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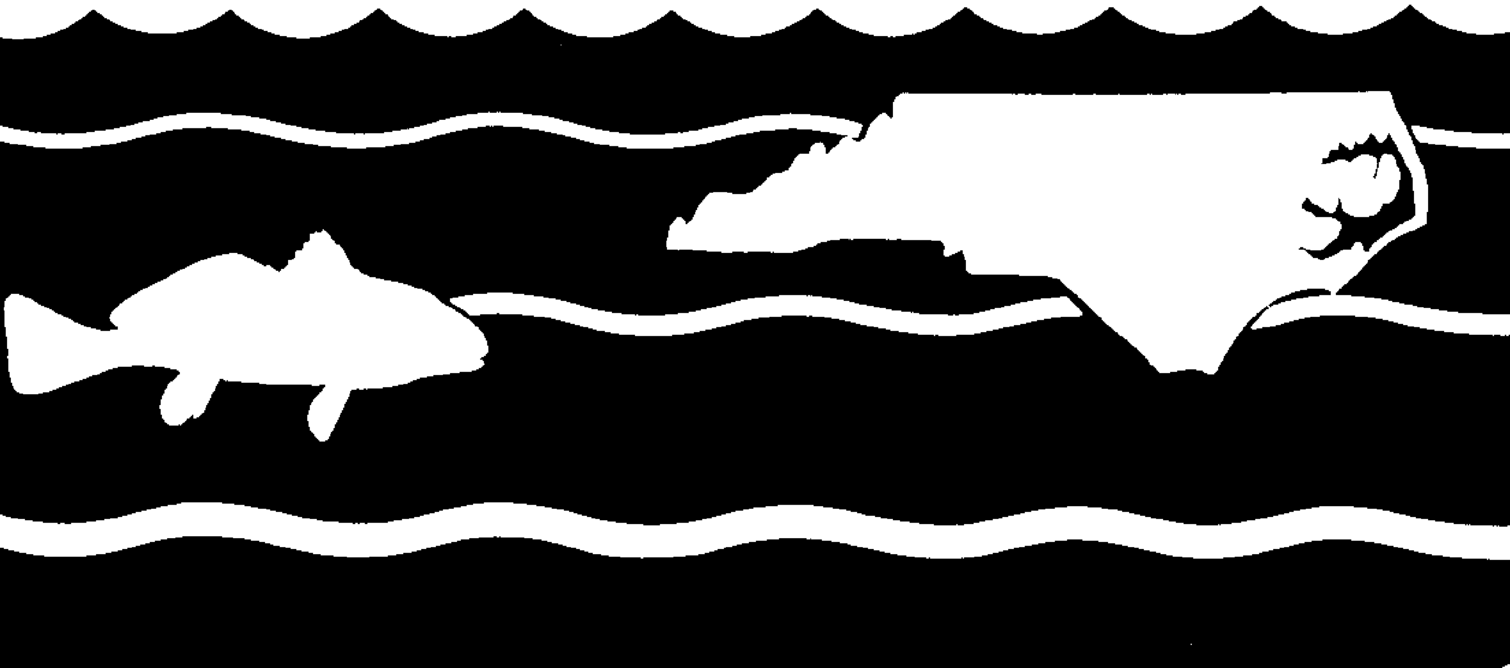

SOME STATISTICAL PROPERTIES OF WAVE-CURRENT FORCE ON OBJECTS

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SOME STATISTICAL PROPERTIES OF WAVE-CURRENT FORCE ON OBJECTS

INTRODUCTION

When wave encounters current, its characteristics change; if the current is in the direction of wave propagation, wave amplitude decreases and its length increases, but if the current opposes the wave, the wave steepens and shortens. These changes are due to the interactions between current and wave as was explained in Ref. (5). In a random wave field, component wave amplitude and wave length experience similar changes resulting in modification of frequency and wave-number spectra of surface waves (3). Anticipating that fluid force, being directly related to fluid field kinematics, would be similarly affected by the presence of current, properties of the spectrum of fluid force on cylinder were investiaged (10). Two cases were examined; that is, when current was simply superimposed on waves and when wave-current interactions were considered. It was shown that drastic changes indeed took place in fluid force spectrum particularly when interactions are considered.

Safety analysis of flexible ocean structures requires consideration of both catastrophic and fatique failure. The probability function and expected rate of threshold crossings of stresses induced in the structures are the two quantities widely used in this connection (2,4,9).

In this report, the influence of current on such statistical quantities as the mean, standard deviation, skewness, probability function and expected rate of threshold crossings of random fluid force are studied both with and without wave-current interactions considered. The effects of current and wave-current interactions on these quantities are clearly noted. For simplicity, only deep water stationary random waves under the influence of a steady current uniformly distributed in depth are considered. The fluid force is evaluated at an element of a cylinder of unit diameter and unit length situated immediately beneath the mean water level.

SPECTRA OF WAVE FIELD KINEMATICS

In evaluating fluid force, the Morrison's formula is used in this study. That is, fluid force is considered to consist of two parts, the inertia component, linearly proportional to fluid particle acceleration, and the drag component, nonlinearly related to fluid particle velocity. In subsequent computation of the statistical properties of fluid force, the quantities σ_v, the standard deviation of fluid particle velocity and σ_a, $\sigma_{\dot{a}}$, those of fluid particle acceleration and its derivative are required. These quantities in turn are determined from their respective spectra. Thus the influence of current on wave frequency spectrum and spectra of fluid particle velocity, acceleration and its derivative are discussed first.

It was shown (3) that under the action of a steady current, the frequency spectrum of the surface waves of a stationary gravity wave field in deep water is given by

$$\phi(n) = \frac{4\phi^*(n)}{\left[1+\left(1+\frac{4Un}{g}\right)^{1/2}\right]\left[\left(1+\frac{4Un}{g}\right)^{1/2}+\left(1+\frac{4Un}{g}\right)\right]} \qquad (1)$$

in which n is total frequency, U is current velocity, g is gravitational acceleration and $\phi(n)$ and $\phi^*(n)$ are respectively the frequency spectrum of surface waves with and without including the influence of current. In this study $\phi^*(n)$ is taken to be

$$\phi^*(n) = \frac{\alpha g^2}{n^5} \exp\left(-\beta\left(\frac{n_o}{n}\right)^4\right) \tag{2}$$

the Kitaigorodskii-Pierson-Moskowitz spectrum, in which α and β are non-dimensional constants equal to 0.8×10^{-2} and 0.74 respectively and $n_o = g/\overline{W}$ with $\overline{W}$ the mean wind speed. The spectra $\phi(n)$ for various values of current speed U are plotted in Fig. 1. It is seen that when the current is in the direction of the waves, that is, when U is positive, the surface spectrum is lowered. On the other hand, under adverse current, the surface wave spectrum increases in magnitude. When the current speed is negative, there is a cut-off frequency in the surface wave spectrum determined by the condition $1 + \frac{4Un}{g} = 0$ beyond which no waves can exist.

The spectra of fluid particle velocity, acceleration and its derivative at mean water level are given by

$$\phi_{VV}(n) = n^2\phi(n) \tag{3}$$

$$\phi_{aa}(n) = n^4\phi(n) \tag{4}$$

and

$$\phi_{\dot{a}\dot{a}}(n) = n^6\phi(n). \tag{5}$$

The standard deviations of σ_v of fluid particle velocity and σ_a, $\sigma_{\dot{a}}$ of fluid particle acceleration and its derivative are respectively obtained from

$$\sigma_v = \left[\int_n \phi_{VV}(n)\,dn\right]^{1/2} \tag{6}$$

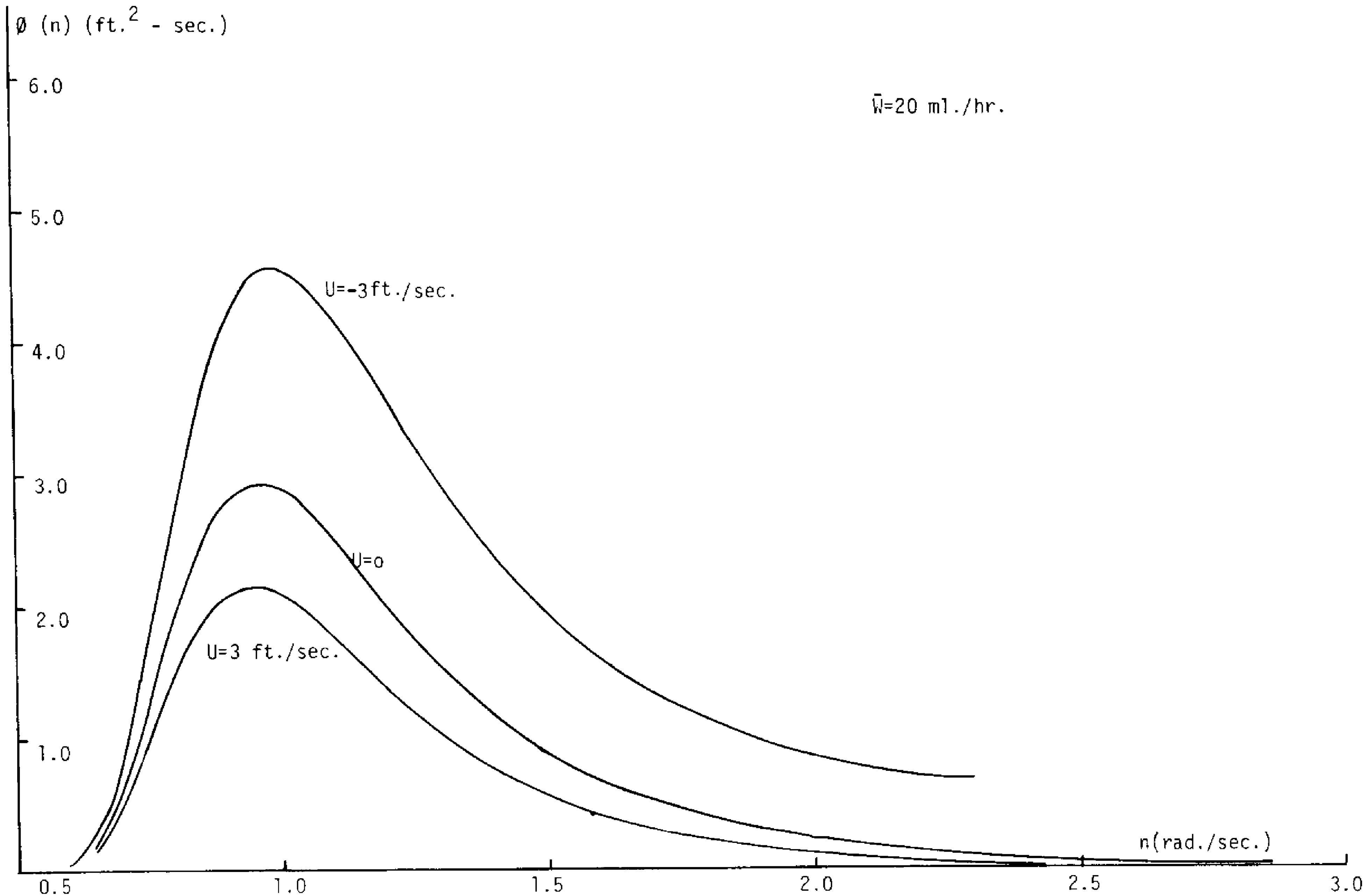


FIGURE 1. SURFACE WAVE FREQUENCY SPECTRUM

$$\sigma_a = [\int_n \phi_{aa}(n)dn]^{1/2} \qquad (7)$$

and

$$\sigma_{\dot{a}} = [\int_n \phi_{\dot{a}\dot{a}}(n)dn]^{1/2} \ . \qquad (8)$$

Fig. 2 shows the effect of wave-current interactions on the fluid particle velocity spectrum $\phi_{VV}(n)$ under various current conditions. Spectra of fluid particle acceleration and its derivative exhibit similar characteristics and are not shown.

STATISTICAL MOMENTS OF FLUID FORCE

The fluid force on the cylinder of unit length, according to Morrison's formula is

$$F(t) = C_D V(t)|V(t)| + C_M a(t) \qquad (9)$$

in which a(t) and V(t) are respectively the fluid particle acceleration and velocity with V(t) = v(t) + U, v(t) being the oscillatory wave induced particle velocity. In Eq. 9, $C_D = \rho K_D D$ and $C_M = \rho K_M \frac{\pi D^2}{4}$, with ρ, density of water, D, diameter of cylinder and K_D = 0.5 to 0.7 and K_M = 1.4 to 2.0 the drag and inertia coefficients (6). In this study, the values of K_D and K_M are chosen to be 0.5 and 1.4 respectively, and for convenience, but without loss of generality, ρ is set equal to unity.

The expressions of the various moments of the random process F(t) were derived by Borgman (1) using the moment generating function. These quantities are rederived in the Appendix in a more direct manner. The

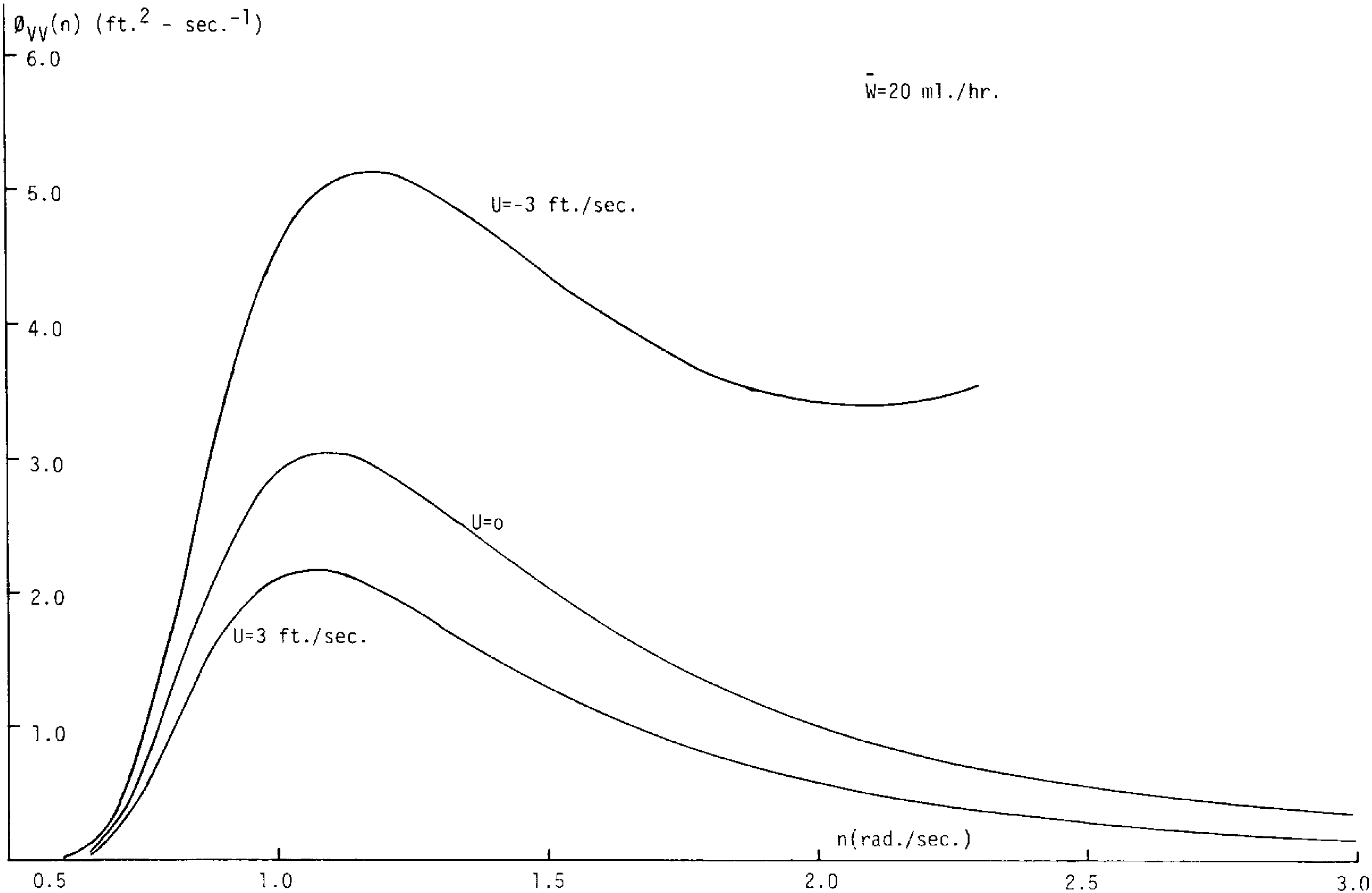


FIGURE 2. VELOCITY FREQUENCY SPECTRUM

expected value or the mean of the random force F(t) is

$$E[F] = 2C_D\sigma_v^2[\gamma T(\gamma) + (1 + \gamma^2)P(\gamma)] \qquad (10)$$

in which E [.] denotes the expected value of the random quantity enclosed in the bracket, $\gamma = U/\sigma_v$ is a parameter measuring the strength of current, $T(\gamma) = \frac{1}{\sqrt{2\pi}} \exp(-\gamma^2/2)$ and $P(\gamma) = \int_0^\gamma T(x)dx$ is the error function (7). In Eq. 10, the argument t of F(t) is omitted for convenience. Under given wind and current conditions, the expected value of the fluid force can be computed from Eq. 10 using Eqs. 1 through 4 and Eqs. 6 and 7. If wave-current interactions are ignored, that is, if it is assumed that current does not affect wave characteristics, E[F] can still be obtained from Eq. 10 in conjunction with Eqs. 2, 3, 4, 6, 7 with $\phi(n)$ replaced by $\phi^*(n)$ in Eqs. 3 and 4.

In Fig. 3, the absolute value of E[F] is plotted as a function of current speed U with and without wave-current interactions considered and with wind speed as parameter. In the former case, the curve is skewed, indicating that when $U > o$, waves are dampened thus reducing the value of the expected fluid force whereas, when $U < o$, waves are amplified causing an increase in expected fluid force. When interactions are neglected, the curve is necessarily symmetrical. Figure 3 also shows that the magnitude of the expected fluid force increases with increase in current and wind speed.

To further investigate the effect of wave-current interactions on E[F], the ratio of E[F] with and without interactions considered is presented

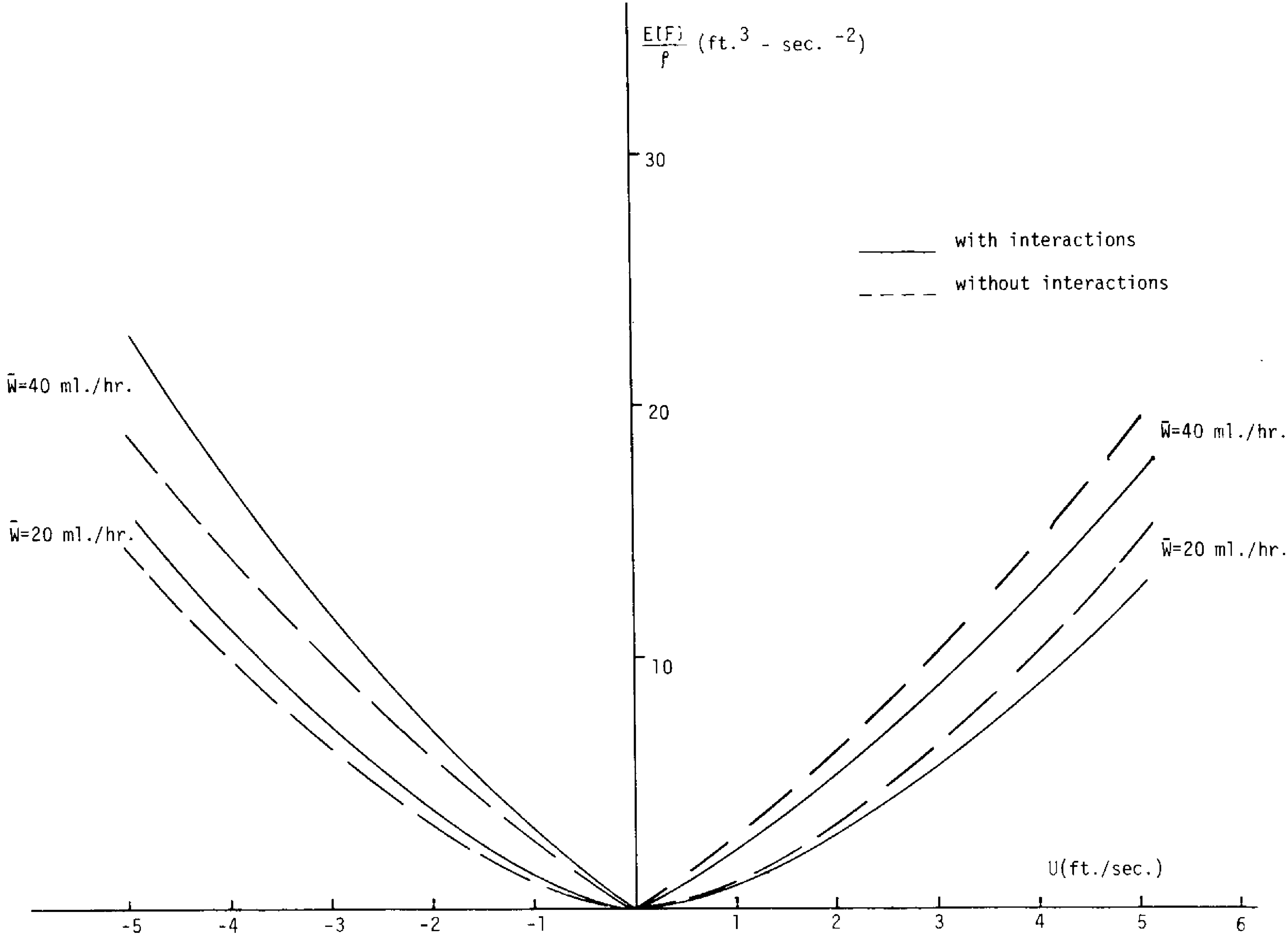


FIGURE 3. MEAN OF FLUID FORCE

in Fig. 4 against current speed with wind speed as parameter. It is seen that under moderate current conditions increase in wind speed tends to lessen the effect of interactions. As current speed increases, current predominates the expected value of F(t) thus reducing the importance of interactions and the lesser the wind speed, the earlier current overpowers the expected fluid force and the sooner the effect of interactions diminishes. Finally, for the range of wind and current speed considered, the maximum effect of interactions on D[F] is about 20% when $U < o$ and 12% when $U > o$. That is, the phenomenon of interactions is more pronounced when $U < o$ as indicated in Figs. 1 and 2.

The second moment of the fluid force F(t) is

$$E[F^2] = C_M^2\sigma_a^2 + C_D^2\sigma_v^4(\gamma^4 + 6\gamma^2 + 3) \tag{11}$$

and the standard deviation σ_F of F(t) is

$$\sigma_F = \{E[F^2] - E^2[F]\}^{1/2}. \tag{12}$$

Plotted in Fig. 5 is the ratio of σ_F with and without current, as a function of current speed. When no interactions are considered the curve is symmetrical and, with interactions the curve is skewed. It is noted that for moderate positive values of current speed, the standard deviation of fluid force drops slightly below that when there is no current. For all the cases presented in the figure, the weaker the wind condition, the more susceptible it is to the influence of current except when $U < o$ and when wave-current interactions are taken into account. This is due to the fact that waves produced by lesser wind undergo excessive breaking in adverse current thus reducing the value of σ_F. The influence of interactions on σ_F is also clearly exhibited

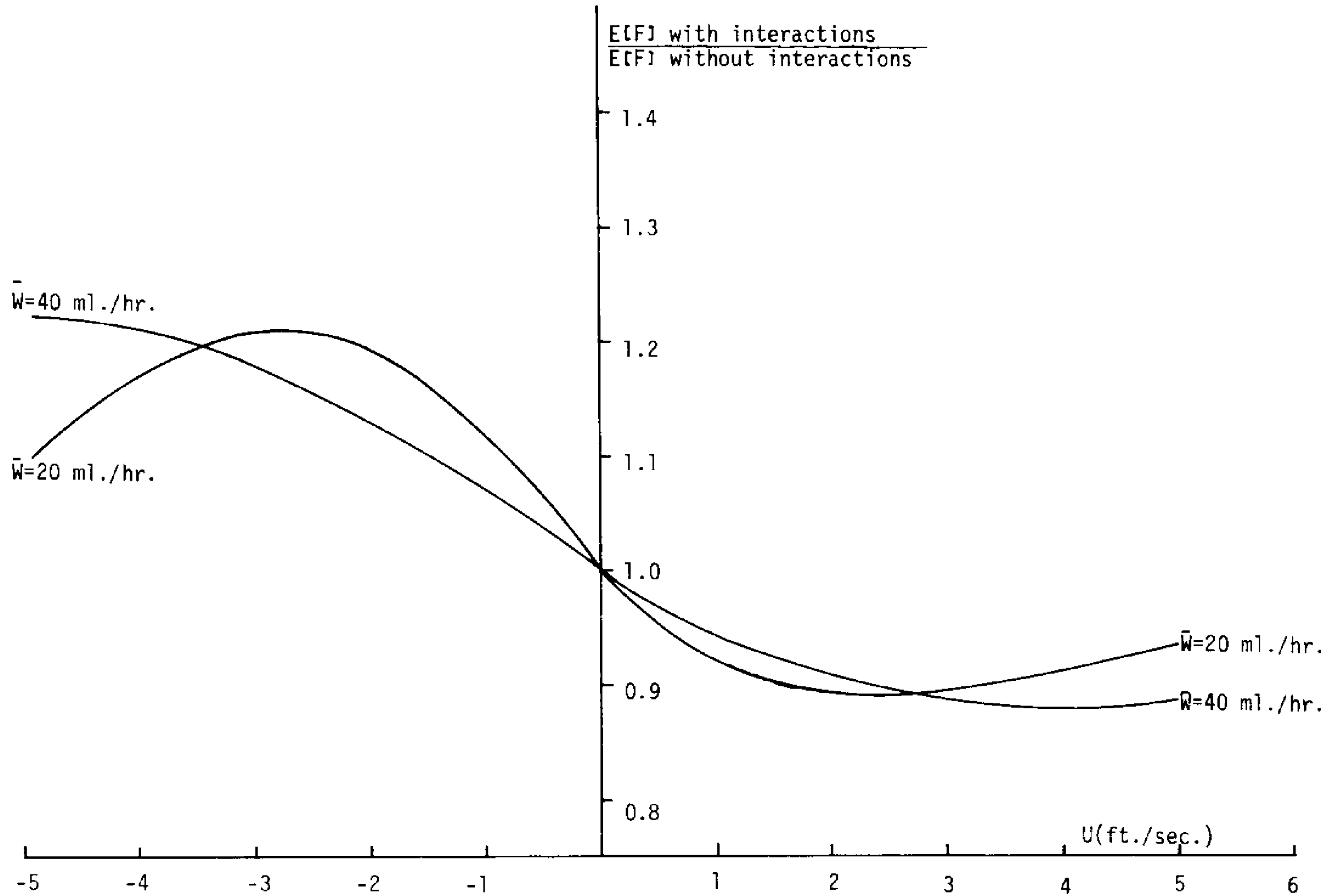


FIGURE 4. RATIO OF MEAN FORCE WITH AND WITHOUT INTERACTIONS

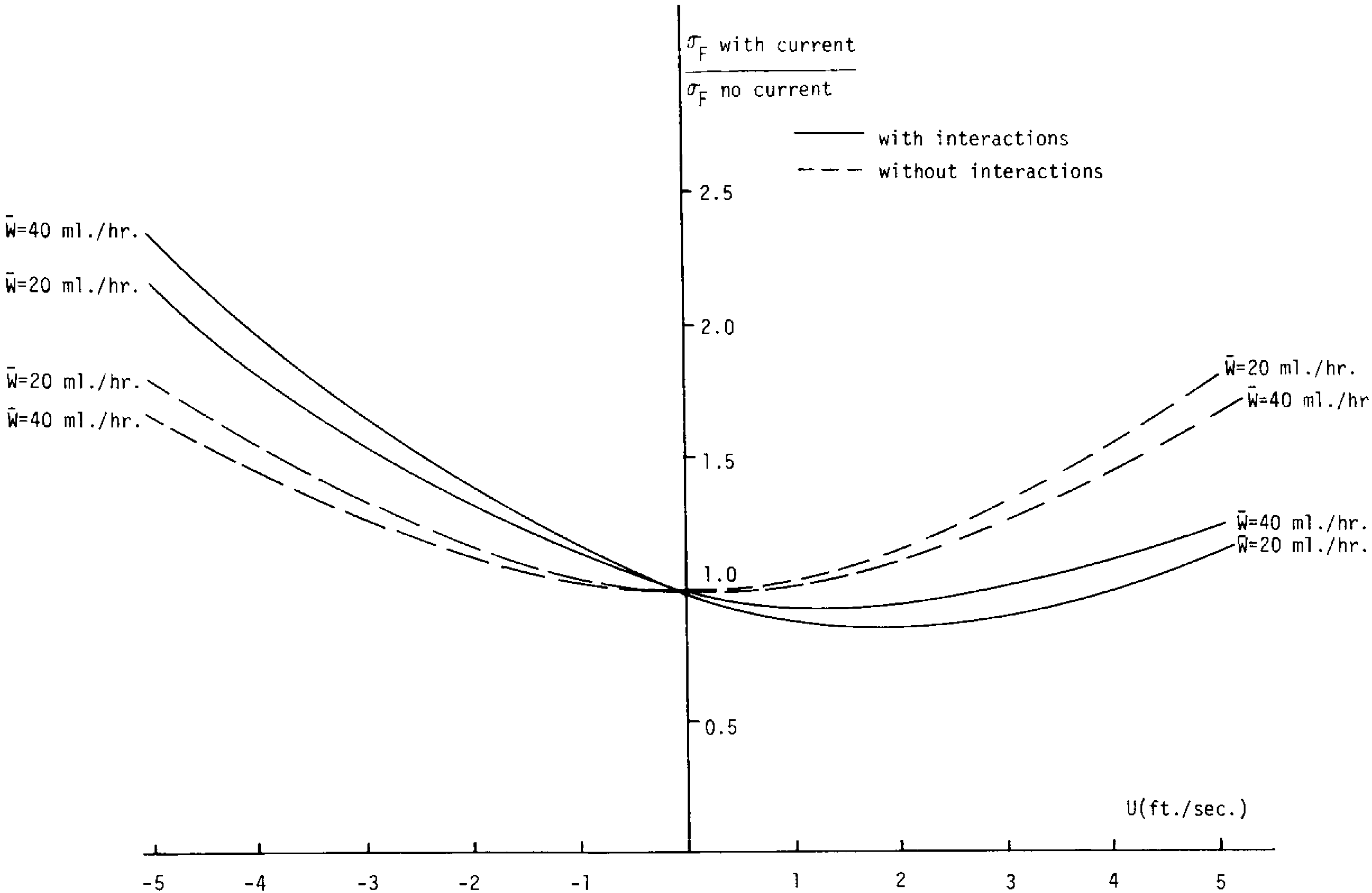


FIGURE 5. RATIO OF STANDARD DEVIATION OF FORCE WITH AND WITHOUT CURRENT

in the figure. That is, for $U > o$, the stronger the wind, the less important is interactions phenomenon, but the situation is reversed when $U < o$. This can again be explained by the wave breaking phenomenon. Finally, for the wind and current speeds considered the maximum effect of interactions on σ_F can be as high as 50% indicating that interactions are more important for σ_F than for E[F].

While the expected value and standard deviation of F(t) give indications of the average value and spread of the probability density function of F(t), its skewness may be measured by

$$\gamma_1 = E[(F - E[F])^3]/\sigma_F^3 \tag{13}$$

in which the numerator is the central third moment of F(t) and is related to E[F], σ_F and $E[F^3]$ as

$$E[(F - E[F])^3] = E[F^3] - 3\sigma_F^2 E[F] + 3\sigma_F E^2[F] - E^3[F]. \tag{14}$$

In Eq. 14 the term $E[F^3]$ is the third moment of F(t) and is given by

$$E[F^3] = C_M^3\sigma_a^3\{\frac{\gamma T(\gamma)}{4}\ [\frac{\gamma^4 + 14\gamma^2 + 33}{\lambda^3} + \frac{12}{\lambda}]$$

$$+ P(\gamma)[\frac{\gamma^6 + 15\gamma^4 + 45\gamma^2 + 15}{4\lambda^3} + \frac{3(1 + \gamma^2)}{\lambda}]\} \tag{15}$$

in which $\lambda = C_M\sigma_a/2C_D\sigma_v^2$.

In Fig. 6, the absolute value of γ_1 is plotted. Several trends are noted in the figure. First, for a given current speed, the stronger the wind, the more skewed is F(t) but less is the influence of interactions. Secondly, skewness of F(t) increases with increase in magnitude of current

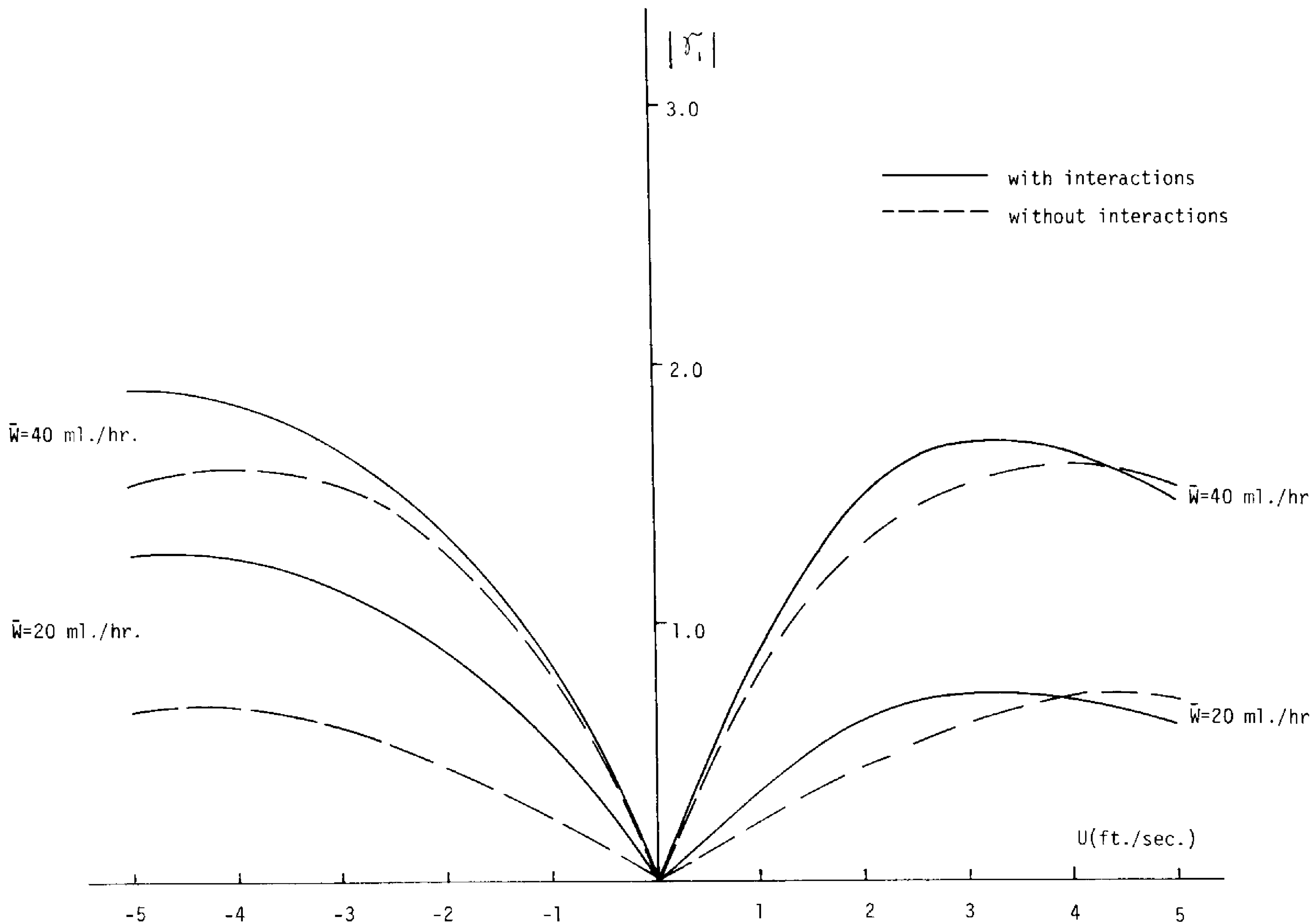


FIGURE 6. SKEWNESS OF FORCE

speed until current predominates when the force F(t) becomes almost deterministic. Finally, under moderate current condition, wave-current interactions tend to make F(t) more skewed than when no interactions are present and negative current renders F(t) more skewed than positive current does unless interactions are ignored in which case the curve is symmetrical.

PROBABILITY FUNCTION OF FLUID FORCE

It was pointed out by Pierson and Holmes (6) both by theory and field observations that the force $F(t) = C_D V|V| + C_M a$, being a linear combination of the Gaussian inertia force and non-Gaussian drag force, is non-Gaussian. It was further noted that the parameter $\lambda = C_M \sigma_a / 2C_D \sigma_v^2$ is a measure of the relative importance of the inertia and drag components of fluid force and therefore also serves as an indicator of the degree of closeness of F(t) to a Gaussian process. Thus, the smaller the value of λ, the more important is the drag force relative to the inertia force and the more F(t) deviates from Gaussian. Although there was no current involved in the work of Pierson and Holmes (8), the same conclusion was noted by Borgman (1) when the fluid particle velocity V(t) may possess a non-zero mean. It is therefore of interest to investigate, before taking up the subject of the probability function of the fluid force F(t), the influence of wind and current on the quantity λ.

In Fig. 7, the quantity λ is plotted against current speed. It is seen that when no interactions are included, λ is independent of current condition. When interactions are considered, current invariably renders the process F(t) less Gaussian, more so when the current speed is negative

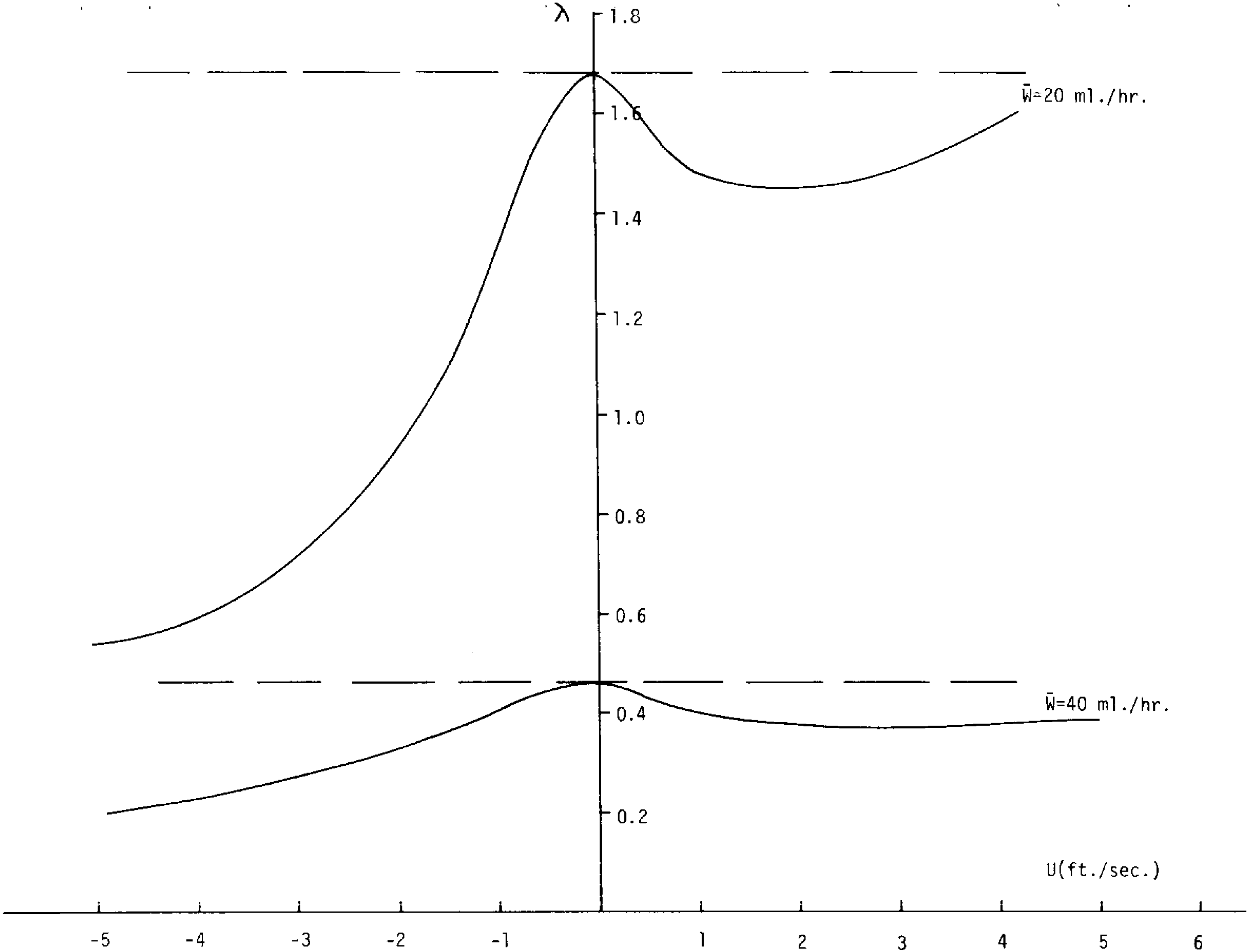


FIGURE 7. PARAMETER $\lambda = C_M \sigma_a / 2 C_D \sigma_v^2$

than when it is positive. It is also observed that the higher the wind speed, the farther removed is F(t) from being Gaussian but the effect of interaction on the value of λ is also reduced. It should be mentioned here that neither the moments nor the quantity λ can provide complete and accurate information regarding the probability structure of the random process F(t). A more complete description of the process F(t) is contained in the probability density function.

The expression for the probability density function of F(t) was derived by Borgman (1) although no numerical result was given. In this study, for ease of reference, the probability density function $f_X(x)$ of X = F(t) is rederived, using a slightly different approach, in the Appendix. The probability density function of X = F(t) as derived, is

$$f_X(x) = \frac{\sqrt{2\lambda}}{2\pi C_M \sigma_a} \left\{ \int_0^{\infty} \exp\left[-\frac{1}{2} \left(\frac{x}{C_M \sigma_a} + s^2\right)^2 - \lambda\left(s + \frac{\gamma}{\sqrt{2\lambda}}\right)^2 \right] ds \right.$$

$$\left. + \int_0^{\infty} \exp\left[-\frac{1}{2} \left(\frac{x}{C_M \sigma_a} - s^2\right)^2 - \lambda\left(s - \frac{\gamma}{\sqrt{2\lambda}}\right)^2 \right] ds \right\}. \qquad (16)$$

The integrals in Eq. 16 can not be carried out exactly and are therefore performed numerically.

Presented in Fig. 8a, are the probability density functions of X = F(t) with current speed U = 0, and U = 3 ft./sec. When U = 3 ft./sec., both the cases with and without wave-current interactions considered are given and for all cases considered the corresponding Gaussian approximation is also presented. When fluid force X = F(t) is assumed to be Gaussian, the probability density function $f_X(x)$ is given by

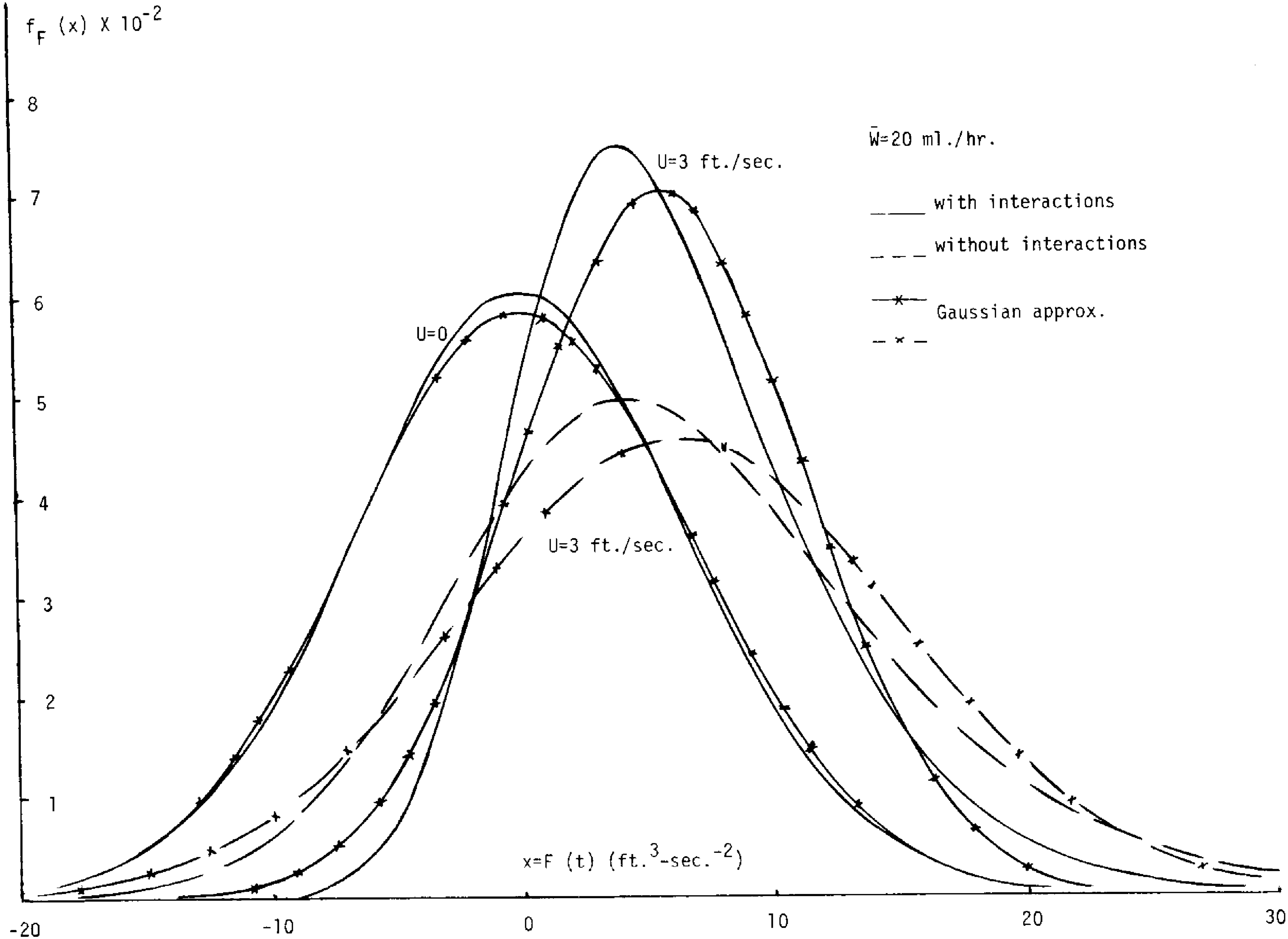


FIGURE 8a PROBABILITY DENSITY FUNCTION OF FORCE where U = 0 and U = 3 ft./sec.

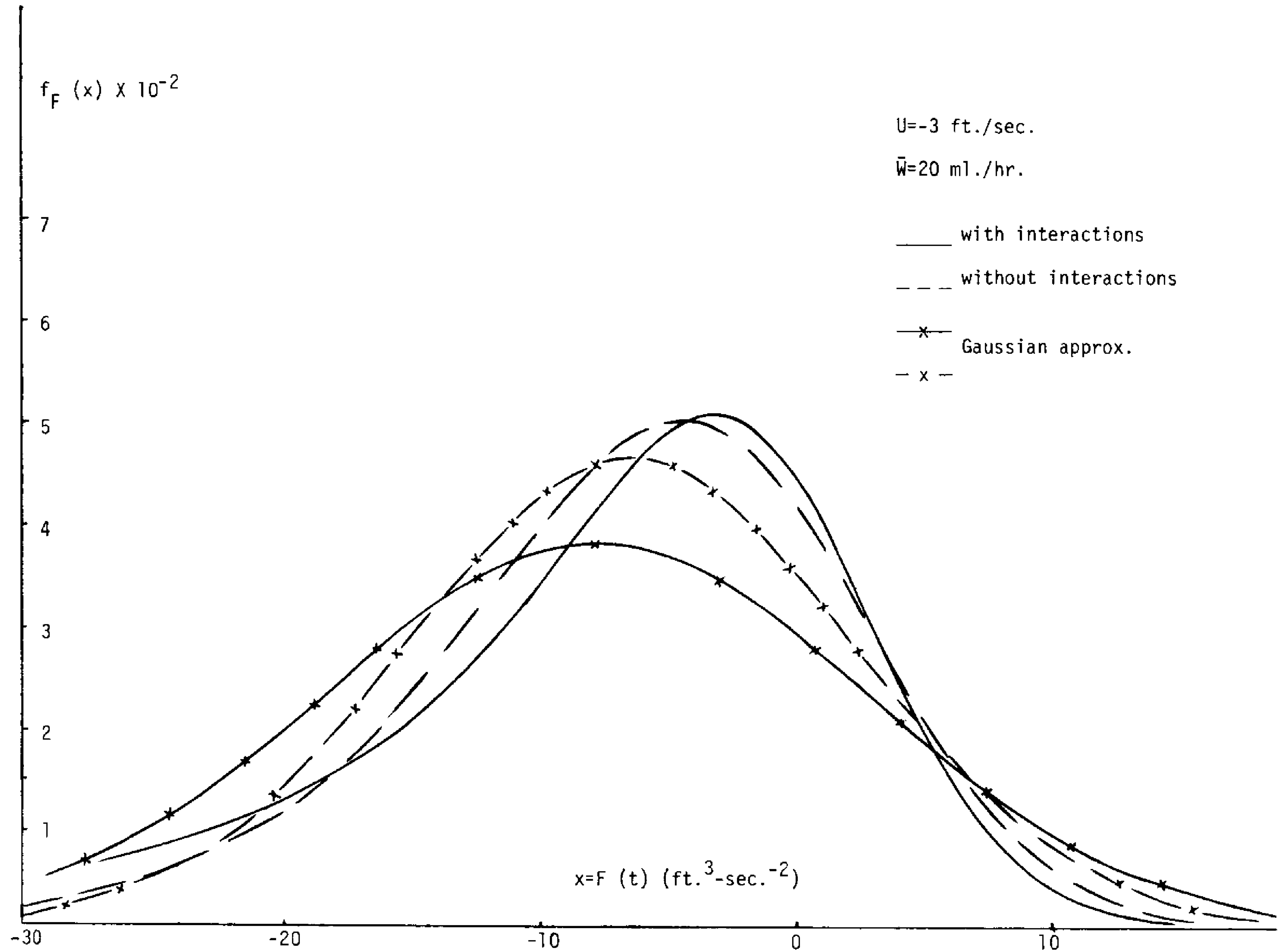


FIGURE 8b PROBABILITY DENSITY FUNCTION OF FORCE where U = -3 ft./sec.

$$f_X(x) = \frac{1}{2\pi\sigma_F} \exp\left[-\frac{1}{2} \left(\frac{x-E[F]}{\sigma_F}\right)^2 \right] \tag{17}$$

with E[F] and σ_F given in Eqs. 10 and 12 respectively. It is seen that when U = 0, $f_X(x)$ is but slightly non-Gaussian. However, when current is present, $f_X(x)$ becomes skewed and therefore non-Gaussian. In Fig. 8b, the case of U = -3 ft./sec. is shown. When no interactions are taken into account, $f_X(x)$ and its Gaussian approximation are merely the mirror image of those of the case U = 3 ft./sec. presented in Fig. 8a. When wave-current interactions are considered, it is seen that $f_X(x)$ deviates appreciably from its Gaussian approximation.

The departure of $f_X(x)$ from Gaussian can also be seen by plotting the probability distribution function $F_X(x)$ of X = F(t) on Gaussian (normal) probability paper. The cases of U = 3 ft./sec. and U = -3 ft./sec. are presented respectively in Figs. 9a and 9b. That fluid force is non-Gaussian is clearly noted.

EXPECTED RATE OF THRESHOLD CROSSINGS OF FLUID FORCE

The expression of the expected number fluid force crosses threshold level x, from below, per unit time, denoted by $E[N_+(x)]$ and derived in the Appendix, is

$$E[N_+(x)] = \frac{I_1 + I_2}{C_M^2 (2\pi)^{3/2} |S|^{1/2}} \tag{18}$$

in which

$$I_1 = \frac{|S|}{C} \int \exp\left[-\frac{S_{vv}}{2|S|}(z - U)^2 \right] \exp\left[-\frac{S_{aa}}{2|S|}\left(\frac{x - C_D |z| z}{C_M}\right)^2 \right]$$

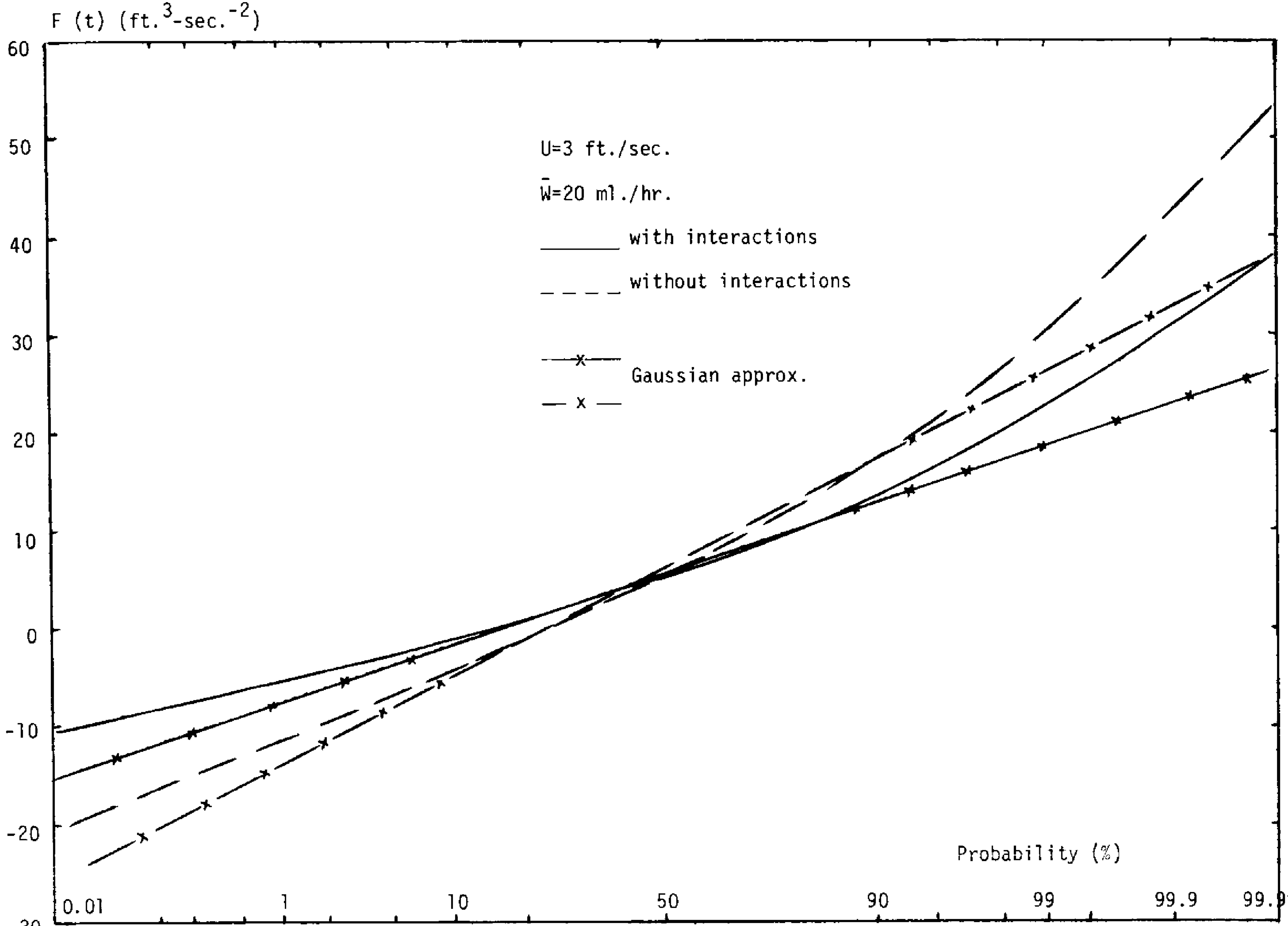


FIGURE 9a PROBABILITY DISTRIBUTION FUNCTION OF FORCE where U = 3 ft./sec.

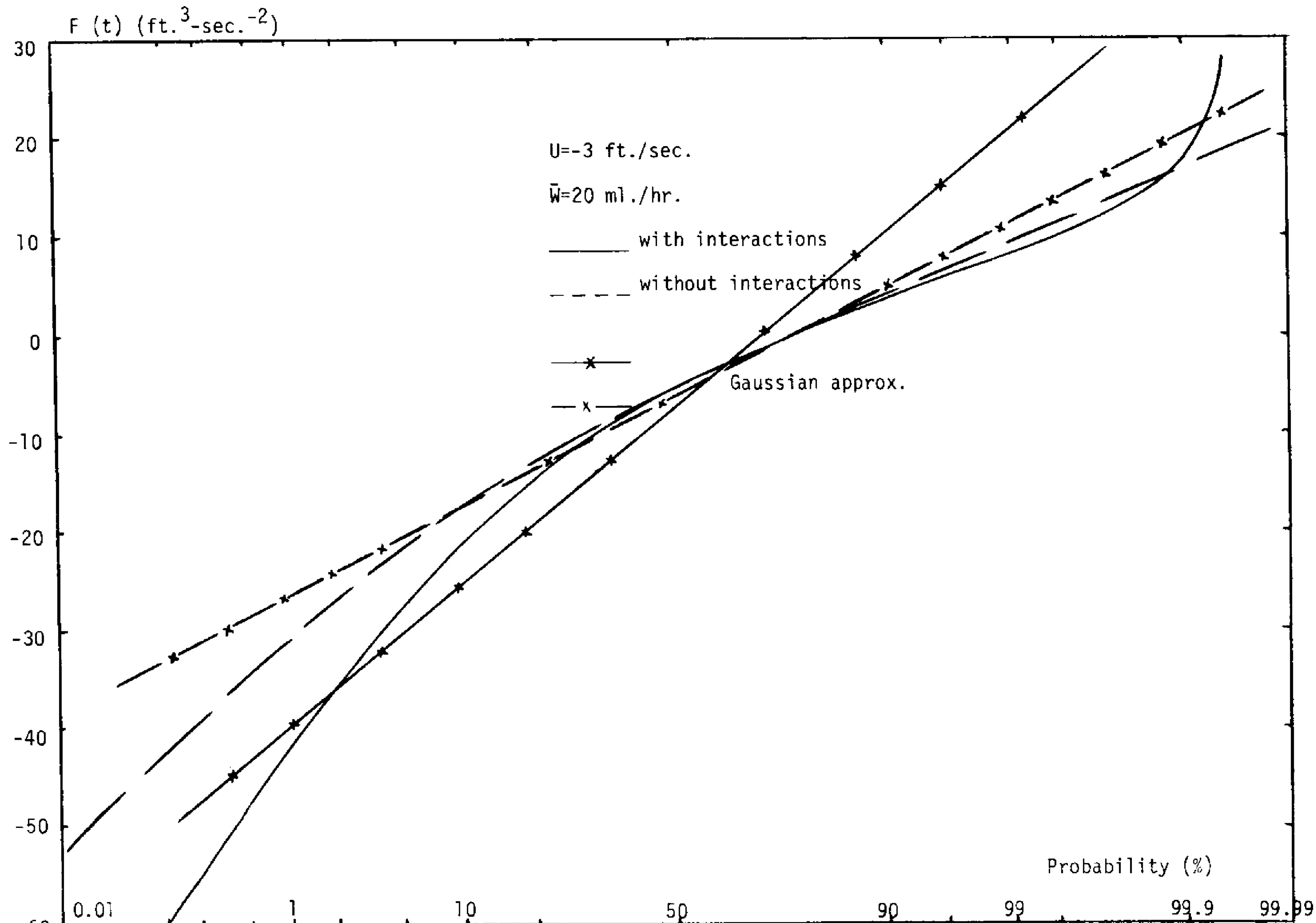


FIGURE 9b PROBABILITY DISTRIBUTION FUNCTION OF FORCE where U = -3 ft./sec.

$$\exp[-\frac{CB}{2|S|}(B - \frac{A}{C})]dz \tag{19}$$

and

$$I_2 = \sqrt{\frac{|S|}{C}} \int dz(B-\frac{A}{2C})\exp[-\frac{1}{2\sigma_v^2}(z - U)^2]$$

$$\exp[-\frac{1}{2\sigma_a^2}(\frac{x-C_D z|z|}{C_M})^2][\frac{1}{2} - P(\frac{C}{|S|}(B - \frac{A}{2C}))] . \tag{20}$$

In these expressions,

$$A = 2S_{v\dot{a}}(z - U)/C_M$$

$$B = 2C_D|z|(x - C_D|z|z)/C_M \tag{21}$$

$$C = S_{\dot{a}\dot{a}}/C_M^2$$

and

$$|S| = \sigma_v^2\sigma_a^2\sigma_{\dot{a}}^2 - \sigma_a^6$$

$$S_{vv} = \sigma_a^2\sigma_{\dot{a}}^2$$

$$S_{v\dot{a}} = \sigma_a^4 \tag{22}$$

$$S_{aa} = |S|/\sigma_a^2$$

$$S_{\dot{a}\dot{a}} = \sigma_v^2\sigma_a^2$$

The integration with respect to z in Eqs. 19 and 20, of course, can only be performed numerically.

Computed and presented in Fig. 10a and 10b are the quantity $E[N_+(x)]$

for wind speed $\overline{W}$ = 20 mi./hr. for the cases when the current is in the direction of (U = 3 ft./sec.) and opposite to (U = -3 ft./sec.) the waves. In each case, two separate conditions are examined. That is, when wave-current interactions are considered and ignored. From these two figures, it is immediately apparent that wave-current interactions have profound effects on the quantity $E[N_+(x)]$.

In dealing with random phenomenon, Gaussian assumption is often involved for reasons of mathematical expediency. It is therefore of interest to see how Gaussian approximation affects the quantity $E[N_+(x)]$. It is shown in the Appendix that under the assumptions made,

$$E[N_+(x)] = \frac{\sigma_{\dot{F}}}{2\pi\sigma_F} \exp\left[-\frac{1}{2}\left(\frac{x-E[F]}{\sigma_F}\right)^2\right] \tag{23}$$

and

$$\sigma_{\dot{F}} = [C_M^2\sigma_{\dot{a}}^2 + 4C_D^2\sigma_a^2\sigma_v^2(1+\gamma^2)]^{1/2} \tag{24}$$

Using the Gaussian Assumption of fluid forces, the quantity $E[N_+(x)]$ is computed for $\overline{W}$ = 20 mi./hr. for all the cases covered in figures 10 a and 10b, and is presented in the same figures. In addition, in Figure 10c, the exact value of $E[N_+(x)]$ and its Gaussian approximation are given for the case when there is no current. That Gaussian assumption is inadequate to use in the evaluation of fatigue damage when current is involved is obvious from these figures. This is especially so when wave-current interactions are included and when current speed is negative. The deviation of the approximate $E[N_+(x)]$ from its exact value is due largely to the fact that while the processes F(t) and $\dot{F}(t)$ are truly uncorrelated, they are actually not statistically independent, a fact which the Gaussian assumption of the processes fails to reflect.

CONCLUDING REMARKS

In this report, it is shown that in a random wave field, the presence of even a moderate current can affect the statistical properties of fluid force studied here to an appreciable extent especially when the phenomena of wave-current interactions are considered. Gaussian assumption, while convenient to use, is decisively inadequate in the evaluation of probability function and expected rate of threshold crossings of fluid force, quantities one must contend with in the evaluation of the safety of ocean structures against catastrophic and fatigue failure.

FIGURE 10a EXPECTED RATE OF THRESHOLD CROSSINGS OF FORCE where U = 3 ft./sec.

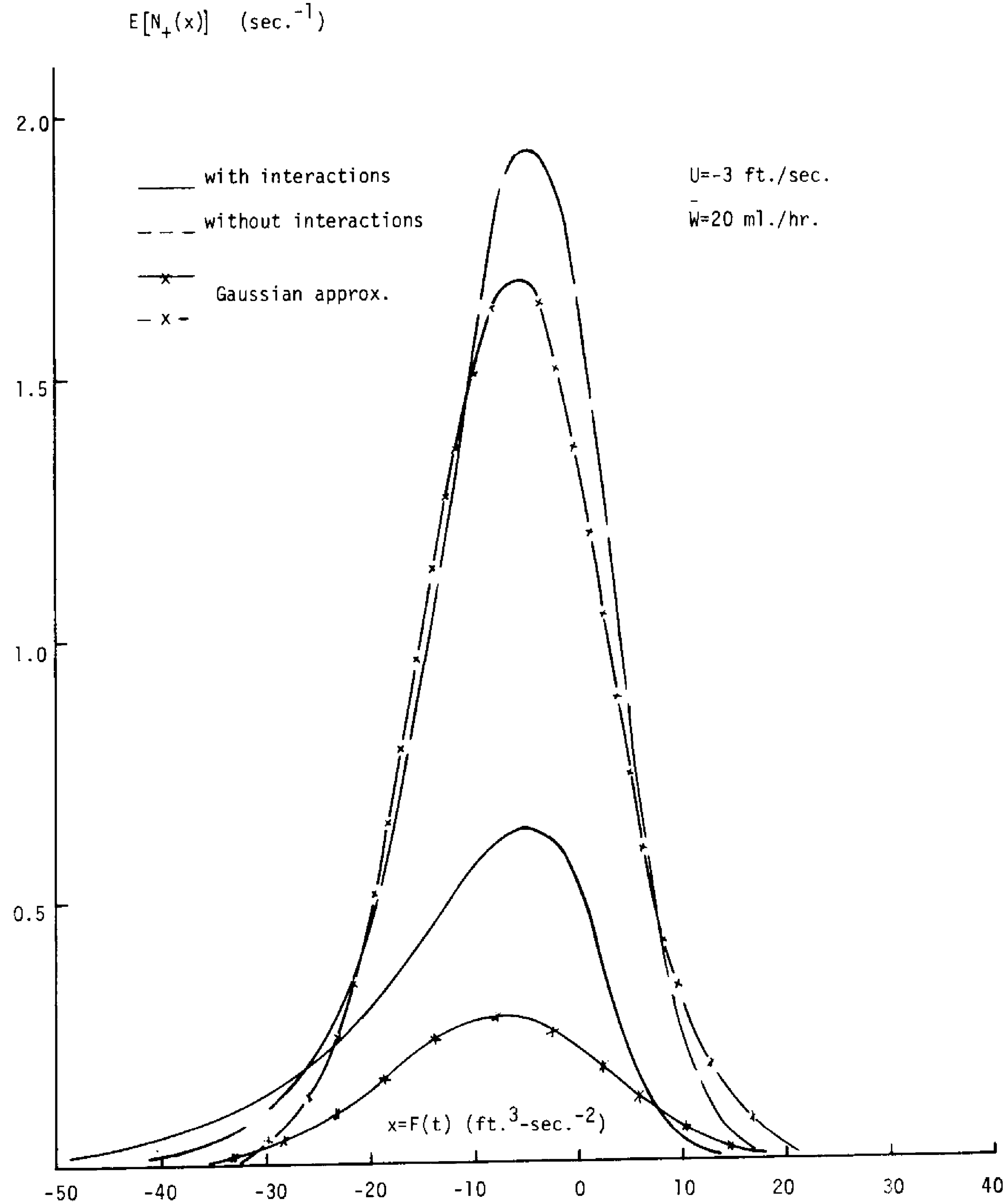


FIGURE 10b EXPECTED RATE OF THRESHOLD CROSSINGS OF FORCE where U = -3 ft./sec.

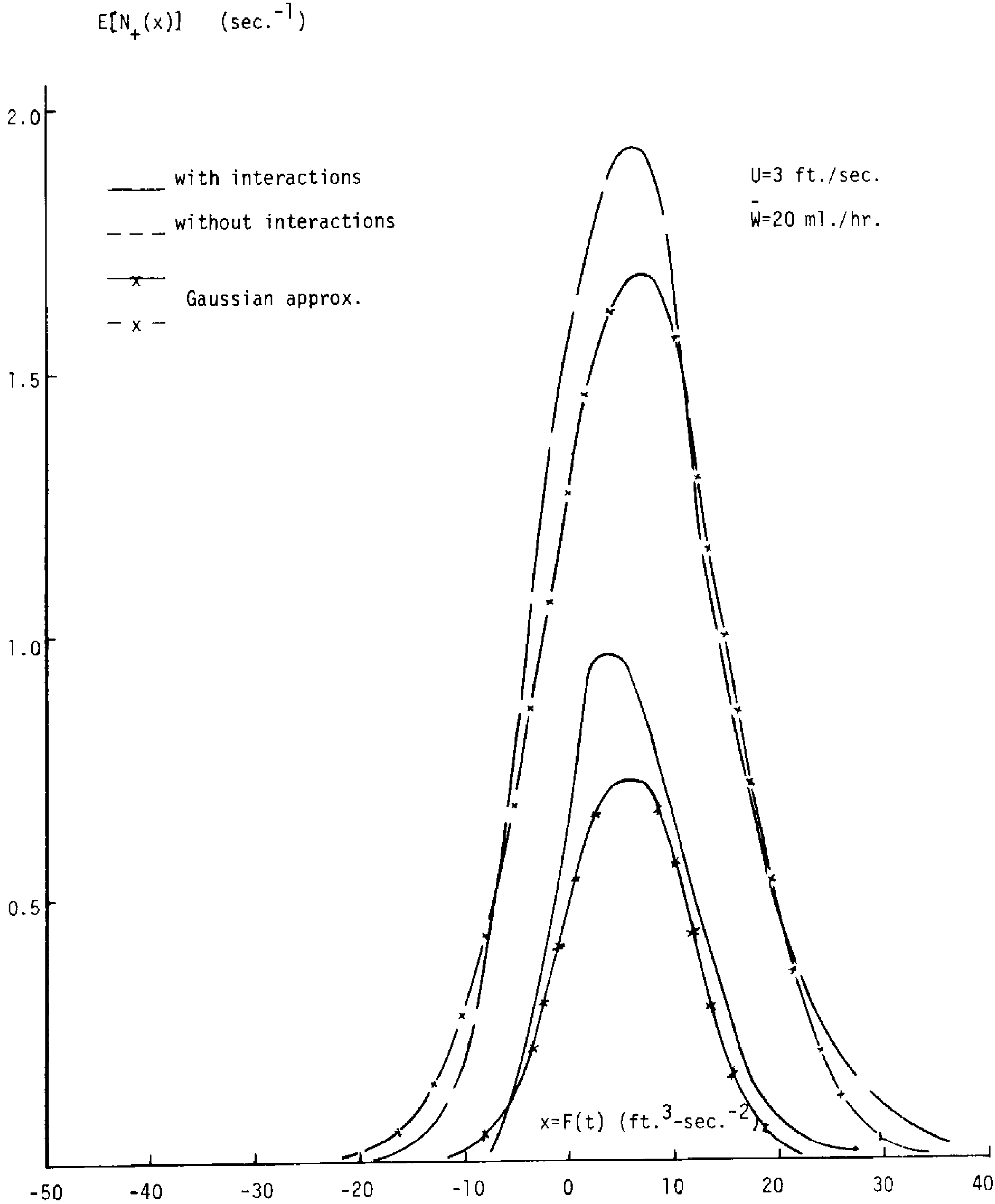
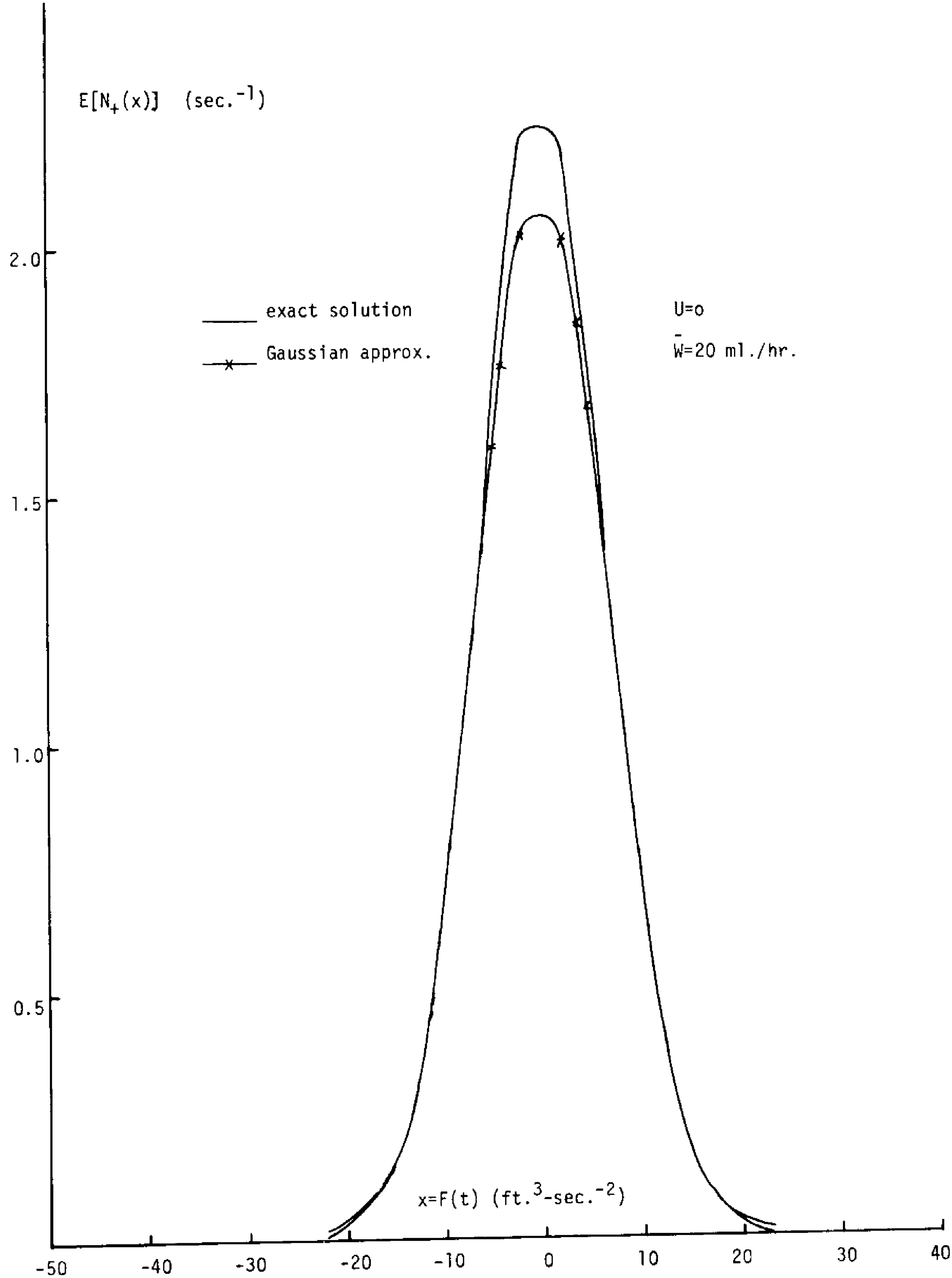


FIGURE 10c EXPECTED RATE OF THRESHOLD CROSSINGS OF FORCE where U = 0



APPENDIX

The expected value, standard deviation, skewness, probability function and expected rate of threshold crossings of fluid force F(t) defined in Eq. 9 are derived in the following.

The expected value of F(t) given in Eq. 9 is obtained by taking the expected value of the quantities on both sides of the equation. That is,

$$E[F] = C_M E[a] + C_D E[V|V|] \qquad \text{(A-1)}$$

in which the argument t of F(t), V(t) and a(t) is omitted for simplicity. Assuming that the sea is Gaussian and a(t) has zero mean value, then E[a] = o and

$$E[V|V|] = \quad V|V|f_V(V)dV \qquad \text{(A-2)}$$

in which

$$f_V(V) = \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left[-\frac{1}{2}\left(\frac{V-U}{\sigma_v}\right)^2\right] \qquad \text{(A-3)}$$

is the probability density function of the stationary random process V(t). The integration in Eq. A-2 can be carried out by making the transformation of the variable of integration $v = (V - U)/\sigma_v$ and separating the integral into two integrals with limits of integration from − to −γ and from −γ to respectively. Within these limits of integration the absolute sign in Eq. A-2 can be eliminated and the integration can be performed without difficulty. The expected value of F(t) is finally as shown in Eq. 10 of the text.

The second moment of F(t) is obtained by first squaring both sides of Eq. 9 and taking the expected value. That is,

$$E[F^2] = C_D E[(v + U)^4] + C_M^2 E[a^2] + 2C_D C_M E[a(v + U)|v + U|]. \quad \text{(A-4)}$$

The expected value in the first term on the right hand side of Eq. A-4 can be further reduced to

$$E[(v + U)^4] = E[v^4] + 4UE[v^3] + 6U^2 E[v^2] + 4U^3 E[v] + U^4$$

$$= 3\sigma_v^4 + 6U^2\sigma_v^2 + U^4 \quad \text{(A-5)}$$

in which zero mean properties of the Gaussian process v(t) have been used. The second term on the right hand side of Eq. A-4 is simply

$$C_M^2 E[a^2] = C_M^2 \sigma_a^2. \quad \text{(A-6)}$$

To evaluate $E[a(v + U)|v + U|] = E[a\ V|V|]$, note that the processes V(t) and a(t) are stationary and therefore are uncorrelated. Being that V(t) and a(t) are jointly Gaussian, they are statistically independent, giving

$$E[a(v + U)|v + U|] = E[aV|V|] = E\ [a]E[V|V|]. \quad \text{(A-7)}$$

But since $E[a] = o$, the third term on the right hand side of Eq. A-4 is therefore equal to zero. The final expression of $E[F^2]$ is given in Eq. 11.

The third moment of F(t) can be similarly obtained by first forming the product $F^3(t)$ from Eq. 7 and taking the expected value. That is

$$E[F^3] = C_M^3 E[a^3] + 3C_M^2 C_D E[a^2 V|V|] + 3C_D^2 C_M E[aV^4] + C_D^3 E[V^5|V|] \quad \text{(A-8)}$$

in which the first term on the right hand side of Eq. A-8 is obviously zero. The expected values in the second and third terms in Eq. A-8 can be shown, by using the argument leading to Eq. A-7, to be respectively

$$E[a^2V|V|] = E[a^2]E[V|V|] = 2\sigma_a^2\sigma_v^2[\gamma T(\gamma) + (1 + \gamma^2)P(\gamma)] \quad \text{(A-9)}$$

and

$$E[aV^4] = E[a]E[V^4] = o. \quad \text{(A-10)}$$

The term $E[V^5|V|]$ can be evaluated in the same manner as $E[V|V|]$ was evaluated in Eq. A-2. The final result is given in Eq. 15.

The probability density function of F(t) may be constructed by using the standard method of transformation of random variables (7). Thus, let $Q = C_D V|V|$ and $W = C_M a$. The joint probability density function of Q and W can be expressed in terms of that of V and a as

$$f_{QW}(q,w) = f_{Va}(\pm \frac{|q|}{C_D}, \frac{w}{C_M}) / |J| \quad \text{(A-11)}$$

in which $|J| = 2C_D C_M |V|$ is the Jacobian of transformation and

$$f_{Va}(V,a) = \frac{1}{2\pi\sigma_v\sigma_a} \exp\{-\frac{1}{2}[(\frac{V-U}{\sigma_v})^2 + (\frac{a}{\sigma_a})^2]\} . \quad \text{(A-12)}$$

X = F(t) = Q + W, being the sum of the two random variables Q and W, has probability density function (7),

$$f_X(x) = \int f_{QW}(y, x-y)dy$$

$$= \frac{1}{4\pi C_D C_M \sigma_v \sigma_a} \exp\{-\frac{1}{2}[(\frac{x}{C_M\sigma_a})^2 + \gamma^2]\}$$

$$\sqrt{\frac{C_D}{y}}\ \exp\ [\ -\ \frac{1}{2}(\frac{y}{C_D\sigma_v^{\ 2}} + \frac{y^2}{C_M^{\ 2}\sigma_a^{\ 2}})]$$

$$\{\ \exp[\frac{Y}{\sigma_v}\ \frac{y}{C_D} + \frac{xy}{C_M\sigma_a}] + \exp\ [\ -\ \frac{Y}{\sigma_v}\ \frac{y}{C_D} - \frac{xy}{C_M\sigma_a}]\}dy. \qquad \text{(A-13)}$$

The integration, however, can only be performed numerically. Thus, the singularity of the integrand at $y = 0$ must be removed. By letting $\sqrt{y} = s$, one obtains, after rearranging, expression of $f_X(x)$ as given in Eq. 16 of the text.

To compute the extected rate of threshold crossings of the fluid force $F(t)$, the joint probability density function of $F(t)$ and its derivative process $\dot{F}(t)$ is required. It is easy to verify that

$$\dot{F}(t) = 2C_D|V(t)|a(t) + C_M\dot{a}(t). \qquad \text{(A-14)}$$

The joint probability density function of $F(t)$ and $\dot{F}(t)$ can be obtained again by the standard method of transformation of random variables (7). Thus, let $X = F(t)$, $Y = \dot{F}(t)$, introduce an auxiliary random variable $Z = V(t)$ and first determine the joint probability density function $f_{XYZ}(x,y,z)$ of the random variables X,Y, and Z. That is

$$f_{XYZ}(x,yx,z) = f_{Va\dot{a}}(V,a,\dot{a})/|J| \qquad \text{(A-15)}$$

in which $|J| = C_M^2$ is the Jacobian of transformation and the arguments $V,a,\dot{a}$, in $f_{Va\dot{a}}(V,a,\dot{a})$ are to be replaced by

$$V = z$$

$$a = \frac{x-C_D|z|z}{C_M}$$

and

$$\dot{a} = [y - \frac{2C_D|z|}{C_M}(x - c_D|z|z)]/C_M. \qquad \text{(A-16)}$$

In Eq. A-15, the function $f_{Va\dot{a}}(V,a,\dot{a})$ is the joint probability density function of $V(t)$, $a(t)$ and $\dot{a}(t)$. Assuming that they are jointly Gaussian, $f_{Va\dot{a}}(V,a,\dot{a})$ has the expression (4)

$$f_{Va\dot{a}}(v,a,\dot{a}) = \frac{1}{(2\pi)^{3/2}|S|^{1/2}}$$

$$\exp[\frac{-1}{2|S|}(S_{vv}(V-U)^2 + 2S_{V\dot{a}}(V-U)\dot{a} + S_{aa}a^2 + S_{\dot{a}\dot{a}}\dot{a}^2)]. \qquad \text{(A-17)}$$

The quantities appearing in Eq. A-17 are all defined in Eq. 22 of the text.

The joint probability density function $f_{XY}(x,y)$ of $X = F(t)$ and $Y = \dot{F}(t)$ is the marginal density to the joint probability density function $f_{XYZ}(x,y,z)$

$$f_{XY}(x,y) = \int f_{XYZ}(x,y,z)dz \qquad \text{(A-18)}$$

and the expected rate of threshold crossings, from below, of the process $X = F(t)$, at threshold level x,

is

$$E[N_+(x)] = \int yf_{XY}(x,y)dy$$
$$= \int dz \int yf_{XYZ}(x,y,z)dy. \qquad \text{(A-19)}$$

The integral with respect to y can be achieved explicity in closed form, giving the result shown in Eq. 18 of the text.

In evaluating the quantity $E[N_+(x)]$, if the processes $X = F(t)$ and $Y = \dot{F}(t)$ are assumed to be jointly Gaussian, then by stationary assumption of the processes, they are uncorrelated and therefore statistically independent. Thus

$$f_{XY}(x,y) = f_X(x) f_y(y) \qquad \text{(A-20)}$$

in which

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_F} \exp\left[-\frac{1}{2}\left(\frac{x-E[F]}{\sigma_F}\right)^2\right] \qquad \text{(A-21)}$$

and

$$f_y(y) = \frac{1}{\sqrt{2\pi}\sigma_{\dot{F}}} \exp\left[-\frac{1}{2}\left(\frac{y-E[\dot{F}]}{\sigma_{\dot{F}}}\right)^2\right] \qquad \text{(A-22)}$$

are Gaussian probability density function of $X = F(t)$ and $Y = \dot{F}(t)$ which are determined completely by their respective expected values $E[F]$ and $E[\dot{F}]$ and standard deviations σ_F and $\sigma_{\dot{F}}$. The quantities $E[F]$ and σ_F are given in Eqs. 10 and 12 derived earlier in the Appendix. It can be similarly shown that $E[\dot{F}] = o$ and

$$\sigma_{\dot{F}} = [C_M^2\sigma_{\dot{a}}^2 + 4C_D^2\sigma_a^2\sigma_v^2(1 + \gamma)]^{1/2}. \qquad \text{(A-23)}$$

Using Eq. (A-19),

$$E[N_+(x)] = \int y f_{XY}(x,y)\,dy = f_X(x) \int y f_Y(y)\,dy$$

$$= \frac{\sigma_{\dot{F}}}{\sqrt{2\pi}} f_X(x) \qquad \text{(A-24)}$$

giving the expression shown in Eq. 23.

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NOTATION

The following symbols are used in this report:

A = quantity used in Eqs. 19 and 20 (see Eq. 21);

$a(.)$, $\dot{a}(.)$ = fluid particle acceleration and its derivative;

B, C = quantities used in Eqs. 19 and 20 (see Eq. 21);

C_D, C_M = coefficients of drag force and inertia force (see Eq. 9);

D = diameter of cylinder;

$E[.]$ = expected value of the quantity enclosed in the bracket;

$E[F]$, $E[F^2]$, $E[F^3]$ = first, second and third moment of fluid force $F(t)$ (Eqs. 10, 11 and 15);

$F(.)$ = fluid force on cylinder of unit length (Eq. 9);

$\dot{F}(.)$ = derivative of $F(t)$ (Eq. A-14);

$f_{QW}(.,.)$ = joint probability density function of the random quantities Q and W (Eq. A-11);

$f_V(.)$ = probability density function of fluid particle velocity $V(t)$ (Eq. A-3);

$f_{Va}(.,.)$ = joint probability density function of fluid particle velocity $V(t)$ and acceleration $a(t)$ (Eq. A-12);

$f_{Va\dot{a}}(.,.,.)$ = joint probability density function of fluid particle velocity $V(t)$, acceleration $a(t)$ and its derivative $\dot{a}(t)$ (Eq. A-17);

$F_X(.)$ = probability distribution function of $X = F(t)$ (Eqs. 16 and A-21);

$f_x(.)$ = probability density function of $X = F(t)$ (Eqs. 16, 17, A-13);

$f_{XY}(.,.)$ = joint probability density function of $X = F(t)$ and $Y = \dot{F}(t)$ (Eqs. A-18 and A-20);

$f_{XYZ}(.,.,.)$ = joint probability density function of $X = F(t)$, $Y = F(t)$ and $Z = V(t)$ (Eq. A-15);

f_Y = probability density function of $Y = \dot{F}(t)$ (Eq. A-22);

g = gravitational acceleration;

I_1, I_2 = quantities defined in Eqs. 19 and 20;

$|J|$ = Jacobian of transformation of random variables (Eqs. A-11 and A-15);

K_D, K_M = drag and inertia coefficients;

n = frequency;

$n_o = g/\overline{W}$;

$P(\gamma) = \quad T(x)dx$, the error function;

$Q = C_D V|V|$, a random quantity;

q = dummy variable;

$|S|$, S_{aa}, $S_{\dot{a}\dot{a}}$, $S_{v\dot{a}}$, S_{vv} = quantities defined in Eq. 22;

s = dummy variable;

$T(\gamma) = \frac{1}{2\pi} \exp(-\frac{1}{2}\gamma^2)$;

t = time;

U = current speed;

$V(.)$ = fluid particle velocity;

$v(.) = V(.) - U$, oscillatory part of fluid particle velocity;

$W = C_M a$;

$\overline{W}$ = mean wind speed;

w = dummy variables;

$X = F(t)$, a random quantity;

x = dummy variable;

$Y = \dot{F}(t)$, a random quantity;

y = dummy variable;

$Z = V(t)$, a random quantity;

α,β = constants in surface wave spectrum (Eq. 2);

$\gamma = U/\sigma_v$;

γ_1 = skewness (Eq. 13);

ρ = density of water;

$\lambda = C_M\sigma_a/2C_D\sigma_v^2$

$\sigma_a, \sigma_{\dot{a}}, \sigma_v$ = standard deviations of fluid particle acceleration, its derivative and velocity (Eqs. 7, 8 and 6);

$\sigma_F, \sigma_{\dot{F}}$ = standard deviations of fluid force and its derivative (Eqs. 12 and 24);

$\phi(.)$, $\phi^*(.)$ = frequency spectra of surface waves with and without the influence of current (Eqs. 1 and 2) and

$\phi_{aa}(.)$, $\phi_{\dot{a}\dot{a}}(.)$, $\phi_{vv}(.)$ = frequency spectra of fluid particle acceleration, its derivative and velocity (Eqs. 4, 5, and 6).

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