SEA GRANT PROJECT OFFICE

THE OVERALL COMPRESSION BUCKLING OF PARTIALLY CONSTRAINED SHIP GRILLAGES

by

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Administrative Statement

D. Faulkner, by developing this discrete beam solution for biaxial compression buckling of an orthoganally stiffened plate grillage, has provided the formulae and method of sufficient accuracy for use in the early stages of ship design. This permits the designer to make a more accurate preliminary selection of structural members thereby minimizing the dependence and time required for the more accurate computer analyses.

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Alfred H. Keil
Director

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NOTATION

1. Material and Fabrication Properties

σ_o, σ_p material tensile yield, proportional limit stress

σ_{om} "mean" yield stress if plate and stiffener material differ $= (\sigma_{os} A_s + \sigma_{op} bt)/(A_s + bt)$

σ'_o yield stress in prevailing conditions, e.g. $= \sigma_o(1-\mu + \mu^2)^{-1/2}$ for plate bending using the Maxwell distortion energy criterion (so called von Mises-Hencky yield condition).

σ_{ps} structural proportional limit in compression $\simeq \sigma_o - \sigma_r = p_r\sigma_o$

p_r σ_{ps}/σ_o stress ratio defining beginning of inelastic effects in compression (typical value 0.5 for welded ship panels)

σ_r longitudinal compressive welding residual stress in the middle zone of the plate

E,G extensional, shear modulii

E_t structural tangent modulus $d\sigma/d\varepsilon$ in compression from "stub column" type tests, otherwise estimate:

(a) for materials having a well defined yield plateau by using the Ostenfeld-Bleich quadratic parabolae

$$\frac{E_t}{E} = \frac{\sigma(\sigma_o-\sigma)}{\sigma_{ps}(\sigma_o-\sigma_{ps})}$$

μ, μ_p elastic, plastic Poisson's ratio

2. Grillage Geometry and Section Properties

x, y	longitudinal, transverse coordinates and stiffeners measured from the corner of the grillage.
w, w_{mn}	normal deflection, central Fourier harmonic
Δ	$w\sqrt{E/\sigma_o}/b = w/t\beta$ plate deflection parameter

Plate element properties:

a, b, t	length, width, thickness
α	a/b plate aspect ratio
β	$b/t\ \sqrt{\sigma_o/E}$ plate slenderness
D	$Et^3/12(1-\mu^2)$ plate flexural rigidity

Stiffener properties:

subscripts "x" and "y" are always first and denote X and Y stiffeners respectively

$d, \bar{z}, A_s, t_w$	depth, centroid height, area, web thickness
I_s, I_t	MI (moment of inertia) in web plane about centroid, toe
I_z, I_o	MI about web, polar MI about toe $I_o = I_t + I_z$
J	St. Venant torsion constant for stiffener $\simeq 1/3 \int t^3 ds$ for thin-walled open cross-sections $\simeq 4A^2/\int (1/t) ds$ for unperforated thin-walled closed cross-sections (reduced when perforated in the ratio area of holes/gross area of wall).
Γ	Longitudinal warping constant = $1/4\ I_z d^2$ for symmetrical stiffeners.

2. <u>Grillage Geometry and Section Properties</u> (Cont'd)

J_t $= J + \pi^2 E(I_z \bar{z}^2 + \Gamma)/G\ell^2$ twisting rigidity

where $\ell = a$ - tripping wave length for panel

$= A/m, B/n$ ... twisting wave lengths for X,Y grillage stiffeners (m,n modes).

r_s $= \sqrt{I_s/A_s}$ stiffener radius of gyration

i $= I_t/I_s$ for computing stiffener-plate properties

$= 1 + (\bar{z}/r_s)^2$

<u>Grillage</u> stiffener-plate properties:

γ_x, γ_y A_{xs}/bt, A_{ys}/at area ratios, stiffener/plate

t_x, t_y $= t(1 + \gamma_x)$, $t(1 + \gamma_y)$ "mean" thicknesses for computing average applied stresses $\sigma_x = N_x/t_x$, $\sigma_y = N_y/t_y$

$\bar{t}$ $= t(1 + \gamma_x + \gamma_y)$ true mean thickness of stiffened plate for weight consideration

t_{xe}, t_{ye} $= t(b_e/b + \gamma_x)$, $t(a_e/a + \gamma_y)$

effective thickness for computing compression stresses for panel or grillage failure

$\sigma_{x,e} = N_x/t_{xe}$, $\sigma_{y,e} = N_y/t_{ye}$

I_e $\approx I_s \left| \frac{\gamma + i\zeta}{\gamma + \zeta} \right|$ effective MI - assumes flange and plate areas concentrated at outer fibers

ζ = effective plate/total plate

r_{ce} $= \sqrt{I_{xe}/bt_{xe}}$ radius of gyration of "x" stiffeners and effective plate.

2. <u>Grillage Geometry and Section Properties</u> (Cont'd)

α_o = $(A/B)[I_{ye}\, b/I_{xe}\, a]^{1/4}$ orthotropic aspect ratio

α_c = $\alpha_o\ [K_{1y1}/K_{1x1}]^{1/4}$ constrained orthotropic aspect ratio (where the K's are defined later).

Γ_{xy} = $\frac{G}{2E}\sqrt{\frac{a}{I_{ex}}\frac{b}{I_{ey}}}\left[\bar{p}\,\frac{J_{xt}}{b} + \bar{q}\,\frac{J_{yt}}{a}\right]$ tripping torsion constant for the grillage deforming in the (m,1) mode where $\bar{q} = (q-1)/(q+1)$ in general, or $2q/(q+1)$ when $m/(q+1)$ is integer; likewise $\bar{p} = (p-1)/(p+1) \simeq 1$ for large p.

= $D_{xy}/\sqrt{D_x\ D_y}$ in orthotropic plate notation. The J_t twisting rigidities are defined above under stiffener properties.

p,q number of X,Y stiffeners ($p \geq 3$, q any value)

m,n positive integers representing the number of half waves of the buckled grillage in the X,Y directions. For general instability only (m,1) modes are considered.

R_x, R_y = $C_x A/E\ I_{xe}$, $C_y B/E\ I_{ye}$ dimensionless spring stiffnesses at X,Y stiffener ends

C_x, C_y moment/slope elastic stiffnesses at X,Y stiffener ends

$\left.\begin{matrix} K_{xm}, K_{1xm}, K_{2xm} \\ K_{yn}, K_{1yn}, K_{2yn} \end{matrix}\right\}$ constraint-mode functions for grillage instability theory for the general (m,n) mode ...

2. <u>Grillage Geometry and Section Properties</u> (Cont'd)

K_{xm} $= 1 + (R_x/2\pi m(q+1))$ [3 cot $\pi m/2(q+1)$ - cot $3\pi m/2(q+1)$] $\sin^2 m\pi/2 + 3R_x{}^2/16\pi^2 m^2$

$\simeq 1 + (8R_x/3\pi^2 m^2) \sin^2 m\pi/2 + 3R_x{}^2/16\pi^2 m^2$

if m exceeds (q+1) significant errors can occur in the approximation and the exact expression above should be used.

K_{1xm} $= 1 + (4 + 13R_x/16)\ R_x/\pi^2 m^2 K_{xm}$ important coefficient in vertical bending terms, and tends to 5.33 as R_x becomes clamped.

K_{2xm} $= 1 + R_x^2/16\pi^2 m^2\ K_{xm}$ coefficient for the twisting terms, and tends to 1.33 as R_x becomes clamped.

Similarly the yn functions are obtained by interchanging R_y for R_x, n for m and q for p. However, we are often only concerned with the (m,1) mode in which case ...

K_{y1} $= 1 + 8R_y/3\pi^2 + 3R_y{}^2/16\pi^2$

K_{1y1} $= 1 + (4 + 13R_y/16)R_y/\pi^2\ K_{y1}$

K_{2y1} $= 1 + R_y{}^2/16\pi^2\ K_{y1}$

K_{xy} $= (\sin m\pi/2 + R_x/8)(\sin n\pi/2 + R_y/8)$

shorthand notation

S_x, S_y $= \sin m\pi x/A,\ \sin n\pi y/B$

S_{2x}, S_{2y}, etc. $= \sin 2m\pi x/A,\ \sin 2n\pi y/B$, etc.

S_r, S_s, etc. $= \sin n\pi r/(p+1)$ for X beams,
$\sin m\pi s/(q+1)$ for Y beams, etc.

Similarly for cosine functions.

[Note: $S_{rb} = S_{y=rb} = \sin n\pi rb/B = \sin n\pi r/(p+1) = S_r$]

3. Stress notation

σ_{pE} $= k\pi^2 D/tb^2$ plate elastic buckling stress

$= 3.62\sigma_o/\beta^2$ pinned long plate $a \geq b$, $\mu = 0.3$

$= 6.3\sigma_o/\beta^2$ clamped long plate $a \geq b$, $\mu = 0.3$

σ_{CE} $= \pi^2 E\, r_{ce}{}^2/a^2$ pinned Euler column stress for unsupported span a.

σ_{oP} $= \dfrac{2\pi^2 E}{t_{xe}B^2}\sqrt{\dfrac{I_{xe}}{a}\dfrac{I_{ye}}{b}}$ orthotropic plate pinned buckling stress when $\Gamma_{xy} = 0$ (twisting ignored) and $\alpha_o \geq 1$.

ψ $= \sigma_o/\sigma_E$ slenderness ratio yield-elastic buckling stress

ϕ $= \sigma_u/\sigma_o$ failure stress ratio

$= \dfrac{\sigma_e}{\sigma_o}\left[\dfrac{\gamma + b_e/b}{\gamma + 1}\right]$ where σ_e is value of edge stress at failure.

τ_o $= \sigma_o/\sqrt{3}$ assumed shear yield stress

p lateral pressure

N_x, N_y in-plane loads

$m_{x\ell}$ $= \sigma_{CE}/(\sigma_{CE} - \sigma_x)$ local magnification factor for X stiffeners between intersections

m_{xo} $= \sigma_{GE}/(\sigma_{GE} - \sigma_x)$ overall magnification factor for grillage bending between supports; assumed to be applicable to X and Y stiffeners providing first mode (1,1) value of σ_{GE} is used.

ABSTRACT

An explicit discrete beam solution has been derived for biaxial compression buckling of an orthogonal stiffened-plate grillage with sides and ends elastically restrained against rotation. A method is provided for quickly estimating these constraints, and for allowing for coupled buckling effects from adjoining structure. Twisting of the stiffeners about their plate connection is included, as are inelastic effects. Accuracy is considered to be well within the 10-15 per cent required for most design purposes. Typical ship examples indicate side constraints to be nearer clamped than pinned, which can approximately double the buckling stresses. The report includes a discussion of edge constraints, inelastic effects and effective plating. The theory has been programmed for design use, and safety is discussed, albeit tentatively, since there is a dearth of experimental confirmation.

INTRODUCTION AND BACKGROUND

Explicit analysis equations and graphs have existed for some while for estimating the overall elastic stability of a ship's deck or shell structure between say transverse bulkheads. The structure can be treated either as a uniform equivalent orthotropic plate whose flexural and torsional rigidities in orthogonal directions represent the combined strength of stiffeners and plating [1,2]*, or as grillages of discrete beams in which the plating is represented by effective flanges acting with the stiffeners according to simple beam theory [3]. Solutions can be obtained quickly by the designer using slide-rule or desk calculating machines.

The most serious limitation in these methods is that they are confined to the classical boundary conditions of simple support or clamped edges. Simple supports are usually assumed on the premise that the results will be conservative, and that in any case great accuracy should not normally be required in good design because as a mode of failure general instability should be avoided by using suitably large safety margins. There are sound reasons for this, not the least being the finality of collapse and the known sensitivity of collapse loads to overall shape imperfections which commonly arise during the fabrication of lightweight deck and hull grillages. Furthermore, this philosophy would lead to inelastic collapse at loads which are known to be somewhat

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*References - listed at end of report.

less affected by variations in assumed rotational boundary constraints than are elastic collapse loads.

However, in spite of this knowledge being readily available for many years, recent evidence [4,5] has indicated that in certain existing frigates and destroyers the longitudinal compressive stresses experienced in steel superstructure decks (which are integral members in the ships' cross-section resisting vertical bending) may have approached or even exceeded the level required for overall buckling assuming simply supported boundaries. Furthermore, these stresses are appreciably below the elastic limit for the material. Independent studies by the present author confirmed these low general instability load factors, and also the equally low safety margins against inter-beam collapse [6].

With these findings the authors' immediate concern was to

(a) discover why these decks had not yet shown signs of failing in a general manner, and

(b) formulate for the Royal Navy a more precise philosophy than exists at present for the design of orthogonal grillages to withstand high compressive loads.

As mentioned in the discussion of reference [5], the main reason for (a) was considered to be the beneficial effects arising from the rotational constraints which exist at the unloaded sides of the deck grillages. For "long" orthotropic plates having negligible torsional rigidities it can easily be demonstrated that clamping all edges increases the elastic general instability stress 2.24 times above the simple support value. Releasing the rotational constraints on the loaded ends reduces the buckling load, but this only becomes appreciable at low aspect ratios. For ratios greater than about 1, it is therefore the constraint on the unloaded sides which are most important. The "orthotropic aspect ratio" of many deck grillages is greater than unity, and the constraints provided by adjoining structure at the unloaded sides can be demonstrated to be appreciable, and so it may be expected that actual elastic buckling stresses will be appreciably greater than those calculated assuming simple supports.

The subject appears also to be of considerable interest in merchant ships [7,8]. Schultz [8] has demonstrated that the deck fields adjacent to hatch openings of transversely framed ships about 100m and more long often show theoretical pinned general instability stresses appreciably lower than for local plate buckling, and comparable with primary ship bending stresses. Using orthotropic plate theory [7], he demonstrated that buckling is strongly dependent upon the amount of elastic restraint provided by the stiffened side shell plating. Schultz further showed that the torsional rigidity of the deck beams should not be ignored. Jojic [9] also considers general instability to be important in some ship side structures.

It is the purpose of the present paper to examine this topic, and to present sufficiently reliable and easily usable data to get the designer into the "right street" during the relatively early stages of design. Reference can then be made to more exact computer theories for greater precision should this be thought necessary. Such theories should ideally be developed to take into account the de-stabilizing effects that often arise from compression (and sometimes from shear) in adjoining structure. The input data associated with these more widespread structural elements is likely to be lengthy, and would perhaps only be justified in certain cases. There are similar reasons why these powerful computer programs require appreciable effort if they are used for parametric studies to generate design data. For example, the permutations and combinations required to investigate a practical range of grillage aspect ratios, flexural and torsional rigidities, and boundary constraints is very large. Considerable time is necessary to digest, analyse and present the results in useful form.

The purpose of the present paper is therefore to review previous work and develop the most promising method of analysis which would lead to an explicit solution and formulae of sufficient accuracy for use in early design and yet retaining enough generality to cater for partial elastic constraints at least on the important unloaded edges.

CHOICE AND SCOPE OF THEORY BY REFERENCE TO PREVIOUS WORK

A literature survey was aided by Abrahamsen's useful review of orthogonally stiffened plate fields [10]. It appears that Svennerud, who took Vedeler's earlier work [3] as a starting point, was the first to put forward an approximate energy method for the general instability of regularly stiffened panels with partially constrained unloaded sides [11]. He later [12] improved the accuracy by more precisely taking into account the "end fixation" of the cross stiffeners. Referring to Figure 1, he considered the deflection curve after critical failure to be of the form

$$w = w_o \text{ Sin } \frac{m\pi x}{A} f(y)$$

where for practical purposes he chose as an approximate y function the deflection curve of a beam with a uniformly distributed load and with "degrees of fixation" f_A and f_B. These fixities are not defined, but it is clear from the analysis that they are the ratio of the (unknown) end moments to the "fixed-end moments" of the clamped uniformly loaded beam. This concept bedeviled naval architecture for long enough. Quite apart from its use being fraught with difficulty when applied to beams having unequal boundary constraints, especially in the presence of unsymmetrical loads, the concept has little physical significance which the naval architect soon realises when he has to choose two numbers f_A and f_B between 0 and 1.5. The only viable concept for defining elastic rotational constraints is the moment-slope relationship which can quickly be estimated for practical structures [13] as discussed in Appendix III.

Further limitations in the value of this earlier work is that it ignores completely the internal energy stored in the rotational springs and serious errors arise when the number of stiffeners in either set is small which precludes its use for many structures. As shown in Appendix I, the spring energy can be very appreciable in many ship cases. Later similar work by Ahmed [14] shares this same deficiency, and both methods ignore torsion. Nevertheless, they represented significant advances towards the solution of a difficult problem.

Schultz has extended Bleich's isotropic plate solution [15] to orthotropic plates [7], but some accuracy may be lost when there are only a few members in either set of beams. More recently, Smith [5] has provided a discrete beam matrix solution using lumped masses and finite elements. It can cater for plate grillages with a high degree of irregularity, and in which any degree of rotational and normal constraint can be provided at the unloaded edges, the loaded edges being considered simply supported. Buckling loads are calculated automatically on a computer by iteration to any required accuracy. Unfortunately, the solution as it stands is not suitable for generating analysis equations or design data.

However, it is considered to be the most suitable program available for checking specific designs where torsion can be neglected. It was used for examining the accuracy of the present more approximate but explicit analysis developed by the author which also allows for the effect of torsion to be examined explicitly.

Smith and Faulkner [16] demonstrated that accurate results can be achieved with orthotropic plate theory when considering the closely related eigenvalue problem of vibration of partially constrained uniform grillages. A solution was presented for the case of elastically constrained sides, which can also be used to estimate static buckling loads. It was not possible to derive equivalent formulae for orthotropic plates having elastic support at the ends as well as the sides. However, it was noticed that the mode shapes corresponded exactly with the characteristic values and mode shapes for uniform beams. It would therefore appear possible to derive such formulae using Hearmon's method [17], employing characteristic functions. This course was pursued far enough to realise that not only was the mathematics messy, but recourse to characteristic value tables would still be necessary; thus defeating the requirement for an explicit formula.

Structures have also been treated by Smith [18] as plate beam or folded plate systems which include panels of orthotropic or isotropic plating with single direction stiffeners in which the stiffening members such as tee-bars, angle bars or top-hats are treated either as beams or as assemblies of plate strips. In examining local vibration, solutions may thus be obtained which take rigorous account of coupling between the plating and stiffeners. Solutions may also be obtained for orthogonally stiffened two and three dimensional shells; stiffeners in one direction being treated discretely while those in the perpendicular direction are represented together with the plating as equivalent orthotropic plates.

This short survey would be incomplete without reference to the latest work of Chang et al.[19,20]. They provide a general computer solution for biaxial stressing N_x and N_y which caters for any "general" boundary conditions. The solution is not directly helpful to the designer, and of course provides no explicit formulae which this study seeks. The same comment applies to a recent important paper by Gisvold and Moe [21], who demonstrate the use of nonlinear programming to solve combined in-plane as well as lateral load problems.

Three additional papers dealing with transversely stiffened plates deserve mention. Budiansky and Seide [22] consider flexure and torsion and provide a series solution which they reduce to closed form to provide useful formulae and charts for pinned stiffeners. Weiss [23] solves the coupled differential equations for an orthotropic plate having transverse stiffeners elastically restrained at their ends. Chang mentions Sin-Ben Tzu [24] who considers mixed stiffeners and refers to several papers published in Russia and China covering the case when the load exists on the heavy beams. Perhaps the best-known method for considering the buckling of plate elements supported by the flexural and torsional characteristics of transverse stiffeners is that developed by Bleich [15].

FINAL CHOICE OF THEORY

The orthotropic plate theory is attractive, but is not easily solved for the general case of partial constraints on all four edges. Moreover, although it has been demonstrated [16] that when the number of transverse members is three or less orthotropic theory can provide sufficiently accurate answers (within, say, 10-15 per cent), this is not always the case and the theory can be appreciably optimistic. There are many such grillages in frigates, and in one case it was found that orthotropic theory was optimistic by about 40 per cent. There is a tendency to increase frame spacing to reduce costs, and so deck grillages frequently have only a few beams or perhaps only one beam between deep frames or bulkhead boundaries. For this reason, an alternative to the orthotropic plate approach was sought, though a solution has been presented earlier for the practically important case of hinged loaded edges and partially constrained sides [16].

The author has often found energy methods to be surprisingly versatile, and so, as the main method of solution, he chose the energy test for stability, using Rayleigh's principle. This solution leads to explicit equations which were used in gross-panel strength analyses developed for early design use [25,26]. Moreover, the energy approach allows sideways bending of the stiffeners to be more conveniently considered. This movement is imposed on the stiffeners when any twisting occurs due to their weld attachment to relatively rigid plating (in-plane). By using an iterative procedure, the solutions can, if required, take account of:

(a) reduced stiffness and effective width associated with large deflections of the plating, e.g., in its post-buckled state

(b) reduced spring stiffness at the unloaded edges due to axial force effects in adjacent structure

(c) inelastic effect (an alternative approach is also suggested later).

This approach was therefore pursued and developed to include the energy associated with rotation of the stiffeners about their toe connections with the plating. This is considered to be more realistic than the conventional treatment of torsion effects and, although it is usually argued that these can safely be ignored for open-section stiffeners, it will be seen that their inclusion imposes no special difficulty. Moreover, Mansour has recently concluded [27] that torsion effects become much more pronounced as the compressive in-plane load approaches its critical value; and of course it is necessary to include torsion when considering closed-section stiffeners or sandwich structures as might be used, for example, in GRP decks. The author was encouraged in developing the method to find that it predicted the critical loads for elastically constrained struts within two per cent;

and, finally, for regular grillages it was found to check very well with Smith's more exact theory.

The inclusion of shear deformations was considered, but was abandoned because of the considerable algebraic complexity which results. The reduction in buckling loads caused by shear deformations is only likely to be large for deep grillages, where general instability is an unlikely mode of failure anyway. An approximate method of allowing for shear would be to reduce the flexural rigidities of each member as for a strut by

$$\left(1 + \rho \frac{\pi^2}{\ell^2}\right)^{-1}$$

where ρ is the ratio of effective flexural rigidity to effective shear rigidity, and ℓ is the half wave length of the buckled deformation for the member.

The analysis was first developed to cater for two important cases for which explicit solutions have been obtained for N_x loading.

Case 1 - Loaded ends simply supported, sides having equal elastic rotational constraints; extension to unequal constraints is then discussed.

Case 2 - Loaded ends having equal elastic rotational constraints, sides simply supported.

The first case would cover the majority of single skin ship grillages, whereas the second case would be useful perhaps where the loaded edge boundaries are provided by stiff transverse bulkheads, whose stiffeners are continuous with the axially loaded grillage stiffeners. This would be especially important for low values of orthotropic aspect ratio where the effect of loaded edge constraint is known to be most marked. The important case of pxo grillages falls into this category and is considered. Graphs are presented which show the effect of a wide range of rotational constraints in raising buckling stresses for a variety of torsional rigidities. The case of loaded ends clamped and sides elastically constrained is also catered for by making use of isotropic plate correlation as discussed in Appendix I.

OUTLINE OF THEORY AND RESULTS

Assumptions

a) The material is elastic and obeys Hooke's law;

b) Bernoulli-Euler bending theory, in which shear deformation is ignored;

c) St. Venant torsion theory with Prandtl membrane analogy solution for thin-walled stiffeners;

d) Twisting of the plating is ignored since numerical examples have shown that for ships' decks and shells this strain energy is very small compared with other terms. The main effect of the plate is to act as effective flange for the stiffeners, and this effect is included in their combined rigidities;

e) Sideways bending and longitudinal warping of the stiffeners is considered, but transverse warping is neglected. The toe connection of the stiffeners to the plating is assumed not to move sideways due to the high in-plane rigidity of the plating;

f) Local instability of stiffener cross sections is excluded. These forms of buckling usually occur at stresses well above yield;

g) Loss of in-plane stiffness of the plating, arising from initial stresses and deformations, buckling or shear lag is allowed for insofar as cross-section properties include an effective width of plating;

h) The rotational constraints at the stiffener ends are considered to be linear elastic rotational springs whose stiffness is unaffected by the axial force. A conservative method of allowing for such interaction is suggested later;

i) For convenience in defining the buckled deformation, there is assumed to be only one neutral surface for bending on the two perpendicular directions;

j) The axial stresses σ_x, σ_y are assumed to be constant across the effective cross section of the grillage;

k) The stiffeners and effective plating are assumed to be essentially straight initially;

l) The direct energy of elastic compression of the stiffeners is ignored.

Buckled form

It has been abundantly verified [28] that sufficient accuracy for practical purposes can usually be obtained by taking for the special deformation that which is represented mathematically by the simplest algebraic expression satisfying the geometrical and equilibrium boundary conditions imposed in the actual structure. From orthotropic plate theory results for pinned and clamped edges [1,2], it is known that the buckled form consists of one half wave across the width, and m half waves along the length where m is a positive integer number depending on the panel aspect ratio, rigidity ratio and constraints. There seems no reason to suppose that for intermediate constraints the same will not be true. However, to examine this point and to provide general terms for a grillage bending solution, the energy expressions were derived for the most general (m,n) buckled deformation:

$$w = w_{mn} \sin(m\pi x/A) \sin(n\pi y/B)(1+G_1 \sin(n\pi x/A))(1+G_2 \sin(n\pi y/B)$$

where the constants G_1,G_2 are to be found from the moment-slope boundary conditions.

If there were just a few longitudinal stiffeners, or if they were of differing rigidities, then a double trigonometric series involving m and n half waves would be recommended. For transverse stiffeners alone a single series in m with n = 1 would be adequate. In such cases the energy approach would lead to a system of homogeneous linear equations in the unknown deflection coefficients w_{mn}, and by equating the determinant of this system of equations to zero in the usual way we would obtain an equation for determining the critical load. These cases are outside the scope of the present paper and the reader is referred to the second edition of reference [1] or to reference [3] for series solutions where the supports are simple and torsion can be ignored.

General biaxial stress solution

The reader is referred to Appendix I for the key stages in the development of the theory. It is a conventional energy solution using Rayleigh's principle [28] as a practical test for elastic stability. Apart from the inclusion of the elastic springs, the only novel feature is the inclusion of the sideways torsion and bending of the stiffeners about their enforced line of attachment with the plating. The biaxial stress solution takes the form for the general (m,n) mode of buckling:

$$\sigma_{xGE(y)} \; [1 + \frac{n^2 A^2 \; t_{ye} \; K_{2yn}}{m^2 B^2 \; t_{xe} \; K_{2xm}} \; (\frac{\sigma_y}{\sigma_x})] = \sigma_{xGE} \tag{12}$$

where (σ_y/σ_x) is applied stress ratio and:

$$\sigma_{xGE} = \frac{1}{2} \sigma_{op} [K_{1xm} \; m^2/\alpha_o{}^2 + 2n^2 \; \Gamma_{xy} \; K_{2xm} \; K_{2yn} + K_{1yn} \; n^4 \alpha_o{}^2/m^2]/K_{2xm} \tag{13}$$

where σ_{xGE} is the critical grillage elastic buckling stress under uniaxial compression and the use of subscript (y) in $\sigma_{xGE(y)}$ is simply to denote the critical value of σ_x in the presence of σ_y. The reason that more importance is attached to σ_x than to σ_y is because this may be considered to be the primary load, for example, as induced by ship bending. All terms are defined in the notation, the K's being constraint-mode functions. The first subscript is a number indicating the type of function, the second is x or y denoting the stiffener set and constraints involved, and the third is the mode number m or n. These K's depend upon

- the rotational constraints at the stiffener ends R_x, R_y
- the mode wave numbers m,n

The two special cases of importance when m = n = 1 are plotted as K_1, K_2 versus R in Figure 2. σ_{op} is the minimum "long" orthotropic plate pinned buckling stress when twisting can be ignored and $\alpha_c \geq 1$.

It can be shown that either one or both of m or n is always unity, the (m,1) mode prevailing when $\alpha_c > 1$ where α_c is a "constrained" orthotropic aspect ratio

$$\alpha_c = \alpha_o[K_{1y1}/K_{1x1}]^{\frac{1}{4}} \tag{15}$$

α_o is the conventional pinned orthotropic aspect ratio. At lower values of α_c we may expect the (1,1) mode, except where $\sigma_y > \sigma_x$ in which case the (1,n) mode is possible. This is of no practical interest.

Under biaxial conditions to find the critical σ_x stress in the presence of a smaller σ_y ($\leq 0.3\sigma_x$ is suggested), the value of wave number for the lowest critical stress is

$$m_{(y)} \simeq \alpha_c \sqrt{1-2\sigma_y/\sigma_x} \tag{17}$$

The nearest integer value should be substituted in equations (12,13) to find the lowest $\sigma_{x,GE(y)}$.

The solution made use of a transformation from a summation for each set of stiffeners into a more convenient integral. The conditions under which the identity (eq.(3), Appendix I) is satisfied are examined in Appendix II, where it is found to be true with one minor approximation which arises in the three components of the twisting strain energy. There are no approximations in the vertical bending terms, which dominate the solution with open section stiffeners.

Also, as discussed in Appendix II, if $\frac{m}{q+1}$ is an integer, then the last term inside the square bracket of equation (13) disappears and is to be omitted, since all the Y beams will then lie on nodal lines and will not be bent. The integer m is still of course retained for the first term, and the lowest buckling load may arise when $m = q + 1 = A/a$. The theory presented does of course allow this special case of inter-frame or "panel" collapse to be examined, but the author discusses this important mode of failure more fully elsewhere [25] since it is more seriously affected by residual welding stress actions.

The theory has been checked where possible against numerous special cases which may be considered "exact" from references [15,29,30,31] and has been found to agree within 5 per cent (see Appendix I). In all cases the theory was slightly optimistic as expected using Rayleigh's principle. Calculations for certain ship decks were checked against a more exact computer theory [5] and agreement was everywhere within 10 per cent. This agreement is considered reasonable, and lends confidence in the wider use of the theory for early design purposes. Nevertheless, experimental confirmation would be welcome.

Confirmation or otherwise would be particularly welcome for even mode ($m = 2, 4, 6 \ldots$). Adamchak has pointed out two inaccuracies which arise in this case, one of which may be serious. With even modes it can be shown that the displacement form equation (1) is not symmetrical. This is unlikely to be serious, however, since the solution depends upon the total strain energy. Greater concern may arise in that the boundary equilibrium conditions have been stated in absolute physical terms. For example, for a typical rth beam of the x-set the analysis in Appendix I requires:

$$\left|\frac{\partial w}{\partial x}\right| = \left|\frac{EI_x}{C_x}\ \left(\frac{\partial^2 w}{\partial x^2}\right)\right|$$

This avoidance of considering the algebraic sign of the terms is of no consequence for odd modes, but may be insufficient for even modes where physical interpretation involves the idea of negative springs.

Stating the rotational boundary conditions with complete rigor to overcome this objection complicates the analysis appreciably, and this is being investigated by Adamchak. Meanwhile, it should be noted that this deficiency can only affect even modes. Even then, the errors which may be introduced will depend upon the level of rotational constraint at the ends of stiffeners deforming in these modes. Even modes can only arise in longitudinal X-stiffeners (for the transverses n = 1 throughout), and then only is the grillage is "long" with

$$\alpha_c \geq 2$$

since this is the lowest even mode. Under these conditions it has been shown that the effect of end constraint R_x is very small. The more important side constraints R_y can introduce no error since n is odd (=1). Moreover, R_x constraints will often be small in continuous structures, due to buckling actions in adjoining "in line" grillages.

Thus it seems unlikely that serious errors would arise even if the objection is a valid one, and perhaps this is why agreement with more exact computer solutions is good. However, it would be prudent to treat even mode solutions with caution, or to avoid the doubt by assuming $R_y = 0$ is such cases.

Uniaxial stressing

For the special case when $\sigma_y = 0$, equation (13) represents the solution, and its minimum value is given by substituting the nearest integer value of

$$m = \alpha_c$$

in equation (13). Treating m as a variable provides a lower bound locus to the family of buckling coefficients for $\alpha_c \geq 1$

$$\min \sigma_{xGE} = \sigma_{op}[(K_{1xm} K_{1y1})^{\frac{1}{2}}/K_{2xm} + \Gamma_{xy} K_{2y1}] \quad (16)$$

in which K_{1xm} and K_{2xm} are evaluated for a given R_x with $m = \alpha_c$ from the equations in the notation, and the K_{1y1} and K_{2y1} can be evaluated for a given R_y from the notation or from Figure 2. As a particular example, consider the grillage for which $\alpha_c \simeq 1$. Then it follows m = n = 1, and the ratios min σ_{xGE}/σ_{op} calculated from equation (16) for a range of R_x, R_y and Γ_{xy} = 0,1 are given in Tables 1 and 2.

Table 1, $\Gamma_{xy} = 0$

R_x \ R_y	0	6	∞
0	1.0	1.6	2.3
6	1.4	2.3	3.4
∞	1.7	3.2	4.0

Table 2, $\Gamma_{xy} = 1$

R_x \ R_y	0	6	∞
0	2.0	2.7	3.6
6	2.4	3.4	4.7
∞	2.7	4.3	5.3

R = 6 corresponds to a reasonable value for many ship grillages [13], and values of zero and infinity correspond to pinned and clamped conditions. $\Gamma_{xy} = 0$ would approximate to a single skin grillage having open cross-section stiffeners, whereas $\Gamma_{xy} = 1$ would be appropriate for an isotropic plate or may be a reasonable approximation for a cellular structure. The tabled results illustrate:

- the high sensitivity of overall buckling stresses to the rotational constraints, particularly those at the ends of the unloaded transverse beams.
- practical ship constraints do indeed offer considerable opposition to overall buckling, especially in single skin grillages having low torsional stiffness.

Similar examples at higher α_c values would lay even more emphasis on the importance of the constraints at the unloaded edges, and on their diminished importance at the loaded edges. The reverse is true for $\alpha_c < 1$ and this now leads naturally to the consideration of two special cases.

Case 1, $\sigma_y = 0$, "long" grillages ($\alpha_c \leq 1$) with sides elastically constrained

As $R_x = 0$, $K_{1xm} = K_{2xm} = 1$ and equation (13) with n=1 becomes

$$\sigma_{xGE} = \frac{1}{2}\sigma_{op} \left[\frac{m^2}{\alpha_o^2} + 2\Gamma_{xy} K_{2y1} + \frac{\alpha_o^2}{m^2} K_{1y1}\right] \qquad (17a)$$

As before, the last term disappears if m/(q+1) is an integer, since the Y beams then all lie on nodal lines. Rearranging the terms

$$\sigma_{xGE} = \sigma_{op} \left[\frac{1}{2} \sqrt{K_{1y1}} \left(\frac{m^2}{\alpha_c^2} + \frac{\alpha_c^2}{m^2}\right) + K_{2y1} \Gamma_{xy}\right]$$
$$= \sigma_{op}\left[\sqrt{K_{1y1}}\, \kappa + K_{2y1} \Gamma_{xy}\right] \qquad (17b)$$

where κ is plotted against α_c and m in Figure 3, and can be shown to be equivalent to $\frac{k - 2}{2}$ where k is the well-known buckling stress coefficient for a pinned isotropic plate [32] having side aspect ratio α_c. R. Smith [33] appears to have been the first to have spotted this useful correlation when applying orthotropic theory to predict the buckling strength of plywood panels. Wittrick [2] extended the conclusions to cover all symmetrical combinations of clamped and pinned edge supports, and for biaxial stressing. The curves for clamped ends is also shown in Figure 3. Inspection of equation (17b) shows the minimum value of the critical stress occurs when $m = \alpha_c$, as derived before, and the minimum value for "long" grillages ($\alpha_c \geq 1$) is:

$$\min \sigma_{xGE}/\sigma_{op} = \sqrt{K_{1y1}} + K_{2y1} \Gamma_{xy} \qquad (18)$$

The right-hand side is plotted in Figure 4 against side constraint R_y for a variety of Γ_{xy}, so that the designer may quickly assess their importance for any particular grillage.

<u>Case 2, $\sigma_y = 0$, "wide" grillages ($\alpha_c \leq 1$) with sides elastically constrained</u>

Again, referring to equation (13) and substituting $R_y = 0$, $K_{1yn} = K_{2yn} = 1$, $n = 1$, and making $m = 1$ by the previous argument, the minimum buckling stress is

$$\min \sigma_{xGE} = \frac{1}{2} \sigma_{op}\left[\left(\frac{K_{1x1}}{\alpha_o^2} + \alpha_o^2\right)/K_{2x1} + 2\Gamma_{xy}\right] \qquad (19)$$

For the special but practically important case of a pxo plated grillage or panel, the buckling stress becomes

$$\min \sigma_{xGE} = \sigma_{CE} \left[\frac{K_{1x1}}{K_{2x1}} + \frac{A^2 Db}{B^2 EI_{xe}} \left(\frac{GJ_{xt}}{Db} + 2 + \frac{A^2}{B^2 K_{2x1}}\right)\right] \tag{20}$$

where $\sigma_{CE} = \pi^2 EI_{xe}/bt_{xe}A^2$ Euler column stress for span A and $A^2 Db/B^2 EI_{xe} \leq 1$ (see Appendix I). As the panel becomes very wide A/B → 0 and the buckling stress becomes

$$\min \sigma_{xGE} = \sigma_{CE} \left(\frac{K_{1x1}}{K_{2x1}}\right) \tag{21}$$

which agrees almost precisely with the exact solution for an elastically constrained strut [29].

<u>Biaxial compression, "long" grillages ($\alpha_c \geq 1$) with sides elastically constrained</u>

Equation (12) simplifies slightly:

$$\sigma_{xGE(y)} = \left[1 + \frac{A^2 t_{ye}}{m^2 B^2} \frac{K_{2y1}}{t_{xe}} \left(\frac{\sigma_y}{\sigma_x}\right)\right]^{-1} \sigma_{xGE} \tag{22}$$

where σ_{xGE} is the (m,1) mode solution for uniaxial compression as defined in Case 1 by equation (17a), etc. It can be shown that either one or both of m or n is always unity, the (m,1) mode prevailing when $\alpha_c > 1$. At lower values the (1,1) mode prevails, except where $\sigma_y > \sigma_x$ in which case the (1,n) mode is possible. This is of no practical interest.

To find the critical σ_x stress in the presence of a smaller σ_y ($\leq 0.3\sigma_x$ is suggested), the value of the wave number for the lowest critical stress is from equation (17)

$$m_{(y)} \simeq \alpha_{c1} \sqrt{1 - 2\sigma_y/\sigma_x} \tag{23}$$

The nearest integer value should be substituted to find the lowest buckling stress $\sigma_{xGE(y)}$.

ESTIMATION OF CONSTRAINTS R_x, R_y

A fuller basis for the analysis is given in Appendix III. Essentially, it is assumed that in calculating the rotational constraint provided by the adjoining structure it will be sufficiently accurate for most purposes to consider only that limited part of the stiffened structure to which the grillage stiffener is directly connected in the plane of bending. It further assumes that the remote ends of this limited structure is constrained by a rotational spring $C\ell/EI = 6$ which is typical for ship structures. Then, referring to Figure 5, the constraint which this "first remove" structure exerts on the end o of a grillage beam is given by

$$C_o = 3.6\, E \sum_{i=1}^{3} \frac{I_i}{\ell_i} \tag{27}$$

Hence

$$R_x, R_y = 3.6\, \frac{\ell}{I} \sum_{i=1}^{3} \frac{I_i}{\ell_i} \tag{28}$$

where ℓ, I refer to the grillage beam being considered, and ℓ_i, I_i refer to a typical adjoining beam stiffener.

Some or all of this adjoining structure may itself be liable to buckle in an overall manner under the forces applied. Then it is suggested the constraint it exerts on the grillage Y beam is reduced to C_{yr} given by

$$\frac{C_{yr}}{C_y} + \frac{\sigma_x}{\sigma_{xGE}} = 1 \tag{29}$$

where σ_{xGE} is the critical stress for the adjoining structure assuming it is simply supported. The Appendix outlines how this should be calculated, and suggests a simple approximation when the adjoining structure takes the form of a side wall or deck extension having the same spacing and frame size as for the Y beams of the grillage.

Unequal constraints

Where opposite edges of a grillage are unequally restrained it is possible, by analogy with the isotropic plate buckling problem, to apply the Lundquist-Stowell approximation [34]. This involves obtaining two solutions for grillages with equal edge restraints on opposite edges, using each value in turn and taking the average of the critical stresses so derived. This and inelastic effects will be discussed further.

INELASTIC EFFECTS

In view of the restricted value of σ_y stresses, it will be assumed that plasticity effects can occur only in the x beams. By ignoring plasticity effects in twisting (Γ_{xy} is in any case small for open cross sections of present interest), it is then possible to keep the "buckling coefficient" entirely elastic inside the square bracket terms of the various buckling equations. This will in any case be slightly conservative, as was also found by Bleich [15] in discussing inelastic effects in plate buckling. The elastic buckling stresses σ_{xGE} are proportional to $\sqrt{(EI_{xe})(EI_{ye})}$, and so with a final assumption that the stiffness of the x stiffeners is reduced in the ratio E_t/E the general statement of the buckling stresses becomes:

$$\sigma_{xG} = \sqrt{\frac{E_t}{E}}\ \sigma_{xGE} \qquad (30)$$

This is clearly less conservative than using the plasticity reduction factor E_t/E which is very widely used for primary modes of failure. Vedeler [reference 3, and in discussion at the Numerical Methods Symposium, Oslo, 1963] seems to favour the use of E_t/E, perhaps for its greater simplicity, Schlicher [35] evidently did also, but it appears his main reason was that his findings showed the choice made little difference, and so he opted for the conservative approach. The author is inclined to think there were too many imponderables clouding the issue of these early findings, and has confidence in recommending the use of equation (30) in the belief that inelastic effects will not appear in the transverse beams until after collapse has been initiated.

It is important to recognise that E_t is the "structural tangent modulus" and should ideally be derived from stub column type tests on typical elements of the x stiffener-plate combination arranged so that all the effects of cold rolling and welding residual stresses are present. In practice this is seldom possible. Numerous investigations, notably at Lehigh [36,37], have shown that inelastic effects appear in compression testing when the average applied stress = $\sigma_p - \sigma_r$, where σ_p is the material proportional limit and σ_r is a significant residual compression stress. This σ_r in the case of a welded stiffened panel would be the average longitudinal compression stress in the middle zone of the plate. The author has discussed this problem more fully elsewhere for welded ship panels [25], and has proposed using a structural proportional limit stress in compression given by

$$\sigma_{ps} = \sigma_p - \sigma_r = p_r \sigma_o \qquad (31)$$

Typically, σ_r may be expected to be $0.25\sigma_o$ or more in ship structures, even after stress shake-out at sea, but it can vary appreciably as it is dependent upon many factors associated with the welding requirements and site conditions. For many purposes a value of $p_r = 0.5$ is recommended, and is in keeping with Column Research Council findings [38]. Not only does σ_r lower the proportional limit, but it also "rounds off" the structural stress-strain curve in the yield region, thereby giving rise to early tangent modulus effects even in metals having a pronounced yield.

To make use of these findings in equation (30), it is advisable to distinguish between

a) flat yield materials

b) strain-hardening materials.

For the former it is convenient to use the Bleich formulation of the Ostenfeld quadratic parabolae [15]. These define the tangent modulus for any given applied stress σ in terms of the yield stress σ_o and the proportional limit stress. With the author's proposal in equation (31) this relation is:

$$E_t = \frac{\sigma(\sigma_o - \sigma)}{\sigma_{ps}(\sigma_o - \sigma_{ps})} = \frac{\sigma_G(1 - \sigma_G)}{p_r(1 - p_r)} \tag{32}$$

where $\phi_G = \sigma_{xG}/\sigma_o$ the required failure stress ratio for inelastic grillage instability. Substituting equation (32) in (30) leads to the required result

$$\phi_G = \frac{\sigma_{xG}}{\sigma_o} = [1 + p_r(1 - p_r)(\sigma_o/\sigma_{xGE})^2]^{-1} \tag{33}$$

in the range $0 \leq \sigma_o/\sigma_{xGE} \leq 1/p_r$. At lower values of σ_{xGE} collapse will be elastic. Hence, it is seen that $\psi = \sigma_o/\sigma_{xGE}$ is an important parameter to compute, where the elastic buckling stress σ_{xGE} is calculated from the appropriate equation in the report. ψ is also a useful parameter when considering structural safety, and values not more than 1/3, for example, would be recommended for highly stressed designs where there was any doubt about the overall flatness of the as-built grillage.

The one exception to the use of equation (33) which the author would advise is for the special case of the pxo grillage with $\alpha_c \leq 1$ as discussed in Case 2, equations (20) and (21). Since there are then no Y beams present the direct use of the E_t/E plasticity reduction factor is recommended. Proceeding in exactly the same fashion, it can be shown that in this case

$$\phi = 1 - p_r(1 - p_r)\psi \qquad (34)$$

in the range $0 \leq \psi \leq 1/p_r$.

The use of the above approach based on the Ostenfeld-Bleich quadratic parabolae could be used for materials exhibiting no yield plateau, for example, by replacing σ_o by, say, the 0.2 per cent proof-stress. However, if strain hardening were pronounced, as with aluminum and magnesium alloys, this approach would very probably be conservative. Use of the Ramberg-Osgood three parameter stress-strain relations [39], as given in the Notation, is then a more rational choice for dealing with inelastic effects. However, it is then more difficult to incorporate residual stress effects, making use of equation (27), since there is strictly no proportional limit. An approximation which is sometimes used for σ_p when using these equations is to take 0.0001 offset.

Quite apart from any numerical difficulties which could arise, there may be some objection to using this approach in the same way when accounting for residual stress action in strain hardening materials. This method is as far as the author is aware untried, and may be no better than, say, adopting a reduction in base stress σ_o (perhaps pro rata with σ_r) and n.

EFFECTIVE PLATING

In computing stiffener-plate properties, it is necessary to assess how much plating acts with the stiffener in flexure and in compression. There is an almost total lack of uniformity in this subject, as shown by a recent review [40]. For single skin ship grillages of conventional stiffener/plate area ratios the choice is often not too important, though for cellular structure and for least-weight structures the choice becomes very important. Some guidance based on simplifications from reference [25] is given below for those who may not be bound by codes, or who perhaps have not made their personal preferences.

a) When N_x acts alone $b_e = b/\beta$; $a_e = a/2$.

b) When N_x and N_y act take b_e as above; a_e is a_{em} wide plate strength equation.

$a_{em} = a_{em}/a = 0.9/\beta^2 + 1.9(1 - 0.9/\beta^2)/\alpha\beta$.

c) For grillage bending stresses under pressure $b_e/b = a_e/a = 1/2$ unless the plate compression stresses are greater than plate buckling stress, in which case use least value of above or $b_{em}/b = (2\beta - 1)/\beta^2$ and a_{em}.

The advice for b_e when N_x acts is based on plate stiffness using the reduced effective width concept b_e' and allowing for residual stress action [25]. An approximate expression is

$$\frac{b_e'}{b} = \frac{1}{\beta}\sqrt{\frac{\sigma_o}{\sigma_e}} \cdot R_r$$

where σ_e is the "edge stress" in the effective plate and X stiffeners at failure, and $R_r \leq 1$ is a somewhat complex residual stress reduction factor depending on E_t (which itself strictly depends on σ_e) and plate compression residual stress σ_r. To avoid an iterative solution for σ_e, we note that the radical in the above equation is > 1, and so we may approximately assume the last two terms cancel. Thus the reduced effective width $\simeq b/\beta$. A similar approximation for conventional effective width (stress distribution) is $(2\beta - 1)/\beta^2$.

For the special case of "panel" collapse where m = A/a, or for pxo grillages, residual stress actions often assume greater importance. This is because there are no stabilising Y beams and the strength and stiffness of the plating associated with the X stiffeners therefore becomes more important. In this case an iterative solution is advised and it is suggested that reference [25] be consulted, especially for welded panels. It should also be consulted for slender cross sections or where low weight is sought.

EFFECT OF PRESSURE

Under combined pressure and in-plane loads the overall and local bending stresses should be magnified (or attenuated, if tensile) by factors m_{xo} and $m_{x\ell}$ respectively, where the terms are defined in the Notation. The use of magnification factors for ship deck and bottom calculation has been substantiated by

Smith [5].

These magnification factors would also allow the effect of as-built lack-of-flatness to be examined, providing these distortions were analysed into harmonics (m,n) and then magnified by using the appropriate σ_{xGE}. If measured distortions were found to give rise to high bending stresses, then it would be possible to lay down a lack-of-flatness standard for construction, for example, based on the avoidance of yield at the extreme design load.

If computer solutions to grillage bending analysis are not available, then Clarkson's book will be found most useful [45].

SHIP DECK EXAMPLE

Consider the upper deck of a 505-foot guided missile destroyer with the following particulars:

A, B	= 228 in., 192 in.
$p \times q$	= 9 x 4
EI_{xe}/b , EI_{ye}/a	= 18,000, 18,000 ton in.
t_{xe}	= 0.494 in.
C_y	= 24,000 ton in./radian
C_x, Γ_{xy}	= 0
min σ_o steel	= 20 ton/in.^{2}

Hence, the derived parameters are

$$R_y = C_y B/EI_{ye} = 5.6$$

$$\alpha_o = \frac{A}{B}\left[\frac{I_{ye}}{I_{xe}}\frac{b}{a}\right]^{1/4} = 1.188$$

$$\sigma_{op} = \frac{2\pi^2 E}{t_{xe} B^2}\sqrt{\frac{I_{xe} I_{ye}}{a \; b}} = 19.52 \text{ ton/in.}^2$$

It is required to examine the uniaxial grillage buckling strength. Assume first the stresses are elastic, and that n=1. Equation (17) with $\Gamma_{xy} = 0$ becomes

$$\sigma_{m1} = \frac{1}{2}\sigma_{op}\left[\frac{m^2}{\alpha_o^2} + k_1 \frac{\alpha_o^2}{m^2}\right]$$

where K_1 = 2.5 from Figure 2 for $R_y = 5.6$.

$$\therefore \sigma_{11} = 9.76\ [0.709 + 2.5 \times 1.411]$$
$$= 41.3 \text{ ton/in.}^2$$
$$\sigma_{21} = 9.76\ [2.836 + 2.5 \times 0.353]$$
$$= 36.8 \text{ ton/in.}^2$$

The third mode is greater, and so σ_{21} is the lowest. The more exact computer solution [5] provides $\sigma_{21} = 38.1$ ton/in.^{2} The agreement is very good.

For comparison the hinged solution was evaluated to be 20.7 ton/in.^{2} (21.0 ton/in.^{2} by reference [5]). Therefore, the side constraints, even with a modest value $R_y = 5.6$, nearly double the elastic buckling stresses.

Inelastic effects can be examined using equation (29). Assume as in reference [6] that the mean yield is 1.645 standard deviations higher than the "minimum." The mean yield assuming c.o.v. = 6% is 22.2 ton/in.^{2} Assume $p_r = 0.5$ to allow for residual welding stress effects, then

$$\sigma_{xG} = \sigma_o/[1 + \frac{1}{4}(\sigma_o/\sigma_{xGE})]$$
$$= 22.2/[1 + \frac{1}{4}(22.2/36.8)]$$
$$= 19.3 \text{ ton/in.}^2$$

Hence, this may be considered to be a reasonable estimate on which to base the grillage collapse load for the deck. A fuller discussion of statistical aspects appears in references [6,46].

CONCLUSIONS

Discrete beam explicit equations have been derived for buckling stresses for biaxially compressed orthogonally stiffened-plate grillages having opposite boundaries equally elastically constrained against rotation. By using the Lundquist-Stowell plate buckling approximation, the solution for the completely general case where opposite edges are unequally restrained can be obtained by a double calculation. A listing of the equations appears at the end of these conclusions.

Other novel features of the solution include:

- a more realistic treatment than hitherto of bending-torsion twisting of the stiffeners about their plate connection, with provision made for open or closed cross-section stiffeners having intact or perforated walls;
- allowance for inelastic effects in flat yield materials, and a suggested treatment for strain hardening materials;
- a quick method for estimating the initial unloaded value of the constraints R at stiffener ends, and for estimating their reduction under load as caused by coupled buckling actions in adjoining structure;
- in computing stiffener-plate properties reference was made to a parallel study of effective widths for stiffened plates which include the degrading effects from welding stresses [25]. This sophistication is recommended for slender cross sections or where low

weight is sought requiring low safety;

- shear deflections can be approximately allowed for, but it is pointed out they only become important when flexural buckling is unlikely;
- the theory was extended to include the effect of lateral pressure from the point of view of estimating total stresses in the structure. This makes use of magnification concepts using the elastic buckling stresses, and thereby allows the effect of as-built lack-of-flatness to be included.

Numerous accuracy checks were made where possible and, in spite of the several approximations involved, accuracy was everywhere well within the 10-15 per cent required for most design purposes. For simplicity in estimating mode number m, it was assumed $\sigma_y/\sigma_x \leq 0.3$, which is believed to cover all likely ship applications. Where required, this restriction can be removed by working from the general buckling stress relationship in equations (12,13).

Some general conclusions which arise from the results of the theory are:

1) The parameter $\alpha_c = \alpha_o[K_{1y1}/K_{1x1}]^{1/4}$ which is termed the "constrained" orthotropic plate aspect ratio is important in the same way as the orthotropic ratio $\alpha_o = A/B[I_{ye}b/I_{xe}a]^{1/4}$, or the isotropic plate aspect ratio $\alpha = a/b$. It divides grillages into two broad types

- "long" when $\alpha_c \geq 1$
- "wide" when $\alpha_c < 1$

Many longitudinally framed ship grillages familiar to the author fall in the long range $1 < \alpha_c < 3$, whilst transversely framed ship grillages are mostly wide.

For long grillages the lowest biaxial buckling stresses occur with m half waves in the longitudinal direction and n=1 half wave transversely. For wide grillages the lowest buckling stresses may arise in the (1,n) mode if σ_y dominates; in ship grillages generally $\sigma_y << \sigma_x$ and then only the (1,1) mode is of interest.

In long grillages the effect of side constraints R_y at the ends of the transverses is marked. In wide grillages the end constraints R_x for the longitudinals assume greater importance. For "square" grillages ($\alpha_c \simeq 1$) both constraints are important, and clamping the edges can increase the buckling stresses for a torsionally weak grillage by about 4:1.

2) For long grillages the lowest uniaxial buckling stress occurs with $m \simeq \alpha_c$. Providing σ_y is small ($< 0.3\sigma_x$ suggested), the lowest mode occurs with $m_{(y)} \simeq \alpha_c\sqrt{1 - 2\sigma_y/\sigma_x}$. The side constraints shorten the buckled wave lengths in long grillages, whereas the presence of a transverse load lengthens them. This is analogous to long isotropic plate behaviour.

3) Under uniaxial conditions it was demonstrated that for long grillages with constrained sides but pinned ends (Case 1) buckling stresses can quickly be determined using readily available isotropic plate buckling coefficients for a given aspect ratio $\alpha_c = \alpha$, as in Figure 3. Wittrick also showed a similar relation existed for biaxial buckling, and so it would appear feasible to provide further design data by extending the

presentation of buckling coefficients for any R_x,R_y constraints under uniaxial or biaxial compression.

4) As mentioned, constraints can appreciably raise elastic buckling stresses, particularly for single-skin grillages of low torsional rigidity. For long grillages the side constraints are most important, and there is now no doubt that they have prevented overall buckling of RN frigate decks at sea. Typical deck grillages have R_y = 6 or more, and these sort of values nearly double the buckling stresses. Practical constraints are closer to "clamped" than to "pinned" conditions.

5) It follows that any design methods which assume the edges to be pinned will be considerably and unnecessarily pessimistic. It may be argued by analogy with isotropic plate behavious [see, for example, reference 25] that the side constraints would diminish as load increases due to inelastic effects, and therefore pinned assumptions are justified. This analogy is incorrect, since the discrete nature of the transverse stiffeners ensures that they are not severely stressed until collapse deformations build up. The same is not necessarily true, of course, for the end constraints, but these only assume practical importance in the less usual wide grillages.

Unidirectional grillages are really outside the scope of the paper, although advice regarding their treatment has been offered. Reference [41] is probably the best compilation of plate and stiffened-plate buckling data readily available to naval architects, and should be consulted for cases outside the scope of this report.

Equation listing

For convenience a summary of equation numbers for elastic buckling stresses is:

(12,13) general biaxial solution

(13) general uniaxial solution

(17,18) uniaxial solution with $R_x = 0$, $\alpha_c \geq 1$

(19) uniaxial solution with $R_y = 0$, $\alpha_c \leq 1$

(20,21) uniaxial solution for pxo grillage

(22,17a) biaxial solution with $R_y = 0$, $\alpha_c \geq 1$

Inelastic effects are allowed for using:

(33) all pxq grillages

(34) pxo grillages equations (20,21)

Elastic constraints are evaluated from:

(27,28) stable adjoining structure

(29) instable adjoining structure

RECOMMENDATIONS

1) Enough confidence has been established in the theory to recommend its use in design.

2) However, since overall collapse would be catastrophic, and is suspected of being sensitive to shape imperfections (whose effects can be examined using the theory presented), it is felt most desirable to obtain some experimental confirmation.

3) The theory can be used in many systematic studies. Three examples recommended are:

- parametric studies to establish or disprove the belief that the possibility of general instability in highly stressed designs can be sufficiently removed with only minor weight penalty in the frames.

- gross panel synthesis studies at various structural loading indices. These should include the effect of lateral pressure.
- parametric studies to establish important variables and to show their effect on weight or cost (if possible).

Ideally, the theory should be incorporated, along with predictions for other modes of failure, in a synthesis routine for, say, amidship section design of longitudinally framed ship.

4) For design purposes, it is necessary to consider safety. Where it is known that scantling cross-section areas are within a variance of, say, 3-4 per cent, and where yield or proof stress variances are typically 6-8 per cent, it is suggested that, under extreme loads based on currently available ship bending data, a deterministic safety factor using inelastic results should be no less than two. An alternative criterion akin to that used to prevent general instability in submarine hulls is to make $\sigma_{xGE}/\sigma_o \geq 3$. Further advice should await experimental model test results, along with more reliable sea load data, and in due course safety should be considered statistically [6].

5) For this purpose, it is necessary to collect frequency distribution data on variations in:

- plate thickness and stiffener cross section
- material properties, notably σ_o, σ_p, E, E_t
- residual stress effects, notably σ_r and as-built overall and panel distortions.

This should be pursued actively now.

6) If for some reason overall instability appears as the active constraint, for example, in deck design, even with the present theory which allows the strengthening effects at the boundaries to be considered, the designer should bear in mind that minor structure, such as partition bulkheads under the deck, can often provide enough support to justify a reduction in the unsupported spans A, B. This may remove the problem, but it is necessary to check that this support does not occur along or close to instability nodal lines where it can do no good. In particular, avoid transverse support at A/m positions, where m is for lowest buckling load. Longitudinal support is generally more effective.

7) If measured grillage or panel lack-of-flatnesses were found on analysis to give rise to high bending stresses, then it would be possible to lay down lack-of-flatness standards for construction, for example, based on the avoidance of yield at the extreme design load. This would be exactly analogous to submarine hull circularity specifications.

8) Design data for buckling coefficient κ could be usefully extended as discussed in paragraph 3 of Conclusions and in Appendix I, using isotropic plate correlations.

APPENDIX I

THEORY OUTLINE

General case

Refer to Fig. 1, and to the Notation where a convenient shorthand notation is outlined for the trigonometric functions. In addition, the subscript "e" for effective stiffener-plate properties will not be used but is implied. The assumptions are given in the text and the most general buckled deformation assumed is:

$$w = w_{mn}\ S_x S_y (1 + G_1\ S_x)(1 + G_2\ S_y) \tag{1}$$

For a typical r^{th} X beam the equilibrium conditions to be satisfied at x = 0, A are:

$$\left|\frac{\partial w}{\partial x}\right| = \left|\frac{EI_x}{C_x}\ \left(\frac{\partial^2 w}{\partial x^2}\right)\right|$$

which leads to $G_1 = R_x/2\pi m$. Likewise $G_2 = R_y/2\pi n$, and so the buckled form is

$$w = w_{mn}\ S_x S_y \left(1 + \frac{R_x}{2\pi m}\ S_x\right)\left(1 + \frac{R_y}{2\pi n}\ S_y\right) \tag{2}$$

The internal strain energy V stored in the structure in the buckled form is considered to be

$V_B = V_{Bx} + V_{By}$ vertical bending of x and y stiffeners

$V_s = V_{sx} + V_{sy}$ rotation of the elastic springs

$V_T = V_{Tx} + V_{Ty}$ torsion

$V_\Gamma = V_{\Gamma x} + V_{\Gamma y}$ longitudinal warping

$V_{BZ} = V_{BZx} + V_{BZy}$ sideways bending

A typical approach is to consider flexure of a typical r^{th} beam of the x set, which will involve integration along the length, and then sum all p beams of the x set. Since the assumed buckled form in its most general case is symmetrical the results for the y set can then be written down directly by inspection. The process is illustrated as follows. Consider vertical bending of a typical r^{th} beam of the x set.

$$V_{Br} = \frac{1}{2} EI_x \int_o^A (\partial^2 w/\partial x^2)^2 \, dx \qquad \text{see footnote*}$$

$$= \frac{1}{2} EI_x \, S_r{}^2 \, (1 + R_y \, S_r/2\pi n)^2 \, (m\pi/A)^4 \int_o^A w_{mn}^2$$

$$(-S_x + R_x \, C_{2x}/\pi m)^2 \, dx$$

$$\text{where} \int_o^A = \frac{A}{2} \left[1 + \frac{8R_x}{3\pi^2 m^2} \sin^2 \frac{m\pi}{2} + \frac{R_x{}^2}{\pi^2 m^2}\right]$$

$$= \frac{A}{2} \, (K_{xm} + 13 \, R_x{}^2/16\pi^2 m^2)$$

Summing for all p beams in the x set.

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*Use of $\int \frac{M^2}{2EI} dx$ where $M = pw + \Sigma$ normal force moments, would in theory give greater accuracy, since w is represented more accurately by the assumed curve than is $\partial^2 w/\partial x^2$. However, in the case of a grillage the ensuing algebra is much more complex.

$$\therefore V_{Bx} = (\pi^4 EI_x m^4 w_{mn}^2/4A^3)(K_{xm} + 13 R_x^2/16\pi^2 m^2)$$

$$\sum_{r=1}^{p} S_r^2(1 + R_y S_r/2\pi n)^2$$

It is convenient to transform the summation into trivial integral forms. An intuitively correct transform is

$$\sum_{r=1}^{p} \sin^c\left(\frac{n\pi r}{p+1}\right) = \frac{p+1}{B}\int_o^B \sin^c\left(\frac{n\pi y}{B}\right) dy \qquad (3)$$

where n and c have positive integer values. The conditions under which this identity is satisfied are examined in Appendix II, where it is found to be applicable to the present problem with one proviso, which will be discussed later. The relevant summations are also quoted there, which leads to

$$\sum_{r=1}^{p} = \frac{p+1}{B}\int_o^B S_y^2(1 + R_y S_y/2\pi n)^2\, dy$$

$$= \frac{p+1}{2}[1 + (8R_y/3\pi^2 n^2)\sin^2 n\pi/2 + 3R_y^2/16\pi^2 n^2]$$

$$= (p+1)K_{yn}/2$$

Hence, the total bending strain energy for the x set of beams is

$$V_{Bx} = \pi^4 EI_x AB(m^4 w_{mn}^2/A^4 8b)[K_{xm} + 13R_x^2/16\pi^2 m^2]K_{yn} \qquad (4)$$

The K terms depend upon the rotational constraints R_x, R_y and are defined in the notation. By symmetry the equation for the y set is obtained by replacing x,m,A,B,b with y,n,B,A,a respectively. Henceforth, only the basic equations and solutions will be quoted for the strain energy and work equations. For spring energy

$$V_{sx} = \sum_{r=1}^{p} C_x \left(\frac{\partial w}{\partial x}\right)^2_{y=rb,x=o}$$

$$= \pi^4 \, EI_x AB(m^4 w_{mn}^2/A^4 8b) \; [4R_x/\pi^2 m^2] K_{yn} \qquad (5)$$

as expected this is of the same form as equation (4) for the bending strain energy of the x beams. The difference lies only in the last bracket terms. Inserting a typical value $R_x = 2\pi$ for the dimensionless spring stiffness (see Appendix III) these bracket terms are 3.70 and 2.55 for equation (4) and (5) respectively, and so it can be seen that the spring strain energy is a sizeable proportion of the beam bending energy. Furthermore, in many ships, and certainly in warship decks, the Y beam bending energy is appreciably larger than any other component of total strain energy; and so it follows that the spring energy absorbed in the structure surrounding the unloaded edges cannot be ignored.

Adding (4) and (5), and then Y-beam equivalents

$$V_B + V_s = \frac{\pi^4 E}{8AB} \sqrt{\frac{I_x \, I_y}{a \; b}} \; w_{mn}^2 \left[\frac{m^4 K_{1xm}}{\alpha_o^2} + n^4 \alpha_o{}^2 K_{1yn}\right] K_{xm} \, K_{yn} \qquad (7)$$

and we now start to recognise a similar form to the orthotropic plate bending terms.

For torsion

$$V_{Tx} = \sum_{r=1}^{p} \int_{o}^{A} \frac{1}{2} GJ_x \, (\partial^2 w/\partial x \partial y)^2_{y=rb} \, dx$$

$$\simeq \frac{\pi^4 G}{8AB} \left[\frac{\bar{p} J_x}{b}\right] (w_{mn} mn)^2 \, K_{xm} \, K_{yn} \, K_{2xm} \, K_{2yn} \qquad (8)$$

It can be shown [13] that when there is an enforced axis of rotation the longitudinal warping constant Γ can be considered together with sideways bending.

$$V_{BZx} = \sum_{r=1}^{p} \int_{o}^{A} \frac{1}{2} EI_{xz} \, (\partial^2 \bar{v}/\partial x^2)^2_{y=rb} \, dx$$

where $\bar{v} = \bar{Z}_x \, (\partial w/\partial y)_{y=rb}$

Hence, it follows

$$V_{BZx} + V_{\Gamma x} = \sum_{r=1}^{p} \int_{o}^{A} \frac{1}{2} E(I_{xz} \, \bar{Z}_x^2 + \Gamma_x)(\partial^3 w/\partial x^2 \partial y)^2_{y=rb} \, dx$$

$$\simeq \pi^6 E \, (I_{xz} \, \bar{Z}^2 + \Gamma_x)(w_{mn}^2 \, m^4 n^2/A^3 B \, 8b) \, K_{xm} \, K_{yn} \, K_{2xm} \, K_{2yn} \qquad (9)$$

Adding (8) and (9) together with the corresponding Y beam components, and rearranging we obtain

$$V_T + V_{BZ} + V_\Gamma \simeq \frac{\pi^4 E}{8AB} \sqrt{\frac{I_x \; I_y}{a \; b}} \, w_{mn}^{\;2} \, [2\Gamma_{xy} m^2 n^2] K_{xm} \, K_{yn} \, K_{2xm} \, K_{2yn} \qquad (10)$$

The approximation sign arises from ignoring $2/K_{xm}(q+1)$ and $2/K_{yn}(p+1)$ in the twisting terms as being small compared with K_{2x} and K_{2y} respectively. Also $3R_x{}^2/4\pi^2 m^2 K_{xm}$ and $3R_y{}^2/4\pi^2 n^2 K_{yn}$ have been ignored in the sideways bending and longitudinal warping terms. There are negligible approximations in the vertical bending terms* which dominate the potential energy in grillages with open section stiffeners. In taking advantage of this grouping of terms that vary in exactly the same way with elastic constraints R_x, R_y (same grouping of the K's) it should be recognised that the tripping-torsion constant Γ_{xy} contains tripping rigidities J_{xt} and J_{yt} which contain the mode integers m and n. For example

$$J_{xt} = J_x + \pi^2 m^2 E(I_{xz}\,\overline{Z}_x{}^2 + \Gamma_x)/GA^2$$

It would be convenient to consider the dimensionless twisting rigidity Γ_{xy} as a constant independent of wave length, and so numerical studies were made with open and closed section stiffeners on typical grillages. These showed that the last term in J_{xt} had only a very small effect on the value of m for the lowest buckling load. As m has to take the nearest integer value to that derived algebraically (by the usual process of minimising) it therefore follows that a small error

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*There is however one odd condition to take note of when the Y beams lie on nodal lines, and this is discussed in Appendix II and in the main body of the report. If p becomes very small, errors will occur in the vertical bending terms of the x set, and for $p \leq 3$ ref. [1] should be consulted when $K_y = 0$.

in the choice of m for evaluating J_{xt} will almost certainly not affect the final value of m for the lowest buckling load. Moreover, experience indicates a lack of sensitivity of buckling load to mode number in the narrow region of change from one value of m to another, so an incorrect choice therefore is not important in such cases. The solution was therefore allowed to proceed assuming Γ_{xy} independent of m. The algebraic expression so derived for the value of m for the lowest critical load is then used for computing J_{xt} and hence Γ_{xy}. The procedure then becomes at worst a first iteration, which may upset the purist, but is justified in the authors view by its convenience to designers with negligible loss in accuracy. The same argument would of course arise for J_{yt} in the general case when there are n half waves across the grillage. However, we may anticipate our main interest when n=1 and so no problem arises.

Neglecting the elastic compression of the beams, the external works done in compression over the whole effective cross-sections are, with the usual small deflection approximation, given by:

$$W_x = \sum_{r=1}^{p} \int_0^A \sigma_x b t_{xe} \frac{1}{2} (\partial w/\partial x)^2_{y=rb} \, dx$$

$$W_y = \sum_{s=1}^{q} \int_0^B \sigma_y a t_{ye} \frac{1}{2} (\partial w/\partial y)^2_{x=sa} \, dy$$

Adding

$$W = (\pi^2 w_{mn}^2/8AB)[\sigma_x m^2B^2t_{xe} K_{2xm} + \sigma_y n^2A^2t_{ye}K_{2yn}]K_{xm} K_{yn} \tag{11}$$

By Rayleigh's principle [27] the practical test for stability is that for all possible deformations $V \geq W$. It follows therefore that equating these two energies will provide an upper bound to the critical loads; or exact solutions in the unlikely event of the assumed buckled form being exact. Equating (11) with the sum of (7) and (10) leads to the relationship for the critical stresses

$$\sigma_{xGE(y)}[1 + \frac{n^2A^2 \; t_{ye} \; K_{2yn}}{m^2B^2 \; t_{xe} \; K_{2xm}} (\frac{\sigma_y}{\sigma_x})] = \sigma_{xGE} \tag{12}$$

where

$$\sigma_{xGE} = \frac{1}{2} \sigma_{op}[K_{1xm} m^2/\alpha_o^2 + 2n^2\Gamma_{xy} K_{2xm} K_{2yn} + K_{1yn}n^4\alpha_o^2/m^2]/K_{2xm} \tag{13}$$

At first sight the terms may appear to have lost their symmetry, but this is not so. The arrangement focuses attention on σ_x rather than σ_y, and equation (13) in fact is the general buckling stress σ_{xGE} when $\sigma_y = 0$. The use of subscript (y) in $\sigma_{xGE(y)}$ is simply to denote the critical value of σ_x in the presence of σ_y, and we know that in most if not all ship cases σ_y will be small compared with σ_x. Note, the K_1 and K_2 terms (m=n=1) are plotted against rotational constraint R in Fig. 2.

Minimizing eq. (12,13) with regard to m and n for any given combination of σ_x, σ_y would provide two eighth order equations in m and n to solve for finding the lowest critical loads. However, it seems reasonable to suppose from the isotropic plate buckling analogy, that perhaps either one of m or n is always unity, or both are = 1. This is indeed so, though the mathematical proof is complex. However, the truth of the statement can be demonstrated more easily for the cases of practical interest where either σ_x or σ_y predominate over the other. In ship structures generally $\sigma_x \gg \sigma_y$, and so the square bracket term in eq. (13) is the one of interest. It will be noted that n is everywhere in the numerator except for certain terms in K_{2yn}. However, the effect of the K_2 terms is very weak compared with the K_1 terms. This follows from their limited range from 1(pinned) to 1.33 (clamped), as compared with 1 to 5.33 for K_1. Thus the product $n^2 K_{2yn}$ in the middle term will be dominated by the n^2 in the numerator terms. Hence, it follows by inspection that in this case of small σ_y n will always be unity.

Noting also that K_{2xm} is near unity and hardly sensitive to m it follows that at least as a very good first approximation to finding the critical value of m when $\sigma_y = 0$, we only require to minimize

$$\frac{m^2}{\alpha_o^2} K_{1xm} + \frac{\alpha_o^2}{m^2} K_{1y1} \qquad (14)$$

We can examine the nature of K_{1xm} by rearranging it using the unmoded coefficients K_{1x1} and K_{x1} which are independent of m, that is (see notation):

$$K_{1xm} = 1 + (K_{1x1} - 1)\ K_{x1}/(m^2 + K_{x1} - 1)$$

or

$$= (K_{1x1}\ K_{x1} + m^2 - 1)/(K_{x1} + m^2 - 1)$$

we see that

a) at low values of constraint R_x, K_{1x1} reduces to unity and so K_{1xm} tends to unity, irrespective of m; at higher values of R_x, K_{x1} dominates since it tends to infinity as R_x does, and so K_{1xm} tends to K_{1x1} irrespective of m.

b) Moreover, the m^2 terms in K_{1xm} are in the denominator and will tend to be weak in relation to the m^4 obtained by cross-multiplying.

This rather suggests that a very good approximation to the critical value of m is given by treating K_{1xm} as independent of m, i.e. $\approx K_{1x1}$. Inserting this in eq. (14) and differentiating in the usual way, we obtain

$$m = \alpha_o \sqrt[4]{K_{1y1}/K_{1x1}} = \alpha_c \qquad (15)$$

For convenience, we shall define this as the constrained orthotropic plate ratio α_c, and its significance is that for values of $\alpha_c > 1$ we may expect buckling in the (m,1) mode.

For $\alpha_c \leq 1$ we may expect the (1,1) mode, except where $\sigma_y > \sigma_x$ in which case the (1,n) mode is possible. This will not be considered since it is of no practical interest.

In using eq. (15) the nearest integer value (increased preferred) should be used. It can be substituted directly in eq. (13) to evaluate the lowest value of critical stress σ_{xGE} and as a check values either side can be investigated to make sure the lowest value has indeed been found. This would also guard against any approximations involved in eq. (15). Providing $\alpha_c \geq 1$ substituting m from eq. (15) directly into eq. (13) provides a lower bound tangent to the family of buckling coefficients in exactly the same way as for isotropic plates. This yields for $\alpha_c \geq 1$

$$\min \sigma_{xGE} \simeq \sigma_{op}[(K_{1xm} K_{1y1})^{\frac{1}{2}}/K_{2xm} + \Gamma_{xy} K_{2y1}] \qquad (16)$$

though as just described a more exact solution is obtained by substituting the nearest integer value of eq. (15) in eq. (13).

To find from eq. (12,13) the critical stress under biaxial conditions with σ_y present, it can be shown by analogy with isotropic plate results for biaxial buckling [1,42] that providing σ_y/σ_x is small (≤ 0.3 is suggested) the value of wave number for the lowest buckling stress is given approximately by

$$m_{(y)} \simeq \alpha_c \sqrt{1 - 2\sigma_y/\sigma_x}$$

The nearest integer value should be substituted in eq. (12,13) to find the lowest $\sigma_{xGE(y)}$.

Accuracy checks

Two special cases 1, 2 when $\sigma_y = 0$ have been discussed in the main body of the report. They are not only important practically, but they also provide certain checks on the accuracy of the theory as follows:

Case 1, $R_x = 0$, $\alpha_c \geq 1$.

a) When $R_y = 0$, $\sigma_{xGE} = \sigma_{op}(1 + \Gamma_{xy})$, which is the well known result for a pinned orthotropic plate. Putting $\Gamma_{xy} = 1$, $EI_x/b = EI_y/a = D$, and $t_x = t$, we obtain the well known result for long pinned isotropic plates $\sigma_{PE} = 4\pi^2 D/tB^2$.

b) When $R_y \neq 0$, but $\Gamma_{xy} = 1$, $EI_x/b = EI_y/a = D$, and $t_x = t$ we obtain for the isotropic $\sigma_{PE} = 2\pi^2 D(K_{1y1} + K_{2y1})/tB^2$ where K_{1y1} and K_{2y1} are given by the equations in the notation, but where R_y is now given by CB/D where C is the moment/slope elastic constraint per unit length of the unloaded edge. σ_{PE} has been evaluated for the complete range of R_y from 0-∞ and found to agree within 5 per cent with the theory given in references [15] and [31] being always slightly greater as expected using Rayleigh's theory. This check is considered to be most searching, and suggests that the assumed buckling form is a good one. In the particular case of $R_y \to \infty$, that is unloaded edges clamped, we obtain $\sigma_{PE} = 6.74\ \pi^2 D/tB^2$ which agrees within about 3 per cent with the well known buckling coefficient 6.97 for long plates with all edges clamped.

c) When $\Gamma_{xy} = 0$, we obtain for the orthotropic plate having zero torsional rigidity, and having flexural rigidities EI_x/b and EI_y/a, $\sigma_{xGE} = K_{1y1}\,\sigma_{op}$. This tends to $2.31\,\sigma_{op}$ as $\beta \to \infty$ which agrees well with the known result $\sqrt{5}\sigma_{op} = 2.24\,\sigma_{op}$ for a long orthotropic plate when all edges are clamped.

d) Putting $m = q + 1$ and ignoring Γ_{xy} we obtain the Euler column buckling stress for buckling of the X beams between the Y beams.

<u>Extension of correlations</u>

In the main body of the report a reference is made in Case 1 to Smith-Wittrick correlations between orthotropic plate and isotropic plate buckling coefficients using $\alpha_o \equiv \alpha$. When $R_x = 0$ no difficulty arises since α_c then $= \alpha_o\, 4\sqrt{K_{1y1}}$ and is not a function of wave length.

With $R_x = 0$ the constrained orthotropic plate aspect ratio should strictly be $\alpha_c = \alpha_o\,[K_{1y1}/K_{1xm}]^{1/4}$ for correlation purposes and therefore depends upon m since K_{1xm} contains m. Hence a trial and error procedure must be used to determine κ. Then the intermediate curves could be drawn in Fig. 3 between $R_x = 0$ and $R_x = \infty$ (this latter clamped case is again independent of m since $K_{1xm} = 5.33$ for all m). The data could then be plotted against $\alpha_c = \alpha_o\,[K_{1y1}/K_{1x1}]$ to provide easily used design data.

Case 2, $R_y = 0$, $\alpha_c < 1$

a) For the isotropic plate substitute $\sigma_{op} = 2\pi^2 D/tB^2$, $\alpha = A/B$ and $\Gamma_{xy} = 1$ to give

$$\sigma_{PE} = (\pi^2 D/tB^2)[K_{1x1}\ B^2/K_{2x1}\ A^2 + 2 + A^2/K_{2x1}\ B^2]$$

and this has been found to give reasonable agreement with an isotropic plate analysis [15]. In the particular case when $A/B \to 0$, $\sigma_{PE} = (K_{1x1}/K_{2x1})\pi^2 D/tA^2$ which agrees within 2 per cent with the exact solution for an elastically constrained wide plate or strut [29]. As $\alpha \to \infty$, $K_{1x1}/K_{2x1} \to 4$, which leads to $\sigma_{PE} = 4\pi^2 D/t^2 A^2$ the correct solution for a clamped wide plate.

b) For the special but practically important case of a pxo plated grillage we may take $EI_y/a = D$ and $GJ_{ty}/2a = D$ the flexural and twisting rigidities for an isotropic plate and obtain

$$\sigma_{xGE} = \sigma_{CE}\ [\frac{K_{1x1}}{K_{2x2}} + \frac{A^2 Db}{B^2 EI_{xe}}\ (\frac{GJ_{xt}}{Db} + 2 + \frac{A^2}{B^2 K_{2x1}}\ )] \qquad (20)$$

when the ends are simply supported $K_{1x1} = K_{2x1} = 1$ and the expression reduces to one given in reference [30] except that the torsional rigidity GJ_{xt} of the stiffeners is ignored. Equation (20) should only be used for $A^2 Db/B^2 EI_x \leq 1$. As this ratio or $A/B \to 0$ the expression degenerates for a "wide" pxo grillage to $\sigma_{CE}\ (K_{1x1}/K_{2x1})$ which agrees within 2 per cent with the exact solution for an elastically constrained strut [29].

In addition to these checks, calculations were also carried out for certain ship decks and checked against a more exact computer solution [5] and agreement was everywhere within 10 per cent. The explicit relations which have been derived are therefore felt to be sufficiently reliable for most design purposes.

APPENDIX II

VALIDITY OF IDENTITY

It is required to examine under what conditions the following identity holds true

$$\sum_{r=1}^{p} \sin^c\left(\frac{n\pi r}{p+1}\right) = \frac{p+1}{b}\int_o^B \sin^c\left(\frac{n\pi y}{B}\right)dy$$

where c is the positive integers 1,2,3 and 4

<u>Summations</u>

Let $S_c = \sum_{r=1}^{p} \sin^c r\theta$ where $\theta = \frac{n\pi}{p+1}$

$$S_1 = \sum \sin r\theta, \quad C_1 = \sum \cos r\theta$$

$$C_1 + i\,S_1 = \sum e^{ir\theta} = \frac{e^{i\theta} - e^{(p+1)i\theta}}{1 - e^{i\theta}}$$

$$= \frac{e^{i\theta} - e^{ni\pi}}{i - e^{i\theta}}$$

$$= -1 \quad n \text{ even}$$

$$= \frac{1 + e^{i\theta}}{1 - e^{i\theta}} = -1 + \frac{2}{1 - e^{i\theta}} \qquad n \text{ odd}$$

Hence $S_1 = 0$ $\qquad$ n even

$$= \frac{2\sin\theta}{(1 - \cos\theta)^2 + \sin^2\theta}$$

$$= \cot \theta/2 = \cot n\pi/2(p+1) \qquad n \text{ odd}$$

$$\simeq 2(p+1)/n\pi \qquad \text{for large } p$$

$$S_2 = p/2 - 1/2 \sum_{r=1}^{p} \cos 2r\theta$$

But $C_1 = -1$ n even, $= 0$ n odd

$\therefore$ since 2n is even $S_2 = (1/2)(p+1)$ all n.

$$S_3 = 3/4 \sum_{r=1}^{p} \sin r\theta - 1/4 \sum_{r=1}^{p} \sin 3r\theta$$

$$= 0 \qquad n \text{ even}$$

$$= (3/4) \cot n\pi/2(p+1) - (1/4) \cot 3n\pi/2(p+1) \qquad n \text{ odd}$$

$$\simeq 4(p+1)/3n\pi \qquad \text{for large } p$$

Now $\sin^4\theta = (\cos 4\theta - 4\cos 2\theta + 3)/8$

$$\therefore \quad S_4 = \left(\sum_{r=1}^{p} \cos 4r\theta - 4 \sum_{r=1}^{p} \cos 2r\theta + 3p\right)/8$$

$$= (-1 + 4 + 3p)/8$$

$$= 3(p+1)/8 \qquad \text{all } n$$

Integrals

These are standard and the results are quoted.

Let $I_c = \int_0^B \sin^c(n\pi y/B)\,dy$

$$I_1 = 2B/n\pi \quad n \text{ odd}, \quad = 0 \quad n \text{ even}$$

$$= (2B/n\pi) \sin^2(n\pi/2) \qquad \text{all } n.$$

$I_2 = B/2$ all n

$I_3 = 4B/3n\pi$ n odd, $= 0$ n even

$= (4B/3n\pi)\sin^2(n\pi/2)$ all n

$I_4 = 3B/8$ all n

For cosine functions the corresponding integrals for all n are $I_1 = 0$, $I_2 = B/2$, $I_3 = 0$, $I_4 = 3B/8$.

Summarizing the comparisons the identity is exactly true unless c and n are both odd (or unless n/(p+1) is integer in which case the summation is zero, and this case is discussed later in this appendix). It is approximately true in the latter case if cn/p + 1 is small. Applying these findings to the theory outlined in appendix I, it is found that although n is usually odd (unity), c is usually even (2 or 4), and so the identity is then exact. Fortunately where c is also odd (e.g. = 3 for one of the three terms in the strain energy of bending of the X beams in Case 1), it will be found that p refers to the number of longitudinal stiffeners, and in most ship grillages p + 1 is usually large compared with c(=3). So the identity is then nearly true leading to a small acceptable error in that particular term. As it is one of many terms the overall error may be assumed to be negligible. For $p \leq 3$ reference [1] should be consulted when $R_x = 0$.

If n/(p+1) is an integer the left hand side of the identity is always zero. The right hand side will only be zero if c is odd and n even and so the identity is not then generally true.

In the present analysis this situation only arises in the bending of the Y beams and their end springs when c=2 and these beams form node lines for the buckled deformation, viz. $m/(q + 1)$ is an integer. In this case V_{By} and V_{Sy} are both zero, and so therefore is the last term in equations (13,17) for the buckling stresses. This term should then be omitted, and the lowest buckling stress may then be given by putting $m = q + 1$ in the first terms.

For the twisting of the Y beams the summations involve cosine^2 functions. We can make use of previous results by expanding as follows:

$$\sum_{s=1}^{q} \cos^2 \frac{m\pi s}{q + 1} = \sum_{s=1}^{q} (1 - \sin^2 \frac{m\pi s}{q + 1})$$

$$= q - \frac{q + 1}{2} \text{ in general}$$

$$= \frac{q - 1}{2}$$

$$\text{or } = q \text{ when } \frac{m}{q + 1} \text{ is integer,}$$

in which case the $\sin^2$ summation is zero. Thus

$$\sum_{s=1}^{q} \cos^2 \frac{m\pi s}{q + 1} = (q + 1)\,\bar{q},$$ where $\bar{q}$ is a modifying factor applied to the twisting rigidity of the Y stiffeners which becomes important for small values of q, the number of Y beams. Cos^2 terms also occur in the twisting of the X beams, but in most ship cases no $\bar{p}$ modification is necessary. This is partly because p is large and hence $\bar{p} = (p - 1)/(p + 1)$

tends to unity, and also because in this case it only occurs in one rather minor term of three. Assuming $\bar{p} = 1$ therefore incurs negligible error, and is certainly convenient.

APPENDIX III

ESTIMATION OF CONSTRAINTS

General remarks

The dimensionless rotational constraints R_x and R_y at the ends of the X and Y beams are defined as

$$R = C\ell/EI$$

where ℓ and EI refer to the beam considered and C = moment/slope, elastic stiffness provided by the adjoining structure and the connections thereto. It has been shown [43] that well designed welded stiffener connections suffer negligible body strains and faithfully transmit rotation, at least until the members connected suffer permanent damage. They may therefore conveniently be assumed to be "rigid". For orthogonal structures C will then depend upon the

(a) flexural stiffnesses of connected beams in the plane of bending

(b) torsional stiffnesses of connected beams perpendicular to the plane of bending

(c) "sway" at stiffener connections in the plane of bending

(d) moment/slope conditions at remote boundaries of the adjoining structure.

Torsional rigidities for open section stiffeners are very much smaller than flexural rigidities (1/500 is a typical ratio in Naval ships), and so (b) can be then ignored. Torsion effects may become significant where heavy closed sections occur near the beam ends, and their contribution to C can be estimated quite quickly using the St. Venant theory. Ignoring torsion is of course conservative in the present context and so the analysis proceeds on this assumption. In orthogonal inter-connected plated structures, such as those found in ships "sway" can be ignored, due to the high in-plane rigidity of plating and egg-box type construction. Lastly, assumptions made at the outer boundaries can be shown to have only a very minor effect on the constraints at the grillage being considered, even if these boundaries are only one or two structural elements away from the grillage.

It is reasonable therefore to concentrate on (a) and a method of evaluating R_x, R_y has been developed by the author [13] which is in a sense the antithesis of moment distribution in that it works invwards from the outer boundaries to the edge of the grillage. The solution is explicit and by making the assumption that we can idealise the boundary of the adjoining structure to one or at most two joints away from the grillage, the boundary constraints are quickly determined. Elastic small deflection beam theory is assumed throughout.

Stiffness of a single beam

Refer to figure 5 which shows the most general orthogonally connected adjoining structure joined to and lying in the plane of a typical grillage beam. Consider first of all a typical "second remove" beam ji having an elastic rotational spring C_j connected to end j. If end i is acted upon moment M_i this will cause a rotation θ_i in the same sense whose value can readily be shown to be given by

$$\left.\begin{aligned} \frac{M_i}{\theta_i} &= 4K_j \frac{EI_j}{\ell_j} \\ \text{where} \quad K_j &= \frac{k_j + 3}{k_j + 4} \\ k_j &= \frac{C_j \ell_j}{EI_j} \end{aligned}\right\} \qquad (24)$$

When $C_j = o$ end j is pinned and we obtain $M_i/\theta_i = 3EI_j/\ell_j$; when $C_j = \infty$ end j is clamped and the coefficient is 4. Both results are well known. Thus even in a single beam a wide variation in constraint at one end causes only a small difference in the rotation at the other. It follows that this influence will be appreciably further reduced at the ends of any beams joined to i.

Multi-connected beams

If we consider now a typical "first remove" beam io, then the rigid joint assumption implies that all members meeting at end i share the same rotation θ_i. It therefore follows from statics that the total moment exerted by beams i1, ij and i3 at the end i of beam io, is

$$M_i = 4 \sum_{j=1}^{3} K_j \frac{EI_j}{\ell_j} \cdot \theta_i$$

Therefore, the rotational spring exerted by the structure connected to end i of beam io has a stiffness $C_i = M_i/\theta_i$ given by

$$C_i = 4 \sum_{j=1}^{3} K_j \frac{EI_j}{\ell_j} \qquad (25)$$

Extending this process to all the beams connected to end o of the grillage, it therefore follows that the rotational spring provided by all adjoining structure at end o of the grillage beam is given by

$$C_o = 4 \sum_{j=1}^{3} K_i \frac{EI_i}{\ell_i}$$

where

$$K_i = \frac{k_i + 3}{k_i + 4}$$

$$k_i = \frac{C_i \ell_i}{EI_i} \qquad (26)$$

and C_i is given by the summation (25) for the second remove members.

Approximation

Analysis of many ship structures has shown that K_j, which has a range of 3/4 when pinned to 1 when clamped, is typically about 0.9 ($k_j = 6$). By assuming this value at second remove joints (or 0.75 or 1 where conditions are known to be pinned or clamped precisely) negligible error will arise in calculating the rotational stiffness C_o. Indeed, calculations for even moderately redundant frameworks show that high accuracy (well within 5 per cent) can be obtained even if the first remove joints are taken as the boundary of the adjoining structure with $K_i = 0.9$. The "limited-frame" concept recommended recently [44] considers these joints to be fixed, which is slightly optimistic.

We can therefore conclude that for most purposes it is sufficient to consider only first remove members which are directly connected in the plane of bending with the grillage beams, and which have rotational springs $K_i = 0.9$ at their first remove joints. Then

$$C_o = 3.6\ E \sum_{i=1}^{3} \frac{I_i}{\ell_i} \tag{27}$$

and $$R_x, R_y = \frac{C_o}{EI} = 3.6\ \frac{\ell}{I} \sum_{i=1}^{3} \frac{I_i}{\ell_i} \tag{28}$$

Where highest accuracy is required, the stiffnesses of the second remove members can be considered and equations (26) and (25) should be used where K_j is taken = 0.9 (or 0.75 or 1 where ends j are known to be virtually pinned or clamped). When considering second remove joints it is possible for the members adjoining one side of the grillage to form closed cells with those from the other side. In such cases the second remove members will be common to both sides, but little loss in accuracy will result if this is ignored and the calculations for both sides of the grillage beams proceed independently.

In applying the method to the calculation of R_x (or R_y) for calculating general instability stresses it must be remembered that these constraints have to be equal at both ends of the beam for all beams of the set. In this symmetrical situation the relation between C and f, the "degree of fixity", as used for example in references [3] and [12], is

$$f = \frac{k}{k+2}\ , \text{ where } k = \frac{C\ell}{EI}$$

Ship's beams have typical k values in the range 4 to 10, which indicate constraints nearer clamped than simply supported.

<u>Effect of axial compression</u>

The analysis thus far assumes the adjoining structure provides constant linear springs C_x, C_y to the grillage boundaries which are unaffected by the magnitude of the axial force. Clearly, if the adjoining structure was itself also liable to buckle in an overall manner then the constraint C it could offer to the grillage would diminish as the axial stresses approached the critical stress σ_{cr} for the adjoining structure. The relative wave lengths are also important. By analogy with results obtained from the analysis for flange and wall buckling of isotropic plate box girders in bending, and for struts supported at a variety of spacings, it would appear to be reasonable and a little conservative to reduce C_y to C_{yr} by the linear interaction formula

$$C_{yr} = C_y \left(1 - \frac{\sigma_x}{\sigma_{xGE}}\right) \qquad (29)$$

where σ_x is the stress in the adjoining structure, and σ_{xGE} its buckling value assuming edges pinned. This will slightly invalidate the assumption of spring linearity used when evaluating internal strain energy in Appendix I, but by ignoring this the buckling stress would if anything be under-estimated slightly. The agreement with another more complex reduction factor was shown to be very good for vibration problems [16].

If, for example, the main structure was a deck, and the adjoining structure were deck side extensions or the side walls of the ship or superstructure, σ_{xGE} could be conservatively estimated by assuming the unloaded edges to be simply supported and that the edge stress was uniform across extensions or the walls. With similar frame spacing to the Y beams of the grillage this would for "long" structures ($\alpha_o \geq 1$) lead approximately to

$$\frac{C_{yr}}{C_y} \simeq 1 - \left[\frac{B_A}{B}\right]^2 \sqrt{\frac{I_y}{I_{yA}}} \qquad (30)$$

where B_A is the width of the adjoining grillage and B the width of the grillage being considered.

For calculating C_x it would in general be unwise to assume any constraint provided by similar structure lying in the plane of compression. R_x should be determined only for those adjoining stiffeners which lie normal to this plane and which have no tendency to buckle.

In certain cases the tendency of the adjoining structure to general instability may be so strong that, but for their attachment to the grillage, they would become unstable at a stress lower than the buckling stress for the grillage with sides simply supported. In these cases the grillage restrains the adjoining structure. The value of R_y is then negative and the buckling stress is reduced below its value for simple supports. This is allowed for using the interaction formula. Negative values of R_y are permissible in the equations and notation used for evaluating K_{1y1} and K_{2y1} used for evaluating σ_{xGE}.

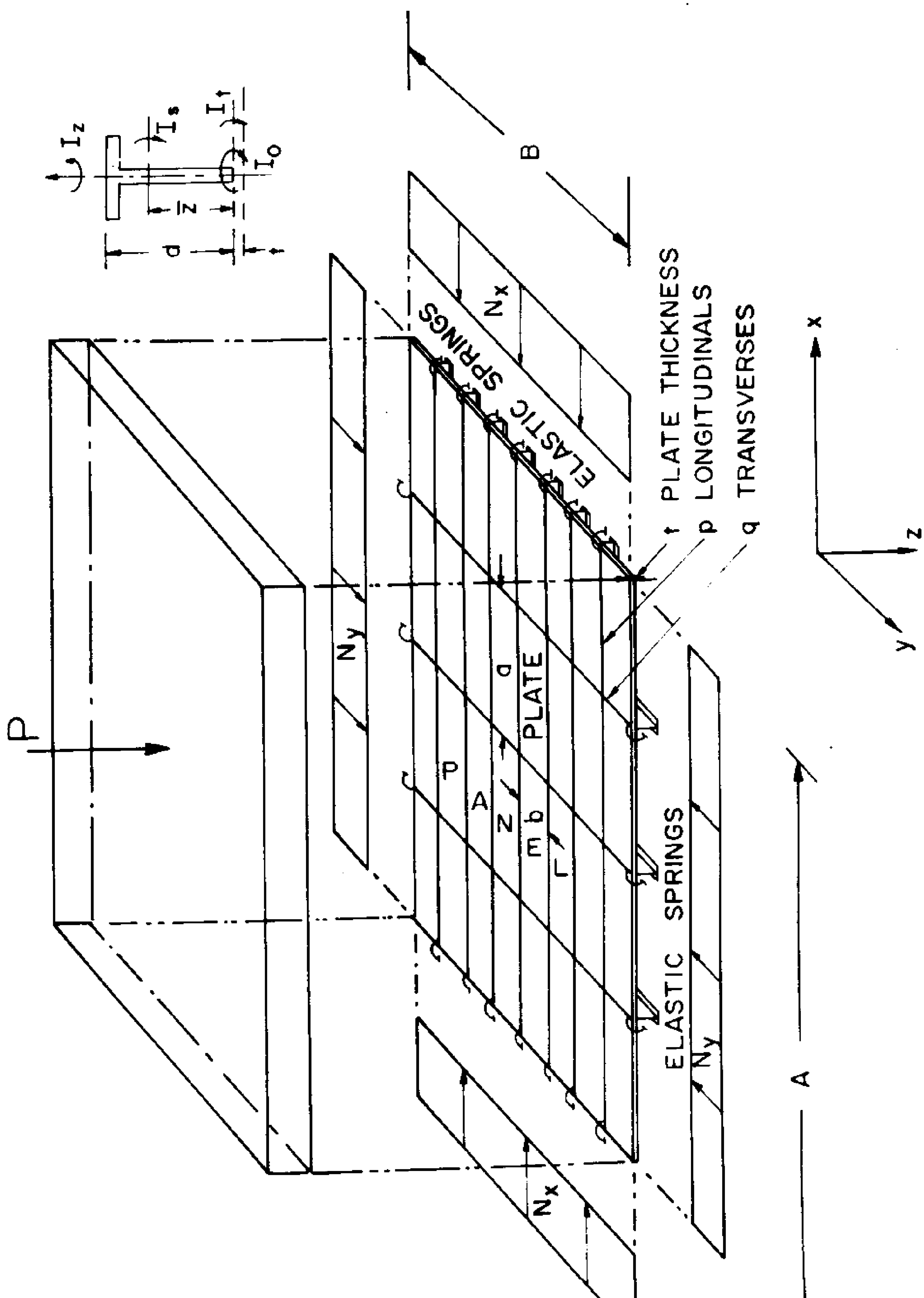


FIG. 1 GRILLAGE IDEALIZATION AND LOADS

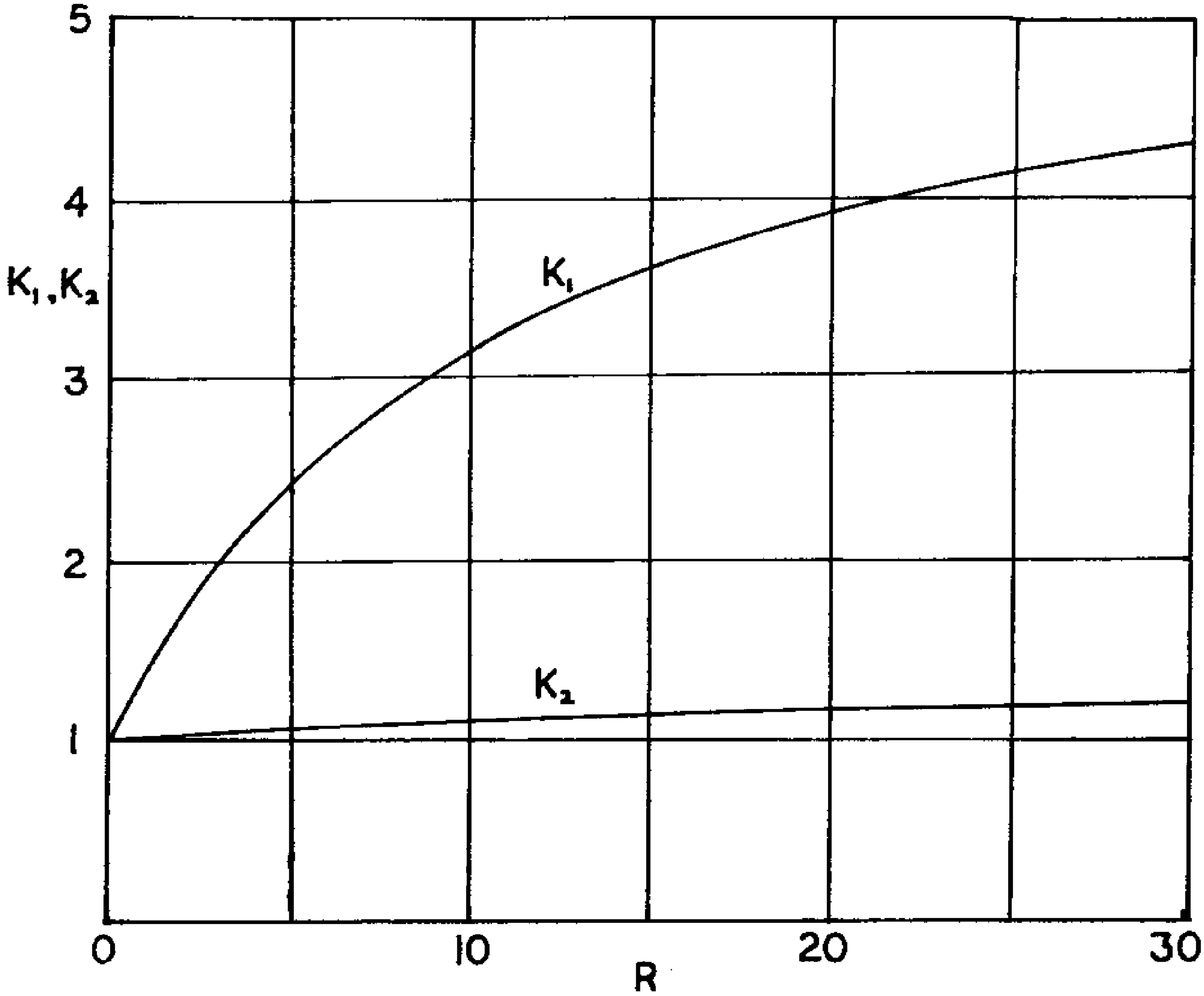
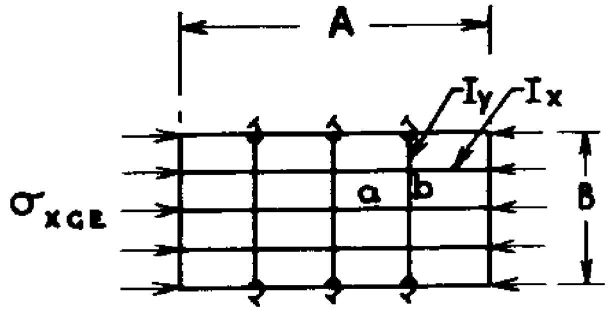


FIG. 2 VARIATION OF K_1 and K_2 WITH R



$$\sigma_{xGE} = \frac{2\pi^2 E}{t_x b^2}\sqrt{\frac{I_x I_y}{ab}}\left\{\sqrt{K_{1y1}}\,k + K_{2y1}\Gamma_{xy}\right\}$$

PROVIDING $\frac{m}{q+1}$ IS NOT INTEGER

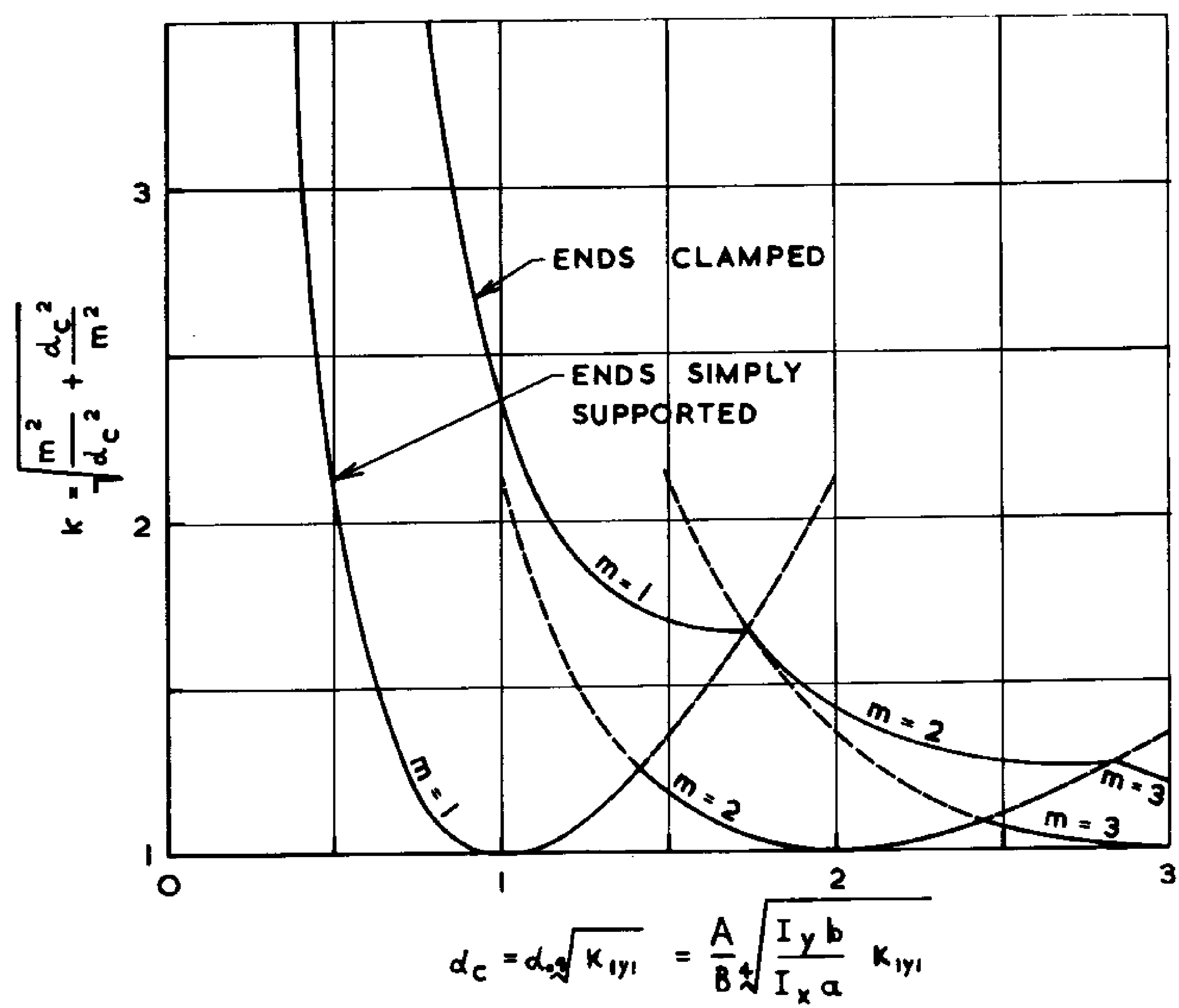


FIG. 3 VARIATION OF κ WITH α_c AND m

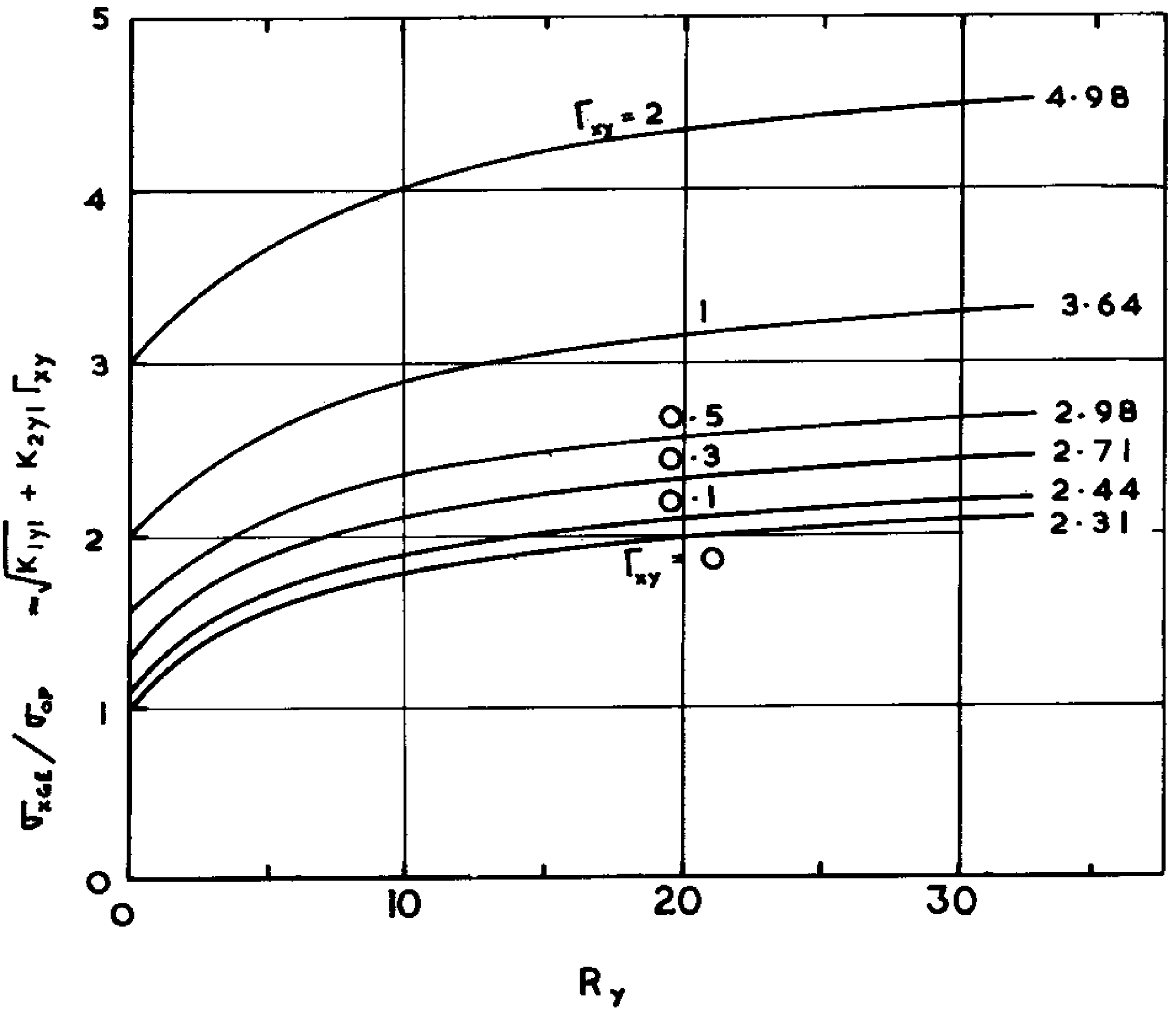


FIG. 4 VARIATION OF min σ_{xGE} WITH

R_y AND Γ_{xy} FOR CASE 1

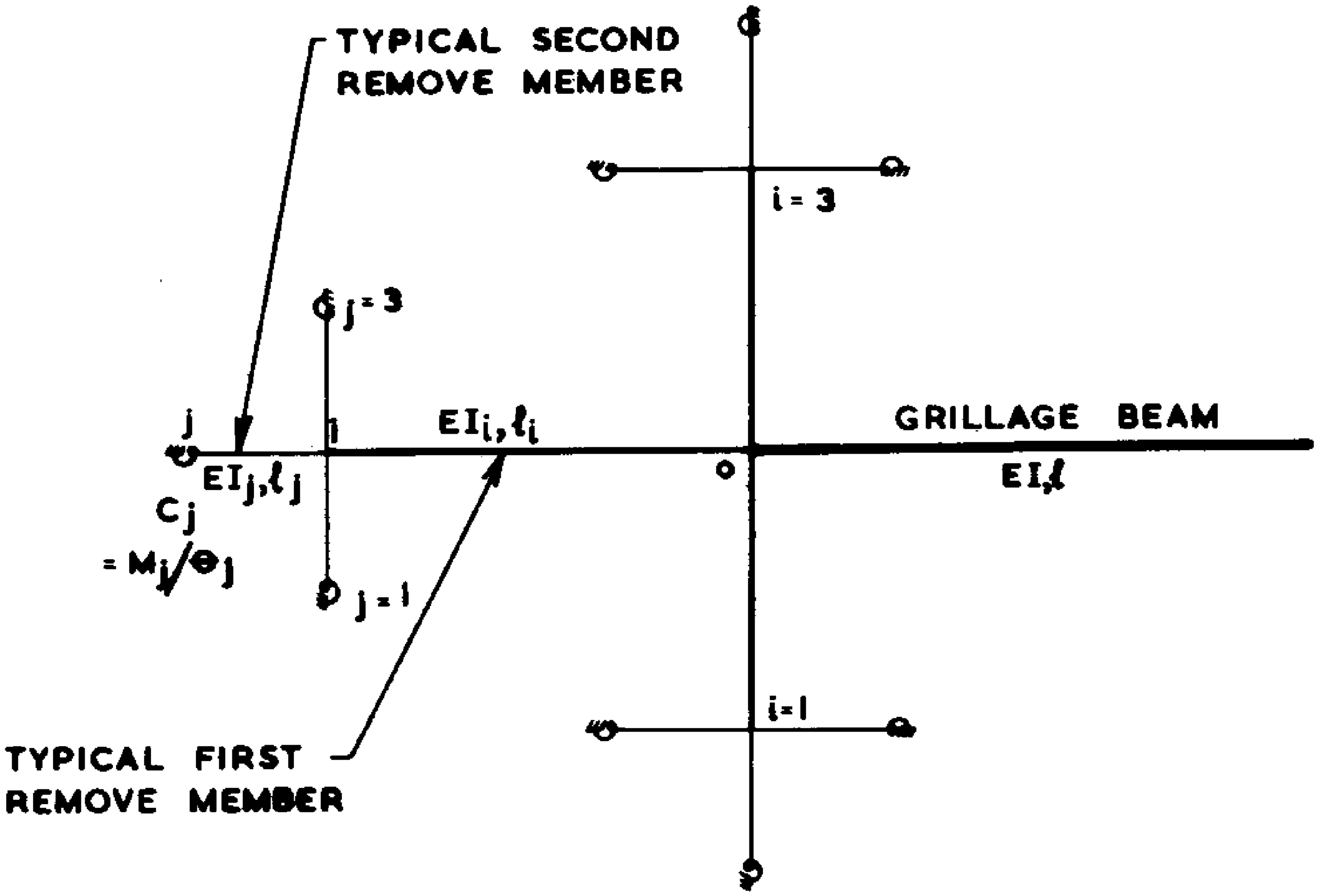


FIG. 5 DEFINITIONS FOR MULTI-CONNECTED BEAMS

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