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SEA GRANT

DEL-SG-16-88

A FUNCTION THEORY FOR THIN ELASTIC SHELLS

$1.00

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August 1988

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This publication is the result of research sponsored by NOAA, Office of Sea Grant, Department of Commerce and by the National Science Foundation through Grant No. DNS 8601318 (Project NO. R/OE-2). The U.S. Government is authorized to produce and distribute reprints for governmental purposes, notwithstanding any copyright notation that may appear hereon.

The University of Delaware Sea Grant College Program is supported cooperatively by the National Oceanic and Atmospheric Administration, U.S. Department of Commerce, and by the State of Delaware.

O. Introduction

It is well known in the theory of elastic shells that a first order approximation using the shell thickness as an expansion parameter leads to the membrane theory of shells. The membrane equations have as solutions the *generalized analytic functions*. These functions have been exhaustively studied by Ilya N. Vekua [6], [7] and his students. R.P. Gilbert and J. Hile [3] introduced an extension of these systems to include elliptic systems of $2n$ equations in the plane and named the solutions of these systems *generalized hyperanalytic functions*.

It is shown in this paper that the next order approximation to the shell, which permits, moreover, the introduction of bending, may be described in terms of the generalized hyperanalytic functions. It is strongly suspected that the higher order approximations may also be described in terms of corresponding hypercomplex systems.

1. Notation and Formulation

In this work we follow the notation used by Dikmen [1], we view a shell as a three dimensional body, which we try to reduce to two dimensional consideration by introducing a suitable reference surface. A set of curvilinear coordinates θ^i , $(i = 1, 2, 3)$ are chosen so that a *reference surface* within the shell may be represented by $\theta^3 = 0$. For purposes of defining our notation let σ be a surface embedded in $\mathbf{R}^3$ which we represent in the form $\mathbf{r} = \mathbf{r}(\theta^\alpha)$. The vectors $\mathbf{a}_\alpha := \mathbf{r}_{,\alpha} = \dfrac{\partial \mathbf{r}}{\partial \theta^\alpha}$ are the *base vectors*. The first fundamental form of the surface is given by $a_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta$ where the $a_{\alpha\beta} := \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ are the covariant components of the metric tensor. If the $A^{\alpha\beta}$ are the cofactors of the $a_{\alpha\beta}$ we define the contravariant metric tensor as $a^{\alpha\beta} := A^{\alpha\beta} / a$ where a is the determinant of $a_{\alpha\beta}$. The second fundamental form of the surface is defined through

$b_{\alpha\beta} d\theta^\alpha d\theta^\beta := \mathbf{a}_3 \cdot \mathbf{a}_{\alpha,\beta} d\theta^\alpha d\theta^\beta = - d\mathbf{r} \cdot d\mathbf{a}_3$, with $\mathbf{a}_3 = \dfrac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}$. The Christoffel symbols are given as usual by $\Gamma_{\alpha\beta\gamma} := \frac{1}{2}(a_{\alpha\gamma,\beta} + a_{\beta\gamma,\alpha} - a_{\alpha\beta,\gamma})$, and covariant differentiation is defined by

$$T_{\alpha|\gamma} := T_{\alpha,\gamma} - \Gamma^\lambda_{\alpha\gamma} T_\lambda . \tag{1.1}$$

The position vector of a generic point on the shell at time t is given by $\mathbf{x} = \mathbf{x}(\theta^\alpha , \xi , t)$ where $(\theta^1 , \theta^2) \,\varepsilon\, \overline{\Omega}$ is a point on the reference surface and $\xi \varepsilon [0,h]$, in particular $\mathbf{x}(\theta^\alpha, 0 , t) = \mathbf{r}(\theta_\alpha, t)$. The methods of Dikmen [1] and Vekua [6] use the shell thickness variable ξ to expand $\mathbf{x}(\theta^\alpha, \xi , t)$. (Vekua expands in terms of Legendre polynomials $P_n(\xi)$ and Dikmen uses powers of ξ.)

$$\mathbf{x}(\theta^\alpha, \xi , t) = \mathbf{v}(\theta^\alpha, t) + \sum_{n=1}^{\infty} \xi^n d^{(n)}(\theta^k, t) \tag{1.2}$$

In terms of the reference surface ($\xi = 0$) we now express a position vector which is *near* the reference surface as $\mathbf{x} = \mathbf{r} + \xi\, \mathbf{a}_3$. In terms of the three dimensional basis vectors $\mathbf{g}_i^*$ we may compute the stress vector $\mathbf{t}$ on a surface as $T^{ij} \hat{n}_i\, \mathbf{g}_j^* = \mathbf{t}$, where T^{ij} is the Cauchy stress tensor, and $\hat{\mathbf{n}}$ is the unit normal to the surface, and $\hat{n}_i$ a covariant component, i.e. $\hat{\mathbf{n}} = \hat{n}_i\, g^{*i} = \hat{n}^i g_i^*$. The equations of motion take the form

$$\mathbf{T}^i_{,i} + \rho^* g^{*1/2} f^* = \rho^* g^{*1/2}\, \dot{\mathbf{v}}^* , \tag{1.3}$$
$$\mathbf{g}_i^* \times \mathbf{T}^i = 0 \tag{1.4}$$

where the vectors $\mathbf{T}^i$ are defined by

$$\mathbf{T}^i := g^{*1/2}\, T^{ij} \mathbf{g}_j^* \,,\; g^* = \det(g_{ij}) \tag{1.5}$$

and f^* is the body force acting on each unit of mass. The moments of (1.3) are investigated by multiplying both sides by ξ^n and integrating over the shell thickness from $\xi = a(\theta^\alpha)$ to $\xi = b(\theta^\alpha)$. One obtains [1]

$$\underset{n}{M}{}^{\alpha}_{|\alpha} + \rho \, \overline{l} \, \underset{n}{m} - = 0 \, , \tag{1.6}$$

with

$$\underset{n}{\mathbf{M}}{}^{\alpha} \, a^{1/2} : = \int_0^h \mathbf{T}^{\alpha} \, \xi^n d\xi \tag{1.7}$$

$$\underset{n}{\mathbf{m}} \, a^{1/2} : = n \int_0^h \mathbf{T}^3 \, \xi^{n-1} d\xi \tag{1.8}$$

$$\rho \, \underset{n}{l} \, a^{1/2} : = \int_0^h \rho^* f^* g^{*\,1/2} \, \xi^n d\xi + [\xi^n (- \mathbf{b}_{,\alpha} \mathbf{T}^{\alpha} + \mathbf{T}^3)]_{\xi = b}$$

$$- [\xi^n (- \mathbf{a}_{,\alpha} \, \mathbf{T}^{\alpha} + \mathbf{T}^3)]_{\xi = a} \; , \tag{1.9}$$

and

$$\underset{N}{\overline{\mathbf{l}}} = \underset{n}{l} - (\underset{n}{k} \, \dot{\mathbf{v}} + \sum_{l=1}^{\infty} \underset{n+l}{k} \; \underset{l}{\dot{\mathbf{w}}}) \, . \tag{1.10}$$

By introducing components, i.e. $\mathbf{N}^{\alpha} = N^{\alpha i} \mathbf{a}_i$, $\underset{n}{\mathbf{M}} = \underset{n}{M}{}^{\alpha i} \mathbf{a}_i$, $\underset{n}{\mathbf{m}} : = \underset{n}{m}{}^{i} \, \mathbf{a}_i$ etc., where we denote $\underset{o}{M}{}^{\alpha}$ by N^{α}. We may rewrite (1.3), with n=0, in the form

$$N^{\alpha\beta}_{|\alpha} - b^{\beta}_{\alpha} \, N^{\alpha 3} + \rho \overline{f}^{\beta} = 0 \, , \tag{1.11}$$

$$(n = 1, 2, ...)$$

$$N^{\alpha 3}_{|\alpha} - b_{\alpha\beta} N^{\alpha\beta} + \rho \overline{f}^{\beta} = 0 \, . \tag{1.12}$$

The angular momentum equations (1.10), moreover, take the form [ ] pg. 61

$$\underset{n}{M}{}^{\alpha\beta}_{|\alpha} - b^{\beta}_{\alpha} \, \underset{n}{M}{}^{\alpha 3} + \rho \overline{l}^{\beta}_{n} = \underset{n}{m}{}^{\beta} \, , \tag{1.13}$$

$$(n = 0, 1, 2, \; \cdots)$$

$$\underset{n}{M}{}^{\alpha 3}_{|\alpha} + b_{\alpha\beta} \, \underset{n}{M}{}^{\alpha\beta} + \rho \overline{l}^{3\beta}_{n} = \underset{n}{m}{}^{3} \, . \tag{1.14}$$

It is well known [1], [6] that if one can ignore all of the above quantities for $n \geq 1$ one obtains the so-called *membrane theory* of shells and that these equations may be treated by a reduction to the generalized analytic function theory. In the next section we shall consider the situation where the $\underset{n}{M}{}^{\alpha\beta}$, $\underset{n}{m}{}^{\beta}$, $\overline{l}^{\beta}_{n}$, etc. may be ignored for all $n \geq 2$. This permits one to introduce *bending* into our analysis. We shall show that in this instance the system of equations may be reduced to the so-called *generalized hyperanalytic system* [6], [7].

2. Derivation of Equations for n = 2:

Following Dikmen [1] we try to introduce bending into our model by retaining only those quantities for $n < 2$. Dropping the index 1, in component form these become

$$N^{\alpha\beta}_{|\alpha} - b^{\beta}_{\alpha} N^{\alpha 3} + \rho \bar{f}^{\beta} = 0 \ , \ (\beta=1,2)$$

$$N^{\alpha 3}_{|\alpha} + b_{\alpha\beta} N^{\alpha\beta} + \rho \bar{f}^{3} = 0 \ , \tag{2.1}$$

$$M^{\alpha\beta}_{|\alpha} - b^{\beta}_{\alpha} M^{\alpha 3} + \rho \bar{l}^{\beta} = m^{\beta} \ , \ (\beta = 1,2)$$

and

$$M^{\alpha 3}_{|\alpha} + b_{\alpha\beta} M^{\alpha\beta} + \rho \bar{l}^{3} = m^{3} \ . \tag{2.2}$$

As is remarked in Dikmen [1] if the coordinate system happens to be a system of normal coordinates in the elastic configuration the angular momentum equations take on the particularly simple form

$$\mathbf{N}^{\alpha} \times \mathbf{a}_{\alpha} + \mathbf{m} \times \mathbf{a}_3 - b^{\gamma}_{\alpha} \mathbf{M}^{\alpha} \times \mathbf{a}_{\gamma} = 0 \ , \tag{2.3}$$

and

$$\mathbf{M}^{\alpha} \times \mathbf{a}_{\alpha} + \frac{1}{2} \underset{2}{\mathbf{m}} \times \mathbf{a}_3 - b^{\gamma}_{\alpha} \underset{2}{\mathbf{M}}{}^{\alpha} \times \mathbf{a}_{\gamma} = 0 \ . \tag{2.4}$$

If we ignore the effect of the terms $\underset{2}{\mathbf{m}}$ and $\underset{2}{\mathbf{M}}{}^{\alpha}$ in equations (2.4) which is similar to the approach used in membrane theory we have that $M^{\alpha\beta} = M^{\beta\alpha}$ and $N^{\alpha 3} = m^{\alpha}$. We will not need these assumptions until late.. From equation (2.1a) we have

$$N^{11}_{|1} + N^{21}_{|2} = N^{11}_{,1} + N^{21}_{,2} + \Gamma^{1}_{11} N^{11} + \Gamma^{1}_{21} N^{21} + \Gamma^{2}_{12} N^{11}$$

$$+ \Gamma^{2}_{22} N^{21} + \Gamma^{1}_{11} N^{11} + \Gamma^{1}_{12} N^{21} + \Gamma^{1}_{21} N^{12} + \Gamma^{1}_{22} N22 \ . \tag{2.5}$$

If the lines of curvature are used as curves on the reference coordinates then in physical components we have

$$N^{\alpha\beta} = \frac{1}{\sqrt{a_{\alpha\alpha}a_{\beta\beta}}} N_{<\alpha,\beta>} \;, \; (\sum \alpha\,,\,\beta) \tag{2.6}$$

and in particular

$$N^{11}_{,1} = \left[\frac{1}{\sqrt{a_{11}a_{11}}} N_{<1,1>}\right]_{,1} = \frac{1}{a_{11}\sqrt{a_{22}}} \left[\sqrt{a_{22}}N_{<1,1>}\right]_{,1}$$

$$+ \left[\frac{1}{a_{11}\sqrt{a_{22}}}\right]_{,1} \left[\sqrt{a_{22}}\,N_{<1,1>}\right] ,$$

$$N^{21}_{,2} = \frac{1}{a_{11}\sqrt{a_{22}}} \left[\sqrt{a_{11}}N_{<2,1>}\right]_{,2} + \left[\frac{1}{a_{11}\sqrt{a_{22}}}\right]_{,2} \left[\sqrt{a_{11}}\,N_{<2,1>}\right] , \tag{2.7}$$

and

$$\Gamma^{\alpha}_{\alpha\beta} = \Gamma^{\alpha}_{\beta\alpha} = \frac{1}{\sqrt{a_{\alpha\alpha}}} \left[\sqrt{a_{\alpha\alpha}}\right]_{,\beta} \;, \; (\sum \alpha) . \tag{2.8}$$

After some computation one obtains

$$N^{\alpha 1}_{\;|\alpha} = \frac{1}{\sqrt{a_{11}}} \left[\frac{1}{\sqrt{a_{11}a_{22}}} \left(\left[\sqrt{a_{22}}\,N_{<1,1>}\right]_{,1} + \left[\sqrt{a_{11}}\,N_{<2,1>}\right]_{,2} \right]\right.$$

$$\left. + (\sqrt{a_{11}})_{,2}\, N_{<1,2>} - (\sqrt{a_{22}})_{,1}\, N_{<2,2>} \right. . \tag{2.9}$$

Using similar steps one also arrives at

$$N^{\alpha 2}_{\;|\alpha} = \frac{1}{\sqrt{a_{22}}} \left[\frac{1}{\sqrt{a_{11}a_{22}}} \left(\left[\sqrt{a_{22}}\,N_{<1,2>}\right]_{,1} + \left[\sqrt{a_{11}}\,N_{<2,2>}\right]_{,2}\right] \right]$$

$$- \left[\sqrt{a_{11}}\right]_{,2} N_{<1,1>} + \left[\sqrt{a_{22}}\right]_{,1} N_{<2,1>} \Big]\Big] \tag{2.10}$$

Let us now introduce the new unknowns

$$u_1 := \sqrt{a_{22}}\, N_{<1,1>} \ , \ u_2 := \sqrt{a_{22}}\, N_{<1,2>} , \tag{2.11}$$

$$v_1 := \sqrt{a_{11}}\, N_{<2,1>} \ , \ v_2 := \sqrt{a_{11}}\, N_{<2,2>} .$$

Then the equations of linear momentum balance

$$N^{\alpha 1}_{|\alpha} - b^1_\alpha N^{\alpha 3} + \rho \bar{f}^1 = 0 ,$$

and

$$\tag{2.12}$$

$$N^{\alpha 2}_{|2} - b^2_\alpha N^{\alpha 3} + \rho \bar{f}^2 = 0 ,$$

become

$$\frac{1}{\sqrt{a_{11}a_{22}}}\left[u_{1,1} + v_{1,2} + (\sqrt{a_{11}})_{,2}\frac{1}{\sqrt{a_{22}}}u_2 - (\sqrt{a_{22}})_{,1}\frac{1}{\sqrt{a_{11}}}v_2\right]$$

$$= \left(\rho\bar{f}^1 + b_{11}a^{11}m^1\right)\sqrt{a_{11}} , \tag{2.13}$$

and

$$\frac{1}{\sqrt{a_{11}a_{22}}}\left[u_{2,1} + v_{2,2} - (\sqrt{a_{11}})_{,2}\frac{1}{\sqrt{a_{22}}}u_1 + (\sqrt{a_{22}})_{,1}\frac{1}{\sqrt{a_{11}}}v_1\right]$$

$$= (\rho\bar{f}^2 + b_{22}a^{22}m^2)\sqrt{a_{11}} .$$

In order to simplify our problem we assume that $T^{\alpha 3} = T^{3\alpha} = 0$ on the faces of the shell Ω^+ and Ω^-, i.e. there is no shearing but rather loading on these surfaces. Now if we seek solutions that have $T^{\alpha 3} = T^{3\alpha} = 0$ throughout the shell-body then one has [1] pg. 145

$$m^\alpha = 0 \ , \ N^{\alpha 3} = 0 \ , \ M^{\alpha 3} = 0 . \tag{2.15}$$

We now turn to computing the terms in (2.1b). First one has

$$N^{\alpha 3}_{|\alpha} = N^{13}_{,1} + N^{23}_{,2} + (\Gamma^1_{11} + \Gamma^1_{12})N^{13} + (\Gamma^1_{21} + \Gamma^2_{22})N^{23}$$

$$+ \Gamma^3_{11} N^{11} + \Gamma^3_{21} N^{12} + \Gamma^3_{12} N^{21} + \Gamma^3_{22} N^{22} . \tag{2.16}$$

On the reference surface $\xi = 0$ we have $\Gamma^3_{\alpha\beta} = \Gamma^*_{\alpha\beta} = b_{\alpha\beta}$; hence, (2.12) becomes

$$N^{\alpha 3}_{|\alpha} = N^{13}_{,1} + N^{23}_{,\alpha} + (\Gamma^1_{11} + \Gamma^1_{12})N^{13} + (\Gamma^1_{21} + \Gamma^2_{22})N^{23}$$

$$+ \frac{b_{11}}{a_{11}\sqrt{a_{22}}}\left[\sqrt{a_{22}}\, N_{<1,1>}\right] + \frac{b_{22}}{a_{22}\sqrt{a_{11}}}\left[\sqrt{a_{11}}\, N_{<2,2>}\right],$$

which upon using (2.11) we obtain for (2.1b)

$$\frac{b_{11}}{a_{11}}(N_{<1,1>}) + \frac{b_{22}}{a_{22}}(N_{<2,2>}) + \frac{1}{2}\,\rho \bar{f}^3 = 0\,, \tag{2.17}$$

In terms of the unknowns u_1, u_2, v_1, v_2 equation (2.17) becomes

$$\frac{b_{11}}{\sqrt{a_{11}}}\,u_1 + \frac{b_{22}}{\sqrt{a_{22}}}\,v_2 = \frac{1}{2}\,\sqrt{a_{11}a_{22}}\,\rho \bar{f}^3\,. \tag{2.18}$$

Rewriting equations (2.2a) over in terms of the physical coordinates and using similar methods to those for the equations (2.1) we obtain

$$M^{\alpha 1}_{|\alpha} = \frac{1}{\sqrt{a_{11}}}\left\{\frac{1}{\sqrt{a_{11}a_{22}}}\left[(\sqrt{a_{22}}\,M_{<11>})_{,1} + (\sqrt{a_{11}}\,N_{<2,1>})_{,2}\right]\right.$$

$$\left.+ (\sqrt{a_{11}})_{,2}\,M_{<12>} - (\sqrt{a_{22}})_{,1}\,M_{<22>}\right]\Bigg\}, \tag{2.19}$$

and

$$M^{\alpha 2}_{|\alpha} = \frac{1}{\sqrt{a_{22}}}\left\{\frac{1}{\sqrt{a_{11}a_{22}}}\left[(\sqrt{a_{22}}\,M_{<12>})_{,1} + (\sqrt{a_{11}}\,M_{<22>})_{,2}\right]\right.$$

$$\left.- (\sqrt{a_{11}})_{,2}\,M_{<11>} + (\sqrt{a_{22}})_{,1}\,M_{<21>}\right]\Bigg\}, \tag{2.20}$$

with $M_{<\alpha\beta>} = M_{<\beta\alpha>}$. Again introducing new unknowns, namely

$$u_3 = \sqrt{a_{22}}\,M_{<11>}\ \ ,\ \ u_4 = \sqrt{a_{22}}\,M_{<12>}\,,$$

$$v_3 = \sqrt{a_{11}}\,M_{<21>}\ \ ,\ \ v_4 = \sqrt{a_{11}}\,M_{<22>}\,,$$

the equations (2.2a) become

$$\frac{1}{\sqrt{a_{11}a_{22}}}\left\{u_{3,1}+v_{3,2}+\frac{(\sqrt{a_{11}})_{,2}}{\sqrt{a_{22}}}\,u_4-\frac{(\sqrt{a_{22}})_{,1}}{\sqrt{a_{11}}}\,v_4\right\}=-\sqrt{a_u}\,\rho\overline{l}^1\,, \tag{2.21}$$

$$\frac{1}{\sqrt{a_{11}a_{22}}}\left\{u_{4,1}+v_{4,2}-\frac{(\sqrt{a_{11}})_{,2}}{\sqrt{a_{22}}}\,u_3+\frac{(\sqrt{a_{22}})_{,1}}{\sqrt{a_{11}}}\,v_3\right\}=-\sqrt{a_{11}}\,\rho\overline{l}^2\,, \tag{2.22}$$

and

$$\sqrt{a_{11}}\,u_4=\sqrt{a_{22}}\,v_3\,. \tag{2.23}$$

At this point we have eight unknowns and only six equations. One equation is obtained from (2.2b) and another form

$$N^{\alpha\beta}-N^{\beta\alpha}=b_\gamma^\alpha\,M^{\gamma\beta}-b_\gamma^\beta\,M^{\gamma\alpha} \tag{2.24}$$

which comes from linearizing the angular momentum equations [1] pg. 139. From (2.2b) we get

$$b_{11}M^{11}+b_{12}M^{12}+b_{21}M^{21}+b_{22}M^{22}=m^3-\rho\overline{l}^3\,,$$

which becomes, in terms of the new unknowns,

$$\left[\frac{b_{11}}{\sqrt{a_{11}}}\,u_3+\frac{b_{22}}{\sqrt{a_{22}}}\,v_4\right]=\sqrt{a_{11}a_{22}}(m^3-\rho\overline{l}^3)\,, \tag{2.25}$$

and from (2.24) we have

$$\frac{1}{\sqrt{a_{11}a_{22}}}\left(\sqrt{a_{11}}\,u_2-\sqrt{a_{22}}\,v_1\right)=\frac{b_1^1-b_2^2}{\sqrt{a_{22}}}\,u_4=\frac{b_1^1-b_2^2}{\sqrt{a_{11}}}\,v_3\,.$$

Collecting our equations we arrive at the following system

$$u_{1,1}+v_{1,2}+\alpha_{12}u_2-\alpha_{21}v_2=f_1\,, \tag{2.27a}$$

$$u_{2,1}+v_{2,2}-\alpha_{12}u_1+\alpha_{21}v_1=f_2\,, \tag{2.27b}$$

$$u_{3,1}+v_{3,2}+\alpha_{12}u_4-\alpha_{21}v_4=f_3\,, \tag{2.27c}$$

$$u_{4,1} + v_{4,2} - \alpha_{21} u_3 + \alpha_{21} v_3 = f_4 \,, \tag{2.27d}$$

where the unknowns are connected by the relations

$$\beta_{11} u_1 + \beta_{22} v_2 = g_1 \,, \tag{2.28a}$$

$$u_4 - \gamma_{21} v_3 = 0 \,, \tag{2.28b}$$

$$\beta_{11} u_3 + \beta_{22} v_4 = g_2 \,, \tag{2.28c}$$

$$u_2 - \gamma_{21} v_1 = \delta u_4 \,. \tag{2.28d}$$

The nonhomogeneous terms above are given by

$$f_k := a_{11} \sqrt{a_{22}}\, \rho \bar{f}^k \quad , \quad (k = 1,2) \tag{2.29}$$

$$f_{k+2} := -\, a_{11} \sqrt{a_{22}}\, \rho \bar{l}^k \quad , \quad (k = 1,2)$$

$$g_1 = -\, \frac{\sqrt{a_{11} a_{22}}}{2}\, \rho \bar{f}^3 \quad ,$$

$$g_2 = \sqrt{a_{11} a_{22}}\, (m^3 - \rho \bar{l}^3) \quad .$$

The coefficients β_{kk} , γ_{21} , δ , $\alpha_{\mu\nu}$ are given by

$$\alpha_{\mu\nu} := \frac{(\sqrt{a_{\mu\mu}})_{,\nu}}{\sqrt{a_{\nu\nu}}} \quad , \quad \beta_{kk} := \frac{b_{kk}}{\sqrt{a_{kk}}} \quad ,$$

$$\delta := b_1^1 - b_2^2 \quad , \quad \gamma_{21} = \frac{\sqrt{a_{22}}}{a_{11}} \quad . \tag{2.30}$$

The system (2.27) (2.28) may be rewritten in the form

$$I\, \mathbf{u}_{,1} + A\, \mathbf{u}_{,2} + B\, \mathbf{u} + \mathbf{F} \quad , \quad \mathbf{u} := (u_1 \,, u_2 \,, u_3 \,, u_4)^t \quad , \tag{2.31}$$

where A and B are the 4x4 matrices defined below and I is the identity, namely

$$A := \begin{bmatrix} 0 & 1/\gamma_{21} & 0 & -\delta/\gamma_{21} \\ -\beta_{11}/\beta_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\gamma_{21} \\ 0 & 0 & -\beta_{11}\beta_{22} & 0 \end{bmatrix}, \tag{2.32}$$

and

$$B := \begin{bmatrix} \alpha_{21}\beta_{11}/\beta_{22} & \alpha_{12}+(1/\gamma_{21})_{,2} & 0 & -(\delta/\gamma_{21})_{,2} \\ -(\alpha_{21}+(\beta_{11}/\beta_{22})_{,2}) & \alpha_{21}/\gamma_{21} & 0 & -\alpha_{21}\delta/\gamma_{21} \\ 0 & 0 & \alpha_{21}\beta_{11}/\beta_{22} & \alpha_{12}+(1/\gamma_{21})_{,2} \\ 0 & 0 & \alpha_{12}+(\beta_{11}/\beta_{22})_{,2} & \alpha_{21}/\gamma_{21} \end{bmatrix} \tag{2.33}$$

III. Reduction to Normal Form

In this section we shall show that the system (2.31) (2.32) (2.33) may be put into the generalized hyperanalytic form [1], [3], [5], [8]. To this end we consider just the principal part of (2.31), namely $I\,\mathbf{u}_x + A\,\mathbf{u}_{,y} = g$. The matrix $(I - \lambda A)$ has the two purely imaginary eigenvalues $\lambda = \pm i \sqrt{\dfrac{\beta_{11}}{\beta_{22}\gamma_{21}}}$ which are double roots of the characteristic equation. Let us designate these two roots by $\pm ib(x,y)$ where $b > 0$. Then the Jordan cannonical form for A is given by

$$J = \begin{bmatrix} ib & 1 & 0 & 0 \\ 0 & ib & 0 & 0 \\ -0 & 0 & -ib & 1 \\ 0 & 0 & 0 & -ib \end{bmatrix}, \tag{3.1}$$

and $J = Q^{-1}AQ$, where $Q = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4]$. Here $\mathbf{q}_1$ is the eigenvector corresponding to $\lambda_1 = ib$ and $\mathbf{q}_2$ is the generalized eigenvectors which satisfies $(A - ib)\mathbf{q}_2 = \mathbf{q}_1$. Moreover $\mathbf{q}_3$ is the eigenvector corresponding to $\lambda_2 = -ib$ and $\mathbf{q}_4$ the associated generalized eigenvector. It follows that Q is given by

$$Q = \begin{bmatrix} 1 & 0 & 1 & 0 \\ ib/a & -1/a & ib/a & -1/a \\ 0 & 2i/b\delta & 0 & -2i/b\delta \\ 0 & -2/a\delta & 0 & -2/a\delta \end{bmatrix} \tag{3.2}$$

In (3.2) the coefficients a, b, and c are defined to be positive functions of x, y given below

$$b = \sqrt{\frac{\beta_{11}}{\beta_{22}\gamma_{21}}} > 0 \ , \ a = 1/\gamma_{21} > 0 \ , \ c = \beta_{11}/\beta_{22} > 0 \ . \tag{3.3}$$

Q^{-1} is given by

$$Q^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -2ia/b & 0 & ia\delta/b \\ 0 & 0 & -ib\delta & -a\delta \\ 2 & 2ia/b & 0 & -ia\delta/b \\ 0 & 0 & ib\delta & -a\delta \end{bmatrix} . \tag{3.4}$$

Introducing the new unknowns $\mathbf{u} := Q\mathbf{V}$ our differential equation (2.31) takes on the form

$$I\mathbf{V}_{,x} + J\mathbf{V}_{,y} + R\mathbf{V} = \mathbf{S} \ , \tag{3.5}$$

where

$$R := Q^{-1}Q_{,x} + Q^{-1}AQ_{,y} + Q^{-1}CQ \ , \ S := Q^{-1}\mathbf{F} \ . \tag{3.6}$$

By using the complex derivative

$$\frac{\partial}{\partial z} = \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right] \ , \ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]$$

(3.5) may be written as

$$V_{\bar{z}} + \left[I - \frac{ie}{1+b}\right]^{-1} \left[\frac{1-b}{1+b} I + \frac{ie}{1+b}\right] V_z + P\mathbf{V} = \mathbf{t} \ ,$$

with e the nilpotent matrix

$$e = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.8}$$

and $P := (I - \frac{ie}{1+b})^{-1}R$, etc.. If we consider the principal part of (3.8) it is known

(See Gilbert Buchanan [2], or Wendland [8] for example.) that using a solution of the Beltrami equation

$$w_{\bar{z}} + \frac{1-b}{1+b}\, w_z = 0 \tag{3.9}$$

as the new independent complex variable permits us to put the first two equations of (3.7) into hypercomplex form. The equation

$$w_{\bar{z}} + \frac{1+b}{1-b}\, w_z = 0$$

provides us with the proper Beltrami transform for the second pair of equations.

A computationally direct approach for finding the normal form arises by using the eigenvalues of the transposed matrix A^t [4]. The eigenvalues are the same $\pm ib$. Corresponding to $+ib$ we have the eigenvector $\mathbf{Y}^{(1)t} := [0,0,1,-ib/c]$ and the generalized eigenvector

$$\mathbf{Y}^{(2)t} := [2i/b\delta,\ 2/c\delta,\ 2i/b,\ 1/c] \ .$$

We introduce the new complex, unknowns w_1, w_2 then as

$$w_1 := \mathbf{y}^{(1)}\cdot \mathbf{u} \quad \text{and} \quad w_2 := \mathbf{y}^{(2)}\cdot \mathbf{u} \ .$$

The components of $\mathbf{u}$ may then be found from the matrix equation

$$\mathbf{u} = G\,\mathbf{w} := \begin{bmatrix} -\delta/2\ , & -ib\delta/4\ , & -\delta/2\ , & ib\delta/4 \\ -ic\delta/4b\ , & c\delta/4\ , & ic\delta/4b\ , & c\delta/4 \\ 1/2 & 0 & 1/2 & 0 \\ ic/2b & 0 & -ic/2b & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \bar{w}_1 \\ \bar{w}_2 \end{bmatrix} . \tag{3.11}$$

The system (2.31) upon multiplying on the right with the row vector $\mathbf{Y}^{(1)t}$ becomes

$$w_{1,x} + ibw_{1,y} + \mathbf{Y}^{(1)t}BG\,\mathbf{w} + c_1 = 0 \ , \quad c_1 := \mathbf{Y}^{(1)t}\cdot\mathbf{F}$$

or

$$(1+b)w_{1,\bar{z}} + (1-b)w_{1,z} + D_1w_1 + D_2w_2 + E_1\bar{w}_1 + E_2\bar{w}_2 + c_1 = 0 \ . \tag{3.12}$$

The system (2.32) upon multiplying on the right with the row vector $\mathbf{Y}^{(2)t}$ becomes

$$(1+b)w_{2,\bar{z}} + (1-b)w_{2,z} + iw_{1,z} - iw_{1\bar{z}} + \mathbf{Y}^{(2)t} BG\,\mathbf{w} + c_2 = 0$$

Since $w_{1,\bar{z}}$ may be removed using (3.12), this takes on the form

$$(1+b)w_{2,\bar{z}} + (1-b)w_{2,z} + iw_{1,z} + D_3 w_1 + D_4 w_2 + E_3 \bar{w}_1 + E_4 \bar{w}_2 + c_2 = 0 \ . \qquad (3.13)$$

The special case where $b \equiv$constant, which *w.o.l.og.* we take to the 1, is of interest. Here (3.12), (3.13) assume the hyperanalytic principal part, namely

$$w_{1,\bar{z}} + \text{terms without derivatives} \ ,$$

and

$$w_{2,\bar{z}} + \frac{i}{2} w_{1,z} + \text{terms without derivatives} \ .$$

Hence by taking the hypercomplex derivative ∂ to be defined as

$$\partial := \frac{\partial}{\partial \bar{z}} + \frac{i}{2} e \frac{\partial}{\partial z} \ , \ e^2 = 0 \ ,$$

and the hypercomplex unknown as

$$w = w_1 + e w_2 \ ,$$

the principal part becomes ∂w, and the system takes the form

$$\partial w + \sum_{k=1}^{2} \sum_{j=1}^{2} e^{k-1}(A_{kj} w_j + B_{kj} \bar{w}_j) + c = 0 \ , \qquad (3.14)$$

$c := c_1 + e c_2$. If $b \neq$constant then a Beltrami transformation must be used in addition to normalize the principal part. When b may be effectively approximated by polynomials there are efficient methods for analytically constructing this mapping. An exhaustive study of the boundary value problems associated with (3.14) is presented in the books by Wendland [8] and Gilbert-Buchanan [2].

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