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SEA GRANT

Technical Report

DEL-SG-01-88

Starting Fields and Far Fields

in Ocean Acoustics

$2.00

by

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January 1988

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The first author's research was supported in part by NSF Grant DNS8601318 and in part by Sea Grant NA86AA-D-SG040. The second author's research was supported by Sea Grant NA86AA-D-SG040.

I. Introduction

In Section II we continue with our use of transmutation theory [Dugw 86a,b], [Dugw 87], [Giwo 86], [Giwo 87a,b] to show how these ideas may be implemented to use with the parallel approximations to the Helmholtz equation for a stratified media [Tapp 77]

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2}+\frac{\partial^2 p}{\partial z^2}+k^2[n^2(z,r)+i\nu(z,r)]p=0\ , \tag{1.1}$$

namely

$$2ik\frac{\partial \psi}{\partial r}+\frac{\partial^2 \psi}{\partial z^2}+\frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}+k^2[n^2(z,r)-1+i\nu(z,r)]\psi=0\ . \tag{1.2}$$

In the particular case where $q(z,r) := n^2(z,r)+i\nu(z,r)$ *separates* into $n^2(z) + \varepsilon m^2(z,r)$ with $0 < \varepsilon \ll 1$, we obtain an analytical expression for the leading terms in the expansion of the transmutation kernel. Such results are useful for constructing parametrix approximations for the fundamental solution.

In Section III a parabolic approximation to the fundamental singularity (ν=0) is derived and this used to obtain an integral representation for the solution in an exterior region. Such representations are useful for constructing starting fields. The integral representation is then used to construct a parametrix for the finite, variable index ocean (3.26).

Section IV treats the far field in a uniform ocean of finite depth. In a finite ocean it is well-known that certain modes propagate, whereas the others attenuate. Hence, it is not surprising that an expression for the *far field* pattern is obtained which depends on just the propagating modes, namely

$$F_o(r,z;\theta) := \sum_{n=0}^{N} e^{ik\alpha_n r} f_{no}(z,\theta)\ . \tag{4.18}$$

It is shown that any propagating solution $v(\mathbf{x})$ of the Helmholtz equation

$$\overline{\lim_{r\to\infty}} \frac{1}{r} \int_{Z_r} |v(\mathbf{x})|^2 d\mathbf{x} < \infty$$

and the proper boundary conditions on the ocean surface and bottom is a *propagating Herglotz function*. Equations (5.25) and (5.27) imply, in contrast to the result of Colton-Monk for $\mathbf{R}^3$, that any attempt to obtain structure in the target identification problem must be constrained. For $\mathbf{R}_h^3$, however, this is seen to be a best possible theoretical result as F lies in a finite dimensional subspace of $L^2(\partial\mathbf{Z})$. Hence, the target identification problem for underwater acoustics must be perpetually plagued by problems of resolution. These need not be insurmountable providing sufficiently many modes propagate. Numerical experiments are presently being made to determine how many these may need to be and will be published in a sequel to the present work.

II. Transmutation Theory

In this section we use transmutation theory to investigate the parabolic approximation

$$2ik\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + k^2[n^2(z,r) - 1 + i\nu(z,r)]\psi = 0 . \qquad (2.1)$$

to the axially symmetric Helmholtz equation [Desa 77], [Mcda 75], [Tapp 77], [Desa 79].

To simplify expressions we set $q(z,r) := k^2[n^2(z,r)-1+i\nu(z,r)]$, and consider transmutations between equations of the form

$$\mathbf{Q}\psi := \psi_{zz} + q(z,r)\psi - 2ik\,\psi_r \ , \qquad (2.2)$$

and

$$\mathbf{P}\phi : = \phi_{zz} + p(z,r)\psi - 2ik\ \phi_r \qquad (2.3)$$

i.e. we seek operators **B** such that **BP** = **QB**, [Carr 84], [Carr 74], [Dgrw 87], [Gilb 74].

As done earlier [Giwo 87a], [Gelev], [Colt 78], [Rund 84] we seek a representation for **B** as an integral transformation of the form

$$\psi(z,r) = \phi(z,r) + \int_h^z K(s,z,r)\phi(s,r)ds \ . \tag{2.4}$$

By formal differentiation we obtain

$$\mathbf{Q}\psi = \phi_{zz} + \int_h^z K_{22}\phi ds + K_2(z,z,r)\phi(z,r) + \frac{\partial}{\partial z}[K(z,z,r)\phi(z,r)]$$

$$+ q\phi + \int_h^z q(z,r)K(s,z,r)\phi(s,r)ds - \phi_r - \int_h^z K_3\phi(s,r)ds$$

$$- \int_h^z K(s,z,r)\phi_r(s,r)ds = [q(z,r) - p(z,r)] +$$

$$\int_h^z [K_{22} + q(z,r)K - K_3]\phi(s,r)ds + K_2(z,z,t)\phi(z,t)$$

$$+ \frac{\partial}{\partial z}[K(z,z,r)\phi(z,r)] - 2ik \int_h^z K\phi_r(s,r)ds \ .$$

The last integral may be simplified by using

$$\int_h^z K(s,z,r)\phi_r(s,r)ds = \int_h^z K(s,z,t)[\phi_{ss} + p(s,r)\phi]ds$$

$$= \phi_z(z,r)K(z,z,r) - \phi_z(h,r)K(h,z,r) - p(z,r)K_1(z,z,r)$$

$$+ \phi(h,r)K_1(h,z,r) + \int_h^z K_{11}(s,z,r)\phi(s,r)ds \ .$$

By putting these together we obtain the following conditions on $K(s,z,r)$,

$$K_{zz}-K_{ss} + [q(z,r) - p(s,r)]K - 2ik\ K_r = 0 \ , \tag{2.5}$$

$$\frac{\partial}{\partial z}[K(z,z,t)] = p(z,r) - q(z,r) \ , \tag{2.6}$$

with

$$\phi_z(h,r)K(h,z,r) - K_1(h,z,r)\phi(h,r) = 0 \ . \tag{2.7}$$

If we require that $\psi_z(h,r) = 0$ a natural choice is to take $\phi_z(h,r) = 0$ and extend $p(z,r)$, $q(z,r)$ as even functions of $(z-h)$ then we may find $K(s,z,t)$ as a solution of (2.5) satisfying

$$\frac{\partial K}{\partial z}(z,z,t) = \frac{1}{2}[p(z,t) - q(z,t)] = \frac{\partial K}{\partial z}(2h-z,z,t) \ . \tag{2.8}$$

Otherwise, if $\tilde{K}(s,z,t)$ satisfies (2.5) and (2.8) we may set

$$K(s,z,t) = \tilde{K}(s,z,t) - \tilde{K}(2h-s,z,t) \ . \tag{2.9}$$

We now consider the special case where $p(z,r)$ only depends on z,

$$p(z,r) := n^2(z) \ ,$$

and

$$q(z,r) := n^2(z) + \varepsilon m^2(z,r) \ . \tag{2.10}$$

Such instances of $q(z,r)$ appear in ocean acoustics where radial effects are seen to be much smaller than the depth dependent variations. Equation (2.5) now takes on the form

$$K_{zz} - K_{ss} + \varepsilon m^2(z,r)K - 2ik\ K_t \ . \tag{2.11}$$

Since ε is a small parameter we now try an asymptotic expansion

$$K(z,r,t) = \sum_{l=0}^{\infty} \varepsilon^l K^{(l)}(z,r,t) \ , \tag{2.12}$$

and for simplicity introduce characteristic coordinates

$$\xi = \frac{z+s}{2} \ , \ \eta = \frac{z-s}{2} \ .$$

We obtain the system

$$K^{(0)}_{\xi\eta} - 2ik\ K_r^{(0)}\ ,\ K^{(0)}(\xi,0,r) = K^{(0)}(0,\eta,r) = 0\ , \tag{2.13a}$$

$$K^{(1)}_{\xi\eta} - 2ik\ K_r^{(1)} = m^2(z,r)K^{(0)}\ ,$$

$$K^{(1)}(\xi,0,r) = \frac{1}{2}\int_0^{\xi}[1-m^2(t+h,r)]dt\ , \tag{2.13b}$$

$$K^{(1)}(0,\eta,r) = \frac{1}{2}\int_0^{n}[1-m^2(t+h,r)]dt\ ,$$

$$K^{(l+2)}_{\xi\eta} - 2ik\ K_s^{(l+2)} = m^2(z,r)K^{(l+1)}\ , \tag{2.13c}$$

$$K^{(l+2)}(\xi,0,r) = K^{(l+2)}(0,\eta,r) = 0\ .$$

We seek solutions of these equations by an iterative procedure, namely

$$K^{(j)}(\xi,n,r) = \sum_{n=0}^{\infty} K_n^{(j)}(\xi,\eta,r)\ , \tag{2.14}$$

with

$$K_o^{(j)}(\xi,\eta,r) = K^{(j)}(\xi,0,r) + K^{(j)}(0,\eta,r) \tag{2.15}$$

$$K_{n+1}^{(j+1)}(\xi,n,r) = 2ik\int_0^{n}\int_0^{\xi}\frac{\partial K_n^{(j+1)}}{\partial r}(\xi,\eta,r)d\xi\,d\eta\ . \tag{2.16}$$

Since $K_0^{(0)}(s,z,r) \equiv 0$ it follows that $K_n^{(0)}(s,z,r) \equiv 0$ for all n and, hence, $K^{(0)}(s,z,r) \equiv 0$. Equation (2.13b) now becomes

$$\begin{cases} \dfrac{\partial K^{(1)}}{\partial\xi\partial n} - 2ik\dfrac{\partial K^{(1)}}{\partial r} = 0 \\[2ex] K^{(1)}(\xi,0,r) = \dfrac{1}{2}\displaystyle\int_0^{\xi}(1-m^2(t+h,r)]dt \\[2ex] K^{(1)}(0,n,r) = \dfrac{1}{2}\displaystyle\int_0^{\eta}[1-m^2(t+h,r)]dt \end{cases} \tag{2.17}$$

We try to obtain to (2.17) in the form

$$K^{(1)}(\xi,n,r) = \sum_{l=0}^{\infty} \frac{(-2ik\,\xi\eta)^l}{(l\,!)^2} \frac{\partial^l}{\partial r^l}\,\phi(\xi,n,r) \tag{2.18}$$

and notice that if we choose $\phi(\xi,n,t)$ of the form

$$\phi(\xi,n,r) = K^{(1)}(\xi,0,r) + K^{(1)}(0,\eta,r) \tag{2.19}$$

not only does (2.18) satisfy the Goursat conditions, but it satisfies the differential equation as well. In a similar way we may construct solutions to (2.13c). Writing

$$\psi(\xi,\eta,r) := m^2(\xi+\eta+k,r)K^{(j)}(\xi,\eta)$$

we notice that a solution of (2.13c) may be written as

$$K_{l+1}^{(j+2)}(\xi,\eta,r) = \int_0^{\xi}\int_0^{n} \sum_{l=0}^{\infty} \frac{(-2ik\ \xi\eta)^l}{(l\,!)^2} \frac{\partial^l}{\partial r^l}\psi(\xi,\eta,r)\ . \tag{2.20}$$

Hence, the terms to second order in $K(\xi,\eta,r)$ are

$$K(\xi,n,r) = \frac{\varepsilon}{2}\sum_{l=0}^{\infty} \frac{(-2ik\,\xi\eta)^l}{(l\,!)^2} \frac{\partial^l}{\partial r^l}\left[\int_0^{\xi}[1+m^2(t+h,r)]dt\right.$$

$$\left.+\int_0^{\eta}[1+m^2(t+h,r)]dt\right] + \frac{\varepsilon^2}{2}\int_0^{\xi}\int_0^{\eta}\sum_{j=0}^{\infty}\frac{(2ik\,\xi\eta)^j}{(j\,!)^2}$$

$$\cdot\frac{\partial j}{\partial r^j}\left[\sum_{l=0}^{\infty}\frac{(-2ik\,\xi\eta)^l}{(1!)^2}\frac{\partial^l}{\partial r^l}\left[\int_0^{\xi}[1+m^2(t+h,r)]dt + \int_0^{\eta}[1+m^2(t+h,r)]dt\right]\right. \tag{2.21}$$

We now show how one may directly compute a fundamental solution to the equation

$$2ik\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0 \tag{2.22}$$

in the form

$$S(z,\xi;r,\rho) := \sum_{n=0}^{\infty}(-1)^n n! \frac{s_n(z-\xi)}{(r-\rho)^{n+1}} \quad . \tag{2.23}$$

Substituting (2.19) into (2.18) leads to the system

$$s_o'' = 0 \quad ,$$

$$2iks_{n-1} + s_n'' = 0 \quad ,$$

which suggests we choose

$$s_n(z-\xi) := (-2ik)^n \frac{(z-\xi)^{2n+1}}{(2n+1)!} \tag{2.24}$$

and

$$S(z-\zeta,r,\rho) = \frac{e^{\frac{ik}{2}\frac{(z-\zeta)^2}{(r-\rho)}}}{r-\rho}\left[1+\frac{1}{2}\,\gamma\left[1/2\ ,\ \frac{ik(z-\zeta)^2}{2(r-\rho)}\right]\right] \quad , \tag{2.25}$$

where $\gamma(\alpha,z) := \int_0^z e^{-t}t^{\alpha-1}dt$ is the incomplete *gamma* function. In axially symmetric coordinates this becomes

$$S(z,r,\theta;\zeta,\rho,\phi) = \frac{\exp\left[\frac{ik}{2}\frac{(z-\zeta)}{|re^{i\theta}-\rho e^{i\theta}|}\right]}{|re^{i\theta}-\rho e^{i\phi}|}\left[1+\gamma\left[\frac{1}{2},\frac{ik(z-\zeta)^2}{2|re^{i\theta}-\rho e^{i\theta}|}\right]\right] \quad .$$

The form of S follows by considering the power series

$$\psi(z) := \sum_{n=0}^{\infty} \frac{n!z^n}{(2n+1)!} \quad ,$$

which comes about by substituting (2.20) into (2.19). From the Legendre duplication formula for gamma functions we replace the sum in $\psi(z)$ by

$$\psi(z) = \frac{\sqrt{\pi}}{4}\sum_{n=0}^{\infty}\left[\frac{z}{4}\right]^n \frac{1}{\Gamma(n+3/2)} \quad .$$

It is clear that $f(\zeta) = \frac{4}{\sqrt{\pi}}\psi(4\zeta)\zeta^{-1/2}$ satisfies the nonhomogeneous ordinary differential equation

$$\frac{df}{d\zeta} - f = \frac{\zeta^{-1/2}}{\Gamma(1/2)}$$

whose general solution is given by

$$g(\zeta) = e^{\zeta}\left[A + \frac{\gamma(1/2,\zeta)}{\Gamma(1/2)}\right] .$$

A solution to (2.18) satisfying the analytic Cauchy data

$$\psi(z,0) = \psi_o(z)$$

may also be found using the Cauchy Kowolewski theorem by seeking it in the form

$$\psi(z,\rho) = \sum_{n=0}^{\infty} \psi_n(z)\rho^n .$$

We obtain

$$\psi(z,\rho) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{i\rho}{2k}\right]^n \frac{d^{2n}}{dz^{2n}}(\psi_o(z)) . \qquad (2.25)$$

Tappert [Tapp 70] replaces the parabolic equation

$$2ik\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + k^2n^2(z,r\phi)\psi = 0$$

by

$$2ik\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial y^2} + k^2n^2(z,y,r)\psi = 0 ,$$

which motivates us to seek a solution of

$$2ik\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial y^2} = 0 , \quad \psi(z,y,0) = \psi_o$$

by the above method. We obtain

$$\psi(z,r,\phi) = \sum_{n=0}^{\infty} r^n \left[\frac{i}{2k}\right]^n \frac{1}{n!}\Delta^n \psi_o \quad , \tag{2.26}$$

where

$$\Delta := \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} \quad .$$

III. Related Differential Equations

We now return to the acoustic Helmholtz equation for an axially symmetric solution, namely

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k^2 n^2(z) p = 0 \, , \tag{3.1}$$

and make a change of dependent variables $p = r^{-1/2} u(r,z) e^{ikr}$. This leads to the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + 2ik \frac{\partial u}{\partial r} + (k^2[n^2(z)-1] + \frac{1}{4r^2})u = 0 \quad . \tag{3.2}$$

The parabolic approximation is that $\left| \frac{\partial^2 u}{\partial r^2} / \frac{\partial u}{\partial r} \right| \ll 1$ and $| u/r^2 | \ll 1$ which permits us to approximate (3.2) by

$$2ik \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} + k^2[n^2(z)-1]u = 0 \quad . \tag{3.3}$$

Using separation of variables the equation (3.3) may be written in the form

$$u(r,z) = \sum_m \alpha_m Z_m(z) \exp\left(ir(k_m^2 - k^2)/2k\right)$$

where the $Z_m(z)$ are the eigenfunctions appearing in the Sturm Liouville problem $(D := \frac{\partial}{\partial z})$,

$$\begin{aligned} &(D^2 + k^2 n^2(z)) Z_m(z) = k_m^2 Z_m(z) \\ &Z_m(0) = Z_m'(h) = 0 \, . \end{aligned} \tag{3.4}$$

The far-field solutions to (3.1) may be written as [Mcda 75], [Desa 79]

$$p(r,z) = r^{1/2} \sum_m \beta_m \, Z_m(z) e^{ik_m r} \tag{3.5}$$

where the $Z_m(z)$ are the eigensolutions of (3.4). This suggests using a starting field for the PE (3.3) at $r = r_o$,

$$\phi(z) := \frac{1}{r_o^{1/2}} \sum_m \beta_m Z_m e^{ik_m r_o} \ ,$$

giving rise to

$$p(r,z) = \frac{1}{r^{1/2}} \sum_m \alpha_m Z_m \ \exp(i(r-r_o)(k_m^2-k^2)/2k) \ . \tag{3.6}$$

We shall now try to relate the solutions (3.5) and (3.6) by means of an integral transform. To this end we notice that if $u(r,z) = e^{-ikr} U$ then (3.2) implies that

$$\frac{\partial^2 U}{\partial r^2} + \frac{\partial^2 U}{\partial z^2} + k^2[n^2(z) + \frac{1}{4r^2}]U = 0 \ ,$$

and the PE becomes

$$\frac{\partial^2 U}{\partial r^2} + \frac{\partial^2 U}{\partial z^2} + k^2 n^2(z) U = 0 \ . \tag{3.7}$$

Moreover, if we use the rotation and stretching $t = \frac{r}{2ik}$, then (3.3) becomes

$$\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial z^2} + k^2[n^2(z)-1]v = 0 \ .$$

The transformation $v(t,z) = V(t,z)e^{-k^2 t}$ returns

$$\frac{\partial V}{\partial t} + \frac{\partial^2 V}{\partial z^2} + k^2 n^2(z) V = 0 \ . \tag{3.8}$$

If the equations (3.1) and (3.3) are to satisfy the same boundary conditions, i.e.

$$\phi(r,0) = 0 \ , \quad \text{and} \quad \frac{\partial \phi}{\partial z}(r,h) = 0 \ , \tag{3.9}$$

where ϕ is taken to be either p or u, then $U(r,z)$ and $V(r,z)$ also satisfy these boundary conditions.

Bragg and Dettman [Brde 68a], [Brde 68b] have, in a series of papers, investigated mappings between solutions of the equations (3.7) and (3.8), which satisfy the same boundary conditions. In particular, if $U(r,z)$ satisfies (3.7) and

$$U(r_o,z) = \phi(z) \ , \quad U_r(r_o,z) = 0 \tag{3.10}$$

then this is related to the solution of (3.8) with the initial condition

$$V(t_o,z) = \phi(z) \ , \tag{3.11}$$

by

$$U(r,z) = r\Gamma(1/2)\mathbf{L}_s^{-1}\left\{s^{1/2}V(\frac{1}{4s}\ ,\ z)\right\}_{\tau\to r^2} \ . \tag{3.12}$$

On the other hand, if $U(r,z)$ is chosen to satisfy the initial conditions

$$U(r_o,z) = 0 \ , \quad U_r(r_o,z) = \phi(z) \ , \tag{3.13}$$

then the solutions are related by

$$U(r,z) = r\Gamma(1/2)\mathbf{L}_s^{-1}\left\{S^{-1/2}V(\frac{1}{4s},z)\right\}_{\tau\to r^2} \ . \tag{3.14}$$

In both instances $\mathbf{L}_s^{-1}$ is the inverse Laplace transform with respect to the s-variable and its paired variable τ is taken to be r^2 after integration is performed. These ideas are useful for constructing the starting field for the PE. We note that if the pressure at $r = r_o$ is given by $p(r_o,z) = \phi(z)$ then $U(r_o,z) = r_o^{1/2}p(r_o,z) = r_o^{1/2}\phi(z)$; moreover, $V(r_o,z) = e^{-\frac{3i}{2}kr_o} r_o^{1/2}\phi(z)$. It is therefore very easy to associate a particular Cauchy problem of the type investigated with Bragg and Dettman for the case of the Helmholtz equation with the PE. Whereas this procedure is of no real use in computing the far field it may be possible to approximate the near-field for the HE from the starting field for the PE.

Suppose we consider the situation where an object of revolution D about the z-axis is radiating sound into a channel of depth h. See Figure 1 below

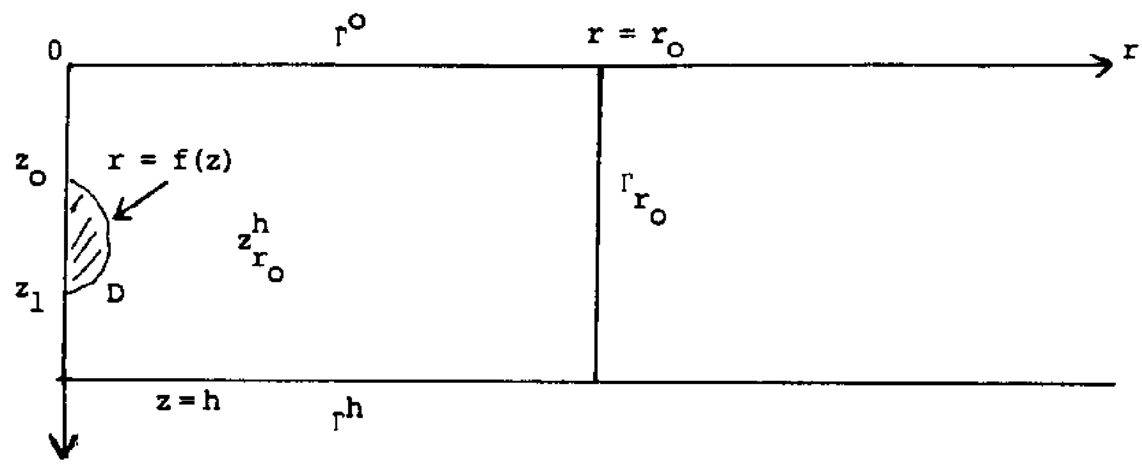

Figure 1

where the objects diameter is situated on the z-axis between z_o and z_1. A generator of the object is given by $r = f(z)$ $z \in [z_o, z_1]$.

On $[0,z_o] \cup [z_1,h]$ $\frac{\partial p}{\partial r} = 0$ because of symmetry, and on the surface $r = f(z)$ we assume $p(r,z)$ is given by $p(f(z),z) = \phi(z)$. As before $p(r,0) = \frac{\partial p}{\partial z}(r,h) = 0$, and at the starting field position $r = r_o$, we assume that the Sommerfeld condition holds, i.e.

$$\frac{\partial p}{\partial r} - ikp = 0(\frac{1}{r^{1/2}}) \quad . \tag{3.15}$$

In terms of our new unknown $V(r,z)$ this last condition remains the same.

The truncated cylinder in Figure 1 is referred to as $Z^h_{r_o} := \{\mathbf{x} : r < r_o \ , \ 0 < z < h\}$, and the slab $0 < z < h$ is called $\mathbf{R}^3_h$. Let $\Phi(r,z;\rho,\zeta)$ be the fundamental solution of (3.1) in $\mathbf{R}^3_h$, i.e.

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial z^2} + k^2 n^2(z)\Phi = \frac{\delta(r-\rho)\delta(z-\zeta)}{2\pi|r-\rho|} , \tag{3.16}$$

with $\Phi(r,0;\rho,\zeta) = \frac{\partial\Phi}{\partial z}(r,h;\rho,\zeta) = 0$, and

$$\frac{\partial\Phi}{\partial r} - ik\Phi = 0(r^{-1/2}) . \tag{3.17}$$

Then if we apply Green's formula to the region $Z_{r_o}^h \,/\, \overline{D}$ we obtain

$$p(r,z) = \int_{\partial D \cup \partial Z_{r_o}^h} \left\{ p(\rho,\zeta)\frac{\partial\Phi}{\partial\nu}(r,z;\rho,\zeta) - \frac{\partial p(\rho,\zeta)}{\partial\nu}\Phi \right\} d\sigma(\rho,\zeta) , \tag{3.18}$$

where $d\sigma(\rho,\zeta)$ is the surface measure. We notice that $\partial Z_{r_o}^h = \Gamma^o \cup \Gamma^h \cup \Gamma_{r_o}$ and that the integrals in (3.18) vanish on $\Gamma^o \cup \Gamma^h$ in lieu of the boundary conditions imposed there on p and Φ. The integral over the lateral surface Γ_{r_o} may be rewritten as

$$2\pi r_o \int_0^h \left[p(r_o,\zeta)\frac{\partial\Phi}{\partial\rho}(r,z;r_o,\zeta) - \frac{\partial\phi}{\partial\rho}(r_o,\zeta)\Phi(r,z;r_o,\zeta) \right] d\zeta$$

$$= 2\pi r_o \int_0^h \left\{ p(r_o,\zeta)\left[\frac{\partial\Phi}{\partial p} - ik\Phi\right] - \Phi\left[\frac{\partial p}{\partial\rho} - ik\rho\right] ds \right. \tag{3.19}$$

By using (3.15), (3.17) and the Schwarz inequality on (3.18) we see that the integral is o(1); hence, we obtain the integral representation

$$p(r,z) = \int_{\partial D} \left\{ p(\rho,\zeta)\frac{\partial\Phi}{\partial\nu}(r,z;\rho,\zeta) - \frac{\partial p}{\partial\nu}\Phi \right\} d\sigma(p,\zeta) , \tag{3.20}$$

holds for $(r,z)\in \mathbf{R}_h^3 \,/\, \overline{D}$.

Indeed, we may use (3.20) to construct the starting field at $r = r_o$ and if r_o is in the middle range use the modal representation for Φ, namely [Ahke 77]

$$\Phi(r,z;\rho,\zeta) = \frac{i}{4}\sum_{n=0}^{\infty} \phi_n(z)\phi_n(\zeta)H_o^{(1)}(ka_n|r-\rho|) \quad \int_o^h \phi_n^2(s)ds . \tag{3.21}$$

The difficulty with (3.20) is that it requires knowledge of both p and $\frac{\partial p}{\partial \nu}$ on ∂D. We can avoid this by using a representation of $p(r,z)$ in terms of a density function $\mu(\rho,\zeta)$ and considering the double-layer representation

$$p(r,z) = \int_{\partial D} \mu(\rho,\zeta) \frac{\partial \Phi}{\partial \nu}(r,z;\rho,\zeta) d\sigma(\rho,\zeta) \quad ,$$

which leads to an integral equation for the density in terms of the boundary data for p. Various authors have used variants on this idea [Cokr 83], [Anck 82], [Urse 73] and considered combinations of double-and single-layer potentials, i.e.

$$p(r,z) = \int_{\partial D} \left[\frac{\partial \Phi}{\partial \nu} - i\eta\ \Phi \right] \mu(\rho,z) d\sigma(\rho,z) \tag{3.22}$$

where $\eta = 0$ is an arbitrary real number chosen so that η Re $k \geq 0$. In this instance we are led to an integral equation of the form

$$(\mathbf{I} + \mathbf{K} + i\eta\ \mathbf{S})\mu(f(z),z) = 2\phi(z) \quad , \quad z \in [z_o, z_1] \quad . \tag{3.23}$$

where $\mathbf{K}$ is the double-layer and $\mathbf{S}$ the single-layer operator. If (3.20) is the Green's function for the uniform ocean a parametrix may be obtained from this by using the transmutation $\mathbf{B} = \mathbf{I} + \mathbf{K}$, where $\mathbf{K}$ is defined in terms of the kernel $K(z,s,k)$ as

$$(\mathbf{K}\phi)(z) := \int_0^z K(z,s,k)\phi(s)ds \tag{3.24}$$

and $K(z,s,k)$ is a solution of [Giwo 86], [Dugw 86b]

$$K_{zz} = K_{ss} + k^2[n^2(z)-1]K = 0 \quad , \tag{3.25}$$

$$K(z, \pm z) = \frac{1}{2} k^2(n^2(z)-1) \quad .$$

The parametrix is given by

$$\Phi(r,z;\rho,\zeta) = \frac{i}{4} \sum_{n=0}^{\infty} \phi_n(\zeta)(\mathbf{B}\phi_n)(z) \frac{H_o^{(1)}(ka_n |r-\rho|)}{\|\phi_n\|^2} \quad . \tag{3.26}$$

and its use in (3.21) instead of Φ leads to an integral equation of the same kind as (3.23).

IV. The Far Field

We consider first the case of a uniform, infinite ocean. In this instance, following Tappert [Tapp 77] we know that locally

$\Phi(r,z,\rho,\zeta) \sim \psi(r-\rho,z-\zeta)H_o^{(1)}(k\,|r-\rho|) \sim \sqrt{\frac{2}{i\pi k\,|r-\rho|}}\,\psi e^{ik|r-\rho|}$, where ψ satisfies the parabolic equation

$$2ik\,\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0 \tag{4.1}$$

Using a Gaussian starting field it may be shown using Fourier transforms [Tapp 77] that

$$\psi = p_o\sqrt{\frac{ik\pi}{2|r-\rho|}}\left[1+\frac{1}{2k^2(r-\rho)^2}\right]^{-1/4}\exp\left[\frac{(z-\zeta)^2}{2(r-\rho)^2\,(1+\frac{1}{2k^2(r-\rho)^2})}\right]$$

$$\times\exp\left[\frac{ik(z-\zeta)^2}{2|r-\rho|\left[1+\frac{1}{2k^2(r-\rho)^2}\right]}+\frac{\pi}{4}-\frac{1}{2}\tan^{-1}(|r-\rho|k)\right]$$

$$\sim p_o\sqrt{\frac{ik\pi}{2|r-\rho|}}\exp\left[-\frac{(z-\zeta)^2}{2(r-\rho)^2}+ik\,\frac{(z-\zeta)^2}{2(r-\rho)}\right].$$

This suggests approximating the $\mathbf{R}^3$- Green's function by

$$\Phi(r,z,\rho,\zeta) \sim P(r,z,\rho,\zeta) := \frac{1}{4\pi|r-\rho|}\,e^{ik|r-\rho|}\left[1+\frac{(z-\zeta)^2}{2|r-\rho|}\right]$$

$$\times\, e^{-\frac{(z-\zeta)^2}{2(r-\rho)^2}}\,. \tag{4.3}$$

A straight-forward computation yields

$$\Delta P + k^2 P = P\left\{\frac{3ik(z-\zeta)^2}{2|r-\rho|^3} + \frac{1}{|r-\rho|^4}\left[\frac{k^2(z-\zeta)^4}{4} - 3(z-\zeta)^2\right]\right.$$

$$\left. + \frac{1}{|r-\rho|^5}(-2ik(z-\zeta)^4) + \frac{(z-\zeta)^4}{|r-\rho|^7}\right\} = 0(\frac{1}{|r-\rho|^4}) \tag{4.4}$$

as $|r-\rho| \to \infty$. Similarly, we get

$$\frac{\partial P}{\partial r} - ikP = P\left\{-\frac{1}{|r-\rho|} - \frac{ik(z-\zeta)^3}{2(r-\rho)^2} + \frac{(z-\zeta)^2}{(r-\rho)^3}\right\} = 0\left[\frac{1}{|r-\rho|^2}\right] . \tag{4.5}$$

This suggests that if we replace Φ by its PE approximation **P** in the Green's representation (3.19), namely

$$p(r,z) = \int_{\partial D}\left\{p(\rho,\zeta)\frac{\partial P}{\partial \nu}(r,z,\rho,\zeta) - \frac{\partial \phi}{\partial \nu}P\right\}d\sigma(\rho,\zeta) \ , \tag{4.6}$$

that the result is valid to within an order of magnitude. Moreover, (4.6) satisfies the radiation condition.

If we consider (4.6) in $\mathbf{R}_h^3$ and use the fact that Φ and p satisfy the same boundary conditions on $z = 0,h$ then for $r > r_o$

$$p(r,z) \sim \int_{\Gamma_{r_o}}\left\{p(\rho,\zeta)\frac{\partial P}{\partial \nu} - \frac{\partial p}{\partial \nu}P\right\}d\sigma(\rho,\zeta) \tag{4.7}$$

Let

$$I_1 := \int_{\Gamma_{r_o}} p(\rho,\zeta)\frac{\partial P}{\partial \nu}d\sigma(\rho,\zeta) \ , \tag{4.8}$$

and

$$I_2 := \int_{\Gamma_{r_o}} \frac{\partial p}{\partial \nu}\mathbf{P}(r,z,\rho,\zeta)\, d\sigma(\rho,\zeta) \ .$$

On the lateral side Γ_{r_o} the normal derivative is

$$\frac{\partial P}{\partial \nu} = \frac{\partial P}{\partial \rho}(r, z, \rho, \zeta)$$

$$= P\left\{ ik - \frac{1}{|\mathbf{x}' - \mathbf{y}'|} - \frac{ik(z-\zeta)^2}{2|\mathbf{x}' - \mathbf{y}'|} + \frac{(z-\zeta)^2}{|\mathbf{x}' - \mathbf{y}'|} \right\}$$

The integral I_1 may be seen to then take the general form, where we allow f to also vary with θ,

$$I_1 := \int_{\Gamma_{r_o}} f \frac{\exp\left[ik|\mathbf{x}'-\mathbf{y}'| - \frac{i(z-\zeta)^2}{2x'-y'} - \frac{(z-\zeta)^2}{2|\mathbf{x}'-\mathbf{y}'|} \right]}{|\mathbf{x}'-\mathbf{y}'|} d\sigma(y) .$$

From the expression

$$P(r, z, \rho, \zeta) e^{-ikr} = \frac{1}{4\pi|\mathbf{x}' - \mathbf{y}'|} \exp\left[ik|\mathbf{x}' - \mathbf{y}'| - r + ik\frac{(z-\zeta)^2}{|\mathbf{x}' - \mathbf{y}'|} - \frac{(z-\zeta)^2}{|\mathbf{x}' - \mathbf{y}'|} \right],$$

we obtain by replacing $p/r = \omega$, $|\omega| < 1$ a convergent Taylor series

$$Pe^{ikr} = \frac{\omega e^{ik\rho[(1-2\omega\cos(\theta-\zeta)+\omega^2)^{1/2}-1]\omega^{-1}}}{4\pi\rho(1-2\omega\cos(\theta-\phi)+\omega^2)^{1/2}} e^{\left[\frac{ik\omega(z-\zeta)^2}{2\rho(1-2\omega\cos(\theta-\phi)+\omega^2)^{1/2}}\right]}$$

$$\times e^{\frac{-\omega^2(z-\zeta)^2}{2\rho^2(1-2\omega\cos(\theta-\phi)+\omega^2)}}$$

$$= f_o(\rho,\phi)\omega + \sum_{n=0}^{\infty} f_n(\rho,\phi,z-\zeta)\omega^{n+1} \quad , \quad (\rho,\phi) \in \partial Z_{R_o} .$$

Hence, the integral I_1 may be evaluated using termwise integration to obtain a power series in ρ/r. Likewise, we may expand $\frac{\partial P}{\partial \nu_y}$ as

$$\frac{\partial P}{\partial \nu_y} = \frac{e^{ikr}}{r} \left(f_o(\rho,\theta) + \sum_{n=1}^{\infty} \tilde{f}_n(\rho,\theta,z-\zeta) r^{-n} \right) .$$

Since $\frac{\partial P}{\partial \nu_y} = 0(1/r^2)$ on $\Gamma^o \cup \Gamma^h$ we conclude $\bar{f}_o(\rho,\phi) = 0$ on $\Gamma^o \cup \Gamma^h$. We summarize the above by the following formula

$$p(r,z,\theta) = \frac{e^{ikr}}{r}\left\{F_o(\theta,k) + \sum_{n=1}^{\infty} \frac{F_n(z,\theta,k)}{r^n}\right\} \tag{4.9}$$

Equation (4.9) is a PE far field approximation for extremely high f- number in $\mathbf{R}^3$, that is when the focussed beam does not see the channel walls. It is really not valid when the effects of the ocean surface and bottom must be considered.

The expression (4.9) as it is a solution to the PE gives rise to the following recursion system for the coefficients F_n,

$$\frac{\partial^2 F_o}{\partial z^2} = 0 \ , \quad \frac{\partial^2 F_1}{\partial z^2} = ik\ F_o \ ,$$
$$\frac{\partial^2 F_{n+2}}{\partial z^2} = 2ik(n+3/2)F_{n+1} - \frac{\partial^2 F_n}{\partial \theta^2} \tag{4.10}$$

We now use the modal expansion to investigate the far field,

$$\Phi(r,z,\rho,\zeta) = \sum_{n=0}^{\infty} \frac{\phi_n(z)\phi_n(\zeta)}{\|\phi_n\|^2} H_o(Ka_n\,|re^{i\theta} - \rho e^{i\phi}|) \ , \tag{4.11}$$

$a_n = [1+(n+1/2)^2/(\frac{\pi}{kh})^2]^{-1/2}$, and where we have indicated $\mathbf{x}' = re^{i\theta}$, $\mathbf{y}' = \rho e^{i\phi}$. Then

$$\Phi(r,z,\rho,\zeta) \sim \sqrt{\frac{2}{i\pi k}}\ |re^{i\theta} - \rho e^{i\phi}|^{-1/2} \sum_{n=0}^{\infty} \frac{\phi_n(z)\phi_n(\zeta)}{\|\phi_n\|^2}\ e^{ika_n|re^{i\theta} - \rho e^{i\phi}|} \tag{4.12}$$

Using (4.12) for $P(r,z,\rho,\zeta)$ in (4.6) yields

$$p(r,z,\theta) = \int_0^h \int_0^{2\pi} \sqrt{\frac{2}{i\pi k}} \left\{ \sum_{n=0}^{\infty} \frac{\phi_n(z)\phi_n(\zeta)}{\|\phi_n\|^2} \left\{ p(\rho,\zeta,o) \right.\right.$$

$$= \frac{\partial R}{\partial \rho} \frac{\partial}{\partial \rho} \left[\frac{e^{ika_n R}}{R^{1/2}} \right] + \frac{\partial p}{\partial \rho} \left| \frac{e^{ika_n R}}{R^{1/2}} \right\} \Bigg|_{\rho = r_o}^{d\phi\, d\zeta} , \tag{4.13}$$

where $R := |re^{i\theta} - \rho e^{i\phi}|$. Let us consider the individual integrals of the type

$$I_{1,n} := \int_0^h \int_0^{2\pi} p(\rho,\zeta,\phi) \frac{\partial R}{\partial \rho} \frac{\partial}{\partial \rho} \left(\left| \frac{e^{ika_n R}}{R^{1/2}} \right] \right|_{\rho = r_o}^{d\phi\, d\zeta} , \tag{4.14}$$

$$I_{2,n} := \int_0^h \int_0^{2\pi} \frac{e^{ika_n R}}{R^{1/2}} \left| \frac{\partial p(\rho,\zeta,\phi)}{\partial \rho} \right|_{\rho = r_o}^{d\phi\, d\zeta} . \tag{4.15}$$

We consider integrals of the type $I_{2,n}$ first, i.e. integrals of the form $\int_0^h \int_0^{2\pi} \frac{e^{ika_n R}}{R^{1/2}} f(\phi,\zeta) d\phi\, d\zeta$, and note that if $\omega := r_o / r$, then

$$\frac{e^{ika_n(R-r)}}{R} = \frac{\omega e^{ikr_o[1-2\omega\cos\gamma + \omega^2)^{1/2} - 1]/\omega}}{r_o[1-2\omega\cos\gamma + \omega^2]}$$

$$= \sum_{m=0}^{\infty} \alpha_{nm}^{(\gamma)} \omega^m .$$

where $\gamma = \theta - \phi$. A similar analysis holds for the integrals of the form $I_{2,n}$; hence, we obtain an expression of the form

$$p(r,z,\theta) = \frac{1}{r^{1/2}} \sum_{n=0}^{\infty} e^{ika_n r} \sum_{m=0}^{\infty} \frac{f_{nm}(z,\theta)}{r^m}$$

$$= \frac{1}{r^{1/2}} \sum_{m=0}^{\infty} \frac{F_m(r,z,\theta)}{r^m} , \quad F_m(r,z,\theta) := \sum_{n=0}^{N} e^{ika_n r} f_{nm}(z,\theta) . \tag{4.17}$$

The term

$$F_o(r,z,\theta) := \sum_{n=0}^{N} e^{ika_n r} f_{no}(z,\theta) \tag{4.18}$$

plays the role of a far-field pattern.

V. The Projection Theorem

From the last section, for large range one has

$$\sum_{n=0}^{N} e^{ika_n r} f_{no}(z,\theta) = r^{1/2} \int_{\partial D} \frac{\partial u(\mathbf{y})}{\partial \nu_y} \sum_{n=0}^{N} \frac{\phi_n(z)\phi_n(\zeta)}{\|\phi_n\|^2} H_o^{(1)}(ka_n |re^{i\theta} - \rho e^{i\phi}|) d\sigma(\mathbf{y}) , \tag{5.1}$$

where $a_o > a_1 > \cdots > a_{N-1} \geq 0$ are the eigenvalues associated with propagating modes. Moreover, we may approximate the terms

$$H_o^{(1)}(ka_n |re^{i\theta} - \rho e^{i\phi}|) = \sqrt{\frac{2}{i\pi k}} |re^{i\theta} - \rho e^{i\phi}|^{-1/2} e^{ika_n |re^{i\theta} - \rho e^{i\phi}|}$$

$$\equiv (1 + \frac{\rho}{2r}\cos(\theta - \phi)) \sqrt{\frac{2}{i\pi k}} e^{ika_n (r - \rho\cos(\theta - \phi))} ; \tag{5.2}$$

hence

$$\sum_{n=0}^{N} e^{ika_n r} f_{no}(z,\theta) = \sqrt{\frac{2}{i\pi k}} \int_{\partial D} \frac{\partial u(\mathbf{x})}{\partial \nu_y} \sum_{n=0}^{N} \frac{\phi_n(z)\phi_n(\zeta)}{\|\phi_n\|^2} \cdot$$

$$\cdot e^{ika_n (r - \rho\cos(\theta - \rho))} d\sigma(\mathbf{y})$$

$$= \sqrt{\frac{2}{i\pi k}} \int_{\partial D} \frac{\partial u(\mathbf{y})}{\partial \nu_y} \sum_{n=0}^{N} e^{ika_n (r - \langle \mathbf{x}_2 , \mathbf{y}_2 \rangle)} \frac{\phi_n(z)\phi_n(\zeta)}{\|\phi_n\|^2} d\sigma(\phi) , \tag{5.3}$$

where $\mathbf{x}_2 , \mathbf{y}_2$ are the z- dimensional vectors $\mathbf{x}_2 := (x_1, x_2)$, $\mathbf{y}_2 := (y_1, y_2)$, and $|x_2| = 1$. Differentiating j- times both sides of (5.3) w.r.t. the range yields the system of equations

$$\sum_{n=0}^{N} (ika_n)^j e^{ika_n r} f_{no}(z,\theta)$$

$$= \sqrt{\frac{2}{i\pi k}} \int_{\partial D} \frac{\partial u(\mathbf{y})}{\partial \nu_y} \sum_{n=0}^{N} (ika_n)^j e^{ika_n r - ika_n \langle \mathbf{x}_2 , \mathbf{y}_2 \rangle} d\sigma(\mathbf{y}) , \tag{5.4}$$

($j = 0,1,2, \cdots, N-1$), which in matrix form may be written as

$$V_N D_N(r)\mathbf{F}(z,\theta) = \sqrt{\frac{2}{i\pi k}}\, V_N D_n(r) \int_{\partial D} \frac{\partial u(\mathbf{y})}{\partial \nu_y} \psi(\mathbf{x}_2 , \mathbf{y}_2 , z,\zeta)\, d\sigma(\mathbf{y}) \quad , \tag{5.5}$$

where V_N is the $(N+1) \times (N+1)$ Vandermonde matrix of constant coefficients

$$A_N := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ ika_o & ika_1 & \cdots & ika_N \\ \\ (ika_o)^N , & (ika_1)^N & & (ika_N)^N \end{bmatrix} , \tag{5.6}$$

$D_N(r)$ is the $(N+1) \times (N+1)$ diagonal matrix

$$D_N(r) := \begin{bmatrix} e^{ika_o r} & 0 & 0 \cdots & 0 \\ 0 & e^{ika_1 r} & 0 \cdots & 0 \\ \\ 0 & 0 & & e^{ika_N r} \end{bmatrix} , \tag{5.7}$$

and $\mathbf{F}$, ψ are the vectors whose transposes are given by

$$\mathbf{F}(z,\theta) := [f_{oo}(z,\theta) \ , \ f_{10}(z,\theta) \ , \ \cdots \ , f_{No}(z,\theta)] \ , \tag{5.8}$$

and

$$\psi(\mathbf{x}_2 , \mathbf{y}_2 , z,\zeta) := \left[\frac{e^{ika_o < \mathbf{x}_2 , \mathbf{x}_2 >}}{\|\phi_o\|^2} \phi_o(z)\phi_o(\zeta) \ , \ \cdots \ , \ e^{ika_N < \mathbf{x}_2 , \mathbf{y}_2 >} \frac{\phi_n(z)\phi_n(\zeta)}{\|\phi_n\|^2} \right] , \tag{5.9}$$

with $|\mathbf{x}_2| = 1$.

Since A_N^{-1} and $D_N^{-1}(r)$ exist multiplication on the left by these matrices leads to the following vector equation for the components of the far field.

$$\mathbf{F}(z,\theta) = \sqrt{\frac{2}{i\pi k}} \int_{\partial D} \frac{\partial u(\mathbf{y})}{\partial \nu_y} \psi(\mathbf{x}_2 , \mathbf{y}_2 , z,\zeta)\, d\sigma(\mathbf{y}) \tag{5.10}$$

Let $\mathbf{Z}_1 := \{(r,z,\theta) : r=1 \ , 0 \le z \le h \ , \ 0 \le \theta \le 2\pi\}$ and $g(z,\theta) \in L_2(\partial \mathbf{Z}_1)$ then multiply (5.6) by $g(z,\theta)$ and integrating over $\partial \mathbf{Z}_1$ leads to

$$\int_{\partial Z_1} \mathbf{F}(z,\theta) g(z,\theta) dz \ d\theta = \int_{\partial D} \frac{\partial u(\mathbf{y})}{\partial \nu_y} \overline{\mathbf{V}(\mathbf{y})}\, d\sigma(\mathbf{y}) \ , \tag{5.11}$$

where

$$\mathbf{V}(\mathbf{y}) := \sqrt{\frac{2}{i\pi k}} \int_{\partial Z_1} g(z,\theta)\psi(\mathbf{x}_2, \mathbf{y}_2, z, \zeta)\, d\sigma(\mathbf{x}) \quad . \tag{5.12}$$

The vector function $\mathbf{V}(\mathbf{y})$ or its representation as a sum of modes

$$v(\mathbf{y}) := \sum_{n=0}^{N} \frac{\phi_n(\zeta)}{\|\phi_n\|^2} \int_{\partial Z_1} g(z,\theta)\phi_n(z) e^{ika_n \langle \mathbf{x}_2, \mathbf{y}_2 \rangle} d\sigma(\mathbf{x}) =: \sum_{n=0}^{N} v_n(\mathbf{y}) \ , \tag{5.13}$$

are to be called *propagating Herglotz function*. The modes $v_n(\mathbf{y})$ are to be called *Herglotz modal solutions*. That the $v_n(\mathbf{y})$ are actually solutions of Helmholtz's equation may be seen directly. If $\Delta_2 := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, then

$$\Delta_2 v_n(\mathbf{y}) = -k^2 a_n^2 \|\mathbf{x}_2\|^2\, v_n(\mathbf{y}) = -k^2 a_n^2 v_n(\mathbf{y}) \quad .$$

Furthermore, since the functions $\phi_n(z)$ satisfy

$$\phi_n''(z) = k^2(a_n^2 - 1)\phi_n \quad ,$$

it follows that $\Delta v_n(\mathbf{y}) = -k^2 v_n(\mathbf{y})$.

As the depth of the ocean h is permitted to increase more terms are permitted in the sum (5.1). In the limit $N \to \infty$ and the system (5.5) would be replaced by an infinite number of equations

$$V_\infty D_\infty(r)\mathbf{F}(z,\theta) = \sqrt{\frac{2}{i\pi k}}\, V_\infty D_\infty(r) \int_{\partial D} \frac{\partial u(\mathbf{y})}{\partial \nu_y} \psi(\mathbf{x}_2, \mathbf{y}_2, z, \zeta)\, d\sigma(\mathbf{y})$$

where $V_\infty := \lim_{N \to \infty} V_N$, $D_\infty(r) := \lim_{N \to \infty} D_N(r)$ are $\infty \times \infty$ matrices and $\mathbf{F}$, ψ are vectors in l_2. Since V_∞^{-1}, and $D_\infty^{-1}(r)$ exist in l_2 we have formally that

$$\mathbf{F}(z,\theta) = \sqrt{\frac{2}{i\pi k}} \int_{\partial D} \frac{\partial u(\mathbf{y})}{\partial \nu_\mathbf{y}} \psi(\mathbf{x},\mathbf{y})\, d\sigma(\mathbf{y}) \tag{5.14}$$

where $\mathbf{F}$ and ψ may be thought of now as functions in $L^2(\partial K_1 \times \partial D)$ and, hence, we shall drop the vector notation in what follows. For finite oceans not all of the above modes propagate; however, for a deep ocean (5.10) will be a good approximation to the far field. Moreover, we also obtain

$$\int_{\partial Z_1} \overline{g(z,\theta)} F(z,\theta) d\sigma(\mathbf{x}) = \int_{\partial D} \frac{\partial u(\mathbf{y})}{\partial \nu_{\mathbf{y}}} \overline{v(\mathbf{y})}\, d\sigma(\mathbf{y}) \tag{5.15}$$

where

$$v(\mathbf{y}) = \sqrt{\frac{2}{i\pi k}} \int_{\partial Z_1} g(z,\theta) \sum_{n=0}^{N} \frac{\phi_n(z)\phi_n(\zeta)}{\|\phi_n\|^2} \cdot e^{ika_l \langle \mathbf{x}_2, \mathbf{y}_2 \rangle}\, dz\; d\theta\,, \tag{5.16}$$

with $0 < h < \infty$, and $|\mathbf{x}_2| = 1$.

We shall call such $v(\mathbf{y})$ that

$$\overline{\lim_{r\to\infty}} \frac{1}{r} \int_{Z_r} |v(\mathbf{x})|^2\, d\mathbf{x} \;<\; \infty \tag{5.17}$$

as *generalized Herglotz wave functions*, and $g(z,\theta)$ is called the *cylindrical Herglotz kernel*.

In what follows we shall consider the far fields which occur from the scattering of the incident wave

$$u^i(\mathbf{x}) := \frac{\phi_o(z)}{\phi_o(z_o)}\, e^{ik\mathbf{x}_2 \cdot \boldsymbol{\alpha}_2} \tag{5.18}$$

off of the soft obstacle D. We denote these far fields as $F(z,\theta;k;\alpha_2)$. The point $(0,0,z_o)$ is taken to lie within D. Essentially we wish to find a solution $u(x)$ of the Helmholtz's equation in $\mathbf{R}_h^3 / \overline{D}$ which vanishes on ∂D and such that $u(\mathbf{x}) = u^i(\mathbf{x}) + u^s(\mathbf{x})$, where $u^s(\mathbf{x})$ is the scattered solution.

Theorem (5.1) Any propagating solution of the Helmholtz equation defined in all of $\mathbf{R}_h^3$, *satisfying*

$$\overline{\lim_{r\to\infty}} \frac{1}{r} \int_{Z_r} |v(\mathbf{x})|^2\, d\mathbf{x} < \infty \;. \tag{5.19}$$

and the boundary conditions $\left|\frac{\partial v}{\partial z}\right|_{z=h} = 0$, $\left|v\right|_{z=0} = 0$ *may be represented in the form (5.12) for some* $g(z,\theta) \in L^2(\partial Z_1)$. *Conversely, any solution in the form (5.12) for some* $g(z,\theta) \in L^2(\partial Z_1)$ *satisfies (5.19).*

Proof. We first prove the direct part. Separation of variables leads to the following expansion for $v(\mathbf{x})$ defined in $\mathbf{R}_h^3$:

$$v(\mathbf{x}_2,z) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} (c_{nm} J_n(ka_n r) + d_{nm} Y_n(ka_n r) e^{im\theta} \phi_n(z) \ .$$

As $Y_m(r)$ is singular at $r=0$ this implies

$D_{nm} = 0$, $(n=0,1,2 \cdots)$, $(m=0, \pm 1, \pm 2, \cdots)$. Hence,

$$v(\mathbf{x}_2,z) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} c_{nm} J_n(ka_n r) e^{im\theta} \phi_n(z) \ . \tag{5.20}$$

The asymptotic condition (5.19) now implies

$$\frac{2\pi}{r} \|\phi_n\|_2^2 \int_0^r \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} |c_{nm}|^2 \ |J_m(ka_{n\rho})|^2 \rho \, d\rho < \infty \ .$$

Note that for any integer $n > N$ with $N := \left[\frac{2kh-\pi}{2\pi}\right]$ $a_n = \left[1 - \frac{(2n+1)^2\pi^2}{4k^2h^2}\right]^{1/2}$ is a pure imaginary. Using

$$J_m(ka_n r) \approx \sqrt{\frac{1}{2\pi k a_n r}} \left[e^{-ika_n r \ + \ i(m+1/2)\pi/2} + e^{ika_n r \ - \ i(m+1/2)\frac{\pi}{2}} \right]$$

we realize that since $\frac{1}{\sqrt{r}} J_m(ka_n r) \to \infty$ as $r \to \infty$ that $c_{nm} = 0$ for $\forall$ $n > N$, $(m=0, \pm 1, \pm 2, \cdots)$. From this (5.20) reduces to the propagating wave form

$$v(\mathbf{x}_2,z) = \sum_{n=0}^{N} \sum_{m=-\infty}^{\infty} c_{nm} J_m(ka_n r) e^{im\theta} \phi_n(z) \ . \tag{5.21}$$

To show $v(\mathbf{x}_2,z)$ may be written in the form (5.16) we rewrite (5.16) in the form

$$V(\mathbf{x}_2,z) = \sqrt{\frac{2}{i\pi k}} \sum_{n=0}^{N} \int_0^h \int_{|\mathbf{x}_2|=1} \frac{\phi_n(z)\phi_n\zeta}{\|\phi_n\|^2} e^{-ika_n r\cos(\theta-\phi)} g(\zeta,\phi)d\phi\, d\zeta$$

where $r = |\mathbf{x}_2|$. Using the Jacobi-Anger expansion one has

$$e^{-ika_n r\cos(\theta-\phi)} = \sum_{m=-\infty}^{\infty} (-i)^m J_m(ka_n r) e^{im(\theta-\phi)}$$

which permits us to rewrite $V(\mathbf{x}_2,z)$ as

$$V(\mathbf{x}_2,z) = \sum_{n=0}^{N} \sum_{m=-\infty}^{\infty} \phi_n(z)e^{im\theta}J_m(ka_n r)\, [\int_0^h \int_0^{2\pi} \frac{(-i)^m}{\|\phi_n\|^2} \sqrt{\frac{2}{i\pi k}}\, e^{-im\phi}$$

$$[\phi_n(\zeta)g(\zeta,\phi)d\phi\, d\zeta] \ . \tag{5.22}$$

If $g(\zeta,\phi)$ is defined to be the function given by the series

$$g(\zeta,\phi) := \sum_{n=0}^{N} \sum_{m=-n}^{n} \frac{i^m c_{nm}}{2\pi\sqrt{\frac{2}{i\pi k}}} \phi_n(\zeta)e^{im\phi} \ , \tag{5.23}$$

then $v(\mathbf{x}_2,z) \equiv V(\mathbf{x}_2,z)$. Moreover, since

$$\int_0^h \int_0^{2\pi} |g(\zeta,\phi)|^2 d\zeta\, d\phi = 2\pi\frac{\sqrt{ik}}{\sqrt{8\pi}} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} |c_{nm}|^2 \cdot \|\phi_n\|^2$$

$$= \frac{h}{2}\,\frac{\sqrt{\pi k}}{\sqrt{2}} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} |c_{nm}|^2 \ , \tag{5.24}$$

we may show that $g(\zeta,\phi) \in L^2(\partial \mathbf{Z}_1)$ if (5.23) is bounded. From (5.17) we have

$$\sum_{n=0}^{N} \sum_{m=-\infty}^{\infty} |c_{nm}|^2 (\frac{1}{r} \int_0^n |J_m(ka_n\rho)|^2 \rho\, d\rho) < \infty \ .$$

Using the asymptotic expansion for large argument of the Bessel function we have

$$\frac{1}{r} \int_0^r |J_m(ka_n\rho)|^2 d\rho = \frac{1}{ka_n\pi} + 0(\frac{1}{r}) \ ,$$

$(n=0,1,\cdots,n)$, $(m=0,\pm 1,\pm 2\cdots)$. This implies $\sum_{n=0}^{\infty}\sum_{m=-\infty} |c_{nm}|^2 < \infty$. □

To prove the converse part we assume $g \in L^2(\partial \mathbf{Z}_1)$. From this we have

$$|V(\mathbf{y})| = \left| \sqrt{\frac{2}{i\pi k}} \int_{\partial \mathbf{Z}_1} g(z,\theta) \sum_{n=0}^{N} \frac{\phi_n(z)\phi_n(\zeta)}{\|\phi_n\|^2} e^{ika_n<\mathbf{x}_2,\mathbf{y}_2>} dz \; d\theta \right|$$

$$\leq \left| \sqrt{\frac{2}{i\pi k}} \right| \cdot \sum_{n=0}^{N} \int_{\partial \mathbf{Z}_1} |g(z,\theta) \cdot \frac{|\phi_n(z)| \; |\phi_n(\zeta)|}{\|\phi_n\|^2} \left| e^{ika_n<\mathbf{x}_2,\mathbf{y}_2>} \right| dz \; d\theta$$

$$\leq \sqrt{\frac{2}{i\pi k}} \sum_{n=0}^{N} \left(\int_{\partial \mathbf{Z}_1} |g(z,\theta)|^2 dz d\theta \right)^{1/2} \left(\int_{\partial \mathbf{Z}_1} \frac{|\phi_n(z)\phi_n(\zeta)|^2}{\|Q_n\|^2} \left| e^{ika_n<\mathbf{x}_2,\mathbf{y}_2>} \right|^2 \right)^{1/2} dz \; d\theta$$

$$\leq C(N) \|g\|_{L^2(\mathbf{JZ})} \; .$$

Consequently (5.17) is valid. The function $Y(\mathbf{y})$ was shown to satisfy the Helmholtz equation. Obviously it satisfies the boundary conditions. □

It follows from the above Theorem and (5.11) if k^2 is a Dirichlet eigenvalue for the Laplacian in D and $v(\mathbf{x})$ a corresponding eigenfunction then

$$\int_{\partial \mathbf{Z}_1} F(z,\theta;k;\alpha_2)\overline{g(z,\theta)} dz \; d\theta = 0 \; . \tag{5.25}$$

Moreover, if k^2 is not a Dirichlet eigenvalue and $v(\mathbf{x})$ the solution of the boundary value problem

$$\left\{ \begin{aligned} &\Delta v + k^2 v = 0 \quad \text{in} \quad D \\ &v(\mathbf{x}) = -\Phi(\mathbf{x},\mathbf{x}^o) \quad \text{on} \quad \partial D \; , \end{aligned} \right\} \tag{5.26}$$

with $\mathbf{x}^o := (0,0,z_o) \in D$, then

$$\int_{\partial D} \frac{\partial u(\mathbf{y})}{\partial \nu_y} \Phi(\mathbf{y},\mathbf{x}^o) d\sigma(\mathbf{y}) = u^{(i)}(\mathbf{x}^o) = 1 \; . \tag{5.27}$$

The equalities (5.20), (5.22) suggest that we follow Colton and Monk [Como 85], [Como 87] and define the set $\mathbf{F}$ of far field patterns corresponding to a fixed domain D and incident fields of the form (5.18)

$$\mathbf{F} := \{F(z,\theta;k;\mathbf{x}^2) : |\alpha_2| = 1\} \ , \tag{5.28}$$

and the set S

$$\mathbf{S} := \{F(z,\theta;k;\alpha_2) - F(z,\theta;k,\alpha_2') : |\alpha_2| = 1\} \ , \tag{5.29}$$

where α_2' is a fixed unit 2- vector. Then as in [Como 85] we notice that if k^2 is a Dirichlet eigenfunction then $\mathbf{F}$ is perpendicular to the Herglotz kernels of the eigenfunctions that are generalized Herglotz wavefunctions. Moreover, if k^2 is not a Dirichlet eigenvalue then $\mathbf{S}$ is perpendicular to the Herglotz kernel of the solution of (5.21). In contrast to the case studied by Colton-Monk $F \in \mathbf{F}$ is a finite dimensional subspace of $L^2(\partial \mathbf{Z}_1)$,

$$L^2(\partial \mathbf{Z}_1) = span\ \mathbf{F} \oplus span\ \{g\} \oplus \mathbf{F}^{\perp} . \tag{5.30}$$

whereas, the latter case becomes

$$L^2(\partial \mathbf{Z}_1) = span\ \mathbf{S} \oplus span\ \{g\} \oplus S^{\perp} . \tag{5.31}$$

This means that as only a finite number of modes propagate any attempt to regain structure in the target identification problem is constrained by the fact that $\dim(\mathbf{F}) < \infty$.

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