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Forced Axisymmetric Vibrations of Underwater Spherical Shells

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Abstract

Axisymmetrical free vibrations of underwater spherical shells have been studied in a previous paper [1]. In this present paper, we shall investigate the axisymmetric forced vibrations of underwater elastic spherical shells. The investigation of harmonic forced vibrations will be based on a single six-order differential equation of motion in normal displacement. Investigation of aperiodic forced vibrations will be based on a pair of coupled equations for general motions in normal and tangential displacements. General expressions of the responses due to all these excitations are derived. Examples are given and results are plotted.

Introduction

In a previous paper [1] axisymmetric free vibrations of underwater spherical shells were investigated. A pair of basic coupled equations of motion based on the bending theory were derived by applying Hamilton's principle. These equations are for general motions in terms of normal and tangential displacements. For harmonic motions, the basic coupled equations were combined into a single six-order nonhomogeneous differential equation of motion in normal dispacement. The interacting problem was then solved by introducing the velocity potential of the water field and by assuming that the normal velocity of the shell was equal to that of the water field at the surface of the shell. The frequency equation was derived and mode shapes were obtained. In the present paper investigations of harmonic and aperiodic forced vibrations of elastic spherical shells in water are based on the single six-order equation and the pair of coupled equations, respectively.

1. Equations of Motion

In reference [1], the general equations of motion for an underwater elastic spherical shell were derived as

$$L_{uu}u + L_{uw}w = -\frac{1-\nu^2}{E}\rho_s R^2 \ddot{u} \tag{1}$$

$$L_{wu}u + L_{ww}w = -\frac{1-\nu^2}{E}\rho_s R^2 \ddot{w} + \frac{1-\nu^2}{Eh}R^2(p_a+f)$$

in which the operators are

$$L_{uu} = -(1+\varepsilon)[(1-\nu) + \nabla^2 - \frac{1}{1-x^2}]$$

$$\begin{aligned} L_{uw} &= -(1-x^2)^{\frac{1}{2}}\frac{d}{dx}[\varepsilon(1-\nu) - (1+\nu) + \varepsilon\nabla^2] \\ L_{wu} &= [\varepsilon(1-\nu) - (1+\nu) + \varepsilon\nabla^2]\frac{d}{dx}(1-x^2)^{\frac{1}{2}} \\ L_{ww} &= \varepsilon\nabla^4 + \varepsilon(1-\nu)\nabla^2 + 2(1+\nu) \end{aligned} \tag{2}$$

For harmonic motions these equations were transformed into

$$L_{uu}U + L_{uw}W = \Omega^2 U$$
$$L_{wu}U + L_{ww}W = \Omega^2 W + \frac{1-\nu^2}{Eh} R^2(P_a+F) \tag{3}$$

which can be reduced into a single equation

$$(\nabla^6 + a\nabla^4 + b\nabla^2 + c)W = \frac{1-\nu^2}{Eh} R^2(d\nabla^2+e)(P_a+F) \tag{4}$$

in which

$$P_a(x) = \omega^2 \rho R \sum_{n=0}^{\infty} \frac{2n+1}{2n+2} C_n P_n(x) \tag{5}$$

$$C_n = \int_{-1}^{1} W(x)P_n(x)dx \tag{6}$$

The frequency equation was derived as

$$\lambda_n^3 - a\lambda_n^2 + b\lambda_n - c - \Omega^2 \frac{R\rho}{h\rho_s} \frac{1}{n+1} (d\lambda_n - e) = 0 \tag{7}$$

The detailed notations in these equations were explained in [1].

2. Harmonic Forced Vibrations

a. Normal Displacement Response

For a given harmonic force excitation

$$f(x,t) = F(x)\cos\omega t \tag{8}$$

which is normal to the surface of the shell with the frequency ω, the equation of motion (4) together with the expression of the hydrodynamic pressure from Eq. (5) becomes

$$[\nabla^6 + a\nabla^4 + b\nabla^2 + c]W(x) - \Omega^2 \frac{R\rho}{h\rho_s} \sum_{n=0}^{\infty} \frac{2n+1}{2n+2} C_n[d\cdot\nabla^2+e]P_n(x)$$
$$= \frac{1-\nu^2}{Eh} R^2[d\cdot\nabla^2+e]F(x) \tag{9}$$

Since the Legendre polynomials $P_n(x)$ are a complete and orthogonal set, and they satisfy the differential equation of the free vibration, we may expand both W(x) and F(x) in the infinite series of $P_n(x)$,

$$W(x) = \sum_{n=0}^{\infty} A_n P_n(x)$$

$$F(x) = \sum_{n=0}^{\infty} F_n P_n(x) \tag{10}$$

in which the coefficients F_n of the known forcing function can be determined by using the orthogonality condition of $P_n(x)$ as

$$F_n = \frac{2n+1}{2} \int_{-1}^{1} F(x)\ P_n(x)dx \tag{11}$$

The substitution of Eq. (10) into Eq. (9) and the use of the identities of $P_n(x)$, together with the definition of Eq. (6) as

$$C_n = \int_{-1}^{1} W(x)P_n(x)dx = \frac{2}{2n+1} A_n \tag{12}$$

result in

$$\sum_{n=0}^{\infty} A_n[-\lambda_n^3+a\lambda_n^2-b\lambda_n+c]P_n(x) - \Omega^2 \frac{R\rho}{h\rho_s} \sum_{n=0}^{\infty} \frac{2n+1}{2n+2} A_n \frac{2}{2n+1}[-d\cdot\lambda_n+e]P_n(x)$$

$$= \frac{1-\nu^2}{Eh} R^2 \sum_{n=0}^{\infty} F_n[-d\cdot\lambda_n+e]P_n(x) \tag{13}$$

Due to the orthogonality condition of $P_n(x)$, we may equate the coefficients of the infinite series to give

$$A_n = I_n F_n \tag{14}$$

in which

$$I_n = \frac{\frac{1-\nu^2}{Eh} R^2(d\lambda_n-e)}{\lambda_n^3-a\lambda_n^2+b\lambda_n-c-\Omega^2 \frac{R\rho}{h\rho_s} \frac{1}{n+1}(d\lambda_n-e)} \tag{15}$$

When a forcing frequency is equal to one of the natural frequencies which satisfy Eq. (7), a corresponding A_n becomes infinite: this is the case of resonance. For ω different from natural frequency, the normal displacement gives

$$w(x,t) = \cos\omega t \sum_{n=0}^{\infty} I_n F_n P_n(x) \tag{16}$$

For illustration let us consider a special case. Assume that around the circle $x = \zeta$ of the shell surface there acts an axisymmetrically distributed line force of the intensity $F(x) = 1$ lb/in. which is normal to the surface and harmonic in time. This force can be expressed as

$$F(x) = \frac{1}{R}\sqrt{1-\zeta^2}\ \delta(x-\zeta). \tag{17}$$

Then the expression for F_n from Eq. (14) becomes

$$F_n = \frac{2n+1}{2R}\sqrt{1-\zeta^2}\ P_n(\zeta) \tag{18}$$

where

$$\int_{-1}^{1} P_n(x)\delta(x-\zeta)dx = P_n(\zeta) \tag{19}$$

has been used. Thus, Eq. (16) gives

$$w(x,t) = \cos\omega t \frac{\sqrt{1-\zeta^2}}{2R} \sum_{n=0}^{\infty} (2n+1) I_n P_n(\zeta) P_n(x) \tag{20}$$

b. Tangential Displacement Response

The tangential displacement due to the harmonic force excitation normal to the shell surface, with the frequency ω, can be derived by introducing the series expansions

$$\begin{aligned} W(x) &= \sum_{n=0}^{\infty} A_n P_n(x) \\ U(x) &= \sum_{n=1}^{\infty} B_n P_n^1(x) \end{aligned} \tag{21}$$

into the first equation of motion (3). With the aid of the identities of the Legendre polynomials $P_n(x)$ and the associated Legendre polynomials $P_n^1(x)$ as

$$\nabla^2 P_n(x) = -\lambda_n P_n(x)$$

$$P_n^1(x) = -(1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_n(x) \tag{22}$$

$$(\nabla^2 - \frac{1}{1-x^2}) P_n^1(x) = -\lambda_n P_n^1(x)$$

we obtain

$$-(1+\varepsilon) \sum_{n=1}^{\infty} B_n [1-\nu-\lambda_n] P_n^1(x) - \sum_{n=1}^{\infty} A_n [\varepsilon(1-\nu)-(1+\nu)-\varepsilon\lambda_n][-P_n^1(x)]$$

$$= \Omega^2 \sum_{n=1}^{\infty} B_n P_n^1(x) \tag{23}$$

Due to the orthogonality condition of $P_n^1(x)$, we may equate the coefficients of the infinite series to give

$$B_n = H_n A_n = H_n I_n F_n \tag{24}$$

in which

$$H_n = \frac{\varepsilon(\lambda_n - 1+\nu)+1+\nu}{(1+\varepsilon)(\lambda_n - 1+\nu)-\Omega^2} \tag{25}$$

and $A_n = I_n F_n$ is given in Eq. (14). Thus, the tangential displacement yields

$$u(x,t) = \cos\omega t \sum_{n=1}^{\infty} I_n H_n F_n P_n^1(x) \tag{26}$$

For illustration let us consider a special case of axisymmetrically distributed line forces acting around the circle $x = \zeta$ of the shell surface. The coefficients of the forcing function F_n are given in Eq. (18). Thus, from Eq. (26) we have

$$u(x,t) = \cos\omega t \, \frac{\sqrt{1-\zeta^2}}{2R} \sum_{n=1}^{\infty} (2n+1) I_n H_n P_n(\zeta) P_n^1(x) \tag{27}$$

3. Aperiodic Forced and Free Vibrations

Introducing the differential operators from Eqs.(2) into Eqs. (1), we rewrite the equations of motion as follows:

$$\begin{aligned} &-(1+\varepsilon)[1-\nu-\frac{1}{1-x^2}+\nabla^2]u(x,t)-(1-x^2)^{\frac{1}{2}}\frac{d}{dx}[\varepsilon(1-\nu)-(1+\nu)+\varepsilon\nabla^2]w(x,t) \\ &\qquad = -A\ddot{u}(x,t) \\ &[\varepsilon(1-\nu)-(1+\nu)+\varepsilon\nabla^2]\frac{d}{dx}(1-x^2)^{\frac{1}{2}}u(x,t)+[\varepsilon\nabla^4+\varepsilon(1-\nu)\nabla^2+2(1+\nu)]w(x,t) \\ &\qquad = -A\ddot{w}(x,t) + A\frac{1}{h\rho_s}[P_a(x,t) + f(x,t)] \end{aligned} \tag{28}$$

in which

$$A = \frac{1-\nu^2}{E} R^2 \rho_s \tag{29}$$

Now, we will solve Eqs. (28) together with the known initial conditions $w(x, 0)$, $\dot{w}(x, 0)$, $u(x, 0)$ and $\dot{u}(x, 0)$. Let us expand w, u, and f in terms of the infinite series of the appropriate Legendre polynomials as

$$\begin{aligned} w(x,t) &= \sum_{n=0}^{\infty} W_n(t)P_n(x) \\ u(x,t) &= \sum_{n=1}^{\infty} U_n(t)P_n^1(x) \\ f(x,t) &= \sum_{n=0}^{\infty} F_n(t)P_n(x) \end{aligned} \tag{30}$$

Applying the orthogonality condition of the Legendre polynomials [2],

$$\int_{-1}^{1}[P_n^m(x)]^2dx = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}$$

$$\int_{-1}^{1} P_n^m(x)P_\ell^m(x)dx = 0 \qquad \text{for} \quad \ell \neq n \tag{31}$$

we can express the coefficients of the initial conditions as

$$W_n(0) = \frac{2n+1}{2}\int_{-1}^{1} w(x,0)P_n(x)dx$$

$$\dot{W}_n(0) = \frac{2n+1}{2}\int_{-1}^{1} \dot{w}(x,0)P_n(x)dx$$

$$U_n(0) = \frac{2n+1}{2}\frac{(n-1)!}{(n+1)!}\int_{-1}^{1} u(x,0)P_n^1(x)dx \tag{32}$$

$$\dot{U}_n(0) = \frac{2n+1}{2}\frac{(n-1)!}{(n+1)!}\int_{-1}^{1} \dot{u}(x,0)P_n^1(x)dx$$

The coefficients of the forcing function give

$$F_n(t) = \frac{2n+1}{2}\int_{-1}^{1} f(x,t)P_n(x)dx \tag{33}$$

The hydrodynamic pressure has been derived in [1] as

$$p_a(x,t) = -\rho R\sum_{n=0}^{\infty}\frac{2n+1}{2n+2}\frac{d}{dt}D_n(t)P_n(x) \tag{34}$$

where

$$D_n(t) = \int_{-1}^{1} \dot{w}(x,t)P_n(x)dx \tag{35}$$

with Eq. (30), $D_n(t)$ can be written as

$$D_n(t) = \frac{2}{2n+1}\dot{W}(t) \tag{36}$$

then Eq. (34) becomes

$$p_a(x,t) = -\rho R\sum_{n=0}^{\infty}\frac{1}{n+1}\ddot{W}_n(t)P_n(x) \tag{37}$$

Substituting Eqs. (30) and (37) into Eqs. (28), and using the identities of Legendre polynomials as shown in Eq. (22) yields

$$-(1+\varepsilon)\sum_{n=1}^{\infty} U_n[1-\nu-\lambda_n]P_n^1(x) - \sum_{n=1}^{\infty} W_n[\varepsilon(1-\nu)-(1+\nu)-\varepsilon\lambda_n][-P_n^1(x)]$$

$$= -A\sum_{n=1}^{\infty} \ddot{U}_n P_n^1(x) \tag{38}$$

$$\sum_{n=1}^{\infty} U_n\lambda_n[\varepsilon(1-\nu)-(1+\nu)-\varepsilon\lambda_n]P_n(x) + \sum_{n=0}^{\infty} W_n[\varepsilon\lambda_n^2-\varepsilon(1-\nu)\lambda_n+2(1+\nu)]P_n(x)$$

$$= -A\sum_{n=0}^{\infty} \ddot{W}_n P_n(x) + A\frac{1}{h\rho_s}\sum_{n=0}^{\infty}[-\frac{R\rho}{n+1}\ddot{W}_n + F_n]P_n(x)$$

Due to the orthogonality conditions of $P_n(x)$ and $P_n^1(x)$, two coupled ordinary differential equations for each pair of W_n and U_n are given from Eqs. (38) as follows, for $n \geq 1$

$$\ddot{U}_n = -\gamma_n U_n + e_n W_n$$

$$K_n\ddot{W}_n = \lambda_n e_n U_n - \sigma_n W_n + \frac{1}{h\rho_s} F_n \tag{39}$$

in which

$$\gamma_n = (1+\varepsilon)(\lambda_n-1+\nu)/A$$

$$e_n = [\varepsilon(\lambda_n-1+\nu)+1+\nu]/A$$

$$\sigma_n = [\varepsilon\lambda_n(\lambda_n-1+\nu)+2(1+\nu)]/A \tag{40}$$

$$K_n = 1 + \frac{R\rho}{h\rho_s}\,\frac{1}{n+1}$$

We shall consider the special cases of $n=0$ and $n=1$ and the general cases of $n \geq 2$ as follows.

Special case of n = 0. For the special case of $n = 0$, there is only one equation in W_0 resulting from the second equation of Eq. (38)

$$K_0\ddot{W}_0 = -\sigma_0 W_0 + \frac{1}{h\rho_s} F_0 \tag{41}$$

Applying the Laplace transform on Eq. (41) yields

$$\overline{W}_0(s) = \frac{1}{s^2+\frac{\sigma_0}{K_0}} \left[\frac{1}{K_0 h\rho_s} \overline{F}_0(s) + s\, W_0(0) + \dot{W}_0(0)\right] \tag{42}$$

Inverting the Laplace transform and making use of the theorem on convolution, we obtain

$$W_0(t) = \frac{1}{K_0 h\rho_s} \int_0^t F_0(\tau)X_0(t-\tau)d\tau + W_0(0)\dot{X}_0(t) + \dot{W}_0(0)X_0(t) \tag{43}$$

where

$$\begin{aligned} X_0(t) &= \frac{1}{\alpha_0} \sin\alpha_0 t \\ \alpha_0^2 &= \frac{\sigma_0}{K_0} = \frac{2(1+\nu)}{(1+\frac{R\rho}{h\rho_s})A} \end{aligned} \tag{44}$$

Note that $A\alpha_0^2$ is the same as the frequency parameter Ω_u^2 for $n=0$ [1] which means the shell vibrates with only one frequency α_0.

Special case of $n=1$. Another special case occurs for $n=1$. Introducing $n=1$ into Eq. (40) gives

$$\begin{aligned} \gamma_1 &= e_1 = \frac{1}{2}\sigma_1 = (1+\varepsilon)(1+\nu)/A \\ K_1 &= 1 + \frac{R\rho}{2h\rho_s} \end{aligned} \tag{45}$$

Then Eqs. (39) become

$$\begin{aligned} \ddot{U}_1 &= -\gamma_1 U_1 + \gamma_1 W_1 \\ K_1\ddot{W}_1 &= 2\gamma_1 U_1 - 2\gamma_1 W_1 + \frac{1}{h\rho_s} F_1 \end{aligned} \tag{46}$$

Combining Eqs. (46) results in

$$K_1\ddot{W}_1 + 2\ddot{U}_1 = \frac{1}{h\rho_s} F_1 \tag{47}$$

Equation (47) indicates that the shell undergoes the rigid body motion which corresponds to the zero frequency parameter Ω_ℓ^2 for $n=1$ [1]. Applying the Laplace transform on Eqs. (47) and the first equation of Eq. (46) results in

$$
\begin{aligned}
K_1 s^2 \overline{W}_1(s) + 2s^2 \overline{U}_1(s) &= \frac{1}{h\rho_s}\overline{F}_1(s) + s[K_1 W_1(0) + 2U_1(0)] \\
&\quad + K_1 \dot{W}_1(0) + 2\dot{U}_1(0) \qquad (48) \\
-\gamma_1 \overline{W}_1(s) + [s^2 + \gamma_1]\overline{U}_1(s) &= sU_1(0) + \dot{U}_1(0)
\end{aligned}
$$

Solving Eqs. (48) for $\overline{W}_1(s)$ and $\overline{U}_1(s)$ gives

$$
\begin{aligned}
\overline{W}_1(s) &= \frac{1}{\Delta}\{\frac{1}{h\rho_s}\overline{F}_1(s)\ (s^2+\gamma_1) + s^3 K_1 W_1(0) + s^2 K_1 \dot{W}_1(0) \\
&\quad + s[K_1 W_1(0) + 2U_1(0)]\gamma_1 + [K_1 \dot{W}_1(0) + 2\dot{U}_1(0)]\gamma_1\} \\
\overline{U}_1(s) &= \frac{1}{\Delta}\{\frac{\gamma_1}{h\rho_s}\overline{F}_1(s) + s^3 K_1 U_1(0) + s^2 K_1 \dot{U}_1(0) \qquad (49) \\
&\quad + s[K_1 W_1(0) + 2U_1(0)]\gamma_1 + [K_1 \dot{W}_1(0) + 2\dot{U}_1(0)]\gamma_1\}
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta &= K_1 s^2 (s^2 + \alpha_1^2) \\
\alpha_1^2 &= \gamma_1 (1 + \frac{2}{K_1}) = \frac{(1+\varepsilon)(1+\nu)}{A}(1 + \frac{2}{1 + \frac{R\rho}{2h\rho_s}}) \qquad (50)
\end{aligned}
$$

Note that $A\alpha_1^2$ is the same as the frequency parameter Ω_u^2 for $n=1$ [1]. Inverting the Laplace transform on Eqs. (49), and using the partial fraction method together with the theorem on convolution, we obtain

$$W_1(t) = \frac{1}{K_1 h\rho_s} \int_0^t F_1(\tau)X_1(t-\tau)d\tau + W_1(0)\dot{X}_1(t) + \dot{W}_1(0)X_1(t)$$
$$+ \frac{2}{K_1} [U_1(0)\dot{Y}_1(t) + \dot{U}_1(0)Y_1(t)]$$
$$U_1(t) = \frac{1}{K_1 h\rho_s} \int_0^t F_1(\tau)Y_1(t-\tau)d\tau + W_1(0)\dot{Y}_1(t) + \dot{W}_1(0)Y_1(t) \tag{51}$$
$$+ U_1(0)\dot{Z}_1(t) + \dot{U}_1(0)Z_1(t)$$

where

$$X_1(t) = \frac{\gamma_1}{\alpha_1^2} [\frac{\alpha_1^2-\gamma_1}{\gamma_1\alpha_1} \sin\alpha_1 t+t]$$
$$Y_1(t) = \frac{\gamma_1}{\alpha_1^2} [- \frac{1}{\alpha_1} \sin\alpha_1 t+t] \tag{52}$$
$$Z_1(t) = \frac{2\gamma_1}{K_1\alpha_1^2} [\frac{K_1\alpha_1^2-2\gamma_1}{2\gamma_1\alpha_1} \sin\alpha_1 t+t]$$

General cases of $n \geq 2$. For $n \geq 2$, Eqs. (39) correspond to the general cases. Applying the Laplace transform to the coupled equations (39) yields

$$-e_n\overline{W}_n(s) + (s^2+\gamma_n)\overline{U}_n(s) = s\,U_n(0) + \dot{U}_n(0)$$
$$[K_n s^2+\sigma_n]\overline{W}_n(s) - \lambda_n e_n\overline{U}_n(s) = \frac{1}{h\rho_s} \overline{F}_n(s) + K_n sW_n(0) + K_n\dot{W}_n(0) \tag{53}$$

The solutions of Eqs. (53) in $\overline{W}_n(s)$ and $\overline{U}_n(s)$ are

$$\overline{W}_n(s) = \frac{1}{\Delta}\{\frac{1}{h\rho_s} F_n(s)\cdot(s^2+\gamma_n) + s^3K_nW_n(0) + s^2K_n\dot{W}_n(0)$$
$$+ s[K_n\gamma_nW_n(0) + \lambda_ne_nU_n(0)] + K_n\gamma_n\dot{W}_n(0) + \lambda_ne_n\dot{U}_n(0)\}$$

$$\bar{U}_n(s) = \frac{1}{\Delta}\{\frac{1}{h\rho_s} e_n\bar{F}_n(s) + s^3K_nU_n(0) + s^2K_n\dot{U}_n(0) \tag{54}$$

$$+ s[K_ne_nW_n(0) + \sigma_nU_n(0)] + K_ne_n\dot{W}_n(0) + \sigma_n\dot{U}_n(0)\}$$

where

$$\Delta = K_n(s^2+\alpha_n^2)(s^2+\beta_n^2)$$

$$\left.\begin{matrix}\alpha_n^2\\ \beta_n^2\end{matrix}\right\} = \frac{1}{2K_n}\{K_n\gamma_n + \sigma_n \pm [(K_n\gamma_n+\sigma_n)^2 - 4K_n(\sigma_n\gamma_n-\lambda_ne_n^2)]^{\frac{1}{2}}\} \tag{55}$$

It can be shown that $A\alpha_n^2$ and $A\beta_n^2$ are the roots of the frequency equation (7), with $A\alpha_n^2$ corresponding to the upper branch Ω_u^2, and $A\beta_n^2$ corresponding to the lower branch Ω_ℓ^2. Again, applying the inverse Laplace transform on Eqs. (54) and using the partial fraction method together with the theorem of convolution, we obtain

$$W_n(t) = \frac{1}{K_nh\rho_s}\int_0^t F_n(\tau)X_n(t-\tau)d\tau + W_n(0)\dot{X}_n(t) + \dot{W}_n(0)X_n(t)$$

$$+ \frac{\lambda_n}{K_n}[U_n(0)\dot{Y}_n(t) + \dot{U}_n(0)Y_n(t)]$$

$$U_n(t) = \frac{1}{K_nh\rho_s}\int_0^t F_n(\tau)Y_n(t-\tau)d\tau + W_n(0)\dot{Y}_n(t) + \dot{W}_n(0)Y_n(t) \tag{56}$$

$$+ U_n(0)\dot{Z}_n(t) + \dot{U}_n(0)Z_n(t)$$

where

$$X_n(t) = \frac{1}{\alpha_n^2-\beta_n^2}[\frac{\alpha_n^2-\gamma_n}{\alpha_n}\sin\alpha_nt - \frac{\beta_n^2-\gamma_n}{\beta_n}\sin\beta_nt]$$

$$Y_n(t) = \frac{e_n}{\alpha_n^2\ \beta_n^2}[-\frac{1}{\alpha_n}\sin\alpha_nt + \frac{1}{\beta_n}\sin\beta_nt] \tag{57}$$

$$Z_n(t) = \frac{1}{K_n(\alpha_n^2-\beta_n^2)} \left[\frac{K_n\alpha_n^2-\sigma_n}{\alpha_n} \sin\alpha_n t - \frac{K_n\beta_n^2-\sigma_n}{\beta_n} \sin\beta_n t\right]$$

Equations (43), (51), and (56) are the most general expressions for the coefficients of the displacements which take into account the effects of the membrane, bending and hydrodynamic pressure. Substituting Eqs. (43), (51) and (56) into Eq. (30), and taking note of $P_0(x) = 1$, $P_1(x) = x$ and Eqs. (32) and (33), result in

$$\begin{aligned}
w(x,t) &= \frac{1}{2h\rho_s}\int_0^t\int_{-1}^1 f(y,\tau)\Big[\frac{1}{K_0} X_0(t-\tau) + 3xy \frac{1}{K_1} X_1(t-\tau) \\
&\quad + \sum_{n=2}^{\infty} P_n(x)P_n(y) \frac{2n+1}{K_n} X_n(t-\tau)\Big]dyd\tau \\
&\quad + \frac{1}{2} \int_{-1}^1\{w(y,0)[\dot{X}_0(t) + 3xy\dot{X}_1(t)] + \dot{w}(y,0)[X_0(t) + 3xyX_1(t)] \\
&\quad + 3xP_1^1(y) \frac{1}{K_1} [u(y,0)\dot{Y}_1(t) + \dot{u}(y,0)Y_1(t)] \\
&\quad + \sum_{n=2}^{\infty} P_n(x)(2n+1)(P_n(y)[w(y,0)\dot{X}_n(t) + \dot{w}(y,0)X_n(t)] \\
&\quad + P_n^1(y) \frac{(n-1)!}{(n+1)!} \frac{\lambda_n}{K_n} [u(y,0)\dot{Y}_n(t) + \dot{u}(y,0)Y_n(t)])\}dy
\end{aligned} \tag{58}$$

$$\begin{aligned}
u(x,t) &= \frac{1}{2h\rho_s} \int_0^t \int_{-1}^1 f(y,\tau)[3yP_1^1(x)\frac{1}{K_1} Y_1(t-\tau) + \sum_{n=2}^{\infty} P_n^1(x)P_n(y)\frac{2n+1}{K_n}Yn(t-\tau)]dyd\tau \\
&\quad + \frac{1}{2} \int_{-1}^1\{3yP_1^1(x)[w(y,0)\dot{Y}_1(t) + \dot{w}(y,0)Y_1(t)] \\
&\quad + \frac{3}{2} P_1^1(x)P_1^1(y)[u(y,0)\dot{Z}_1(t) + \dot{u}(y,0)Z_1(t)]
\end{aligned}$$

$$+ \sum_{n=2}^{\infty} P_n^1(x)(2n+1)\ P_n(y)[w(y,0)\dot{Y}_n(t) + \dot{w}(y,0)Y_n(t)]$$

$$+ P_n^1(y)\ \frac{(n-1)!}{(n+1)!}\ [u(y,0)\dot{Z}_n(t) + \dot{u}(y,0)Z_n(t)]\ ]dy$$

Let us consider the special case of spherical shell vibrations in a vacuum that is based on the membrane theory with homogeneous initial conditions. This special case can be deduced from the present general expressions. Substituting the results of $\rho = \varepsilon = 0$ in Eqs. (40), (44) and (50) and the homogeneous boundary conditions into Eqs. (43), (51), and (56) results in

$$W_0(t) = \frac{1}{h\rho_s\alpha_0} \int_0^t F_0(\tau)\sin\alpha_0(t-\tau)d\tau$$

$$W_1(t) = \frac{1}{3h\rho_s} \int_0^t F_1(\tau)[\frac{2}{\alpha_1}\ \sin\alpha_1(t-\tau)+(t-\tau)]d\tau$$

$$U_1(t) = \frac{1}{3h\rho_s} \int_0^t F_1(\tau)[-\frac{1}{\alpha_1}\ \sin\alpha_1(t-\tau)+(t-\tau)]d\tau$$

$$\begin{aligned} W_n(t) &= \frac{1}{Ah\rho_s(\alpha_n^2-\beta_n^2)} \int_0^t F_n(t)[\frac{A\alpha_n^2+1-\nu-\lambda_n}{\alpha_n}\ \sin\alpha_n(t-\tau) \\ &\qquad - \frac{A\beta_n^2+1-\nu-\lambda_n}{\beta_n}\ \sin\beta_n(t-\tau)]d\tau \end{aligned} \tag{59}$$

$$\begin{aligned} U_n(t) &= \frac{1+\nu}{Ah\rho_s(\alpha_n^2-\beta_n^2)} \int_0^t F_n(\tau)[-\frac{1}{\alpha_n}\ \sin\alpha_n(t-\tau) \\ &\qquad + \frac{1}{\beta_n}\ \sin\beta_n(t-\tau)]d\tau \end{aligned}$$

These results agree with Baker's work [3].

We now proceed to illustrate the forced vibrations. Assume that we have the homogeneous initial conditions,

$$w(x,0) = \dot{w}(x,0) = u(x,0) = \dot{u}(x,0) = 0 \tag{60}$$

and that the loading is applied at time $t = 0$. Two special cases will be considered: one case for a given impulsive force and the other case for a given step force.

Case a. Suppose that around the circle $x = \zeta$ there acts a unit axisymmetrically distributed impulsive force that is normal to shell surface. This force can be expressed as

$$f(x,t) = \frac{1}{R}\sqrt{1-\zeta^2}\ \delta(x-\zeta)\delta(t) \tag{61}$$

Then from Eqs. (58) we obtain

$$\begin{aligned} w(x,t) &= \frac{\sqrt{1-\zeta^2}}{2Rh\rho_s}\left[\frac{1}{K_0} X_0(t) + 3x\zeta \frac{1}{K_1} X_1(t) + \sum_{n=2}^{\infty} P_n(x)P_n(\zeta)\frac{2n+1}{K_n} X_n(t)\right] \\ u(x,t) &= \frac{\sqrt{1-\zeta^2}}{2Rh\rho_s}\left[3\zeta P_1^1(x)\frac{1}{K_1} Y_1(t) + \sum_{n=2}^{\infty} P_n^1(x)P_n(\zeta)\frac{2n+1}{K_n} Y_n(t)\right] \end{aligned} \tag{62}$$

Case b. Suppose that around a circle $x = \zeta$ the shell is subjected to the axisymmetrically distributed step forces that are normal to the shell surface, i.e.

$$f(x,t) = \frac{1}{R}\sqrt{1-\zeta^2}\ \delta(x-\zeta)H(t) \tag{63}$$

where $H(t)$ denotes the Heaviside unit step function. Noting that

$$\int_0^t H(\tau)\sin\alpha_n(t-\tau)d\tau = \frac{1}{\alpha_n}(1-\cos\alpha_n t) \tag{64}$$

we obtain from Eqs. (58)

$$w(x,t) = \frac{\sqrt{1-\zeta^2}}{2Rh\rho_s}\left\{\frac{1}{K_0\alpha_0^2}(1-\cos\alpha_0 t) + 3x\zeta\frac{\gamma_1}{K_1\alpha_1^2}\left[\frac{\alpha_1^2-\gamma_1}{\gamma_1\alpha_1^2}(1-\cos\alpha_1 t) + \frac{t^2}{2}\right]\right.$$

$$+ \sum_{n=2}^{\infty} P_n(x)P_n(\zeta) \frac{2n+1}{K_n(\alpha_n^2-\beta_n^2)}\left[\frac{\alpha_n^2-\gamma_n}{\alpha_n^2}(1-\cos\alpha_n t) - \frac{\beta_n^2-\gamma_n}{\beta_n^2}(1-\cos\beta_n t)\right]\}$$

$$u(x,t) = \frac{\sqrt{1-\zeta^2}}{2Rh\rho_s} \{3\zeta P_1^1(x) \frac{\gamma_1}{K_1\alpha_1^2}\left[-\frac{1}{\alpha_1^2}(1-\cos\alpha_1 t) + \frac{t^2}{2}\right] \tag{65}$$

$$+ \sum_{n=2}^{\infty} P_n^1(x)P_n(\zeta) \frac{e_n(2n+1)}{K_n(\alpha_n^2-\beta_n^2)}\left[-\frac{1}{\alpha_n^2}(1-\cos\alpha_n t) + \frac{1}{\beta_n^2}(1-\cos\beta_n t)\right]\}$$

4. Numerical Examples

Numerical examples for harmonic and aperiodic forced vibrations are presented for a complete spherical steel shell vibrating both in an infinite water field and in a vacuum, for which $E=30\times10^6$ psi, $\nu = 0.3$, $R = 120$ in, $h/R = 0.03$, $\rho_s = 0.7347\times10^{-3}$ $\text{lb.sec}^2/\text{in}^4$ and $\rho/\rho_s = 0.1304$. The results are plotted in the solid and the dotted lines which denote vibrations in water and vacuum respectively.

In Figs. 1 and 2, the responses due to a unit harmonic line force per unit length applied at the equator are plotted versus θ. Fig. 1 shows that the responses due to forces of the same frequency differ greatly when the vibrations are in water and when they are in vacuum. This is because the forces of the same frequency will excite different modes for these two cases. For the sake of comparison, consider the forces of different frequencies: $\Omega^2_{\text{water}} = 0.403$ and $\Omega^2_{\text{vacuum}} = 0.747$, which are respectively the mean values of the natural frequencies corresponding to $n = 3$ and $n = 4$, as shown in Table 1 of [1]. Results are shown in Fig. 2.

The responses due to a unit impulsive line force per unit length applied at the equator are plotted in Figs. 3 and 4. Fig. 3 shows the displacements at the equator versus time, and Fig. 4 shows the displacements versus θ

for $t = 0.0006$ sec. At that instant the equator undergoes the maximum displacement that can occur for the vibrations in water.

The responses due to a unit step line force per unit length applied at the equator are shown in Figs. 5 and 6. Fig. 5 shows the displacements at the equator versus time, and Fig. 6 shows the displacements versus θ for $t = 0.0024$ sec. At that instant the equator undergoes the maximum displacement that can occur for the vibrations in water.

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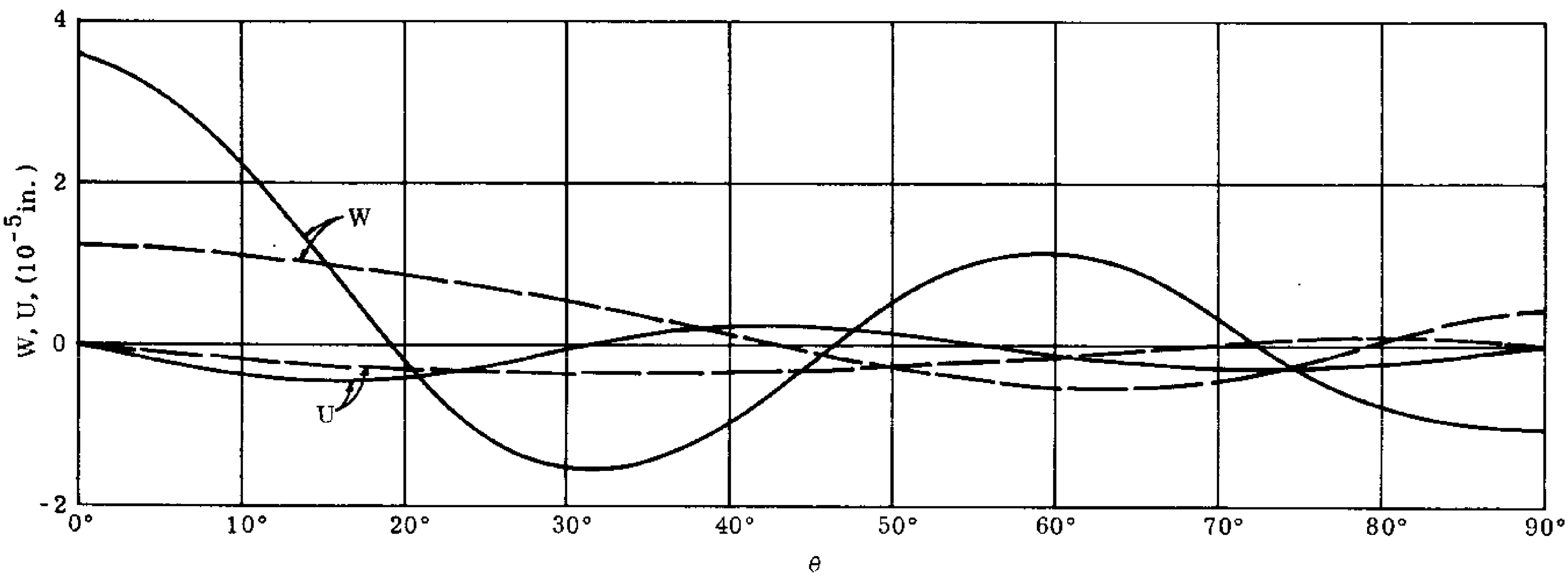

Fig. 1 Responses due to a unit harmonic line force per unit length applied at the equator ($\Omega^2 = 0.64$)

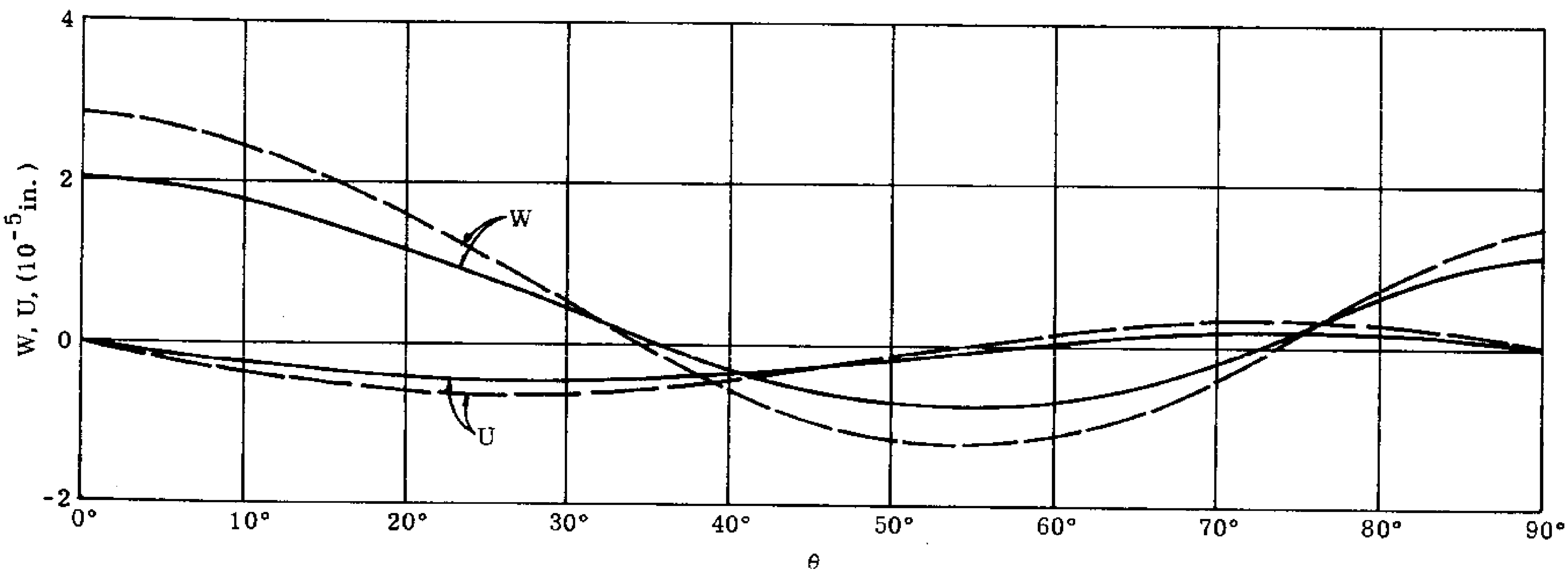

Fig. 2 Responses due to a unit harmonic line force per unit length applied at the equator (Ω^2_{water}=0.403, Ω^2_{air}=0.747)

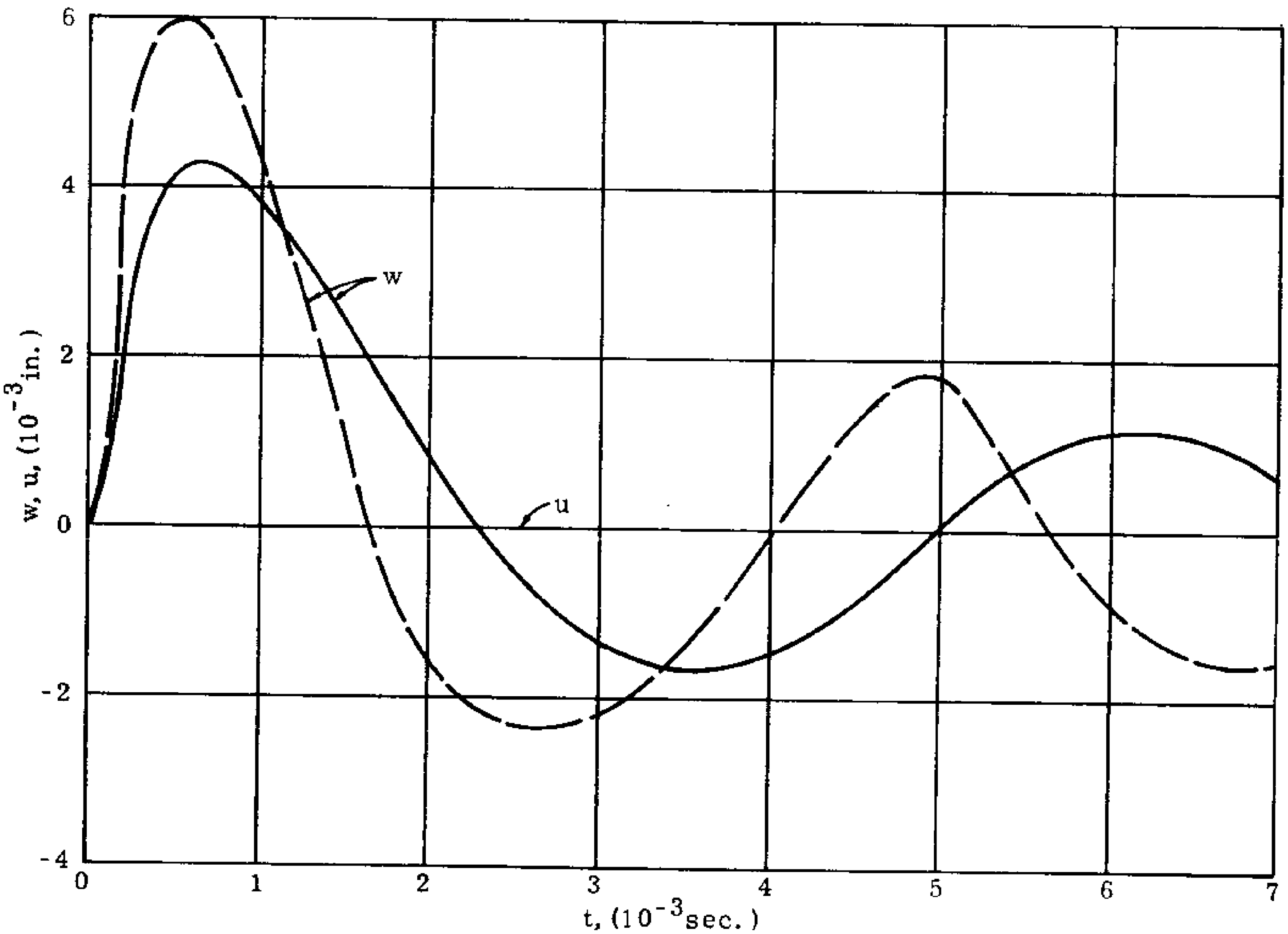

Fig. 3 Responses due to a unit impulsive line force per unit length applied at the equator for displacements at the equator vs time

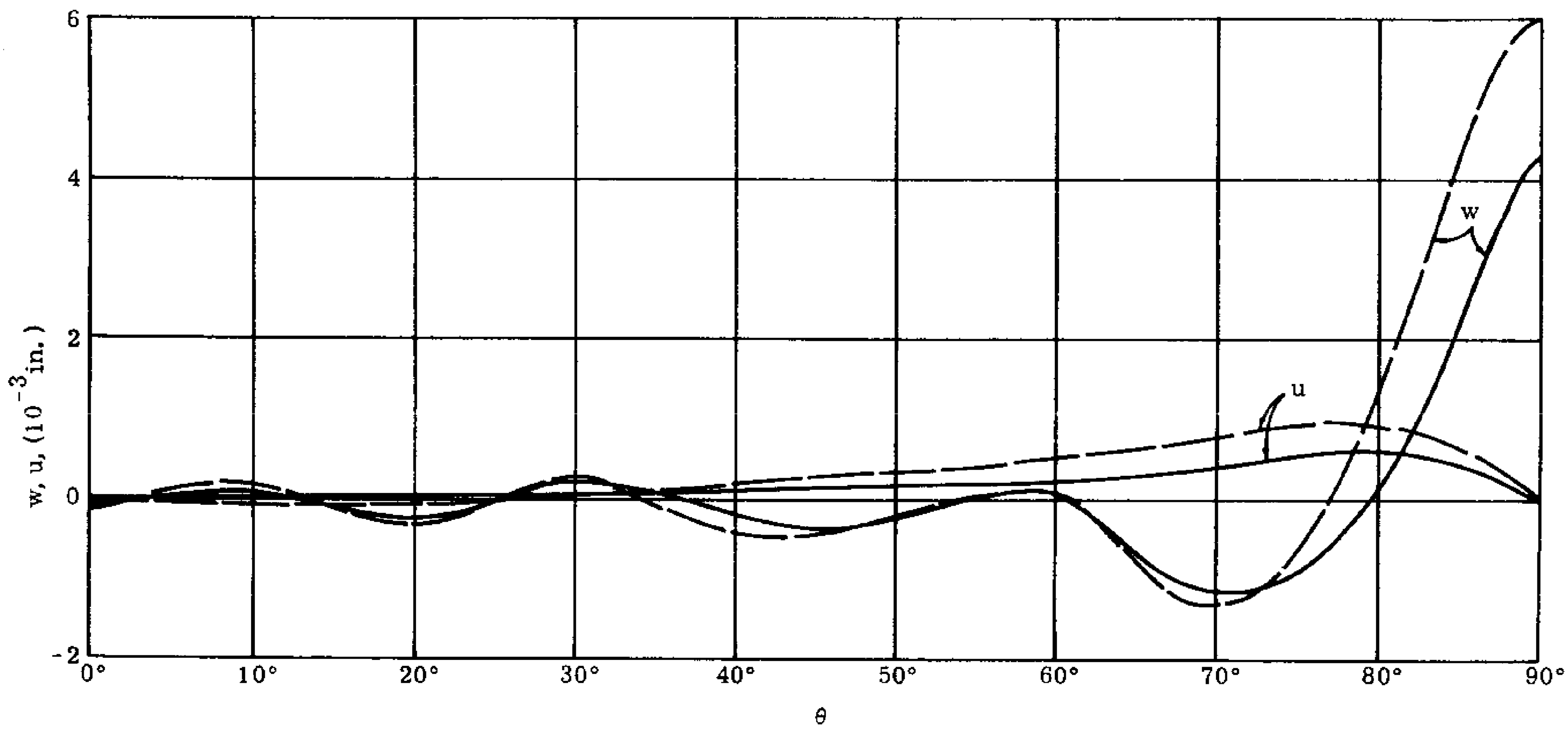

Fig. 4 Responses due to a unit impulsive line force per unit length applied at the equator for displacements vs θ at t=0.0006 sec

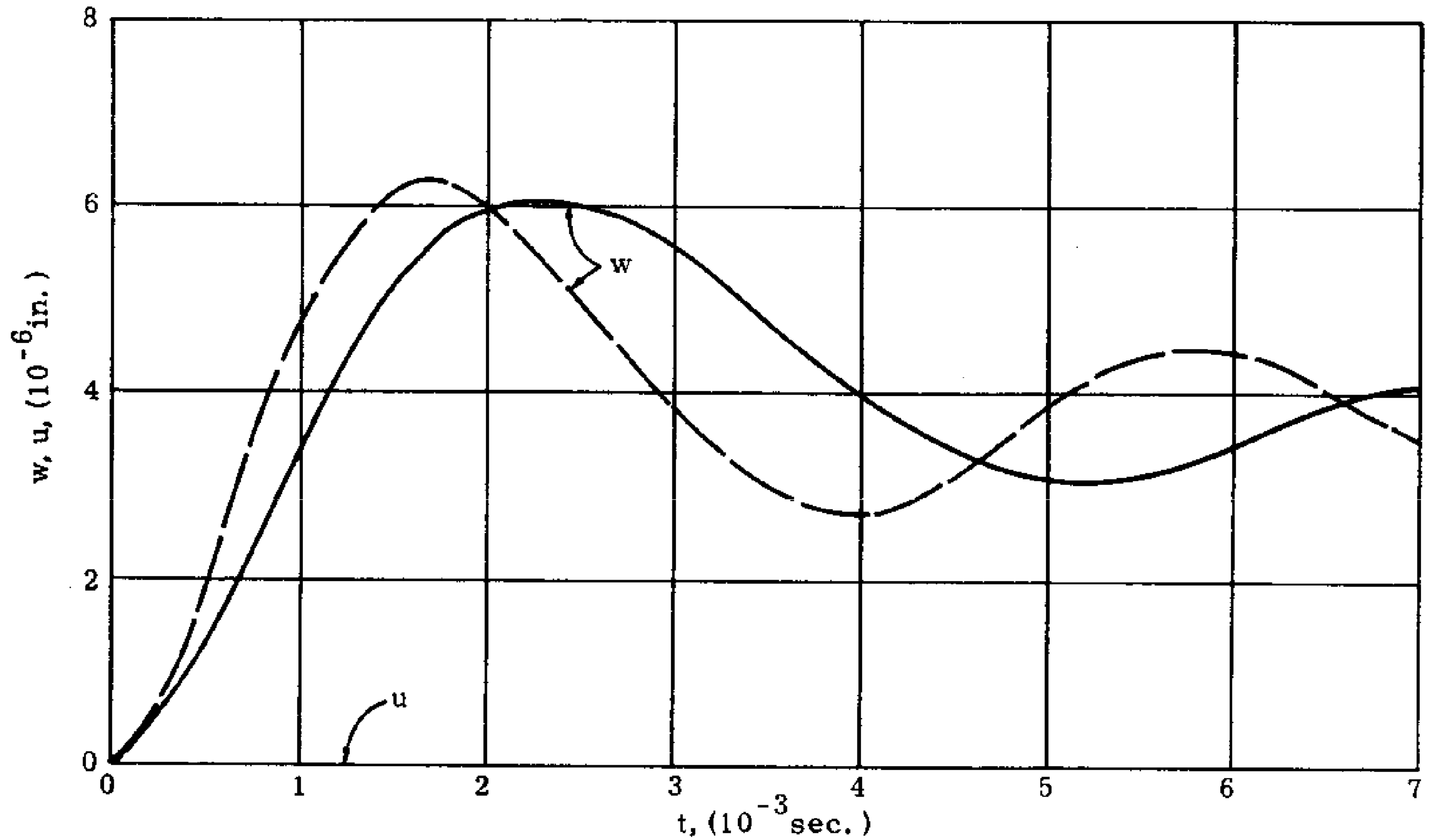

Fig. 5 Responses due to a unit step line force per unit length applied at the equator for displacements at the equator vs time

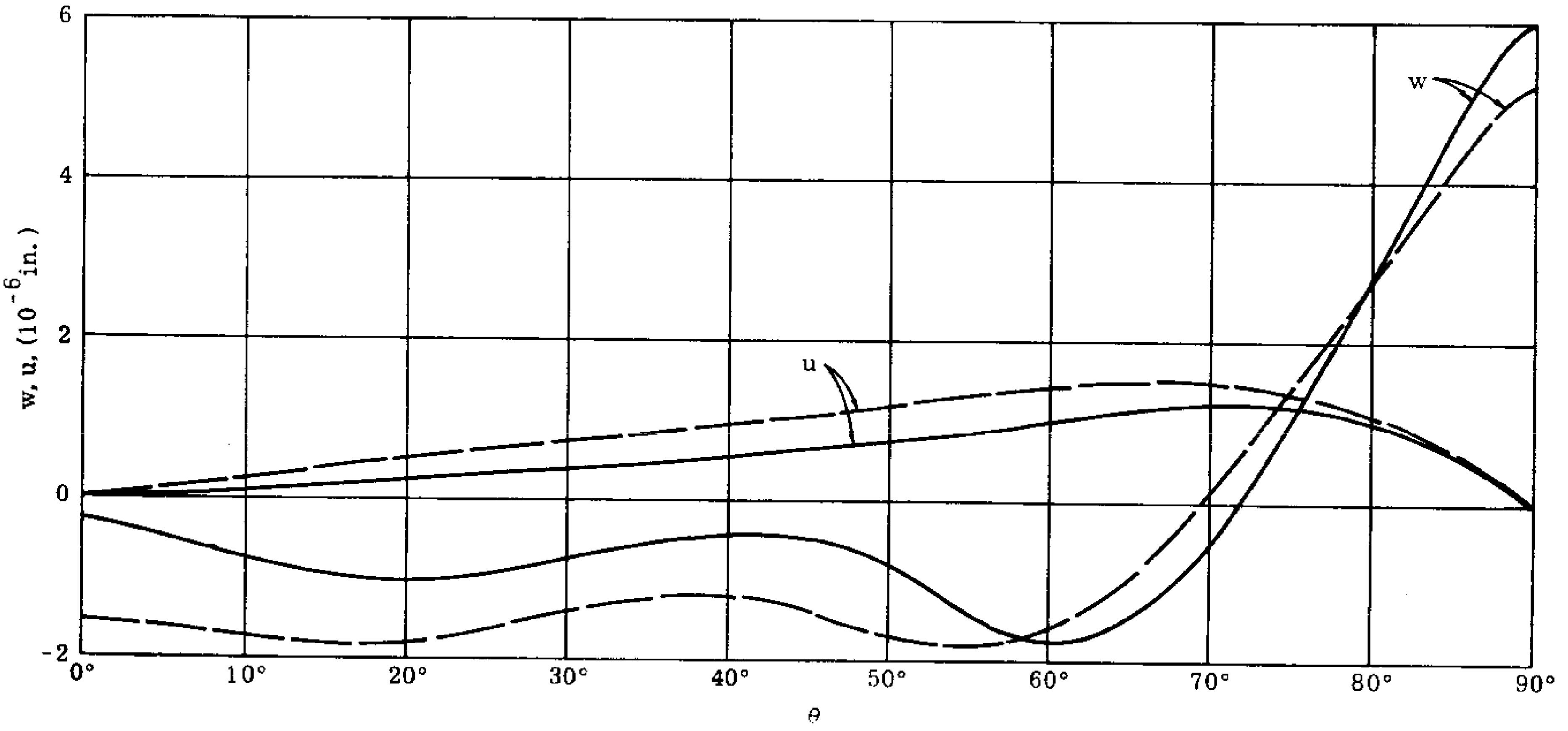

Fig. 6 Responses due to a unit step line force per unit length applied at the equator for displacements vs θ at $t=0.0024$ sec