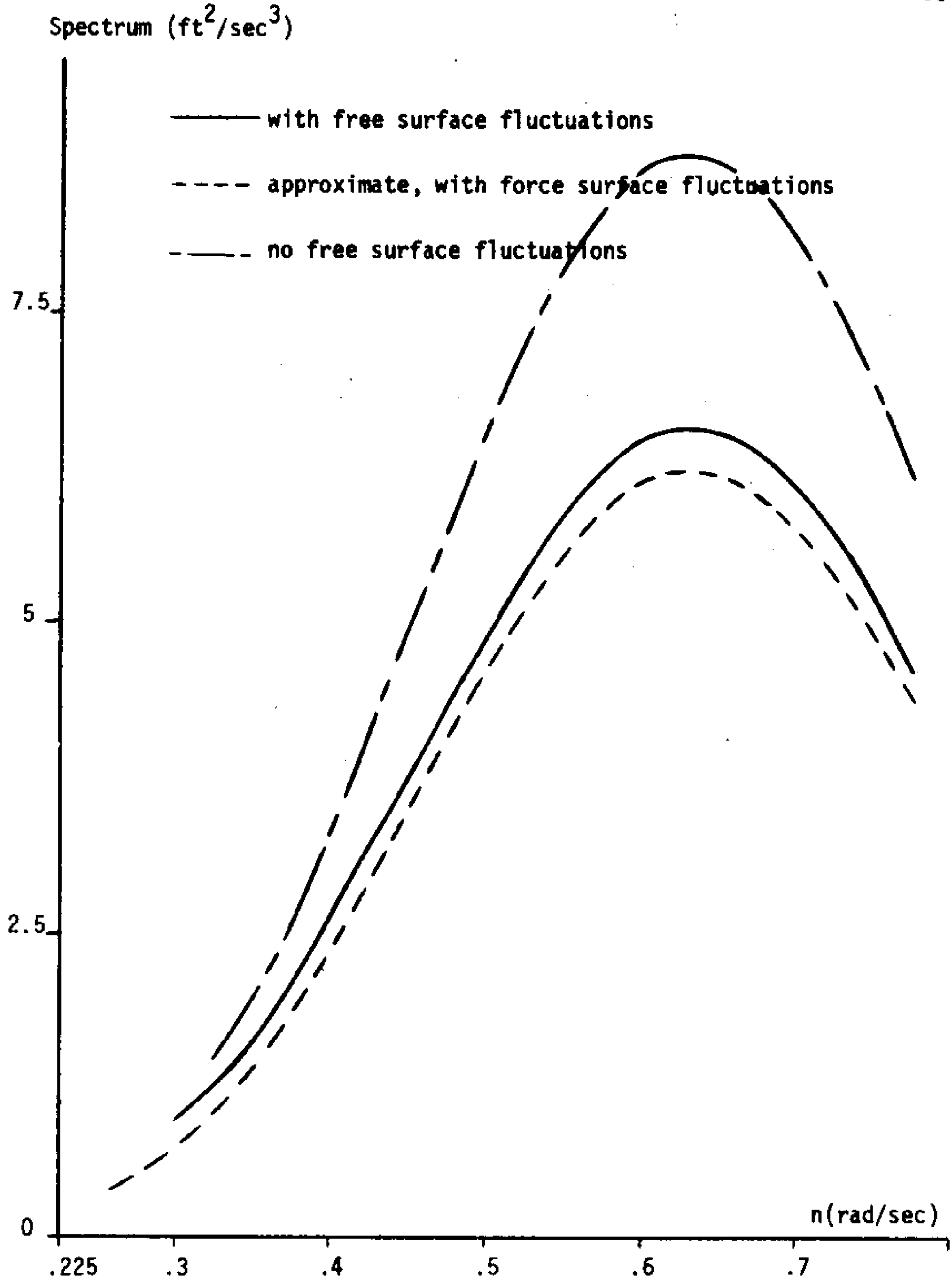

Fig. 3.28 Spectrum of Horizontal Component of Acceleration at $z = -\sigma_\eta = -5.59$ ft., Mean Wind Speed = 40 mph

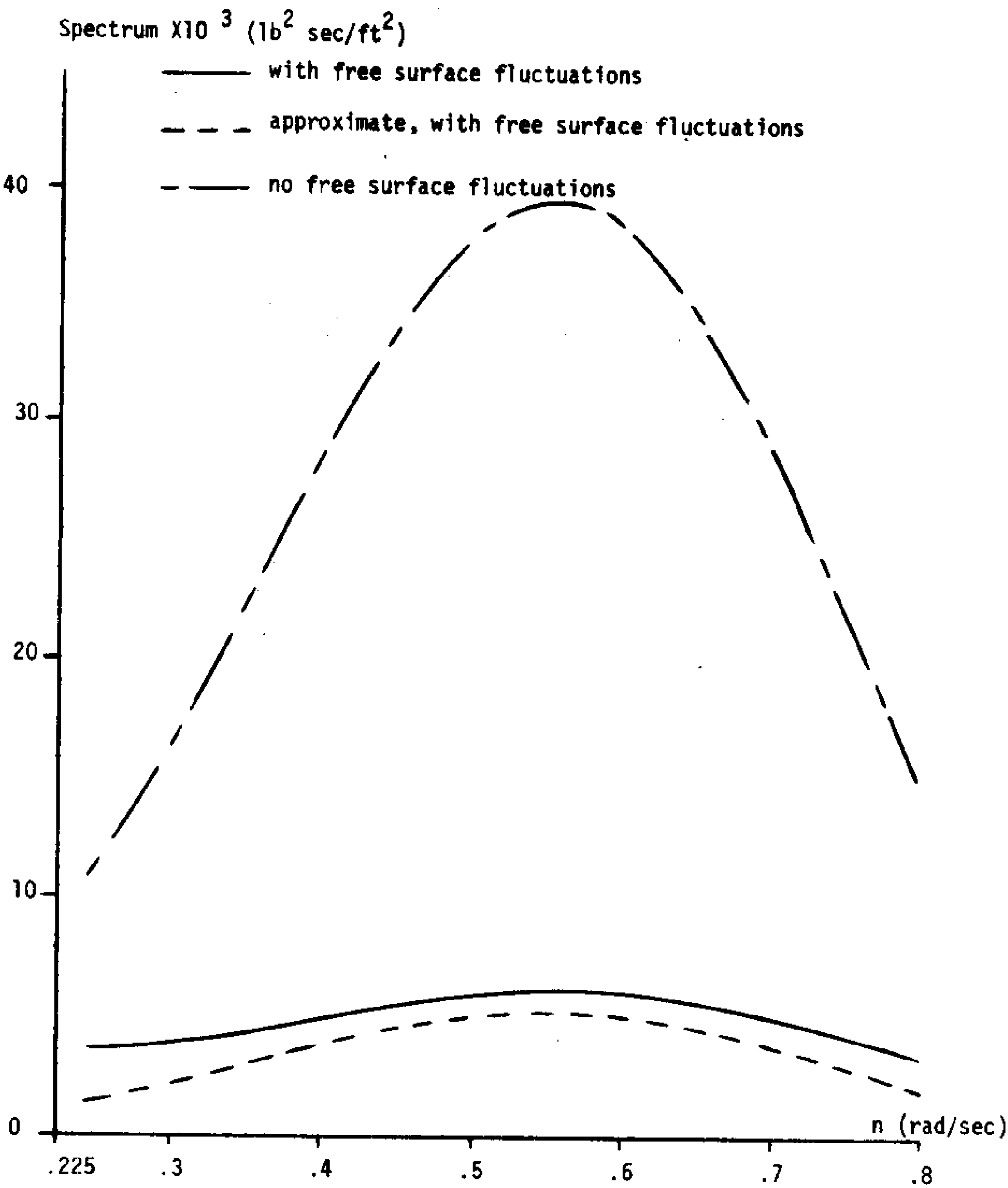

Fig. 3.29 Spectrum of pressure at $z = +\sigma_\eta = +5.59$ ft., Mean Wind Speed = 40 mph

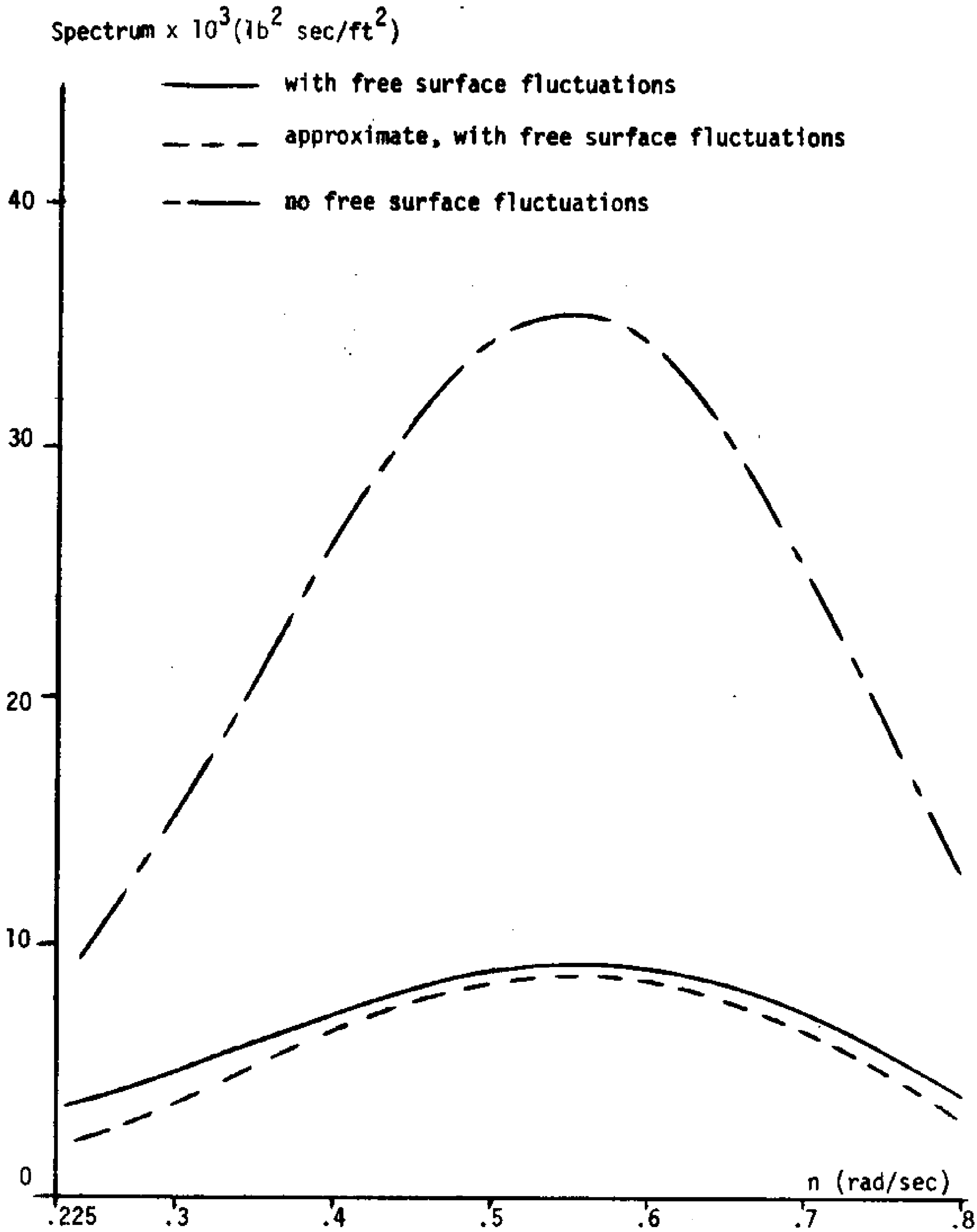

Fig. 3.30 Spectrum of Pressure at z = 0 ft., Mean Wind Speed = 40 mph

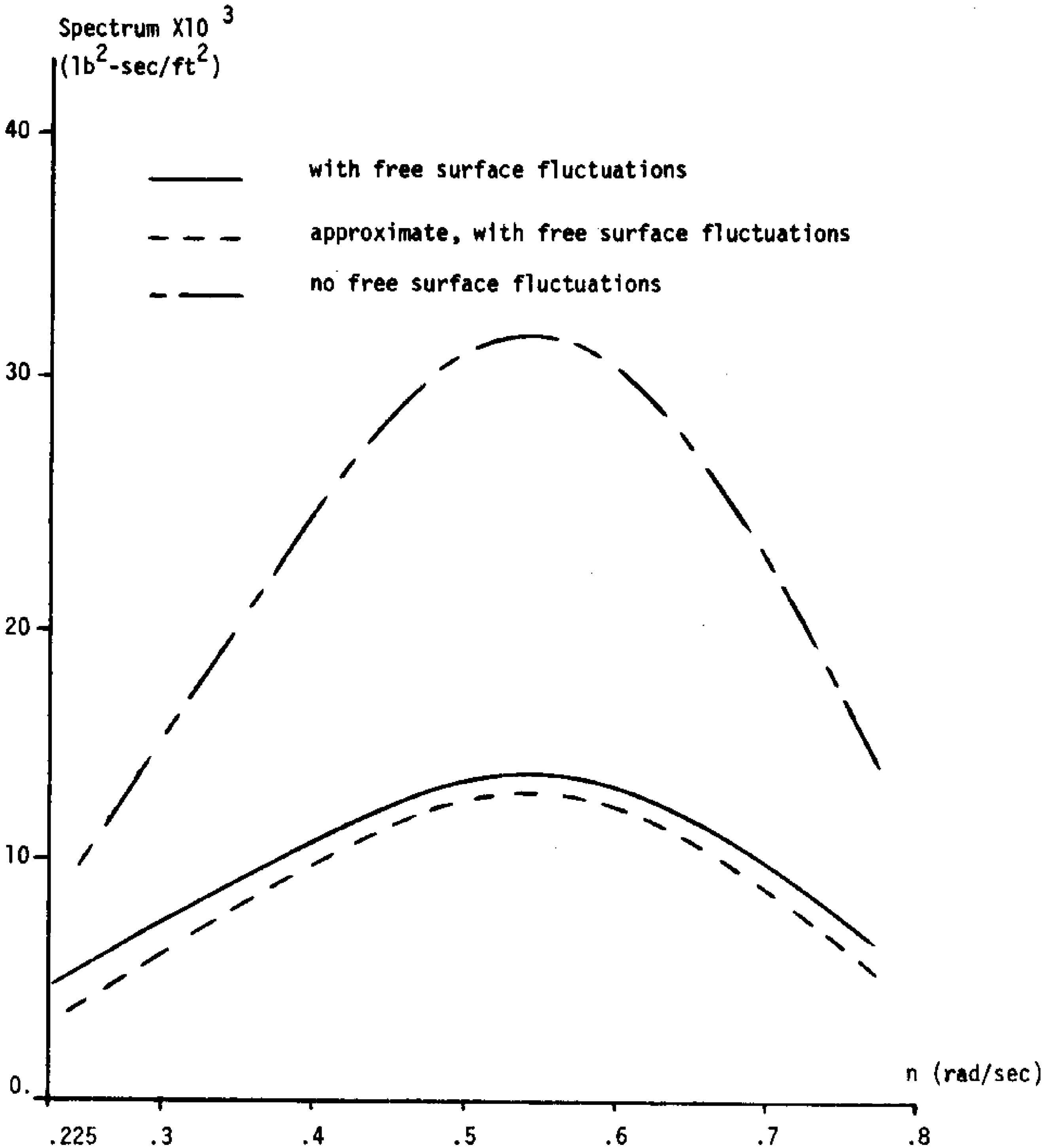

Fig. 3.31 Spectrum of Pressure at $z = -\sigma_\eta = -5.59$ ft., Mean Wind Speed = 40 mph

4. STATISTICAL PROPERTIES OF WAVE FORCE

The purpose of this chapter is to develop expressions of the probability density function, mean, variance, skewness, covariance function, and spectrum of wave force in a random wave field, taking into account the effects of free surface fluctuations pehnomenon.

Following the argument leading to Eqs. (3.1), (3.2), and (3.3), the wave force $\bar{Y}(z,t)$ per unit length of a vertical pile at point (0,z) and time t, is, according to the Morison formula

$$\bar{Y}(z,t) = Y(z,t)H(\eta(t)-z) = [C_D V(z,t)|V(z,t)|+C_M A(z,t)]H(\eta(t)-z) \qquad (4.1)$$

in which $C_D = K_D \rho D/2$, $C_M = K_M \rho \pi D^2/4$, D is pile diameter, and K_D and K_M are experimental coefficients. The values of D = 1 ft. and $K_D = 0.5$, $K_M = 1.4$, $\rho = 1.99$ slug/ft^3 are chosen for this study.

4.1. PROBABILITY LAW OF WAVE FORCE

The probability density function $f_{\bar{Y}}(y)$ of wave force $\bar{Y}(z,t)$ is, by the theorem of total probability

$$f_{\bar{Y}}(y) = [1 - Q(b)]\delta(y) + \int_z^\infty f_{\bar{Y}\eta}(y,\eta)d\eta \qquad (4.1.1)$$

in which $f_{\bar{Y}\eta}(\cdot,\cdot)$ is the joint probability density function of random variables $\bar{Y}(z,t)$ and $\eta(t)$. For convenience, the arguments of $\bar{Y}(z,t)$, $Y(z,t)$, $\eta(t)$, $V(z,t)$, and $A(z,t)$ are all dropped.

To facilitate the determination of $f_{\bar{Y}\eta}(\cdot,\cdot)$, it is convenient to introduce auxiliary random variables. That is

$$Y_o = (C_D V|V| + C_M A)/C_M\sigma_A$$

$$Y_1 = \eta/\sigma_\eta$$

$$Y_2 = C_D V|V|/C_M\sigma_A.$$

Thus, Eq. (4.1.1) can be written as

$$f_{\bar{Y}}(y) = [1-Q(b)]\ \delta\ (y) + \int_{b}^{\infty} f_{Y_oY_1}(y,y_1)dy_1. \tag{4.1.2}$$

The joint probability density function $f_{Y_oY_1}(\cdot,\cdot)$ is the marginal density function of $f_{Y_oY_1Y_2}(\cdot,\cdot,\cdot)$ given by

$$f_{Y_oY_1}(y,y_1) = \int_{-\infty}^{\infty} f_{Y_oY_1Y_2}(y,\ y_1,\ y_2)dy_2 \tag{4.1.3}$$

in which $f_{Y_oY_1Y_2}(\cdot,\cdot,\cdot)$ is the joint probability density function of Y_o, Y_1, and Y_2 and can be written in terms of the joint probability density function $f_{VA\eta}(\cdot,\cdot,\cdot)$ of V, A, and η. That is

$$f_{Y_oY_1Y_2}(y,\ y_1,y_2) = \frac{1}{|J|}\ f_{VA\eta}(v,\ a,\ \eta) \tag{4.1.4}$$

J being the Jacobian function and is

$$J = (2C_DC_M\ |v|)\ /\ C_M^2\ \ \sigma_A^2\ \ \sigma_\eta$$

and the arguments v, a, and η in Eq. (4.1.4) are to be replaced by

$$v = \mathrm{sgn}(y_2)\ \sigma_V\sqrt{2\alpha|y_2|}$$

$$a = \sigma_A(y-y_2)$$

$$\eta = \sigma_\eta\ y_1$$

in which

$$\alpha = C_M\sigma_A/2C_D\sigma_V^{\ 2}.$$

Since V, A, and η are jointly Gaussian and the pairs of random variables (V, A), and (A, η) are statistically independent (see Chapter 2), it may be verified that, after substitution of Eq. (4.1.4) into Eq. (4.1.3),

$$f_{Y_0Y_1}(y,y_1) = \frac{\sqrt{2\alpha}}{2(2\pi)^{3/2}(1-r_{\eta V}^2(0))^{1/2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|y_2|}} \exp\{-\frac{1}{2(1-r_{\eta V}^2(0))}$$

$$(2\alpha|y_2| - 2\mathrm{Sgn}(y_2)r_{\eta V}(0)y_1\sqrt{2\alpha|y_2|} + y_1^2 + (1-r_{\eta V}^2(0))\ (y-y_2)^2)\}dy_2. \quad (4.1.5)$$

Substituting Eq. (4.1.5) into Eq. (4.1.1) and after rearranging

$$f_{\bar{Y}}(y) = [1-Q(b)]\delta(y) + \frac{\sqrt{2\alpha}}{2\sqrt{2\pi}} e^{\frac{1}{2}\alpha^2} \{e^{-\alpha y} \int_0^{\infty} \frac{1}{\sqrt{y_2}} Z(y_2-y+\alpha)Q((\frac{b-r_{\eta V}(0)\sqrt{2\alpha y_2}}{\sqrt{1-r_{\eta V}^2(0)}})$$

$$dy_2+e^{\alpha y}\int_0^{\infty} \frac{1}{\sqrt{y_2}} Z(y_2+y+\alpha)Q((\frac{b+r_{\eta V}(0)\sqrt{2\alpha y_2}}{\sqrt{1-r_{\eta V}^2(0)}})\ dy_2\}. \quad (4.1.6)$$

To verify that

$$I = \int_{-\infty}^{\infty} f_{\bar{Y}}(y)dy = 1, \quad (4.1.7)$$

substitute Eq. (4.1.6) into the above integral

$$I = [1-Q(b)] + \frac{\sqrt{2\alpha}}{2\sqrt{2\pi}} e^{\frac{1}{2}\alpha^2} \{\int_{-\infty}^{\infty} e^{-\alpha y}\int_0^{\infty} \frac{1}{\sqrt{y_2}} Z(y_2 - y + \alpha)$$

$$Q(\frac{b-r_{\eta V}(0)\sqrt{2\alpha y_2}}{\sqrt{1 - r_{\eta V}^2(0)}})\ dy_2\ dy + \int_{-\infty}^{\infty} e^{\alpha y} \int_0^{\infty}\frac{1}{\sqrt{y_2}} Z(y_2+y+\alpha)$$

$$Q(\frac{b + r_{\eta V}(0)\sqrt{2\alpha y_2}}{\sqrt{1-r_{\eta V}^2(0)}})\ dy_2\ dy\}.$$

Since

$$\int_{-\infty}^{\infty} Z(y-y_2) = 1$$

it follows that

$$I = 1-Q(b) + L(b,0,r_{\eta V}(0)) + L(b,0,-r_{\eta V}(0))$$

which is equal to unity (Abramowitz, 1968).

From Eq. (4.1.6), it is seen that when the influence of the free surface fluctuations diminishes ($r_{\eta V}(0)\to 0$), $f_{\bar{Y}}(y)$ approaches

$$f_Y(y) = \sqrt{\alpha/2\pi^2} \quad e^{-\frac{1}{2}y^2} \int_0^{\infty} \frac{1}{\sqrt{y_2}} \; e^{-\alpha y_2 - \frac{1}{2} y_2^2} \cosh(yy_2)dy_2$$

as derived by Borgman (Borgman, 1967).

Numerical results of the probability density function $f_{\bar{Y}}(y)$ are obtained for mean wind speed W = 40 mph and cylinder diameter D = 1 ft. for $z = \sigma_\eta$, 0, $-\sigma_\eta$ and shown in Figs. (4.1), (4.2) and (4.3) respectively.

4.2. MEAN, VARIANCE, AND SKEWNESS OF WAVE FORCE

The first three statistical moments of $\bar{Y}$ are derived herein using three different methods to cross check the results. The variance and skewness are then deduced from the statistical moments.

METHOD 1.

Using the probability density function of wave force,

$$E\{\bar{Y}^j\} = \int_{-\infty}^{\infty} y^j \, f_{\bar{Y}}(y)dy, \; j = 1,2,3.$$

Performing the integration by parts, the following results can be obtained without difficulty except that the algebra is lengthy for the third statisti-

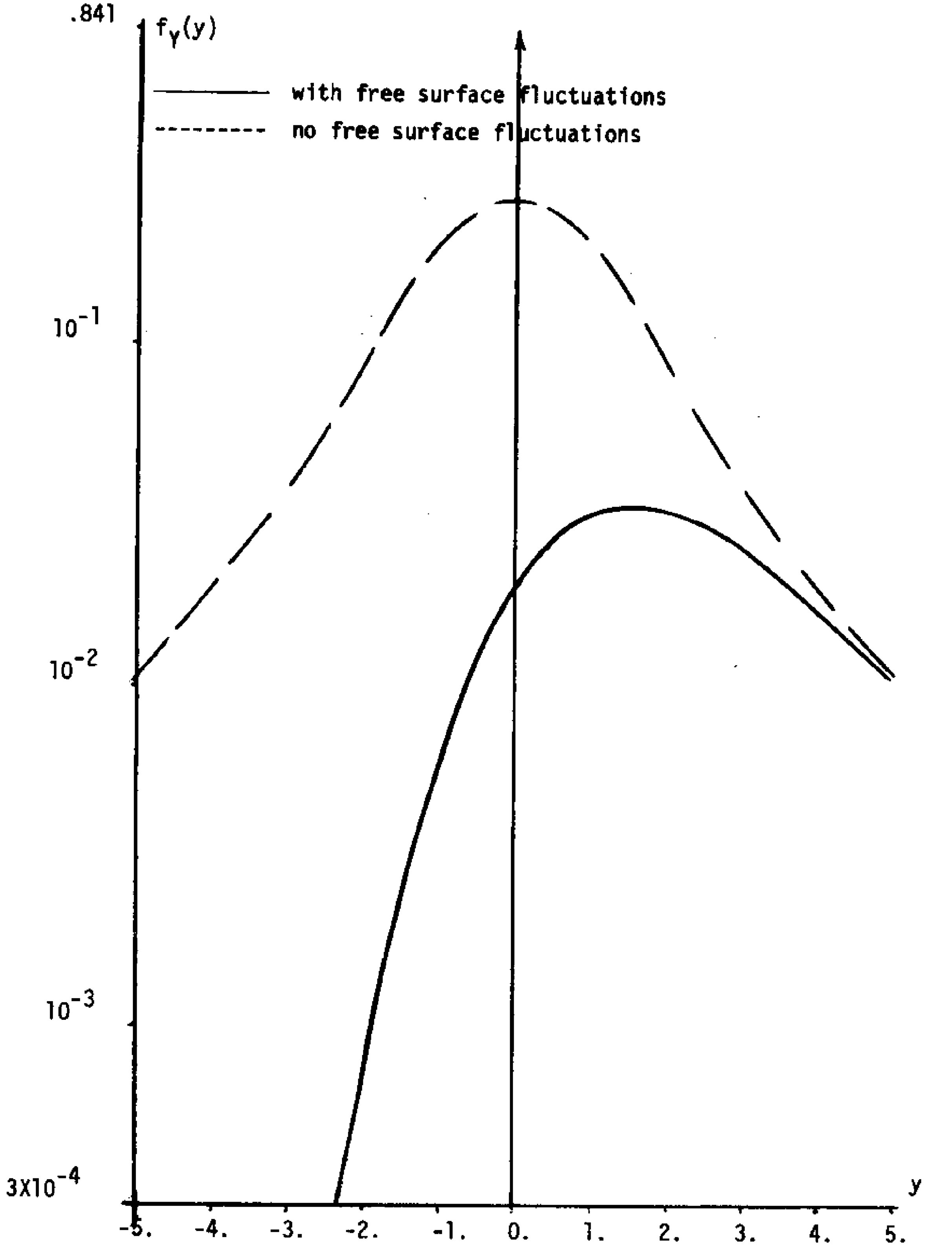

Fig. 4.1 Probability Density Function of Wave Force at $z = + \sigma_\eta = +5.59$ ft., Mean Wind Speed = 40 mph, Diameter of Cylinder = 1 ft.

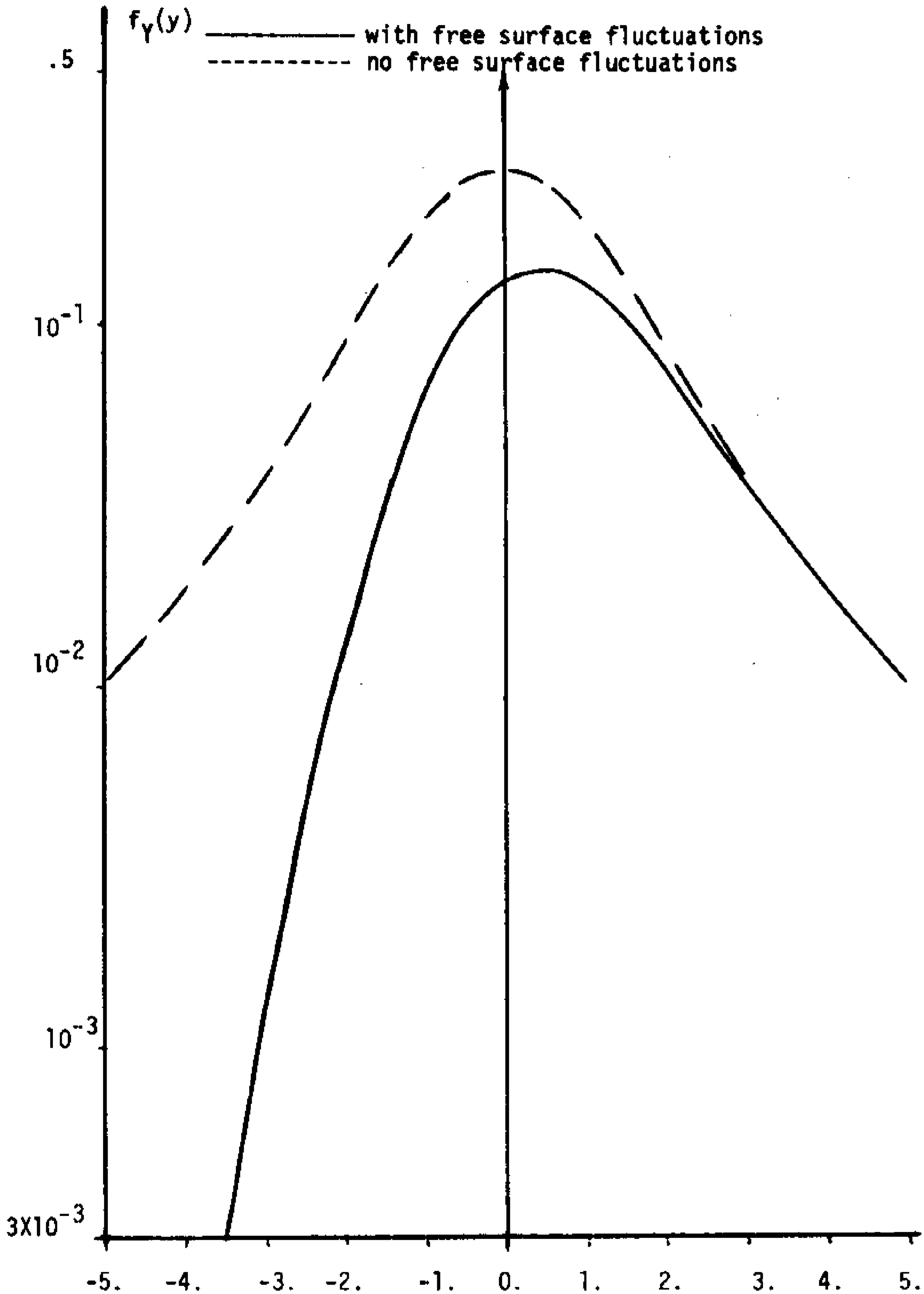

Fig. 4.2 Probability Density Function of Wave Force at z = 0 ft., Mean Wind Speed = 40 mph, Diameter of Cylinder = 1 ft.

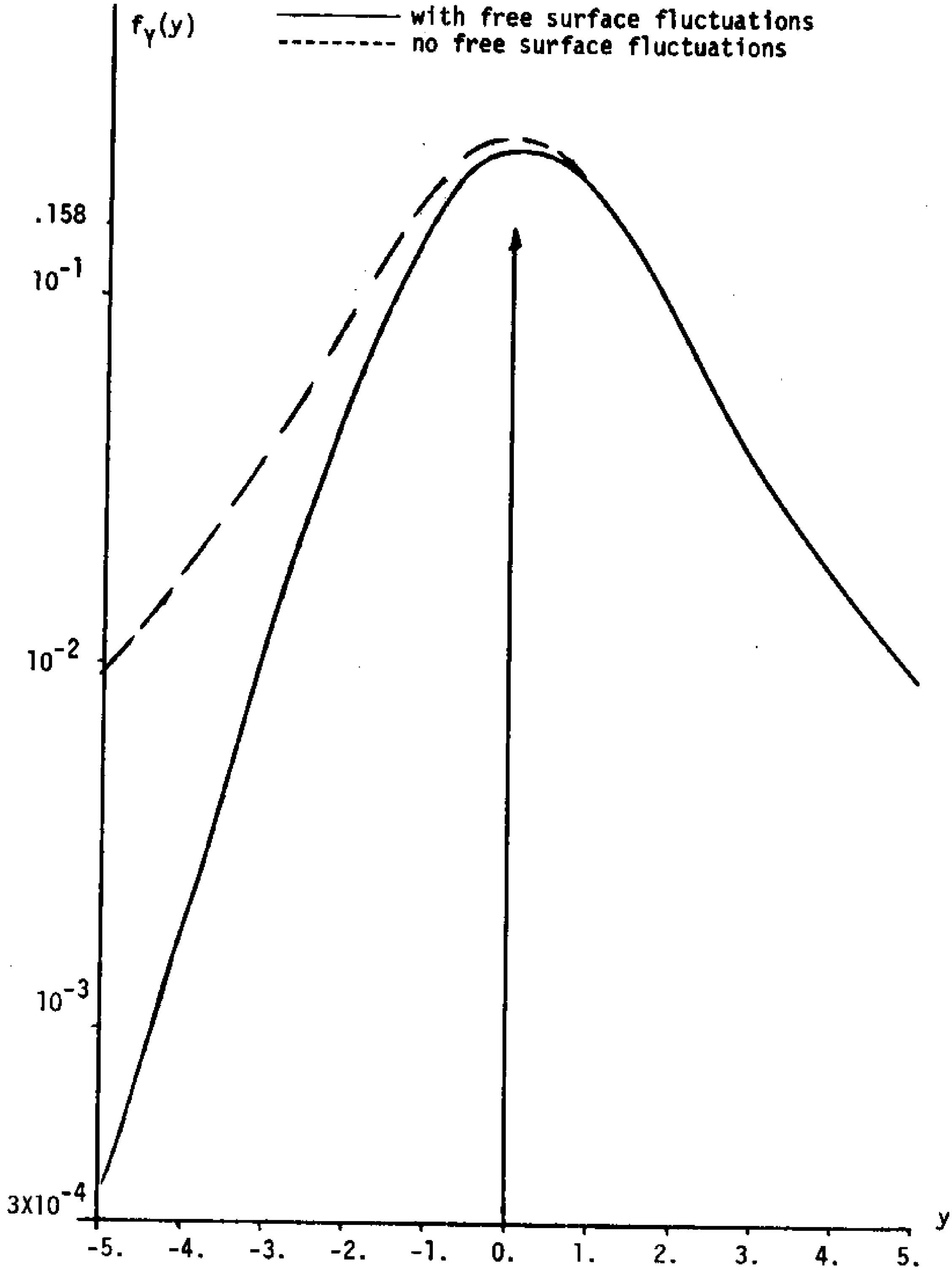

Fig. 4.3 Probability Density Function of Wave Force at z = $-\sigma_\eta$ = -5.59 ft., Mean Wind Speed = 40 mph, Diameter of Cylinder = 1 ft.

cal moment.

$$E\{\bar{Y}\} = C_D\ \sigma_V^2\ \{\frac{2\ r_{\eta V}(0)\sqrt{1-r_{\eta V}^2(0)}}{\sqrt{2\pi}}\ Z\ (\frac{b}{\sqrt{1-r_{\eta V}^2(0)}})$$

$$+ 2L(b,0,r_{\eta V}(0)) + r_{\eta V}^2(0)bZ(b)\ (2Q(\frac{-r_{\eta V}(0)b}{\sqrt{1 - r_{\eta V}^2(0)}})-1)-Q(b)\} \qquad (4.2.1)$$

$$E\{\bar{Y}^2\} = (C_M^2\ \sigma_A^2 + 3C_D^2\ \sigma_V^2)\ Q\ (b) + C_D^2\ \sigma_V^4 r_{\eta V}^2(0)\ (6+r_{\eta V}^2(0)\ (b^2-3))bZ(b) \qquad (4.2.2)$$

and

$$E\{\bar{Y}^3\} = 3C_D C_M^2 \sigma_A^2 \sigma_V^2\ \{\frac{2r_{\eta V}(0)\sqrt{1-r_{\eta V}^2(0)}}{\sqrt{2\pi}}\ Z(\frac{b}{\sqrt{1-r_{\eta V}^2(0)}}) + 2L(b,0,r_{\eta V}(0))$$

$$+ r_{\eta V}^2(0)bZ(b)\ (2Q(\frac{-r_{\eta V}(0)b}{\sqrt{1-r_{\eta V}^2(0)}})\ -1)\ -\ Q(b)\} + C_D^3\sigma_V^6\{-15\ Q(b) + 30L(b,0,r_{\eta V}(0))$$

$$+ \frac{2r_{\eta V}(0)\sqrt{1-r_{\eta V}^2(0)}}{\sqrt{2\pi}}\ Z(\frac{b}{\sqrt{1-r_{\eta V}^2(0)}})\ (8r_{\eta V}^4(0)\ -\ 26r_{\eta V}^2(0) + 33) + \frac{2r_{\eta V}^3(0)\sqrt{1-r_{\eta V}^2(0)}}{\sqrt{2\pi}}$$

$$b^2Z(\frac{b}{\sqrt{1-r_{\eta V}^2(0)}})\ (-9r_{\eta V}^2(0) + 14) + \frac{2r_{\eta V}^5(0)\sqrt{1-r_{\eta V}^2(0)}}{\sqrt{2\pi}}\ b^4Z(\frac{b}{\sqrt{1-r_{\eta V}^2(0)}})+30r_{\eta V}^2(0)$$

$$bZ(\frac{b}{\sqrt{1-r_{\eta V}^2(0)}})\ Q(\frac{r_{\eta V}(0)b}{\sqrt{1-r_{\eta V}^2(0)}})\ (r_{\eta V}^4(0)-3r_{\eta V}^2(0)+3)+10r_{\eta V}^4(0)b^3Z(\frac{b}{\sqrt{1-r_{\eta V}^2(0)}})$$

$$Q(\frac{r_{\eta V}(0)b}{\sqrt{1-r_{\eta V}^2(0)}})\ (-2r_{\eta V}^2(0)+3) + 2r_{\eta V}^6(0)b^5Z(b)\ Q(\frac{r_{\eta V}(0)b}{\sqrt{1-r_{\eta V}^2(0)}})+15r_{\eta V}^2(0)bZ(b)$$

$$(-r_{\eta V}^4(0) + 3r_{\eta V}^2(0)\ -3)-5r_{\eta V}^4(0)b^3Z(b)\ (2r_{\eta V}^2(0)+3)-r_{\eta V}^6(0)b^5Z(b)\}. \qquad (4.2.3)$$

METHOD 2.

In this section the moment generating function $M_{\bar{Y}}(s)$ of wave force $\bar{Y}$ will be presented from which the statistical moments of wave force may be deduced. The moment generating function of wave force is

$$M_{\bar{Y}}(s) = E\{e^{s\bar{Y}}\}.$$

Using Eq. (4.1.6), it may be shown that

$$M_{\bar{Y}}(s) = 1-Q(b)+\frac{\sqrt{2\alpha}}{2\sqrt{2\pi}} \quad e^{\frac{1}{2}s^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{y_2}} \quad e^{-\alpha|y_2|+sy_2}\, Q\left(\frac{b-\mathrm{sgn}(y_2)r_{\eta V}(0)\sqrt{2\alpha|y_2|}}{\sqrt{1-r_{\eta V}^2(0)}}\right)dy_2. \tag{4.2.4}$$

The statistical moments of $\bar{Y}$ can be obtained from $M_{\bar{Y}}(s)$

$$E\{\bar{Y}^j\} = \frac{d^j}{ds^j}M_{\bar{Y}}(s)|_{s=0}, \quad j=1,2,3$$

but the algebra involved is quite lengthy.

METHOD 3.

Using this method, the first three statistical moments are derived without resort to the probability density function or the moment generating function of wave force.

Taking the expected value of both sides of Eq. (4.1)

$$E\{\bar{Y}\} = C_D E\{V|V|H(\eta-z)\} + C_M E\{A\ H(\eta-z)\}. \tag{4.2.5}$$

The expectation in the second term on the right hand side of Eq. (4.2.5) is zero since $r_{\eta A}(0) = E[A] = 0$. The term

$$E\{V|V|H(\eta-z)\} = E\{H(\eta-z)\ E\{V|V|\,|\eta\}\} \tag{4.2.6}$$

in which $E\{V|V||\eta\}$ is the conditional expectation of $V|V|$ given η. It can be shown that

$$E\{V|V||\eta\} = \sigma_V^2(1-r_{\eta V}^2(0))\{(1+\Omega^2)\ (2Q(\Omega)-1)-\Omega Z(\Omega)\} \tag{4.2.7}$$

in which

$$\Omega = -r_{\eta V}(0)\ (\frac{\eta}{\sigma_\eta})\ /\ (1-r_{\eta V}^2(0))^{1/2}.$$

Substituting Eq. (4.2.7) into Eq. (4.2.6) and evaluating the resulting expected values by integration by parts, the result given by Eq. (4.2.1) is retrieved.

The second statistical moment of $\bar{Y}$ can be found in a similar manner from Eq. (4.1)

$$E\{\bar{Y}^2\} = C_D^2E\{V^4H(\eta-z)\} + 2C_DC_ME\{V|V|AH(\eta-z)\} + C_M^2E\{A^2H(\eta-z)\}. \tag{4.2.8}$$

The first expectation on the right hand side of the above equation is

$$E\{V^4H(\eta-z)\} = E\{H(\eta-z)\ E\{V^4|\eta\}\}$$

in which (Papoulis, 1965)

$$E\{V^4|\eta\} = \sigma_V^4\ (1-r_{\eta V}^2(0)\ (3 + 6\Omega^2 + \Omega^4).$$

Therefore

$$E\{V^4H(\eta-z)\} = \sigma_V^4\{r_{\eta V}^2(0)\ b\ (6 + r_{\eta V}^2(0)\ (b^2 - 3))Z(b) + 3Q(b)\}. \tag{4.2.9}$$

The second term on the right hand side of Eq. (4.2.8) vanishes because

$$E\{V|V|AH(\eta-z)\}= E\{A\}\ E\{V|V|H(\eta-z)\} = 0 \tag{4.2.10}$$

since A is zero mean. The third expectation on the right hand side of Eq. (4.2.8) can be written as

$$E\{A^2H(\eta-z)\} = E\{A^2\}\ E\{H(\eta-z)\}$$

$$= \sigma_A^2 Q(b). \qquad (4.2.11)$$

Substitution of Eqs. (4.2.9), (4.2.10), and (4.2.11) into Eq. (4.2.8) yields $E\{\hat{Y}^2\}$ as given by Eq. (4.2.2).

Similarly, the third statistical moment can also be obtained from Eq. (4.1). That is

$$E\{\hat{Y}^3\} = C_M^3 E\{A^3H(\eta-z)\} + 3C_D^2C_M E\{V^4AH(\eta-z)\} + 3C_DC_M^2E\{V|V|A^2H(\eta-z)\} + C_D^3E\{V^5|V|H(\eta-z)\}. \qquad (4.2.12)$$

The expectations in the first and second terms on the right hand side of the above equation are zero because

$$E\{A\} = E\{A^3\} = 0 \text{ and } r_{\eta A}(0) = 0.$$

The expectation in the third term on the right hand side of Eq. (4.2.12) is

$$E\{V|V|A^2H(\eta-z)\} = E\{A^2\}E\{V|V|H(\eta-z)\} = \sigma_A^2 E\{\bar{Y}\}. \qquad (4.2.13)$$

The fourth term on the right hand side of Eq. (4.2.12) can be written as

$$E\{V^5|V|H(\eta-z)\} = E\{H(\eta-z)\ E\{V^5|V|\,|\eta\}\} \qquad (4.2.14)$$

in which

$$E\{V^5|V|\,|\eta\} = \sigma_V^6 \sum_{j=0}^{6} B_j\left(\frac{\eta}{\sigma_\eta}\right)^j \qquad (4.2.15)$$

and B_j are complicated functions of $\bar{\eta}$. Substituting Eq. (4.2.15) into Eq. (4.2.14), performing the necessary integrations by parts, combining the result with Eqs. (4.2.12), (4.2.13), and rearranging, $E\{\bar{Y}^3\}$ of Eq. (4.2.3) is recovered.

Examination of Eq. (4.2.1) indicates that the mean of wave force is non-zero, independent of time t and approaches zero as the point is far removed from the mean water level.

From Eq. (4.2.2), it is seen that

$$\lim_{r_{\eta V}(0) \to 0} E\{\bar{Y}^2\} = C_M^2\,\sigma_M^2 + 3C_D^2\,\sigma_V^4 = E\{Y^2\}$$

the mean square value of Y obtained by Borgman wherein the effect of the free surface fluctuation phenomenon was ignored (Borgman, 1967). Also, far above the mean water level, $E\{\bar{Y}^2\} = 0$ whereas the quantity $E\{Y^2\}$ diverges.

Numerical results of the mean $E\{\bar{Y}\}$, standard deviation $\sigma_{\bar{Y}}$ and skewness $\gamma_{\bar{Y}}$ are obtained and compared with results obtained by Borgman (1967). These quantities are shown in Figures (4.4) and (4.5) for mean wind speed W = 40 mph and cylinder diameter D = 1 foot.

That the quantity $\bar{Y}$ has non-zero mean, and is skewed is readily seen from these figures. The statistical properties of $\bar{Y}$ given in Figs. (4.1) to (4.5) are seen to exhibit the same characteristics as those of $\bar{V}$ and $\bar{P}$.

4.3. COVARIANCE AND SPECTRUM OF WAVE FORCE

The covariance function of wave force can be represented as

$$E\{(\bar{Y}_1 - E\{\bar{Y}_1\})\ (\bar{Y}_2 - E\{\bar{Y}_2\})\} = C_M^2 E\{A_1 A_2 H(\eta_1 - z)H(\eta_2 - z)\} + C_D C_M (E\{V_1|V_1|$$

$$A_2 H(\eta_1 - z)H(\eta_2 - z)\} + E\{A_1 V_2|V_2|H(\eta_1 - z)H(\eta_2 - z)\}) + C_D^2 E\{V_1 V_2|V_1 V_2|H(\eta_1 - z)$$

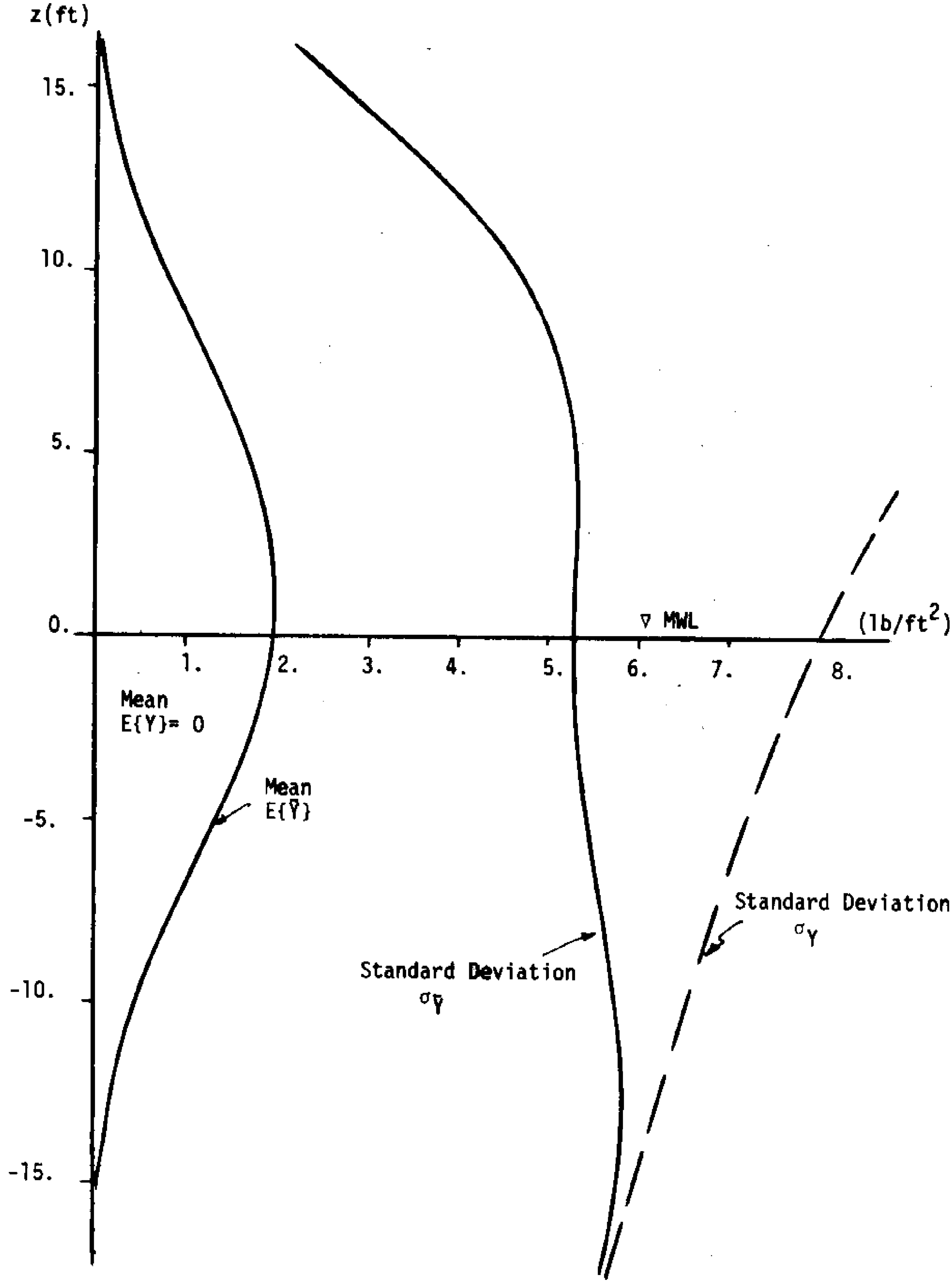

Fig. 4.4 Mean and Standard Deviation of Wave Force, Mean Wind Speed = 40 mph, Diameter of Cylinder = 1 ft.

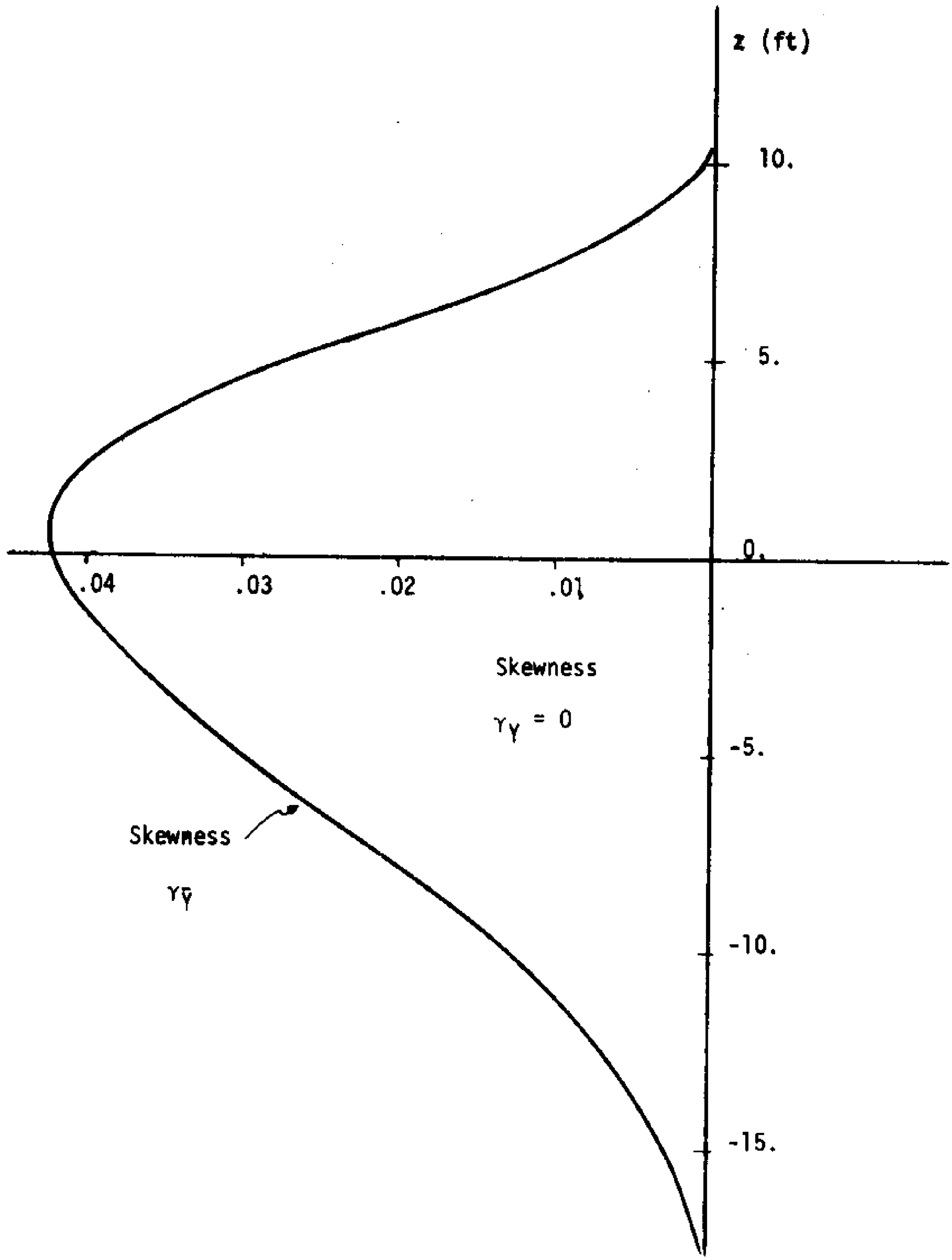

Fig. 4.5 Skewness of Wave Force, Mean Wind Speed = 40 mph, Diameter of Cylinder = 1 ft.

$$H(\eta_2-z)\} - E^2\{\hat{Y}\} \qquad (4.3.1)$$

in which the subscripts 1 and 2 refer to time instants t and t + τ, respectively. The first expectation on the right hand side of the above equation is, utilizing Eq. (3.3.6) and noting that $r_{\eta A}(0) = 0$,

$$E\{A_1A_2H(\eta_1-z)H(\eta_2-z)\} =$$

$$\sigma_A^2\{r_{AA}(\tau)L(b,b,r_{\eta\eta}(\tau)) + r_{\eta A}^2(\tau)Z(\frac{\sqrt{2}\,b}{\sqrt{1+r_{\eta\eta}(\tau)}})/\sqrt{2\pi}\ \sqrt{1-r_{\eta\eta}^2(\tau)}\}. \qquad (4.3.2)$$

It can be shown that the terms enclosed in the parenthesis in the second term on the right hand side of Eq. (4.3.1) cancel and do not contribute to the covariance function. The fourth term on the right hand side of Eq. (4.3.1) may be written as

$$E\{V_1V_2|V_1V_2|H(\eta_1-z)H(\eta_2-z)\} = E\{V_1|V_1|V_2|V_2|E\{H(\eta_1-z)H(\eta_2-z)|V_1V_2\}\} \qquad (4.3.3)$$

in which

$$E\{H(\eta_1-z)H(\eta_2-z)|V_1V_2\} = \int_{BB}^{\infty} Z(s)Q(\frac{AA-r(\tau)s}{\sqrt{1-r^2(\tau)}})ds \qquad (4.3.4)$$

and

$$AA=(z-b_1V_1-b_2V_2)/\Delta_1$$

$$BB=(z-b_2V_1-b_1V_2)/\Delta_1. \qquad (4.3.5)$$

$$r(\tau) = (R_{\eta\eta}(\tau)-b_1R_{\eta V}(\tau) - b_2R_{\eta V}(0))/\Delta_1^2$$

is the conditional correlation coefficient of η_1 and η_2 gives V_1 and V_2

$$b_1=(r_{\eta V}(0) - r_{\eta V}(\tau)\ r_{VV}(\tau))\ \sigma_\eta/(1-r_{VV}^2(\tau))\sigma_V$$

$$b_2 = (r_{\eta V}(\tau) - r_{\eta V}(0)\, r_{VV}(\tau))\sigma_\eta / (1 - r_{VV}^2(\tau))\sigma_V$$

and

$$\Delta_1 = (R_{\eta\eta}(0) - b_1 R_{\eta V}(0) - b_2 R_{\eta V}(\tau))^{1/2} .$$

Substituting Eq. (4.3.4) into Eq. (4.3.3), together with the result of Eq. (4.3.2), Eq. (4.3.1) becomes

$$E\{(\bar{Y}_1 - E\{\bar{Y}_1\})(\bar{Y}_2 - E\{\bar{Y}_2\})\} = C_M^2 \sigma_A^2 \{r_{AA}(\tau) L(b,b,r_{\eta\eta}(\tau)) + \frac{r_{\eta A}^2(\tau)}{\sqrt{2\pi}\sqrt{1 - r_{\eta\eta}^2(\tau)}}$$

$$Z\left(\frac{\sqrt{2}b}{\sqrt{1 + r_{\eta\eta}(\tau)}}\right)\} + C_D^2 \sigma_V^4 \int_{-\infty}^{\infty} y_1 |y_1| Z(y_1) \int_{-\infty}^{\infty} y_2 |y_2| Z\left(\frac{y_2 - r_{VV}(\tau) y_1}{\sqrt{1 - r_{VV}^2(\tau)}}\right) \int_{BB}^{\infty} Z(s)$$

$$Q\left(\frac{AA - r(\tau)s}{\sqrt{1 - r^2(\tau)}}\right) ds\, dy_2 dy_1 - E^2\{\bar{Y}\}. \qquad (4.3.6)$$

The covariance function of wave force $\bar{Y}_1$ and $\bar{Y}_2$ is seen to be independent of time t and is denoted by $R_{\bar{Y}\bar{Y}}(\tau)$.

The spectrum $S_{\bar{Y}\bar{Y}}(n)$ of $\bar{Y}$ is again the Fourier transfrom of $R_{\bar{Y}\bar{Y}}(\tau)$. That is

$$S_{\bar{Y}\bar{Y}}(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{\bar{Y}\bar{Y}}(\tau) e^{in\tau} d\tau . \qquad (4.3.7)$$

It is noted that $R_{\bar{Y}\bar{Y}}(\tau)$ cannot be integrated in closed form and must be performed numerically. Due to the multiple integrals involved, it is desirable to simplify the exact formulation by its approximation.

AN APPROXIMATE REPRESENTATION

The approximate representation of $R_{\overline{YY}}(\tau)$ can be found by Taylor's series expansion of the function about $r_{\eta\eta}(\tau) = r_{\eta V}(\tau) = r_{VV}(\tau) = r_{\eta A}(\tau) = r_{AA}(\tau) = 0$ as previously done for the covariance function of $\overline{U}$ in Chapter 3.

Letting $AR_{\overline{YY}}(\tau)$ denote the approximate covariance function of wave force, it can be shown that by retaining only the first two terms in the expansion,

$$AR_{\overline{YY}}(\tau) = C_M^2\sigma_A^2 Q^2(b) r_{AA}(\tau) + C_D^2\sigma_V^4\{G_0^2 + G_0 G_1 r_{\eta\eta}(\tau) + \left(\frac{2G_1G_2}{\sqrt{1-r_{\eta V}^2(0)}} - \frac{r_{\eta V}(0)G_0G_1}{1-r_{\eta V}^2(0)}\right) r_{\eta V}(\tau) + \left(G_2^2 - \frac{2r_{\eta V}(0)}{\sqrt{1-r_{\eta V}^2(0)}}\, G_1G_2 + \frac{r_{\eta V}^2(0)G_0G_1}{1-r_{\eta V}^2(0)}\right) r_{VV}(\tau)\} - E^2\{\overline{Y}\} \tag{4.3.8}$$

in which

$$G_0 = L(0,b,r_{\eta V}(0)) - L(0,b,-r_{\eta V}(0)) + r_{\eta V}^2(0)\, b\, Z(b)\left[Q\left(\frac{-r_{\eta V}(0)b}{\sqrt{1-r_{\eta U}^2(0)}}\right) - Q\left(\frac{r_{\eta V}(0)b}{\sqrt{1-r_{\eta V}^2(0)}}\right)\right] + 2\, r_{\eta V}(0)\sqrt{1-r_{\eta V}^2(0)}\; Z(b)\; Z\left(\frac{r_{\eta V}(0)b}{\sqrt{1-r_{\eta V}^2(0)}}\right)$$

$$G_1 = \sqrt{1-r_{\eta V}^2(0)}\; Z(b)\; \{(1-r_{\eta V}(0)\;(1-b^2))\left[Q\left(\frac{-r_{\eta V}(0)b}{\sqrt{1-r_{\eta V}^2(0)}}\right) - Q\left(\frac{r_{\eta V}(0)b}{\sqrt{1-r_{\eta V}^2(0)}}\right)\right] + 2r_{\eta V}(0)b\sqrt{1-r_{\eta V}^2(0)}\; Z\left(\frac{r_{\eta V}(0)b}{\sqrt{1-r_{\eta V}^2(0)}}\right)\}$$

$$G_2 = \frac{4}{\sqrt{2\pi}}\; Q\left(\frac{b}{\sqrt{1-r_{\eta V}^2(0)}}\right) + (3-r_{\eta V}^2(0)\;(1-b^2))\; r_{\eta V}(0)\; Z\;(b)\;\left[Q\left(\frac{-r_{\eta V}(0)b}{\sqrt{1-r_{\eta V}^2(0)}}\right)\right.$$

$$-Q\left(\frac{r_{\eta V}(0)b}{\sqrt{1-r_{\eta V}^2(0)}}\right)] + 2r_{\eta V}^2(0)\, b\sqrt{1-r_{\eta V}^2(0)}\; Z(b)\; Z\left(\frac{r_{\eta V}(0)b}{\sqrt{1-r_{\eta V}^2(0)}}\right).$$

The approximate spectrum $AS_{\overline{YY}}(n)$ is the Fourier transform of $AR_{\overline{YY}}(\tau)$ and is

$$AS_{\overline{YY}}(n) = (C_D^2\sigma_V^2 G_0 G_1/\sigma_\eta^2)S_{\eta\eta}(n) + C_D^2\sigma_V^3\left(\frac{2G_1G_2}{\sigma_\eta\sqrt{1-r_{\eta V}^2(0)}} - \frac{r_{\eta V}(0)G_0G_1}{\sigma_\eta(1-r_{\eta V}^2(0))}\right)$$

$$S_{\eta V}(n) + C_D^2\sigma_V^2\left(G_2^2 - \frac{2r_{\eta V}(0)G_1G_2}{\sqrt{1-r_{\eta V}^2(0)}} + \frac{r_{\eta V}^2(0)G_0G_1}{1-r_{\eta V}^2(0)}\right) S_{VV}(n) + C_M^2\; Q^2(b)S_{AA}(n) \quad (4.3.9)$$

in which $S_{\eta V}(n)$ is the cross spectrum of $\eta(t)$ and $V(z,t)$ and is given by Eq. (3.3.10) by replacing U by V.

Numerical results are shown in Figures (4.6) to (4.11). These include the exact and approximate covariance functions and spectrum together with the exact covariance function and spectrum of Borgman (1967). For covariance functions, only the portions corresponding to $\tau \geq 0$ are shown and the spectra are the one-sided spectra. The agreement between the exact and approximate results are in general reasonably satisfactory, although to a lesser extent as compared with the cases of $\bar{V}$, $\bar{A}$, and $\bar{P}$ shown in Figures (3.14) to (3.31). Comparison of the present results with those of Borgman (1967), however, indicates that the latter grossly overestimates these quantities at and above the mean water level.

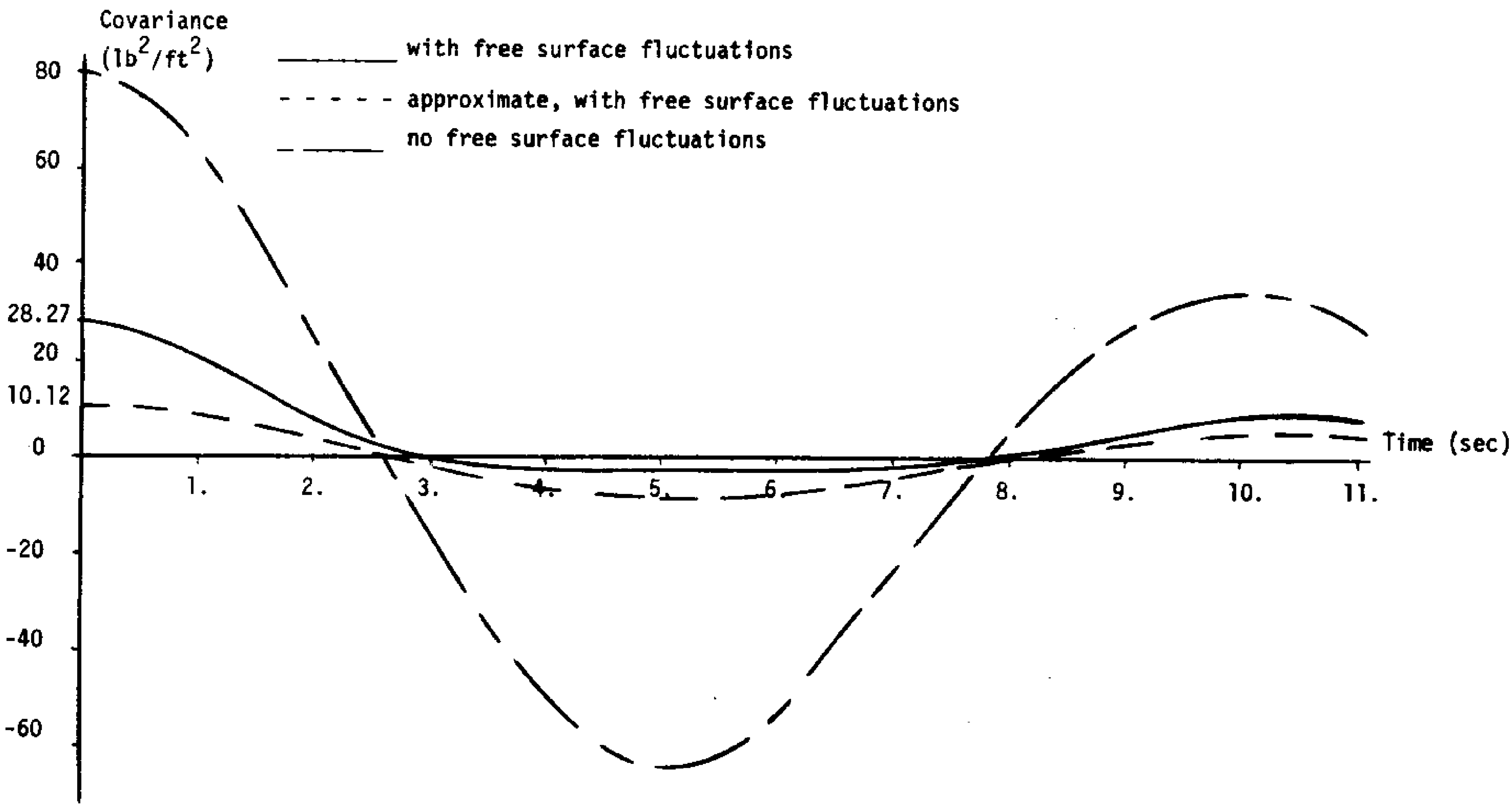

Fig. 4.6 Covariance Function of Wave Force at $z = +\sigma_\eta = +5.59$ ft., Mean Wind Speed = 40 mph, Diameter of Cylinder = 1. ft.

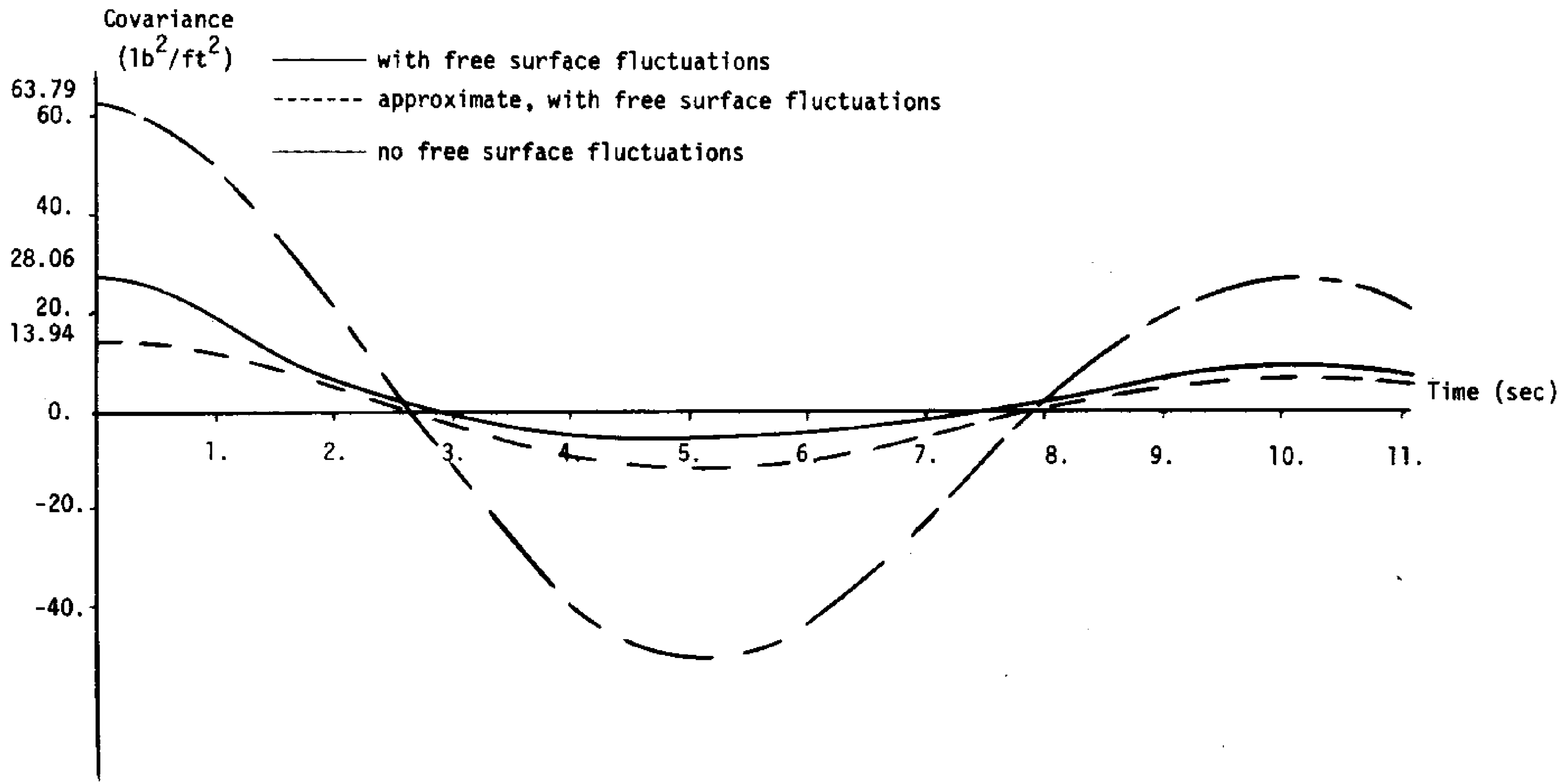

Fig. 4.7 Covariance Function of Wave Force at $z = 0$ ft., Mean Wind Speed = 40 mph, Diameter of Cylinder = 1 ft.

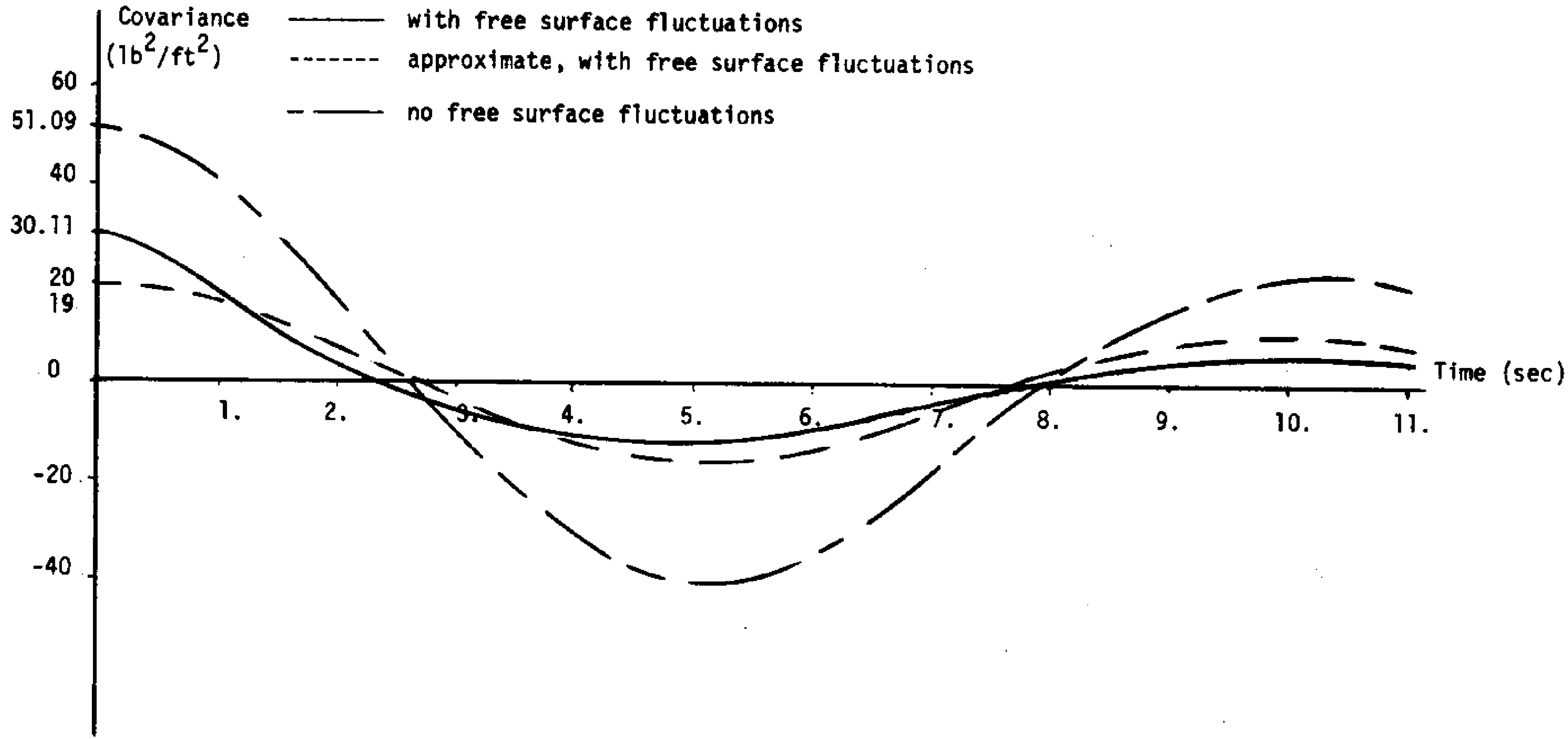

Fig. 4.8 Covariance Function of Wave Force at $z = -\sigma_n = -5.59$ ft., Mean Wind Speed = 40 mph, Diameter of Cylinder = 1 ft.

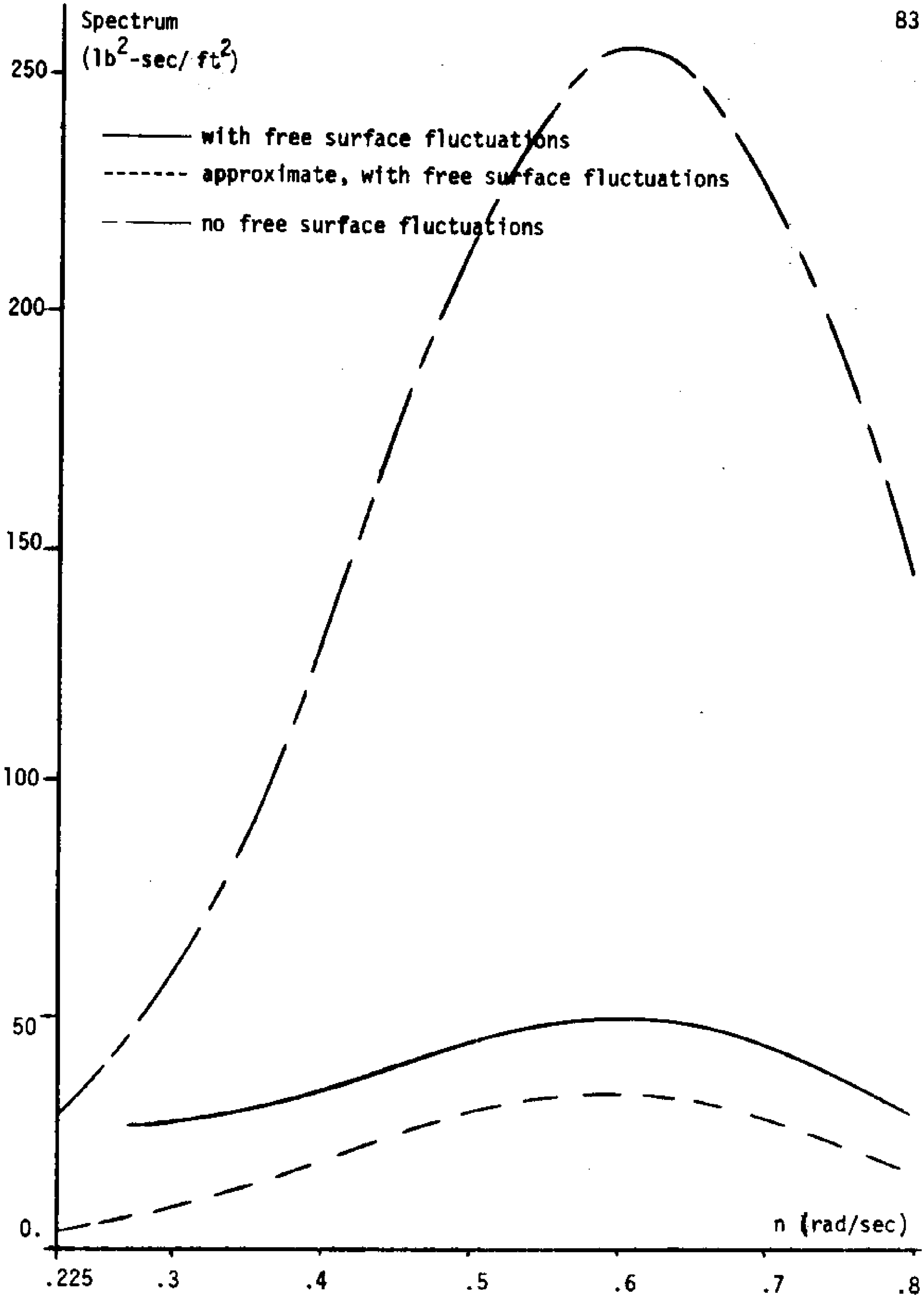

Fig. 4.9 Spectrum of Wave Force at $z = +\sigma_\eta = +5.59$ ft., Mean Wind Speed = 40 mph, Diameter of Cylinder = 1 ft.

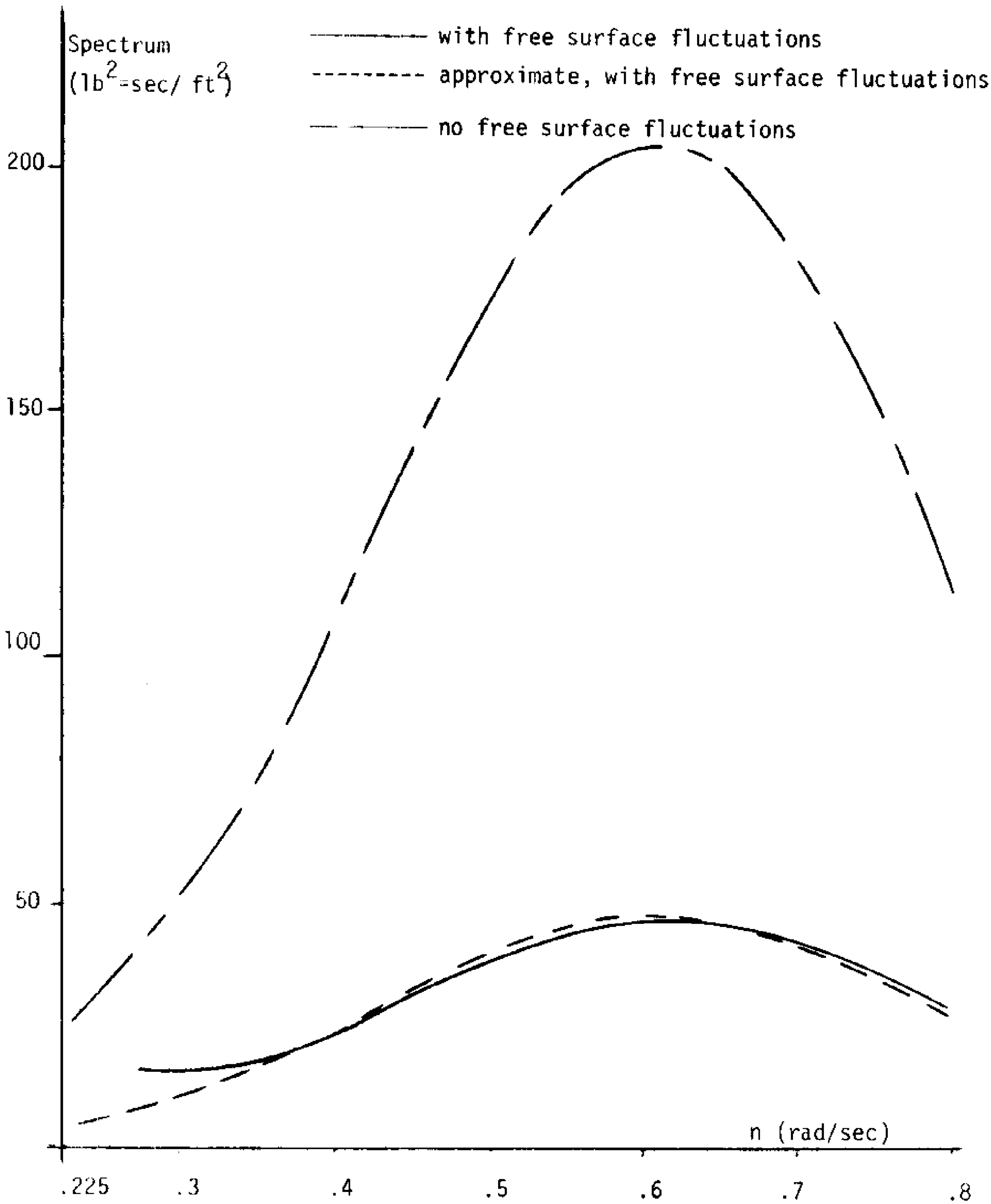

Fig. 4.10 Spectrum of Wave Force at z = 0. ft., Mean Wind Speed = 40 mph
Diameter of Cylinder = 1 ft.

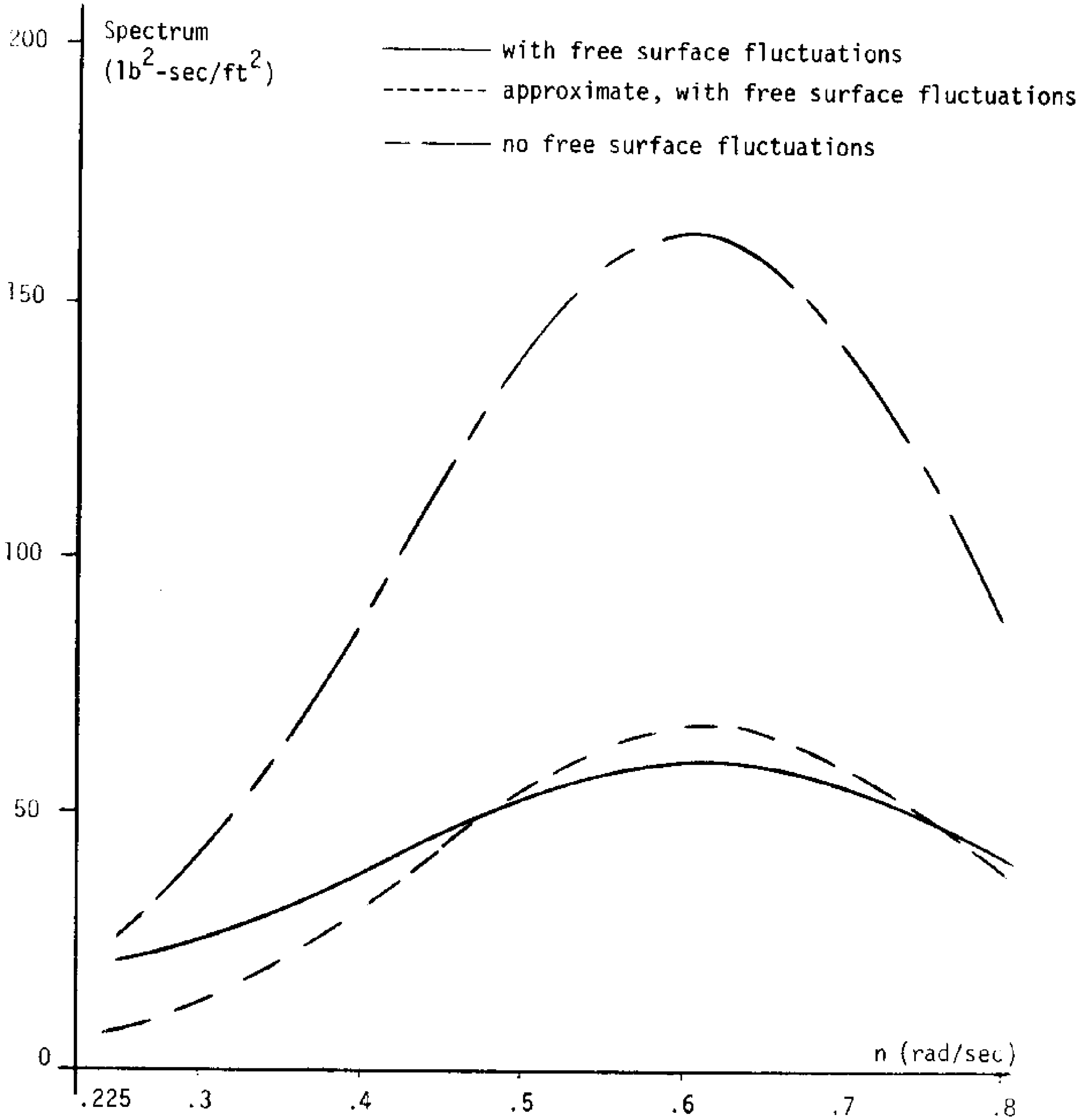

Fig. 4.11 Spectrum of Wave Force at $z = -\sigma_\eta = -5.59$ ft., Mean Wind Speed = 40 mph, Diameter of Cylinder = 1 ft.

5. SUMMARY AND CONCLUSIONS

In this investigation, the statistical properties of wave field kinematics, pressure, and wave force in a random wave field are derived, taking into account the free surface fluctuation phenomenon. Numerical results are also obtained and presented.

The following conclusions arise out of this study:

1. The horizontal component of velocity, acceleration, pressure, and wave force are all non-Gaussian.
2. The horizontal component of velocity, pressure, and wave force all possess non-zero mean and are skewed. The horizontal component of acceleration, however, has zero mean and is not skewed.
3. Far below the mean water level, the mean, variance, and skewness of the horizontal component of velocity, acceleration, pressure, and wave force approach past results. Far above the mean water level, these quantities approach zero whereas past results indicate that they grow indefinitely.
4. The covariance functions and spectra of the horizontal component of velocity, acceleration, pressure, and wave force are derived and approximate expressions are also obtained. The approximations are seen to be adequate. The present covariance function and spectrum converge to past results when the point is far below the mean water level. Far above the mean water level, they deviate drastically from past results.

It may therefore be concluded that the free surface fluctuation phenomenon has a major influence on the statistical properties of the quantities examined and must be considered.

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