DEL-SG-31-89

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$2.00

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October 1989

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Newark, DE 19716

This publication is the result of research sponsored by NOAA Office of Sea Grant, Department of Commerce, under Grant No. NA86AA-D-SG040 (Project No. R/OE-2). The U.S. Government is authorized to produce and distribute reprints for governmental purposes, notwithstanding any copyright notation that may appear hereon.

The University of Delaware Sea Grant College Program is supported cooperatively by the National Oceanic and Atmospheric Administration, U.S. Department of Commerce, and by the State of Delaware.

Generalized Herglotz Functions and Inverse Scattering Problem in a Finite Depth Ocean

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1 Introduction

In an investigation of underwater acoustic wave inverse scattering problem, namely to recover the shape of a scatterer from the far-field pattern of the scattered wave, we have found that, besides the difficulty that the problem is nonlinear and improperly posed, the most difficult point of the problem comes from the fact that the scattered wave field and the far-field is not $1-1$ related [6], [7],[12]. In a finite depth ocean, only a finite number of modes of the scattered wave propagate, the others decrease in an exponential rate. A far-field pattern can carry only the information from the finite propagating modes. Therefore, the same far-field pattern may correspond to very different near fields. In order to recover the shape of the scatterer, which is certainly related to the near field, we need to discover the relation between the near field and the far field under certain restrictions. In [12] we have constructed a proper subset of the incoming waves $u^i(\mathbf{x}, z)$ so that the mapping from the scattered fields to the far field patterns is an injection. Based on this injective result, an algorithm similar to that developed by Kirsch and Kress [10] was presented in [12].

In this paper, we continue our investigation of the problem by using the method of Colton and Monk, that is, we seek an optimal solution in the

[1] This research was supported in part by Sea Grant NA86AA-D-SG040.

orthogonal complement of the closure of the set of far field patterns. [4], [5]

For our purposes, it is convenient to set $\mathbf{R}^3_b = \{(\mathbf{x}, z); \ \ \mathbf{x} = (x_1, x_2) \in \mathbf{R}^2, 0 \leq z \leq h\}$ which is the region corresponding to the finite depth ocean with ocean depth h. Let Ω, an object imbedded in $\mathbf{R}^3_b$, be a bounded, connected domain with C^2 boundary $\partial\Omega$, and an outward unit normal ν. If the object has a sound soft boundary, an incoming wave u^i incident on $\partial\Omega$, will be scattered to produce a propagating wave u^s, as well as a far-field pattern. This problem can be formulated as to find a solution $u \in C^2(\mathbf{R}^3_b \setminus \overline{\Omega}) \cap \mathbf{C}(\mathbf{R}^3_b \setminus \Omega)$ to the Helmholtz equation

$$\triangle_3 \, u + k^2 u = 0, \ in \ \mathbf{R}^3_b \setminus \overline{\Omega}, \tag{1.1}$$

such that u satisfies the boundary conditions

$$u = 0, \ as \ z = 0, \tag{1.2}$$

$$\frac{\partial u}{\partial z} = 0, \ as \ z = h, \tag{1.3}$$

$$u = 0, \ on \ \partial\Omega. \tag{1.4}$$

Here k is a positive constant known as the wave number, and $u = u^i + u^s$, where u^i and u^s are the incident (entire) wave and the scattered wave respectively. The scattered wave has the modal representation

$$u^s = \sum_{n=0}^{\infty} \phi_n(z) u^s_n(\mathbf{x}), \tag{1.5}$$

where

$$\phi_n(z) = sin[k(1 - a_n^2)^{\frac{1}{2}} z], \tag{1.6}$$

$$a_n = [1 - \frac{(2n+1)^2\pi^2}{4k^2h^2}]^{\frac{1}{2}}, \tag{1.7}$$

and the n^{th} mode of u^s, $u_n^s(\mathbf{x})$, satisfies the radiating condition

$$\lim_{r\to\infty} r^{\frac{1}{2}}(\frac{\partial u_n^s}{\partial r} - ika_n u_n^s) = 0, \ r = |\mathbf{x}|, \ n = 0, 1, ..., \infty. \tag{1.8}$$

The inverse problem we are going to consider is that given far field patterns $f(\hat{\mathbf{x}}, z, k)$ for one or several incoming waves, to find the shape of the scattering object Ω. First, we discuss some decomposition properties of the far field pattern, then we present a numerical algorithm to recover the shape of the scatterer. At the last, we apply our algorithm to some numerical examples and recover the shape of some "footballs". For an object with arbitrary shape, the computation becomes very time consuming and more detailed investigation of the computational problem is necessary. We are presently modifying our program to accomplish this and our results will be presented in a subsequent paper.

2 Generalized Herglotz functions and far field patterns for the Helmholtz equation in a finite depth ocean

In this section we discuss the dense and decomposition properties of far field patterns for the Helmholtz equation in a finite depth ocean. An alternate version of these results here may be found in our earlier papers [7], [8].

Let $u = u^i + u^s$ and u^i is the incoming wave and u^s the scattering wave. We assume that $k \neq (2n+1)\pi/2h$ for $n = 0, 1, 2, ...$ For any given incoming wave u^i, u is determined uniquely. [11] Let $G(z, \zeta, |\mathbf{x} - \xi|)$ be the Green's function of Helmholtz equation in $\mathbf{R}_b^3$ satisfying boundary condition (1.2) (1.3) and radiating condition (1.8), $\omega_1 = \{\mathbf{x} = (x_1, x_2) \in \mathbf{R}^2; \ |\mathbf{x}| = 1\}$. If the incoming wave u^i is

$$u^i(\mathbf{x}, z; \alpha, \beta) = \sum_{n=0}^{N} \phi_n(\beta)\phi_n(z)e^{ika_n\alpha\cdot\mathbf{x}}, \ \ \alpha \in \omega_1, \ \beta \in [0, h], \tag{2.1}$$

then the corresponding scattered wave $u^s(\mathbf{x}, z; \alpha, \beta)$ is

$$u^s(\mathbf{x}, z; \alpha, \beta) = \int_{\partial\Omega} (u^s \frac{\partial G}{\partial \nu} - G \frac{\partial u^s}{\partial \nu}) d\sigma$$

$$= \frac{ie^{-i\pi/4}}{4\|\phi_n\|^2} \sum_{n=0}^{N} (\frac{2}{\pi k a_n r})^{\frac{1}{2}} e^{ika_n r} \phi_n(z) \int_{\partial\Omega} \{u^s(\xi, \zeta; \alpha, \beta) \frac{\partial}{\partial \nu_\xi} (e^{-ika_n \mathbf{x}\cdot\xi} \phi_n(\zeta))$$

$$- \frac{\partial u^s(\xi, \zeta; \alpha, \beta)}{\partial \nu_\xi} (e^{-ika_n \mathbf{x}\cdot\xi} \phi_n(\zeta))\} d\sigma_\xi + O(\frac{1}{r^{\frac{3}{2}}}). \tag{2.2}$$

The corresponding far-field pattern can be represented as:

$$F(\mathbf{x}, z; \alpha, \beta) = \sum_{n=0}^{N} \phi_n(z) \int_{\partial\Omega} \{u^s(\xi, \zeta; \alpha, \beta) \frac{\partial}{\partial \nu_\xi} (e^{-ika_n \mathbf{x}\cdot\xi} \phi_n(\zeta))$$

$$- \frac{\partial u^s(\xi, \zeta; \alpha, \beta)}{\partial \nu_\xi} (e^{-ika_n \mathbf{x}\cdot\xi} \phi_n(\zeta))\} d\sigma_\xi, \tag{2.3}$$

where $(\mathbf{x}, z) \in \omega_1 \times [0, h] =: \partial D_1$..

Theorem 2.1: *For any $(\mathbf{x}, z)$, $(\alpha, \zeta) \in \partial D_1$, we have*

$$F(\mathbf{x}, z; \alpha, \zeta) = F(-\alpha, \zeta; -\mathbf{x}, z). \tag{2.4}$$

Proof:

$$F(\mathbf{x}, z; \alpha, \beta) = \int_{\partial\Omega} \left\{ u^s(\xi, \zeta; \alpha, \beta) \frac{\partial}{\partial \nu_\xi} \left[\sum_{n=0}^{N} \phi_n(z) e^{-ika_n \mathbf{x}\cdot\xi} \phi_n(\zeta) \right] \right.$$

$$\left. - \frac{\partial u^s(\xi, \zeta; \alpha, \beta)}{\partial \nu_\xi} \left[\sum_{n=0}^{N} \phi_n(z) e^{-ika_n \mathbf{x}\cdot\xi} \phi_n(\zeta) \right] \right\} d\sigma_\xi$$

$$= \int_{\partial\Omega} \{u^s(\xi, \zeta; \alpha, \beta) \frac{\partial}{\partial \nu_\xi} u^i(\xi, \zeta; -\mathbf{x}, z)$$

$$- \frac{\partial u^s(\xi, \zeta; \alpha, \beta)}{\partial \nu_\xi} u^i(\xi, \zeta; -\mathbf{x}, z)\} d\sigma_\xi$$

$$= - \int_{\partial\Omega} \{u^i(\xi, \zeta; \alpha, \beta) \frac{\partial}{\partial \nu_\xi} u^i(\xi, \zeta; -\mathbf{x}, z)$$

$$- \frac{\partial u^s(\xi, \zeta; \alpha, \beta)}{\partial \nu_\xi} u^s(\xi, \zeta; -\mathbf{x}, z)\} d\sigma_\xi$$

$$= \int_{\partial\Omega} \{u^s(\xi, \zeta; \alpha, \beta) \frac{\partial}{\partial \nu_\xi} u^s(\xi, \zeta; -\mathbf{x}, z)$$

$$-\frac{\partial u^i(\xi,\zeta;\alpha,\beta)}{\partial\nu_\xi}u^i(\xi,\zeta;-\mathbf{x},z)\}d\sigma_\xi$$

$$= -\int_{\partial\Omega}\{u^i(\xi,\zeta;\alpha,\beta)\frac{\partial}{\partial\nu_\xi}u^s(\xi,\zeta;-\mathbf{x},z)$$

$$-\frac{\partial u^i(\xi,\zeta;\alpha,\beta)}{\partial\nu_\xi}u^s(\xi,\zeta;-\mathbf{x},z)\}d\sigma_\xi$$

$$= F(-\alpha,\beta;-\mathbf{x},z).$$

As defined in [6], a generalized Herglotz wave function is such $v(\xi,\zeta)$ that

$$\overline{\lim_{r\to\infty}}\frac{1}{r}\int_{D_r}|v(\xi,\zeta)|^2 d\xi d\zeta < \infty,$$

where $D_r = \{(\mathbf{x},z)\in\mathbf{R}_b^3;\ |\mathbf{x}|\le r\}$. $v(\xi,\zeta)$ may be expressed as

$$v(\xi,\zeta) = \int_{D_1} g(\mathbf{x},z)\sum_{n=0}^{N}\phi_n(z)\phi_n(\zeta)e^{ik\alpha_n\mathbf{x}\cdot\xi}d\sigma_x \tag{2.5}$$

where $g(\mathbf{x},z)\in L^2(\partial D_1)$ is the generalized Herglotz kernel.

Define

$$\mathcal{F} = \{F(\mathbf{x},z;\alpha_n,\beta); n=1,2,...;\beta\in[0,h]\},$$

$$\mathcal{S} = \{F(\mathbf{x},z;\alpha_n,\beta) - F(\mathbf{x},z;\alpha_1,\beta); n=1,2,..,\beta\in[0,h]\}$$

and

$$\mathcal{S}^\perp = \{v\in V^N,(u,v)=0\ for\ u\in\mathcal{S}\},$$

where

$$V^N = L^2[0,2\pi]\times span\{\phi_0,\phi_1,...,\phi_N\}$$

and $\{\alpha_n\}$ has a limit point in $[0,2\pi]$. In the same way as in [8] Th.4.3, we can prove:

Theorem 2.2 If v defined by (2.5) is not an eigenfunction of the Dirichlet problem in Ω, then the set $\mathcal{F}$ is dense in V^N.

Now we consider the dense property of $\mathcal{S}$. If $g\in\mathcal{S}^\perp$, then

$$\int_{\partial D_1}[F(\mathbf{x},z;\alpha_n,\beta) - F(\mathbf{x},z;\alpha_1,\beta)]\overline{g(\mathbf{x},z)}d\sigma_x = 0, \tag{2.6}$$

$$n = 1, 2, ...;\ \beta \in [0, h].$$

By analytic continuation of $F(\mathbf{x}, z; \alpha, \beta)$ with respect to (α, β), we get for every $\alpha \in \omega_1$ that

$$\int_{\partial D_1} [F(\mathbf{x}, z; \alpha, \beta)] \overline{g(\mathbf{x}, z)} d\sigma_x = \sum_{n=0}^{N} c_n \phi_n(\beta),\ \beta \in [0, h], \tag{2.7}$$

where c_n are constants.

Suppose not all c_n equal zero and let

$$U^s(\mathbf{x}, z) = \int_{\partial D_1} u^s(\mathbf{x}, z; \alpha_n, \beta) \overline{g(-\alpha, \beta)} d\sigma_\alpha, \tag{2.8}$$

then the far field pattern of $U^s(\mathbf{x}, z)$ is

$$\begin{aligned} &\int_{\partial D_1} F(\mathbf{x}, z; \alpha_n, \beta) \overline{g(-\alpha, \beta)} d\sigma_\alpha \\ &= \int_{\partial D_1} F(-\alpha_n, \beta; -\mathbf{x}, z) \overline{g(-\alpha, \beta)} d\sigma_\alpha \\ &= \sum_{n=0}^{N} c_n \phi_n(z) \end{aligned} \tag{2.9}$$

Without loss of generality, we assume that $c_n = \phi_n(z_0)$ where $(\mathbf{0}, z_0)$ is in the interior of Ω. It follows that for $|\mathbf{x}| \geq R$, where R is a constant such that $D_R \supset \overline{\Omega}$,

$$U^s(\mathbf{x}, z) = \frac{2}{h} \sum_{n=0}^{N} \phi_n(z) \phi_n(z_0) H_0^{(1)}(k a_n r) + \sigma_N, \tag{2.10}$$

where σ_N contains no propagating modes. However, since $g \in \mathcal{S}^\perp \subset V^N$, so for $|\mathbf{x}| \geq R$, $U^s(\mathbf{x}, z)$ can be expressed as

$$\begin{aligned} U^s(\mathbf{x}, z) &= \int_{\partial D_1} u^s(\mathbf{x}, z; \alpha_n, \beta) \overline{g(-\alpha, \beta)} d\sigma_\alpha \\ &= \sum_{n=0}^{N} \phi_n(z) U_n^s(\mathbf{x}). \end{aligned} \tag{2.11}$$

Hence, we must have

$$U^s(\mathbf{x}, z) = \frac{2}{h} \sum_{n=0}^{N} \phi_n(z) \phi_n(z_0) H_0^{(1)}(k a_n r),\ \ r \geq R. \tag{2.12}$$

The importance of the above result is that even though for a given far field pattern, we usually can not determine a unique near field, we can find a function such as $U^s(\mathbf{x}, z)$ that is uniquely determined by the far field pattern

$$\int_{\partial D_1} F(\mathbf{x}, z; \alpha_n, \beta)\overline{g(-\alpha, \beta)}d\sigma_\alpha.$$

From (2.8), $U^s(\mathbf{x}, z)$ is a solution to the Helmholtz equation in $\mathbf{R}_b^3 \setminus \Omega$. The real analyticity of solutions to the Helmholtz equation follows that $U^s(\mathbf{x}, z)$ can be continuated to $\partial\Omega$ uniquely. A construction of $U^s(\mathbf{x}, z)$ for $r \leq R$ will be given in later section of this paper.

Now we are in the position to prove a theorem which was first presented by Colton and Monk for the $\mathbf{R}^2$ case.[4] [5]

Theorem 2.3: Assume that k^2 is not an eigenvalue of the interior Dirichlet problem and let v be the solution of the Dirichlet problem

$$\triangle v + k^2 v = 0, \ in \ \Omega; \tag{2.13}$$

$$v(r, \theta, z) = \overline{U^s(\mathbf{x}, z)} \ on \ \partial\Omega, \tag{2.14}$$

where $U^s(\mathbf{x}, z)$ is given by (2.8) and (r, θ, z) is the polar coordinates related to $(\mathbf{x}, z)$. Then

(1) If v is an entire Herglotz wave function with Herglotz kernel $g \neq 0$, then $S^\perp \neq \{0\}$;

(2) If v is not an entire Herglotz wave function. $S^\perp = \{0\}$.

Proof: If $g \in S^\perp$, then we have (2.7). i.e.

$$\int_{\partial D_1} [F(\mathbf{x}, z; \alpha, \beta)]\overline{g(\mathbf{x}, z)}d\sigma_x = \sum_{n=0}^{\infty} c_n \phi_n(\beta) \tag{2.15}$$

for every $(\alpha, \beta) \in \partial D_1$. If $c_n = 0$ for $n = 0, 1, ..., N$, then $g = 0$ by Theorem 2.2. If some $c_n \neq 0$, then the far field pattern of $U^s(\mathbf{x}, z)$ defined by (2.8) is not identical to zero and represented by (2.15).

Furthermore,

$$w(\mathbf{x}, z) = -\int_{\partial D_1} \overline{g(-\alpha, \beta)} \sum_{n=0}^{N} \phi_n(z)\phi_n(\beta)e^{ika_n\alpha\cdot\mathbf{x}} d\sigma_\alpha$$

$$= \overline{\int_{\partial D_1} g(\alpha, \beta) \sum_{n=0}^{N} \phi_n(z)\phi_n(\beta)e^{ika_n\alpha\cdot\mathbf{x}} d\sigma_\alpha}$$

$$= \overline{v(\mathbf{x}, z)}, \tag{2.16}$$

which satisfies on $\partial\Omega$ that

$$w(\mathbf{x}, z) = -\int_{\partial D_1} \overline{g(-\alpha, \beta)} u^i(\mathbf{x}, z; \alpha, \beta) d\sigma_\alpha$$

$$= \int_{\partial D_1} \overline{g(-\alpha, \beta)} u^s(\mathbf{x}, z; \alpha, \beta) d\sigma_\alpha$$

$$= U(\mathbf{x}, z). \tag{2.17}$$

Hence,

$$v(\mathbf{x}, z) = \overline{U^s(\mathbf{x}, z)} \quad on \ \partial\Omega. \tag{2.18}$$

It proves the theorem in a reciprocal way.

3 The inverse scattering problem

In this section we will like to reformulate the inverse scattering problem as a problem in constrained optimization based on the preceding analysis. A similar formulation has been carried out by Colton and Monk for an object in $\mathbf{R}^2$. [4] [5]

We assume that Ω is such that $\partial\Omega$ can be parameterized in the form $\rho = \rho(\theta, \phi)$ where $0 < a \leq \rho(\theta, \phi) \leq b$ for $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$, and a, b are some positive constants. Let (ρ, θ, ϕ) be the spherical coordinates with respect to $(\mathbf{x}, z)$. We define the sets U_1 and U_2 by

$$U_1(M) = \{g = \sum_{n=0}^{N} g_n(\theta)\phi_n(z) \mid g_n \in W_2^1[0, 2\pi], \ \|g_n\| \leq M\}$$

$$U_2 = \{\rho \in C([0,2\pi]\times[0,\pi]) \mid \rho(0,\phi) = \rho(2\pi,\phi), 0 < a \le \rho(\theta,\phi) \le b,$$

$$|\rho(\theta_1,\phi_1) - \rho(\theta_2,\phi_2)| \le C_1|\theta_1 - \theta_2| + C_2|\phi_1 - \phi_2|\}$$

where M, C_1, C_2 are positive constants and $W_2^1[0,2\pi]$ denotes a Sobolev space with norm $\|\cdot\|$. Following Colton and Monk [5], we know that $U_1(M)$ and U_2 are compact in $C([0,2\pi]\times[0,h])$ and $C([0,2\pi]\times[0,\pi])$ respectively. Hence, $U(M) = U_1(M)\times U_2$ is compact in $C([0,2\pi]\times[0,h])\times C([0,2\pi]\times[0,\pi])$ by Tikhonov's theorem. If $F(\theta,z)$ is the measured far field pattern corresponding to the incident "plane" wave

$$u^i(\mathbf{x},z) = \sum_{n=0}^{N} \phi_n(\beta)\phi_n(z)e^{ik a_n \alpha\cdot\mathbf{x}}$$

then we define the optimization problem as:

$$\mathcal{J}(F,M,J,L) = \min_{(g,\rho)\in U(M)} \left\{ \sum_{j=1}^{J}\sum_{l=1}^{L} \left| \int_{\partial D_1} f(\mathbf{x},z;\alpha_l,\beta_j)\overline{g(\mathbf{x},z)}d\sigma_x \right.\right.$$

$$\left.\left. -\sum_{n=0}^{N}\phi_n(z_0)\phi_n(\beta_j)\right|^2 + \int_0^{\pi}\int_0^{2\pi} |v(r(\theta,\phi),\theta,z(\theta,\phi)) - U^s(r(\theta,\phi),\theta,z(\theta,\phi))|^2 d\theta d\phi \right\}, \tag{3.1}$$

where

$$r(\theta,\phi) = \rho(\theta,\phi)sin\phi,\ \ z(\theta,\phi) = \rho(\theta,\phi)cos\phi.$$

v is defined by (2.5) and a representation of $U^s(r,\theta,z)$ on $\partial\Omega$ will be given in next section.

Definition 3.1: A function $\rho \in C([0,2\pi]\times[0,\pi])$ is admissible if and only if a pair $(g,\rho)\in U(M)$ minimizes (3.1) over $U(M)$.

It is clear that from the compactness of $U(M)$ and continuity of the integral in (3.1) with respect to ρ and g, there exists at least one admissible solution.

In examining the relationship between admissible solutions and actual solutions of the inverse scattering problem, we can prove the following theorems in a similar way as the corresponding theorem in [4], [5].

Theorem 3.1: Let $\Phi(F)$ be the set of admissible solutions corresponding to the far field pattern F. If $F_j \to F$ in V^N, $\rho_j \in \Phi(F_j)$, then there exists a convergent subsequence of $\{\rho_j\}$ and every limit point llies in $\Phi(F)$.

Proof: Since U_2 is compact, without loss of generality we can assume that $\{\rho_j\}$ converges to $\rho^* \in U_2$. Let (g_j, ρ_j) be the pair such that $(g_j, \rho_j) \to (g^*, \rho^*) \in U(M)$. We need to show that

$$\mathcal{J}(F, M, J, L) = \sum_{j=1}^{J}\sum_{l=1}^{L} | \int_{\partial D_1} F(\mathbf{x}, z; \alpha_l, \beta_j)\overline{g^*(\mathbf{x}, z)}d\sigma_x - \sum_{n=0}^{N} \phi_n(z_0)\phi_n(\beta_j)|^2$$

$$+ \int_0^{\pi}\int_0^{2\pi} |v^*(r^*(\theta,\phi), \theta, z^*(\theta,\phi)) - U^s(r^*(\theta,\phi), \theta, z^*(\theta,\phi))|^2 d\theta d\phi \quad (3.2)$$

where v^* is the generalized Herglotz wave function associated with g^*, $r^* = \rho^* sin\phi$ and $z^* = \rho^* cos\phi$. (We will use similar notations for ρ_j, r_j, z_j as well as $\tilde{\rho}, \tilde{r}, \tilde{z}$ without saying it every time. Now if $\tilde{\rho} \in \Phi(F)$ is with associated pair $(\tilde{g}, \tilde{\rho})$, then

$$\mathcal{J}(F, M, J, L) \le \sum_{j=1}^{J}\sum_{l=1}^{L} | \int_{\partial D_1} F(\mathbf{x}, z; \alpha_l, \beta_j)\overline{g^*(\mathbf{x}, z)}d\sigma_x - \sum_{n=0}^{N} \phi_n(z_0)\phi_n(\beta_j)|^2$$

$$+ \int_0^{\pi}\int_0^{2\pi} |v^*(r^*(\theta,\phi), \theta, z^*(\theta,\phi)) - U^s(r^*(\theta,\phi), \theta, z^*(\theta,\phi))|^2 d\theta d\phi$$

$$= \lim_{j\to\infty} \Bigg\{ \sum_{j=1}^{J}\sum_{l=1}^{L} | \int_{\partial D_1} F_j(\mathbf{x}, z; \alpha_l, \beta_j)\overline{g_j(\mathbf{x}, z)}d\sigma_x - \sum_{n=0}^{N} \phi_n(z_0)\phi_n(\beta_j)|^2$$

$$+ \int_0^{\pi}\int_0^{2\pi} |v_j(r_j(\theta,\phi), \theta, z_j(\theta,\phi)) - U^s(r_j(\theta,\phi), \theta, z_j(\theta,\phi))|^2 d\theta d\phi \Bigg\}$$

$$= \lim_{j\to\infty} \mathcal{J}(F_j, M, J, L)$$

$$\le \lim_{j\to\infty} \Bigg\{ \sum_{j=1}^{J}\sum_{l=1}^{L} | \int_{\partial D_1} F_j(\mathbf{x}, z; \alpha_l, \beta_j)\overline{\tilde{g}(\mathbf{x}, z)}d\sigma_x - \sum_{n=0}^{N} \phi_n(z_0)\phi_n(\beta_j)|^2$$

$$+ \int_0^{\pi}\int_0^{2\pi} |\tilde{v}(\tilde{r}(\theta,\phi), \theta, \tilde{z}(\theta,\phi)) - U^s(\tilde{r}(\theta,\phi), \theta, \tilde{z}(\theta,\phi))|^2 d\theta d\phi \Bigg\}$$

$$= \mathcal{J}(F, M, J, L).$$

It finishes the proof.

Theorem 3.2: Assume that k^2 is not the eigenvalue of the interior Dirichlet problem for Ω and Ω a bounded domain with C^2 boundary $\partial\Omega$: $\rho = \rho(\theta, \phi)$ such that $\rho \in U_2$. Assume the solution of (2.3),(2.4) is an generalized entire Herglotz wave function with Herglotz kernel $g \in W_2^1([0, 2\pi] \times [0, h])$. If F is the far field pattern corresponding to Ω and the incident wave

$$u^i(\mathbf{x}, z) = \sum_{n=0}^{N} \phi_n(\beta)\phi_n(z)e^{ika_n\alpha\cdot\mathbf{x}}.$$

Then there exists a constant $M_0 < \infty$ such that $\mathcal{J}(F, M, J, L) = 0$ for each $M \geq M_0$ and integers J, L. For each J, L, let $\{\rho_j^{J,L}\}$, $j = 1, 2, \cdots, n_{J,L}$. The number of these limit points is finite.

Proof: Since k^2 is not a eigenvalue for the interior Dirichlet problem, the problem (2.3), (2.4) has unique solution. Hence, $g \in U_1(M)$ is uniquely determined for $M \geq M_0$ where M_0 is a positive constant. From (2.7) and (2.13) we see that $\mathcal{J}(F, M, J, L) = 0$ for each $M \geq M_0$ and every integers J and L.

Now let $\{\rho_j^{J,L}\}$ be as defined in the theorem. Then since U_2 is compact, the sequence $\{\rho_j^{J,L}\}$ has a subsequence converging to $\rho^* \in U_2$. Let $g_j^{J,L} \in U_1(M)$ be a function associated with $\rho_j^{J,L}$, then $\{g_j^{J,L}\}$ has a subsequence converging to a limit point $g^* \in U_1(M)$. But it follows from the fact that $\mathcal{J}(F, M, J, L) = 0$ for $M \geq M_0$ and each J, L,

$$\int_{\partial D_1} [F(\mathbf{x}, z; \alpha_i, \beta_l)]\overline{g_j^{J,L}(\mathbf{x}, z)}d\sigma_x = \sum_{n=0}^{N} \phi_n(z_0)\phi_n(\beta_l) \tag{3.3}$$

for $i = 1, 2, ..., J$; $l = 1, 2, ..., L$. Hence,

$$\int_{\partial D_1} [F(\mathbf{x}, z; \alpha_i, \beta_l)]\overline{g^*(\mathbf{x}, z)}d\sigma_x = \sum_{n=0}^{N} \phi_n(z_0)\phi_n(\beta_l) \tag{3.4}$$

for $i = 1, 2, ...; l = 1, 2, ...$ In view of Theorem 2.2 we now can conclude that $g^* = g$.

Now we prove that there is only finite number of limit points ρ^* lying in U_2. Let $\{\rho_i\}, \{g_i\}$ be the convergent subsequences defined above, then since $\mathcal{J}(F, M, J, L) = 0$ for $M \geq M_0$, we have that

$$v_i(\rho_i(\theta, z), \theta, z) = \int_{D_1} g_i(\phi, \zeta) \sum_{n=0}^{N} \phi_n(z)\phi_n(\zeta) e^{ik\alpha_n \rho_i(\theta,z)\cos(\theta-\phi)} d\sigma_\xi = U^s(\rho_i, \theta, z). \tag{3.5}$$

and hence by passing to the limit

$$v(\rho^*(\theta, z), \theta, z) = \int_{D_1} g(\phi, \zeta) \sum_{n=0}^{N} \phi_n(z)\phi_n(\zeta) e^{ik\alpha_n \rho^*(\theta,z)\cos(\theta-\phi)} d\sigma_\xi = U^s(\rho^*, \theta, z). \tag{3.6}$$

If there existed an infinite number of limit points ρ^* in U_2, then from the compactness of U_2 and the Arzela-Ascoli theorem we could find a domain D^* with arbitrary small area such that

$$v_k(\rho, \theta, z) := v(\rho, \theta, z) - U^s(\rho, \theta, z)$$

is an eigenfunction of D^* with eigenvalue k^2. But this is impossible. Hence, there is only finite number of limit points ρ^*.

4 A representation of $U^s(r, z)$

Since $U^s(\mathbf{x}, z) = U^s(r, z)$ for $r \geq R$, we are going to look for a solution of

$$\triangle u + k^2 u = \frac{\delta(r)}{2\pi r}\delta(z - z_0),\ (r, \theta, z) \in D_R, \tag{4.1}$$

$$u = U^s(r, z),\ \ at\ r = R, \tag{4.2}$$

$$\frac{\partial u}{\partial r} = \frac{\partial U^s}{\partial r}(r, z),\ \ at\ r = R, \tag{4.3}$$

$$u = 0,\ \ at\ z = 0, \tag{4.4}$$

$$\frac{\partial u}{\partial z} = 0,\ \ at\ z = h. \tag{4.5}$$

If $u(r, z)$ is a solution to (4.1) – (4.5), then

$$\hat{u}(\mathbf{x}, z) := \begin{cases} u(\mathbf{x}, z), & r \leq R, \\ U^s(r, z), & r \geq R \end{cases}$$

satisfies (1.1) in $\mathbf{R}_b^3 \setminus \Omega$. Hence, the uniqueness to the Helmholtz equation will follow that

$$u(r, z) = U^s(r, z), \quad for\ (\mathbf{x}, z) \in \Omega_R \setminus \Omega. \tag{4.6}$$

We can solve problem (4.1) – (4.5) by a variant of Hankel transform. Let

$$\tilde{u}(s, z) = \int_0^R J_0(sr)u(r, z)rdr, \ \ 0 \leq s < \infty, \ 0 \leq z \leq h, \tag{4.7}$$

then

$$\begin{aligned} &\int_0^\infty J_0(rs)\tilde{u}(s, z)sds \\ &= \int_0^R u(t, z)[\int_0^\infty J_0(ts)J_0(sr)sds]tdt = u(r, z), \\ &0 \leq r < R, \ 0 \leq z \leq h. \end{aligned} \tag{4.8}$$

From (4.1), we have

$$\int_0^R J_0(kar)(ru_r)_r dr + \tilde{u}_{zz}(ka, z) + k^2\tilde{u}(ka, z) = -\delta(z - z_0). \tag{4.9}$$

Since

$$\begin{aligned} &\int_0^R J_0(kar)(ru_r)_r dr \\ &= J_0(kar)ru_r|_0^R - \int_0^R J_0'(kar)karu_r dr \\ &= J_0(kaR)Ru_r(R, z) - J_0'(kaR)kaRu(R, z) \\ &\quad + \int_0^R (karJ_0'(kar))'ud(kar) \\ &= J_0(kaR)Ru_r(R, z) - J_0'(kaR)kaRu(R, z) \\ &\quad -(ka)^2 \int_0^R J_0(kar)urdr \end{aligned}$$

$$= J_0(kaR)Ru_r(R,z) - J_0'(kaR)kaRu(R,z)$$

$$-(ka)^2\tilde{u}(ka,z),$$

(4.9) becomes

$$\tilde{u}_{zz} + k^2(1-a^2)\tilde{u} = f(ka,z) - \delta(z-z_0), \tag{4.10}$$

$$\tilde{u}(ka,0) = 0, \tag{4.11}$$

$$\tilde{u}_z(ka,h) = 0, \tag{4.12}$$

where

$$f(ka,z) = J_0'(kaR)kaRu(R,z) - J_0(kaR)Ru_r(R,z)$$

$$= \frac{2Rk}{h}\sum_{n=0}^{N}\phi_n(z_0)\phi_n(z)\{aJ_0'(kaR)H_0^{(1)}(ka_nR) - a_nJ_0(kaR)[H_0^{(1)}(ka_nR)]'\}$$

Let

$$K(ka,z,z_0) = \frac{sin[k(1-a^2)^{1/2}z_>]cos[k(1-a^2)^{1/2}(z_< - h)]}{(1-a^2)^{1/2}cos[kh(1-a^2)^{1/2}]},$$

where $z_> = max\{z,z_0\},\ and\ z_< = min\{z,z_0\}$. The solution to the problem (4.10) – (4.12) is

$$\tilde{u}(ka,z) = K(ka,z,z_0) + \int_0^h f(ka,t)K(ka,z,t)dt. \tag{4.13}$$

Hence,

$$u(r,z) = \int_0^\infty J_0(kar)\tilde{u}(ka,z)kad(ka),\ (r,z)\in D_R, \tag{4.14}$$

gives a solution to the problem (4.1) – (4.5).

5 Numerical examples

In this section, we present simple examples to show numerical implementation of our algorithm. The data for the inverse problem is the approximate far field pattern corresponding to a finite number of equally spaced incoming plane waves. To provide this data, we have developed a "matching" scheme for the direct problem in a finite depth ocean.[9] The basic idea there is that we use the ray representation to approximate the near field and normal modes representation the far field. First, we write the scattered near field as a finite sum of basic functions $h_n^{(1)}(kr)P_n^{|m|}(cos\phi)e^{im\theta},\ -n \le m \le n;\ n = 0, 1, ..., \infty$, where $h_n^{(1)}$ denote the spherical Hankel function of first kind and $P_n^{|m|}(t)$ are the associated Legendre functions. We find the coefficients in this expression by matching the sum with the incoming wave on the boundary of the scattering obstacle $\partial\Omega$. Then we express the propagating field in a normal modes representation

$$u^s(r,\theta,z) = \sum_{n=0}^{N_1} \sum_{m=-M_1}^{M_1} \beta_{mn}\phi_n(z)H_m^{(1)}(ka_nr)e^{im\theta},$$

and match it with the starting field on a properly chosen cylinder. After the coefficients β_{mn} are determined, the approximate propagating far field pattern is

$$F(\theta,z) = \sum_{n=0}^{N} \sum_{m=-M_1}^{M_1} \beta_{mn}(-i2h)e^{-im\pi/2}\phi_n(z)e^{im\theta}.$$

In this section, we present two simple examples that the boundary of object can be represented as: $\rho = \rho(z,\theta) > 0,\ 0 \le z \le h,\ 0 \le \theta \le 2\pi$. For this case, the function $U^s(r,z)$ is simply that

$$U^s(r,z) = \frac{2}{h}\sum_{n=0}^{N}\phi_n(z_0)\phi_n(z)H_0^{(1)}(ka_nr),\ for\ r \ge \rho(z,\theta) > 0.$$

Also we can match the propagating field with the incident field on the boundary of scattering object directly.

For L incoming waves with direction

$$\alpha_l = \begin{pmatrix} cos(2\pi l/L) \\ sin(2\pi l/L) \end{pmatrix},\ l = 1, 2, ..., L; \tag{5.1}$$

and $\beta_j = jh/J,\ j = 1,2,...,J$, we denote the responding far field pattern as

$$F^{lj}(\theta,z) = \sum_{n=0}^{N} \sum_{m=-M_1}^{M_1} F_{mn}^{lj} \phi_n(z) e^{im\theta},\ l = 1,2,...,L;\ j = 1,2,...,J. \quad (5.2)$$

Also, we write the generalized Herglotz kernel $g \in V^N$ as

$$g^{lj}(\theta,z) = \sum_{n=0}^{N} \sum_{m=-M_1}^{M_1} g_{mn}^{lj} \phi_n(z) e^{im\theta}. \quad (5.3)$$

If the scatterer is represented by

$$\rho_a(\theta,z) = \sum_{l=0}^{l_1} \sum_{j=0}^{j_1} (\gamma_{lj}^1 cosj\theta cos\frac{l\pi z}{h}$$

$$+ \gamma_{lj}^2 cosj\theta sin\frac{l\pi z}{h} + \gamma_{lj}^3 sinj\theta cos\frac{l\pi z}{h} + \gamma_{lj}^1 sinj\theta sin\frac{l\pi z}{h}), \quad (5.4)$$

then we can discrete the inverse optimization problem as:

$$\mathcal{J}_a(F,M,J,L) = \min_{(g_a,\rho_a)\in U(M)} \left\{ \sum_{j=1}^{J} \sum_{l=1}^{L} \left| h\pi \sum_{n=0}^{N} \sum_{m=-M}^{M} F_{mn}^{lj} \overline{g_{mn}} + \sum_{n=0}^{N} \phi_n(z_0)\phi_n(jh/J) \right|^2 \right.$$

$$\left. + \frac{2\pi h}{m_1 m_2} \sum_{p=1}^{m_1} \sum_{q=1}^{m_2} |v_a(\theta_p, z_q) - U^s(\rho_a(\theta_p,z_q),z_q)|^2 \right\}$$

where

$$v_a(\rho(\theta_p,z_q),\theta_p,z_q) = \frac{2\pi h}{m_1 m_2} \sum_{\mu=1}^{m_1} \sum_{\nu=1}^{m_2} g_a(\phi_\mu,\zeta_\nu) \sum_{n=0}^{N} \phi_n(\zeta_\nu)\phi_n(z_q) e^{ika_n \rho(\theta_p,z_q) cos(\phi_\mu - \theta_p)}$$

and $\phi_\mu = 2\pi\mu/m_1,\ \theta_p = 2\pi p/m_1, \zeta_\nu = \nu h/m_2, z_q = qh/m_2$.

Now we present two examples from our computations.

Example 1:

Exact figure: $x_1^2 + x_2^2 = 8.0625 sin(\pi z/h) + 1$, (see figure 1a).

Parameters: $k = 4,\ h = 5,\ N = 5$;

Direct problem: $M_1 = 3,\ m_1 = 16,\ m_2 = 20$;

Number of incoming waves: Figure 1b, $J = 3,\ L = 8$; Figure 1c, $J = 6,\ L = 8$.

Inverse problem: $M_1 = 3,\ m_1 = 16,\ m_2 = 20,\ l_1 = 2,\ j_1 = 2.$

Approximate figure: see figure 1b and 1c;

$\mathcal{J}(F, M, J, L) = 2.3 \times 10^{-4}.$

$\|\rho_a - \rho^*\| = O(10^{-2}).$

Example 2:

Exact figure: $x_1^2 + x_2^2 = 5z^2(10 - z)^2/16^2 + 9$, (see figure 2a).

Parameters: $k = 2,\ h = 10,\ N = 5;$

Direct problem: $M_1 = 3,\ m_1 = 16,\ m_2 = 24;$

Number of incoming waves: $J = 8,\ L = 10;$

Inverse problem: $M_1 = 3,\ m_1 = 16,\ m_2 = 20,\ l_1 = 3,\ j_1 = 3.$

Approximate figure: see figure 2b.

$\mathcal{J}(F, M, J, L) = 1.3 \times 10^{-3}.$

$\|\rho_a - \rho^*\| = O(10^{-2}).$

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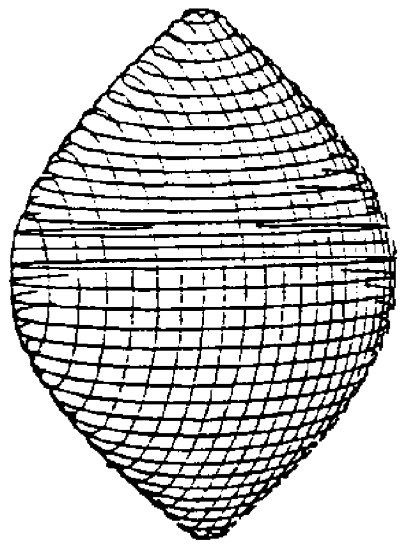

Figure 1a

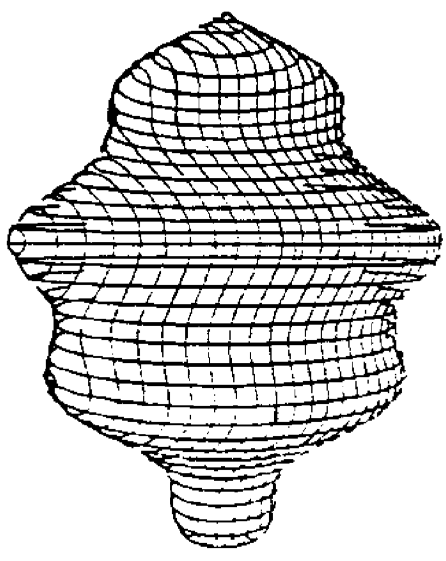

Figure 1b

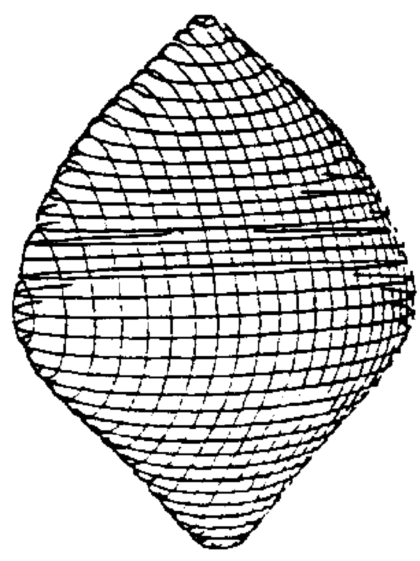

Figure 1c

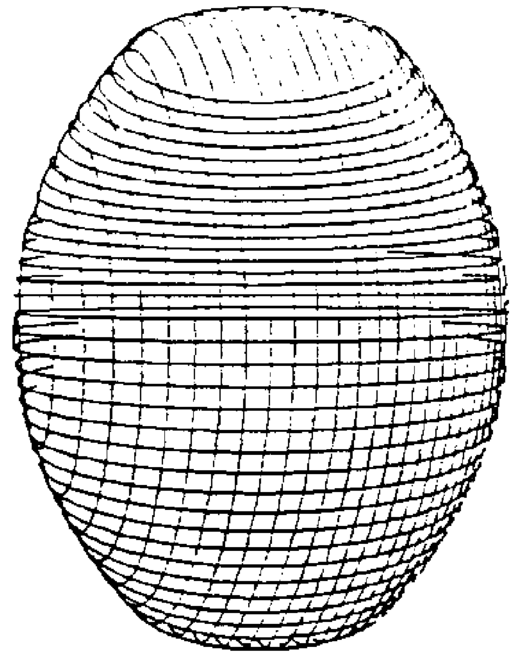

Figure 2a

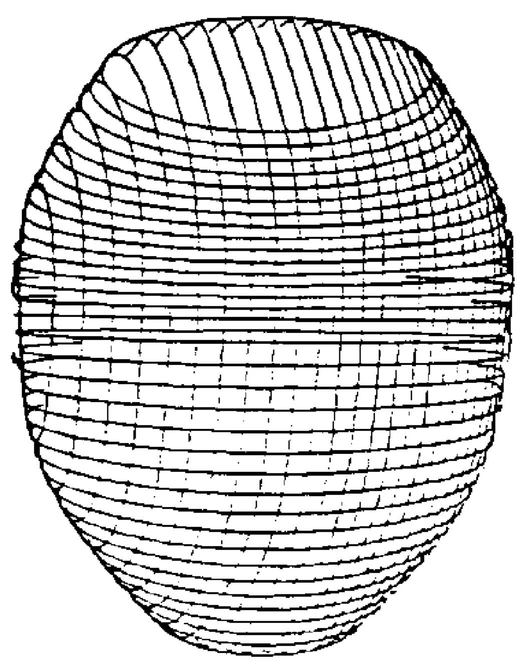

Figure 2b