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DENSE SETS AND THE PROJECTION THEOREM FOR ACOUSTIC HARMONIC WAVES IN A HOMOGENEOUS FINITE DEPTH OCEAN

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Dense Sets and the Projection Theorem for Acoustic Harmonic Waves in a Homogeneous Finite Depth Ocean

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1. Introduction

In a homogeneous constant depth ocean with a free surface and a rigid bottom, the propagating outgoing waves will evanesce except for a finite number of propagating modes[1]. Let $\mathbf{R}^3_b = \{(\mathbf{x}, z);\ \ \mathbf{x} = (x_1, x_2) \in \mathbf{R}^2, 0 \leq z \leq h\}$ be a region corresponding to the finite depth ocean, where h is the ocean depth . Let Ω be an object inbedded $\mathbf{R}^3_b$. The Dirichlet boundary value problem for the scattering of time-harmonic acoustic waves in the ocean can be formulated as finding a solution $u \in C^2(\mathbf{R}^3_b \setminus \overline{\Omega}) \cap C(\mathbf{R}^3_b \setminus \Omega)$ to the Helmholtz equation

$$\triangle_3 u + k^2 u = 0,\ in\ \mathbf{R}^3_b \setminus \overline{\Omega}, \tag{1.1}$$

such that u satisfies the boundary conditions

$$u = 0,\ as z = 0, \tag{1.2}$$

$$\frac{\partial u}{\partial z} = 0,\ as z = h, \tag{1.3}$$

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$$u = 0, \ on \ \partial\Omega. \tag{1.4}$$

Here k is a positive constant known as the wave number, and $u = u^i + u^s$, where u^i and u^s are the incident (entire) wave and the scattered wave respectively. The scattered wave has the modal representation

$$u^s = \sum_{n=0}^{\infty} \phi_n(z) u_n^s(\mathbf{x}). \tag{1.5}$$

where

$$\phi_n(z) = sin[k(1-a_n^2)^{\frac{1}{2}}] \tag{1.6}$$

$$a_n = [1 - \frac{(2n+1)^2\pi^2}{4k^2h^2}]^{\frac{1}{2}} \tag{1.7}$$

and the n^{th} mode of u^s, $u_n^s(\mathbf{x})$, satisfies the radiating condition

$$\lim_{r\to\infty} r^{\frac{1}{2}}(\frac{\partial u_n^s}{\partial r} - ika_n u_n^s) = 0, \ r = \mid \mathbf{x} \mid, \ n = 0, 1, ..., \infty. \tag{1.8}$$

The incident wave u^i is scattered by Ω to produce a scattered "propagating" far-field pattern.[5]. We want to extract information about the far-field in order to use it to investigate the object Ω. This problem has already been investigated in $\mathbf{R}^2$ by Colton and Kirsch[2]. To do this they introduce a certain dense subset of the far-field pattern. Colton and Monk [7],[8] are able to determine the shape of the object by introducing an extremal problem and solving it in projected subspaces. Two kinds of algorithms have been developed for the whole space case [7],[8]; see also [6]. However,in the case of a finite depth ocean, Gilbert and Xu[5] showed that the "propagating" far-field pattern can only carry the information from N+1 propagating modes; here, N is the largest integer less than $\frac{2kh-\pi}{2\pi}$. This loss of information makes this nonlinear, improperly posed ,inverse scattering problem much more difficult to solve than the case mention above.

In this paper, we consider the density properties of the propagating far-field in a proper subspace of $L^2(D_1)$ where D_1 is the unit cylinder. The propagating far field is decomposed into orthogonal components which allows a numerical algorithm to be generated for the express purpose of reconstructing the object Ω. This algorithm along with numerical simulations will be presented in a following paper.

2 Complete Sets in $L^2(\partial\Omega)$

Let $\mathbf{R}_b^3 = \{(\mathbf{x}, z) \in \mathbf{R}^3, 0 \leq z \leq h\}$ where $\mathbf{x} = (x_1, x_2)$, and h is a positive constant. Let Ω be a bounded, connected domain in $\mathbf{R}_b^3$ with C^2 boundary $\partial\Omega$ having an outward unit normal ν. Moreover, define the relative complement of Ω as $\Omega_e := \mathbf{R}_b^3 \setminus \overline{\Omega}$.

Let $J_n(r)$ denote Bessel's function of order n and $H_n^{(1)}(r)$ Hankel's function of the first kind of order n. $\phi_n(z)$ and a_n are defined as (1.6) and (1.7),

Theorem2.1 Let λ be a complex number such that $0 \leq Im\lambda \leq \infty$. Then the set of functions

$$(\frac{\partial}{\partial\nu} + \lambda)[\phi_n(z)J_m(ka_nr)cos(m\theta)] \tag{2.1}$$

$$(\frac{\partial}{\partial\nu} + \lambda)[\phi_n(z)J_m(ka_nr)sin(m\theta)] \tag{2.2}$$

$$n, m = 0, 1, ..., \infty,$$

are complete in $L^2(\partial\Omega)$.

Proof : It suffices to show that if $g \in L^2(\partial\Omega)$, such that

$$\int_{\partial\Omega} g(r,z,\theta)(\frac{\partial}{\partial\nu}+\lambda)[\phi_n(z)J_m(ka_n r)cos(m\theta)]d\sigma = 0, \tag{2.3}$$

$$\int_{\partial\Omega} g(r,z,\theta)(\frac{\partial}{\partial\nu}+\lambda)[\phi_n(z)J_m(ka_n r)sin(m\theta)]d\sigma = 0, \tag{2.4}$$

for $m, n = 0, 1, ..., \infty$, then g is identically zero on $\partial\Omega$.

Let (2.3),(2.3) be true for some $g \in L^2(\partial\Omega)$, and let Ω_{r_0} be a cylinder containing Ω, $\Omega_{r_0} = \{(\mathbf{x}, z) \in \mathbf{R}_b^3, |\, \mathbf{x} \,| = r_0\}$, then for $(\mathbf{x}, z) \in \mathbf{R}_b^3 \setminus \overline{\Omega_{r_0}}$, and $(\xi, \zeta) \in \partial\Omega$, we know that for $r = |\, \mathbf{x} \,| > |\, \xi \,| = r'$, we can expand the Green's function $G(z, \zeta, |\, \mathbf{x} - \xi \,|)$ as

$$G(z,\zeta,|\,\mathbf{x}-\xi\,|) = \frac{i}{4}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\epsilon_m\phi_n(z)\phi_n(\zeta)}{\|\,\phi_n\,\|^2}H_m^{(1)}(ka_n r)J_m(ka_n r')$$

$$[cos(m\theta)cos(m\theta') + sin(m\theta)sin(m\theta')]. \tag{2.5}$$

Here we denote $(\mathbf{x}, z)$ in cylindrical coordinates by (r, θ, z), and (ξ, ζ) by (r', θ', ζ).

From (2.2) and (2.3), we can see that

$$u(\mathbf{x}, z) := \int_{\partial\Omega}(\frac{\partial}{\partial\nu_\xi}+\lambda)G(z,\zeta,|\,\mathbf{x}-\xi\,|)g(r',\zeta',\theta')d\sigma \tag{2.6}$$

is identically zero for $(\mathbf{x}, z) \in \mathbf{R}_b^3 \setminus \overline{\Omega_{r_0}}$. Since u, as defined by (2.6), is a solution of the Helmholtz equation in $\mathbf{R}_b^3 \setminus \overline{\Omega_{r_0}}$, we can conclude by the analyticity of solutions to the Helmholtz eqation [4] that $u(\mathbf{x}, z)$ is identically zero for $(\mathbf{x}, z) \in \mathbf{R}_b^3 \setminus \Omega$.

Let $(\mathbf{x}, z)$ tend to $\partial\Omega$, then in view of the ray representation for the Green's function [1]

$$G(z,\zeta,|\,\mathbf{x}-\xi\,|) = \frac{e^{ik\sqrt{|\mathbf{x}-\xi|^2+(z-\zeta)^2}}}{4\pi\sqrt{|\,\mathbf{x}-\xi\,|^2+(z-\zeta)^2}} + \Phi_1(z,\zeta,|\,\mathbf{x}-\xi\,|), \tag{2.7}$$

where

$$\Phi_1(z,\zeta,|\mathbf{x}-\xi|)=\frac{1}{4\pi}\sum_{n=-\infty,n\neq 0}^{\infty}\{\frac{e^{ik\sqrt{|\mathbf{x}-\xi|^2+(z-\zeta-2nh)^2}}}{\sqrt{|\mathbf{x}-\xi|^2+(z-\zeta-2nh)^2}}$$

$$-\frac{e^{ik\sqrt{|\mathbf{x}-\xi|^2+(z+\zeta-2nh)^2}}}{\sqrt{|\mathbf{x}-\xi|^2+(z+\zeta-2nh)^2}}\}-\frac{e^{ik\sqrt{|\mathbf{x}-\xi|^2+(z+\zeta)^2}}}{4\pi\sqrt{|\mathbf{x}-\xi|^2+(z+\zeta)^2}}$$

and the properties of single and double layer potentials, we know (cf.[3] [4])

$$0=g(\mathbf{x},z)+\int_{\partial\Omega}(\frac{\partial}{\partial\nu_\xi}+\lambda)G(z,\zeta,|\mathbf{x}-\xi|)g(\xi,\zeta)d\sigma,\ (\mathbf{x},z)\in\partial\Omega. \quad (2.8)$$

Now let us denote by $u_+(\mathbf{x}_0,z_0),\ u_-(\mathbf{x}_0,z_0)$,

$$u_+(\mathbf{x}_0,z_0)=\lim_{(\mathbf{x},z)\to(\mathbf{x}_0,z_0)}u(\mathbf{x},z),\quad (\mathbf{x},z)\in\Omega_e,\ (\mathbf{x}_0,z_0)\in\partial\Omega,$$

and

$$u_-(\mathbf{x}_0,z_0)=\lim_{(\mathbf{x},z)\to(\mathbf{x}_0,z_0)}u(\mathbf{x},z),\quad (\mathbf{x},z)\in\Omega,\ (\mathbf{x}_0,z_0)\in\partial\Omega.$$

Similar definitions are made for $(\frac{\partial u}{\partial\nu})_+$ and $(\frac{\partial u}{\partial\nu})_-$. From our knowledge of single and double layer potentials

$$u_+-u_-=2g,\ \ on\ \partial\Omega, \quad (2.9)$$

and

$$(\frac{\partial u}{\partial\nu})_+-(\frac{\partial u}{\partial\nu})_-=-2\lambda g,\ on\ \partial\Omega. \quad (2.10)$$

Since $u_+=(\frac{\partial u}{\partial\nu})_+=0$, we have from (2.9) and (2.10) that

$$(\frac{\partial u}{\partial\nu})_-+\lambda u_-=0,\ on\ \partial\Omega. \quad (2.11)$$

Hence u, as defined by (2.6), is a solution of Helmholtz equation in Ω and continuously assumes the boundary data (2.11) on $\partial\Omega$. It follows that $u=0$ in Ω.

By the relation

$$g = \frac{1}{2}(u_+ - u_-), \quad on\ \partial\Omega$$

we can conclude that $g = 0$ on $\partial\Omega$. It proves the completeness of the set (2.1) and (2.2) in $L^2(\partial\Omega)$.

3 Dense Sets in $L^2(\partial\Omega)$

We modify the notations of [2] to the case of $\mathbf{R}_b^3$, namely let $\mathcal{N}$ be the family of any finite subset of natural numbers containing 0,1,...,N; $D_1 = [0, h] \times [0, 2\pi]$; and

$$H(k, \Omega_e) := \{u : u \in C^2(\Omega_e) \cap C^1(\overline{\Omega_e}),\ u\, satisfies\, (1.1) \sim (1.4)\, and (1.8)\}$$

$$A(k, R_b^3) := \{u :\ u(\mathbf{x}, z) = \int_0^{2\pi} \int_0^h g(\zeta, \theta') \sum_{n \in \Lambda} \phi_n(\zeta)\phi_n(z)e^{ik\alpha_n \mathbf{x}\cdot\mathbf{y}} d\zeta d\theta',$$

$$where\ \ (\mathbf{x}, z) \in R_b^3,\ \mathbf{y} = (cos\theta', sin\theta'),\ g \in L^2(D_1),\ \Lambda \in \mathcal{N}\}.$$

Moreover, we set

$$T_D(k, \Omega_e) = \{u; u = u^i + u^s, u^i \in A(k, R_b^3), u^s \in H(k, \Omega_e), u = 0,\ on\ \partial\Omega\},$$

and

$$\frac{\partial T_D(k, \Omega_e)}{\partial \nu} |_{\partial\Omega} = \{\frac{\partial u}{\partial \nu} |_{\partial\Omega}:\ u \in T_D(k, \Omega_e)\}.$$

We want to prove the following

Theorem 3.1: $\frac{\partial T_D(k,\Omega_e)}{\partial \nu} |_{\partial\Omega}$ is dense in $L^2(\partial\Omega)$.

Proof: Let $g \in L^2(\partial\Omega)$ such that

$$\int_{\partial\Omega} \overline{g} \frac{\partial u}{\partial \nu} ds = 0,\ for\ any\ u \in T_D(k, \Omega_e). \tag{3.1}$$

The Theorem will be proved if we can show that (3.1) implies g=0 on $\partial\Omega$. If u be an arbitrary element of $T_D(k,\Omega_e)$, then from the representation formula [3] we get

$$u(\mathbf{x},z) = u^i(\mathbf{x},z) - \int_{\partial\Omega} G(z,\zeta,|\,\mathbf{x}-\xi\,|)\frac{\partial u}{\partial\nu}d\sigma_\xi, \tag{3.2}$$

where $u^i \in A(k,R_b^3), u = u^i + u^s$.

Let $(\mathbf{x},z) \to \partial\Omega$, (3.2) then implies that

$$u^i(\mathbf{x},z) = \int_{\partial\Omega} G(z,\zeta,|\,\mathbf{x}-\xi\,|)\frac{\partial u}{\partial\nu}d\sigma_\xi, \tag{3.3}$$

and

$$\frac{\partial u(\mathbf{x},z)}{\partial\nu_x} + 2\int_{\partial\Omega}\frac{\partial}{\partial\nu_x}G(z,\zeta,|\,\mathbf{x}-\xi\,|)\frac{\partial u}{\partial\nu_\xi}d\sigma_\xi = 2\frac{\partial u^i(\mathbf{x},z)}{\partial\nu_x}. \tag{3.4}$$

We define the singular operators $\mathbf{S}$,$\mathbf{K}$, $\mathbf{K}'$ from $L^2(\partial\Omega)$ into itself as usual, by setting

$$\mathbf{S}\phi := 2\int_{\partial\Omega} G(z,\zeta,|\,\mathbf{x}-\xi\,|)\phi(\xi,\zeta)d\sigma_\xi,\ (\mathbf{x},z)\in\partial\Omega,$$

$$\mathbf{K}\phi := 2\int_{\partial\Omega}\frac{\partial}{\partial\nu_\xi}G(z,\zeta,|\,\mathbf{x}-\xi\,|)\phi(\xi,\zeta)d\sigma_\xi,\ (\mathbf{x},z)\in\partial\Omega,$$

and

$$\mathbf{K}'\phi := 2\int_{\partial\Omega}\frac{\partial}{\partial\nu_x}G(z,\zeta,|\,\mathbf{x}-\xi\,|)\phi(\xi,\zeta)d\sigma_\xi,\ (\mathbf{x},z)\in\partial\Omega.$$

In view of the representation (2.5) for $G(z,\zeta,|\,\mathbf{x}-\xi\,|)$ and the fact it is symmetric with respect to $(\mathbf{x},z)$ and (ξ,ζ), it may be shown that $\mathbf{K}'$ is the adjoint of $\mathbf{K}$ subject to the pairing

$$<\phi,\psi>:= \int_{\partial\Omega}\phi\psi d\sigma. \tag{3.5}$$

Moreover, it can be seen that $\mathbf{I}+\mathbf{K}+i\mathbf{S}$ is invertible.(cf.[3]). From (3.3) and (3.4) it follows that

$$(\mathbf{I}+\mathbf{K}+i\mathbf{S})\frac{\partial u(\mathbf{x},z)}{\partial \nu} = 2(\frac{\partial u^i(\mathbf{x},z)}{\partial \nu} + iu^i(\mathbf{x},z)),\ (\mathbf{x},z)\in\partial\Omega, \tag{3.6}$$

and hence

$$\frac{\partial u(\mathbf{x},z)}{\partial \nu} = 2(\mathbf{I}+\mathbf{K}'+i\mathbf{S})^{-1}(\frac{\partial u^i(\mathbf{x},z)}{\partial \nu} + iu^i(\mathbf{x},z)),\ (\mathbf{x},z)\in\partial\Omega. \tag{3.7}$$

Substituting (3.7) into (3.1) yields

$$\begin{aligned} 0 &=< \overline{g}, \frac{\partial u}{\partial \nu} >=< \overline{g}, 2(\mathbf{I}+\mathbf{K}'+i\mathbf{S})^{-1}(\frac{\partial u^i}{\partial \nu} + iu^i) > \\ &= 2 < (\mathbf{I}+\mathbf{K}+i\mathbf{S})^{-1}\overline{g}, \frac{\partial u^i}{\partial \nu} + iu^i > . \end{aligned} \tag{3.8}$$

Since $u^i \in A(k,\mathbf{R}^3_b)$, using the Jacobi-Anger expansion,

$$\phi_n(z)e^{ika_n r cos\theta} = \sum_{m=-\infty}^{\infty} i^m \phi_n(z) J_m(ka_n r) e^{im\theta}, \tag{3.9}$$

$$for\ n = 0,1,...,\infty,$$

we conclude that $\phi_n(z)J_m(ka_n r)cosm\theta$ and $\phi_n(z)J_m(ka_n r)sinm\theta$ are elements of $A(k,\mathbf{R}^3_b)$. Hence, from Theorem 2.1, we get

$$(\mathbf{I}+\mathbf{K}+i\mathbf{S})^{-1}\overline{g} = 0;$$

and,

$$g = 0,\ on\ \partial\Omega.$$

4 The Projection Theorem in V^N

Let $N = [\frac{2kh-\pi}{2\pi}]$, where [a] means the interger part of a, and let us introduce the product space

$$V^N := L^2[0,2\pi] \times \overline{span\{\phi_0,\phi_1,...,\phi_N\}} \tag{4.1}$$

From [3],[5] we know that the propagating far-field patterns of acoustic harmonic waves in a homogeneous finite depth ocean are contained in V^N. We will now establish a condition for the far field patterns to be dense in V^N for arbitrary domains.

We define the injections:

(1) $P:\, A^N \subset A(k,\mathbf{R}_b^3) \to V^N$ by g:=Pu where

$$u(\mathbf{x},z) = \int_{D_1} g(\theta,\zeta) \sum_{n\in\Lambda} \phi_n(z)\phi_n(\zeta) e^{ika_n \mathbf{x}\cdot\mathbf{y}} d\sigma \tag{4.2}$$

where $g \in V^N$, $\mathbf{y} = (cos\theta', sin\theta')$, $(\mathbf{x},z) \in R_b^3$. $A^N = \{u \in A(k,\mathbf{R}_b^3):\ g \in V^N)$.

(2) $F:\ A(k,R_b^3) \to V^N$ by $F(\theta,z,k) := Fu^i$ where $F(\theta,z,k)$ is the propagating far-field pattern of u^s for $u = u^i + u^s \in T_D(k,\Omega_e)$.

Let $E_D(k,\Omega) = \{u:\ u \in C^2(\Omega)\cap C(\overline{\Omega})$, u is a solution of Helmholtz equation in Ω and u=0 on $\partial\Omega\}$

We will prove the following result:

Theorem 4.1:

$$V^N = [P(E_D(k,\Omega)\cap A^N)] \oplus \overline{F(A(k,R_b^3))}$$

where $\overline{F(A(k,R_b^3))}$ is the closure of $F(A(k,R_b^3))$ in V^N.

Proof: By the representation of u^s[3], we have

$$u^s(\mathbf{x},z) = \int_{\partial\Omega} (u^s \frac{\partial G}{\partial\nu} - G\frac{\partial u^s}{\partial\nu}) d\sigma,\ (\mathbf{x},z) \in \Omega_e,$$

and

$$0 = \int_{\partial\Omega} (u^i \frac{\partial G}{\partial\nu} - G\frac{\partial u^i}{\partial\nu}) d\sigma,\ (\mathbf{x},z) \in \Omega_e.$$

Let $u = u^i + u^s$; then

$$u^s(\mathbf{x}, z) = \int_{\partial\Omega} (u \frac{\partial G}{\partial \nu} - G \frac{\partial u}{\partial \nu}) d\sigma, \ (\mathbf{x}, z) \in \Omega_e. \tag{4.3}$$

In view of the asympototic behavior of Hankel's function and the representation

$$G(z, \zeta, \mid \mathbf{x} - \xi \mid) = \frac{i}{2h} \sum_{n=0}^{\infty} \phi_n(z) \phi_n(\zeta) H_0^{(1)}(ka_n \mid \mathbf{x} - \xi \mid),$$

we obtain the asymptotic formula

$$u^s(\mathbf{x}, z) = \frac{i}{2h} e^{-i\pi/4} \sum_{n=0}^{N} (\frac{2}{\pi k a_n r})^{\frac{1}{2}} e^{ika_n r} f_n(\theta, z, k) + O(\frac{1}{r^{\frac{3}{2}}}), \tag{4.4}$$

where

$$\begin{aligned} f_n(\theta, z, k) = \phi_n(z) \int_{\partial\Omega} \{ & u(\xi, \zeta) \frac{\partial}{\partial \nu_\xi} (e^{-ika_n \mathbf{x} \cdot \xi} \phi_n(\zeta)) \\ & - \frac{\partial u(\xi, \zeta)}{\partial \nu_\xi} (e^{-ika_n \mathbf{x} \cdot \xi} \phi_n(\zeta)) \} d\sigma_\xi, \\ & \mathbf{x} = (cos\theta, sin\theta). \end{aligned} \tag{4.5}$$

The function $F(\theta, z, k) := \sum_{n=0}^{N} f_n(\theta, z, k) \in V^N$ is another representation of the propagating far-field pattern.

Let $u = u^i + u^s \in T_D(k, \Omega_e)$ and $v \in E_D(k, \Omega) \cap A^N$, then $Pv \in V^N$. From (4.3) and (4.4), we have

$$u^s(\mathbf{x}, z) = -\int_{\partial\Omega} G(z, \zeta, \mid \mathbf{x} - \xi \mid) \frac{\partial u}{\partial \nu} d\sigma, \ (\mathbf{x}, z) \in \Omega_e, \tag{4.6}$$

$$Fu^i(\theta, z, k) = -\sum_{n=0}^{N} \phi_n(z) \int_{\partial\Omega} \frac{\partial u(\xi, \zeta)}{\partial \nu_\xi} (e^{-ika_n \mathbf{x} \cdot \xi} \phi_n(\zeta)) d\sigma_\xi. \tag{4.7}$$

$$\int_{D_1} \overline{Gv(z, \theta)} Fu^i(\theta, z, k) dz d\theta$$

$$= -\int_{D_1} \overline{g(z,\theta)}[\sum_{n=0}^{N} \phi_n(z) \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \phi_n(\zeta) e^{-ika_n \mathbf{x}\cdot\xi} d\sigma_\xi] dz d\theta$$

$$= -\int_{\partial\Omega} \frac{\partial u}{\partial \nu_\xi} [\int_{D_1} \overline{g(z,\theta)} \sum_{n=0}^{N} \phi_n(z) e^{-ika_n \mathbf{x}\cdot\xi} dz d\theta] \phi_n(\zeta)) d\sigma_\xi. \tag{4.8}$$

Since $g(z,\theta) \in V^N$,

$$\int_{D_1} \overline{g(z,\theta)} \phi_n(z) e^{-ika_n \mathbf{x}\cdot\xi} dz d\theta = 0,$$

for any $n = N+1, ..., \infty$, so (4..8) becomes

$$\int_{D_1} \overline{Gv(z,\theta)} F u^i(\theta, z, k) dz d\theta = -\int_{\partial\Omega} \frac{\partial u}{\partial \nu_\xi} \overline{v(\xi,\zeta)} d\sigma_\xi = 0 \tag{4.9}$$

This proves the orthogonality

$$P(E_D(k,\Omega) \cap A^N) \perp \overline{F(A(k, \mathbf{R}_b^3))}$$

Now we prove that

$$P(E_D(k,\Omega) \cap A^N) = \overline{F(A(k, \mathbf{R}_b^3))}^{\perp} \tag{4.10}$$

In fact, if $g \in V^N$ such that

$$\int_{D_1} \overline{g(z,\theta)} F u^i(\theta, z, k) dz d\theta = 0, \; for\, any\, u^i \in A(k, \mathbf{R}_b^3) \tag{4.11}$$

then from (4.9)

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu_\xi} \overline{v(\xi,\zeta)} d\sigma_\xi = 0, \; for\, any\, u \in T_D(k, \Omega_e) \tag{4.12}$$

where

$$v(\xi,\zeta) = \int_{D_1} g(\theta,\zeta) \sum_{n\in\Lambda} \phi_n(z)\phi_n(\zeta) e^{ika_n \mathbf{x}\cdot\mathbf{y}} d\sigma_x \tag{4.13}$$

From theorem 3.1, $\frac{\partial T_D(k,\Omega_e)}{\partial \nu} |_{\partial\Omega}$ is dense in $L^2(\partial\Omega)$, so we can conclude that $v = 0$ on $\partial\Omega$. That is, $v \in E_D(k,\Omega) \cap A^N$ and $g = Pv \in P(E_D(k,\Omega) \cap$

A^N). It proves (4.10). Since V^N is a Hilbert space, (4.10) implies the theorem.

From the decomposition theorem (4.1), we get the following density result:

Corollary 4.2 A sufficient condition for the far-field patterns of the problem (1.1) $\sim$ (1.5) to be dense in V^N is that $E_D(k, \Omega) \cap A^N = \{0\}$. i.e. the eigenfunctions of Dirichlet problem are not elements of the set A^N.

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