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AN ITERATIVE FINITE DIFFERENCE APPROACH
TO OPTION AND PROJECT VALUATION

David Tomberlin
Valentina Bosetti

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TO OPTION AND PROJECT VALUATION

David Tomberlin[1] and Valentina Bosetti[2]

[1]NOAA, National Marine Fisheries Service
Southwest Fisheries Science Center
110 Shaffer Road
Santa Cruz, California 95060

[2]Fondazione Eni Enrico Mattei
Corso Magenta, 63
20123 Milano

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Introduction

Real options models provide a framework for project valuation and discrete decision-making when uncertainty and irreversibility are central to the decision problem. Specifically, real options models allow for the solution of optimal stopping problems to generate expected-value-maximizing decision rules that are functions of underlying stochastic state variables. This paper describes an algorithm developed for the numerical solution of real options models that include 1) interacting options and 2) stochastic processes other than the few that lead to analytical solutions or simpler numerical techniques.

In the context of fisheries, real options models have applicability both for policy design and as positive models of the investment and participation decisions of fishermen. In past work with these models, we have used model specifications that allowed for solution by minimization techniques such as nonlinear least squares (Tomberlin and Bosetti 2004). However, minimization techniques depend upon derivation of closed-form solutions to individual partial differential equations (PDEs), from which a system solution is then derived. Such analytical solutions are available in only a few cases, the canonical example being that of geometric Brownian motion. Many state variables that are likely to be useful in fisheries models, however, do not appear well represented by geometric Brownian motion. Further, models with more than one state variable also make closed-form solution of individual PDEs difficult or impossible. In such cases, an alternative to minimization methods must be found.

Finite difference methods have long been used for the solution of PDEs, and are a natural alternative to minimization methods for solving real options models when closed-form solutions to PDEs are not available. In the case of a single real option, represented by a single partial differential equation, well-known finite difference methods may be applied more or less directly. However, in the case of multiple, interacting stopping problems, these methods will not suffice, as they do not capture the full value of decision flexibility. That is, when exercise of one option confers ownership of further options, the options must be valued simultaneously. For example, suppose a fisherman is considering exit from a limited-entry fishery because other opportunities look more promising. If the fisherman's choice is simply between staying active in the fishery or exiting for good, the optimal stopping formulation is straightforward and standard real options solution techniques apply. However, in fisheries it is usually the case that the decision problem is more complicated than a simple choice between remaining active in a fishery or exiting: the fisherman may be able to choose among several fisheries or to suspend fishing without giving up the right to fish in the future.

In the case of limited-entry fisheries, this suspension option may require ongoing purchase of an annual license to maintain the right to fish in the future. This scenario leads to a decision problem in which the fisherman chooses among three actions: to remain active in the fishery, to suspend fishing while paying license fees, and to exit the fishery permanently. While the fisherman may move freely between active fishing and suspension, the decision to exit is irreversible. The benefit derived from each choice—

active fishing, suspension, and exit—depends on subsequent available options. Hence, these benefits and the value of choice flexibility represented options must be estimated simultaneously.

Below, we develop an iterative method for solving such problems and demonstrate its relevance to fisheries investment and decision analysis. To make the exposition easier to follow, we begin by presenting a simplified model in which the fisherman has to choose between remaining active in a fishery and exiting permanently. We then introduce the option to suspend operations and describe the iterative finite difference algorithm that allows us to estimate option and project value in this more complicated decision environment. The final section of the paper offers an assessment and conclusions. Throughout, we develop the argument in terms of a stochastic process for which analytical solutions are available for individual PDEs within our system, namely, geometric Brownian motion. While our motivation in developing the iterative finite difference approximation is actually to move beyond geometric Brownian motion to more general stochastic processes, developing the presentation in terms of this more tractable stochastic process allows direct comparison of numerical results from our iterative solver with those from a nonlinear least squares technique.

Valuing a Project with a Single Option

We begin by describing two solution techniques for a simple exit decision problem, i.e., a choice between remaining active in a fishery or irrevocably surrendering the right to participate in this fishery. These techniques are nonlinear least squares and finite difference approximation. We suppose throughout that the fisherman's objective is to maximize long-run expected payoff. Our goals are to identify the value of the fishing enterprise, which includes the value of the option to exit, and to identify the threshold value of state variables at which it is optimal to exit.

Here, we take the case of a single state variable, revenue, and assume that revenue can be represented as a geometric Brownian motion:

1) $dR = \alpha R dt + \sigma R dz$

where R = revenue,
$\qquad \alpha$ = the instantaneous drift rate
$\qquad \sigma$ = the instantaneous volatility rate
$\qquad dz$ = a standard Brownian motion

The fisherman's problem is to choose the action that will maximize the sum of current and expected future profits. In continuous time, the Bellman's equation is thus:

2) $F(R,C,L,t) = \max\{\Omega(R,t),(R-C-L)dt+(1+\rho dt)^{-1}E[F(R+dR,C,L,t+dt)\,|\,R]\}$

where F = the value function, defined recursively
Ω = salvage value
C = periodic operating costs
L = periodic license fees
ρ = fisherman's discount rate

Equation 2 presents the fisherman's decision as a choice between stopping, in which case he receives a one-time payment of $\Omega(R,t)$, and continuing to fish, in which case he receives the current R, C, and L as well as an expected future value. We wish to find R_x, the revenue level below which the optimal decision is to exit the fishery irreversibly. Simultaneously, we will derive expressions for the value of the fishing enterprise and for the value of the managerial options embedded in that enterprise, which here is simply the value of the exit option.

We begin by noting that on the continuation range of R (i.e., where fishing has a higher expected payoff than quitting), the Bellman's equation becomes

3) $F(R,t) = (R - C - L)dt + (1 + \rho dt)^{-1} E[F(R + dR, t + dt) \,|\, R]$

Applying Ito's lemma to the right-hand side, re-arranging terms and taking the limit as $dt \to 0$ yields a partial differential equation for the value of the fishing project (see, e.g., Dixit and Pindyck 1994, pp. 105-107):

4) $R - C - L + \alpha R \dfrac{\partial F}{\partial R} + \dfrac{1}{2}\sigma^2 R^2 \dfrac{\partial^2 F}{\partial R^2} - \rho F + \dfrac{\partial F}{\partial t} = 0$

A solution to this second-order ordinary differential equation is an equation $F = g(R, C, L, \alpha, \sigma)$, which can be solved by invoking the following boundary conditions:

5a) $F(R_x, t) = \Omega$
5b) $F'(R_x, t) = 0$

Equation (5a) is the value-matching condition, which says that at the point where the fisherman is indifferent between pursuing and abandoning the project, the expected value of staying in the fishery must equal the salvage value. Equation (5b) is the smooth-pasting condition, the requirement that the change in the values of the fishing project and the salvage value with respect to changes in R must be equal, for otherwise the optimal exit value of revenue would be at some other value of R.

The model (4-5) is analogous to an American put option, which in general does not have a closed-form solution. A solution to this model yields expressions for the expected value of the fishing project, for the value of the exit option embedded in the project, and for R_x. Here, we present two methods for solving the model: a nonlinear least squares approach, which has limited applicability but will provide a check on our numerical

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results as long as we work with geometric Brownian motions; and a finite difference approach, which interests us because it will be generalizable to stochastic processes that may better represent variables of interest.

Nonlinear least squares

If the time horizon over which the decision to be made is quite long (say more than ten years, which will commonly be the case among career fishermen), we can reasonably approximate the exit problem within an infinite time horizon setting, in which case the PDE in equation (4) has a solution of the following form:

$$6) \quad F(R) = \frac{R}{\rho - \alpha} - \frac{C + L}{\rho} + AR^\beta$$

where A is a constant yet to be identified and β is a function of the parameters α, σ, and ρ. The first two terms represent the discounted expected values of infinite revenue and cost streams, i.e, the value of staying in the fishery forever, while the third term represents the value of the exit option. Using this result, conditions (5a-5b) lead to following systems of two nonlinear equations in two unknowns (R_x, A_x):

$$7a) \quad \frac{R_x}{\rho - \alpha} - \frac{C + L}{\rho} + A_x R_x^\beta - \Omega = 0$$

$$7b) \quad \frac{1}{\rho - \alpha} + \beta A_x R_x^{\beta-1} = 0$$

A system this simple may be solved algebraically for the exact solution. In general, such systems of nonlinear equations may be solved by relying on extensions of Newton's method or conversion to minimization problems [4, 5]. We adopt the latter approach, finding $\{R_x, A_x\}$ that minimizes the sum of least squared errors:

$$8) \quad \{R_x, A_x\} = \min_{R,A} [\{\frac{R}{\rho - \alpha} - \frac{C + L}{\rho} + AR^\beta - S\}^2 + \{\frac{1}{\rho - \alpha} + \beta AR^{\beta-1}\}^2]$$

Solution of (8) is extremely fast and reliable for a wide range of parameter values. We present results from applying this model to our study fishery below, but first describe a finite difference method for solving the same model.

Finite difference approximation

The nonhomogeneous portion of (4) represents the project's flow value, while the homogeneous portion represents the exit option. The finite difference method allows us to estimate the value of this exit option by approximating the homogeneous part of (4) over a discrete grid of revenue and time points, and over a finite time horizon ending at a terminal time T_{MAX}. The PDE is approximated as a set of partial difference equations,

which, after the imposition of some boundary conditions, can be solved using standard linear algebra techniques.

First, we restate the homogeneous part of the PDE (4) for ease of reference:

$$4') \quad \alpha R \frac{\partial F}{\partial R} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 F}{\partial R^2} - \rho F + \frac{\partial F}{\partial t} = 0$$

Letting r be the grid index for revenue values R, t be the grid index for time, and ΔR and Δt the corresponding difference values, we begin by making the following substitutions for elements (4'):

$$\frac{\partial F}{\partial R} \approx \frac{F_{t,r+1} - F_{t,r-1}}{2\Delta R}$$

$$\frac{\partial^2 F}{\partial R^2} \approx \frac{F_{t,r+1} - 2F_{t,r} + F_{t,r-1}}{(\Delta R)^2}$$

$$\frac{\partial F}{\partial t} \approx \frac{F_{t+1,r} - F_{t,r}}{\Delta t}$$

which yields a difference equation system of the form

$$9) \quad \alpha r \Delta R \left(\frac{F_{t,r+1} - F_{t,r-1}}{2\Delta R} \right) + \tfrac{1}{2}\sigma^2 r^2 (\Delta R)^2 \left(\frac{F_{t,r+1} - 2F_{t,r} + F_{t,r-1}}{(\Delta R)^2} \right) - \rho F_{t,r} + \frac{F_{t+1,r} - F_{t,r}}{\Delta t} = 0$$

which approximates the homogeneous portion of the PDE we want to solve. Rearranging (9) leads to a system of linear equations of the form

$$10) \quad a_r F_{t,r-1} + b_r F_{t,r} + c_r F_{t,r-1} = F_{t+1,r}$$

where

$$a_r = \frac{\alpha r \Delta t}{2} - \frac{\sigma^2 r^2 \Delta t}{2}$$

$$b_r = 1 + \sigma^2 r^2 \Delta t + \rho \Delta t$$

$$c_r = -\frac{\alpha r \Delta t}{2} - \frac{\sigma^2 r^2 \Delta t}{2}$$

Note that the relationship in (10) between each time t and the subsequent time $t+1$ is defined by as many equations as there are grid points r. The implicit finite difference

approach to this problem proceeds by solving this system recursively, moving backward through time. Each value of $F_{t+1,r}$ is defined in relation to values of F_t, as in (10). However, if we have a set of F_{t+1} values for all r, the values of F_t for all R can be deduced as the solution to a set of linear equations. Thus, F_t is defined implicitly in relation to F_{t+1}. Figure 1 presents a schematic view of this relation.

Figure 1: A finite difference grid showing F_t in relation to F_{t+1}.
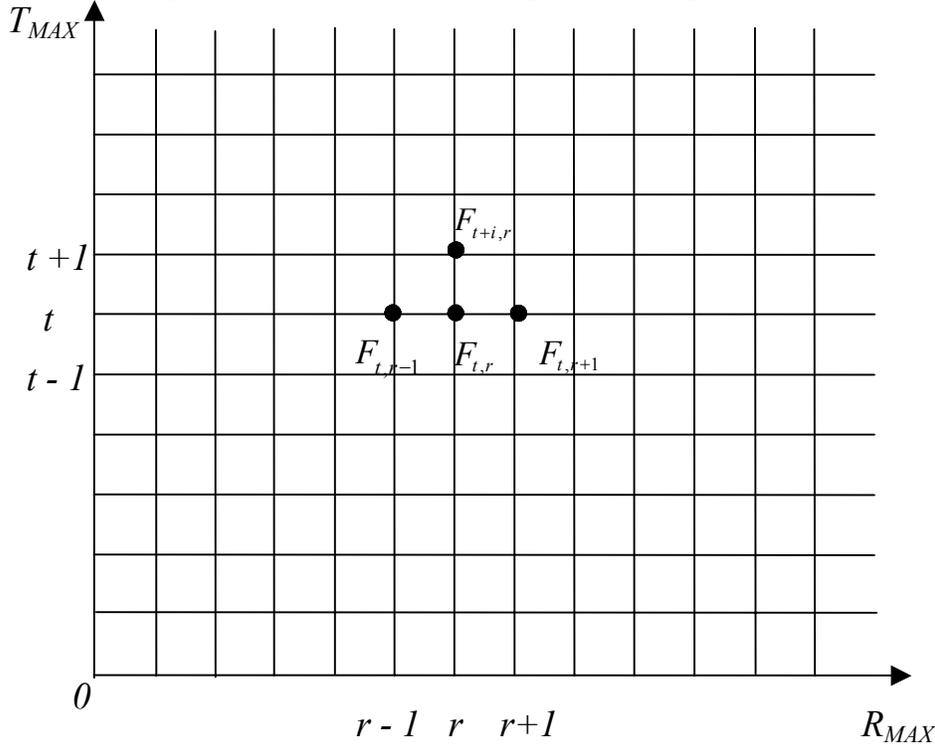
In order to use this recursive approach to solving (10), we require boundary conditions at terminal time T_{MAX} and at the lowest and highest values of revenue considered plausible. At T_{MAX}, we can impose as boundary condition a vector that expresses the value of the static optimization problem faced by a fisherman who has to choose between quitting or accepting the expected value of staying the the fishery forever (since T_{MAX} is by definition the point beyond which quitting is no longer an option). This vector, known as the payoff in the option valuation literature, is simply

$$PAYOFF_r = \max\left(\Omega - \left\{\frac{R_r}{\rho - \alpha} - \frac{C+L}{\rho}\right\}, 0\right)$$

That is, the payoff at terminal time is the greater of a) the value of exercising the exit option (salvage value Ω less the expected net present value of the project given up) or b) zero, the value of allowing the option to expire unexercised.

Another boundary condition applies at the lowest value of revenue on our grid, where we assume that the exit option will surely be exercised. Here, the natural choice for this

lowest value is $R=0$, since R is necessarily non-negative (note the inclusion of cost terms allows for the possibility of operating losses over some range of R). The boundary condition at $R=0$ is simply the payoff to exercising the option when there is no revenue, i.e., the salvage value plus operating cost savings from terminating the fishing enterprise:

$$F_{R_{MIN}} = \Omega + \frac{C+L}{\rho}$$

The final boundary condition applies at the highest value of revenue on our grid, which should be chosen so that it is well beyond the highest values of R that are feasible for R_x. That is, R_{MAX} should be at a level at which we can reasonably suppose that fishermen will *not* exercise the exit option. At this point, which will have to be chosen within the context of a particular problem, the boundary condition becomes

$$F_{R_{MAX}} = 0$$

That is, at very high revenues the value of the option to exit will be zero.

By recursively solving the system of linear equations defined in (10) subject to the boundary conditions at T_{MAX}, R_{MIN}, and R_{MAX}, we can arrive at the value of the exit option and the revenue level R_x below which the option will be optimally exercised for any given time horizon. Choosing a suitably high T_{MAX} yields an approximation to an infinite-horizon decision problem (assuming a positive discount rate), though the computational burden of keeping the grid fine enough for good approximation to the underlying PDE can be substantial. We have found that working with time horizons of 50 to 100 years requires at least 10,000 time steps. This means that the linear system (10) must be solved 10,000 times, which can require many hours on a workstation (depending on the number of grid points in the r-dimension).

Valuing a Project with Multiple, Interacting Options

In most realistic settings, fishermen are likely to have several, often interacting, managerial options available to them. For example, purchase of a commercial fisherman's license may enable entry into several different fisheries among which the fisherman can move at will, subject to the fixed cost of gear purchase and configuration. Similarly, the value of a buyout offer will depend on whether the buyout is absolute or is conditional on the return of the fish stock to some level (in which case the fisherman could exercise a right of re-entry).

Here, we focus on the case of a limited-entry fishery in which a fishing permit, once allowed to lapse, can not be purchased again *ex novo*. The fisherman may choose among three states: active, idle, and retired. Active and idle states require the annual purchase of a fishing permit; each year's permit gives the fisherman the right (but not the obligation) to fish in that year, plus the right to purchase another permit in the following year. A proper valuation of each of these states must account for the associated options, which

are interdependent in the following ways. If an active fisherman exercises the option to idle his boat, he acquires an option to re-enter the fishery the following year and an option to exit permanently. If a fisherman with an idled boat exercises the option to re-activate, he acquires an option to idle the boat again in the future. If a fisherman with an idled boat decides to exit the fishery, he has no further option to re-enter. For the sake of simplicity, we do not consider the possibility of direct exit from active fishing, i.e., a boat must be idled before permanent exit. Including a direct exit option would require estimation of another decision threshold but would not alter the model in any essential way.

The fisherman's decision problem can be represented by separate Bellman's equations for the active and idle boats. The Bellman's equation for the active boat represents the choice of whether to remain active (the continuation decision) or to idle the boat (the stopping decision):

11) $V_A(R,t) = \max\{(R - C - L)dt + (1 + \rho dt)^{-1} E[V_A(R + dR, t + dt) \mid R]$,

$$\Omega - T_I - Ldt + (1 + \rho dt)^{-1} E[V_I(R + dR, t + dt) \mid R] \ \}$$

where the first line is the expected value of maintaining the active project and the second line is the value of choosing to idle (which consists of the salvage value[1] less transition costs T_I and license costs, plus the expected value of the idle boat, which include the value of the option to re-activate the boat in the future). We seek R_I, the value of R at at which the fisherman is indifferent between remaining active and idling (and below which idling is optimal).

In the Bellman's equation for the idle boat, the decision problem is to choose either to remain idle (the continuation decision), to re-activate the boat (one stopping decision), or to exit the fishery for good (another stopping decision). The Bellman's equation in this case becomes:

12) $V_I(R,t) = \max\{(R - C - L)dt + (1 + \rho dt)^{-1} E[V_A(R + dR, t + dt) \mid R] - \Omega - T_A$,

$$- Ldt + (1 + \rho dt)^{-1} E[V_I(R + dR, t + dt) \mid R],$$

$$0 \ \}$$

where the first term is the expected value of re-entering the fishery (the expected value of the active boat less salvage and transition costs T_A), the second term is the value of remaining idle, and the third term is the value of exiting the fishery for good, which we assume is zero. Notice that, in our formulation, the idle boat has already acquired the 'salvage value' represented by the opportunity cost of time spent in the fishery, so that the choice between idling and exiting is not affected by the salvage value. This is simply

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[1] By salvage value here we mean simply whatever value becomes available to the fisherman upon idling the boat, which could be revenues from other fisheries or earnings from non-fishing activities. We do not mean scrap or sale value of the boat, as will become clear below. Salvage value here relates specifically to the opportunity cost of fishing in the study fishery (the one generating revenues R).

an artifact of our model structure, which could easily be changed to include, for example, a scrap value upon exit.

We wish to find R_I, the value of R at which it becomes optimal for an active boat to idle; R_A, the value of R at which it becomes optimal for an idle boat to re-activate; and R_X, the value of R at which it becomes optimal for an idle boat to exit. To identify these decision thresholds, we proceed as before, by defining and solving partial differential equations that hold on the continuation range of each discrete decision.

Above R_I, where the active boat chooses to remain active rather than idle, the PDE representing the continuation value is:

13) $\quad R - C - L + \alpha R \dfrac{\partial V_A}{\partial R} + \dfrac{1}{2} \sigma^2 R^2 \dfrac{\partial^2 V_A}{\partial R^2} - \rho V_A = 0 \quad \forall R \in (R_I, \infty)$

Between R_A and R_X, where the idle boat chooses to remain idle rather than re-enter or exit the fishery, the value of the suspended fishing project must satisfy a PDE given by:

14) $\quad -L + \alpha R \dfrac{\partial V_I}{\partial R} + \dfrac{1}{2} \sigma^2 R^2 \dfrac{\partial^2 V_I}{\partial R^2} - \rho V_I = 0 \quad \forall R \in (R_X, R_A)$

The critical stopping values R_I, R_A, and R_X cannot be determined individually, since they are part of a single decision problem (e.g., the revenue threshold at which idling an active boat becomes optimal will depend on the fixed cost of re-activating the boat once idle). Below, we present two methods for finding these stopping values. As above, we will suppose that revenue follows a geometric Brownian motion, so that we can develop a set of solutions based on a system of nonlinear equations derived from analytical solutions of individual PDEs. This nonlinear systems solution will serve as a benchmark. However, our real interest lies in the development of a solution method that does not rely on such restrictive assumptions regarding the stochastic processes involved. For this purpose, we will employ an iterative finite difference technique, detailed further below.

Nonlinear least squares

Since, in this case, the individual continuation ranges can be represented as PDEs with closed-form solutions, we can proceed as before to arrive at a system of nonlinear equations representing the value-matching and smooth-pasting conditions. Specifically, solutions to the PDEs above will be of the form:

15) $\quad V_A = \dfrac{R}{\rho - \alpha} - \dfrac{C}{\rho} - \dfrac{L}{\rho} + K_I R^{\beta_1}$

16) $\quad V_I = K_A R^{\beta_2} + K_X R^{\beta_1} - \dfrac{L}{\rho}$

Given the value of the 'exited project,' which is zero by construction ($V_x=0$), we have the following value-matching and smooth-pasting conditions:

17) $\quad V_I(R_I) + \Omega - T_I = V_A(R_I)$

17') $\quad V_I'(R_I) = V_A'(R_I)$

18) $\quad V_A(R_A) - \Omega - T_A = V_I(R_A)$

18') $\quad V_A'(R_A) = V_I'(R_A)$

19) $\quad V_X(R_X) = V_I(R_X)$

19') $\quad V_X'(R_X) = V_I'(R_X)$

This system of six equations in six unknowns (R_I, R_A, R_X, K_I, K_A, K_X) is more difficult to solve than the two-equation system of the simple exit problem, but is not conceptually different.

Iterative finite difference approximation

Because, in general, closed-form solutions of the form of (15-16) are not available, we proceed here with a finite difference method that allows us to approximate the values of various project states, the values of real options associated with those states, and the revenue levels at which those options are optimally exercised. The key to this approach is the recognition, which is explicit in the nonlinear least squares formulation above, that the value of entering a new state may include the value of options to leave that state subsequently (e.g., the value of idling a boat includes the value of the option to re-activate the boat or exit fishery, in addition to operating costs savings). This notion is captured in our finite difference approach by including, in the payoff vector for each option, terms for the value of any options acquired upon exercise of the option being valued. In our case, the payoff vector for the entry option includes a term of unknown form for the idling option acquired upon entry, V^I:

$$PAYOFF_A = \max\left(\frac{R}{\rho - \alpha} - \frac{C+L}{\rho} - \Omega + V_I, 0\right)$$

Similarly, the payoff vector for the idling option includes terms for both the option to re-activate the boat and for the option to exit permanently:

$$PAYOFF_I = \max\left(\Omega - \left\{\frac{R}{\rho - \alpha} - \frac{C+L}{\rho}\right\} + V_A + V_X, 0\right)$$

Upon exit, no new options are acquired, and the option to re-activate the boat is given up, so the payoff vector for exercising the exit option (i.e, moving from idle to exited) is:

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$$PAYOFF_X = \max\left(-\frac{L}{\rho} - V_A, 0\right)$$

Because these payoff vectors are each defined in terms of unknown quantities (the unspecified values of options acquired), the finite difference algorithm of the previous section is not directly applicable. Instead, we embed that algorithm in an iterative scheme in which the solution to one PDE provides an estimate of unknown option value(s) that can be used in the solution of another PDE. Each of the PDEs is solved in sequence, and the sequence is repeated until the threshold value for the exercise of each is no longer changing. Figure 2 on the following page shows the flow of the algorithm. Beginning (arbitrarily) with a vector of zeros for the value of the idling option and an optimal exercise threshold of zero for this option, the algorithm proceeds by iteratively estimating the three option values in sequence. With each pass through the estimation process for the three options, estimates of the option values are updated and used in the subsequent iteration $k+1$. The process continues until until the estimates of all three thresholds have converged to stable values.

It's worth pointing out that our convergence criterion is specified in terms of the estimated threshold values rather than the payoff or option value vectors. We adopt this approach because it is simpler to understand and assess than would be a point-by-point comparison of the payoff or option value vectors. Graphical analysis of the output makes it clear that little information is lost by taking this simplifying step.
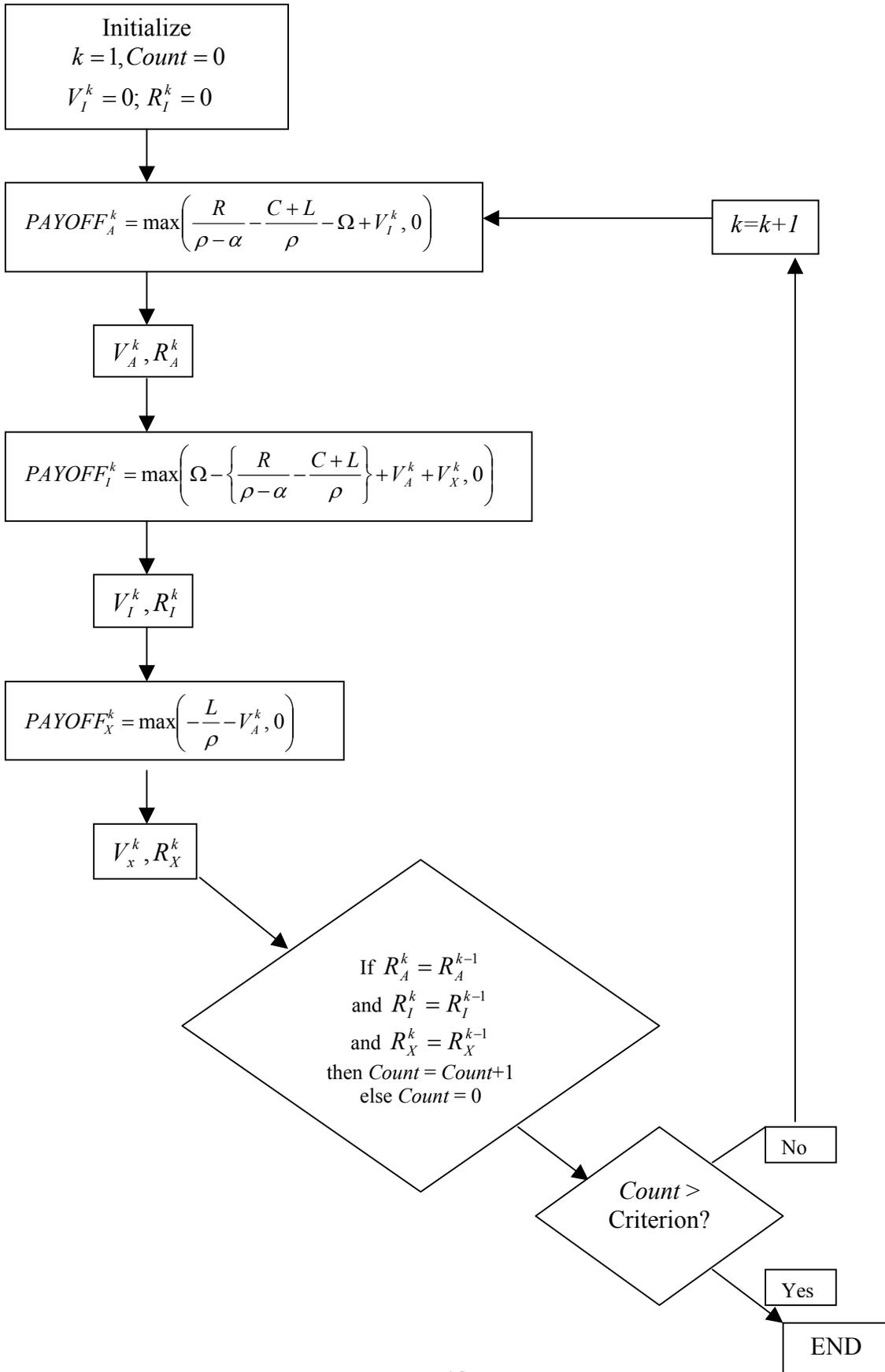
Table 1 presents results of applying the algorithm to (13-14), using parameters based on data from the California commercial salmon fishery. Because the finite difference routines operate on a discrete grid, the threshold value reported here is the average of the last revenue level at which the option will still be held and the first at which it will no longer be held. The grid size here, dR=0.07, is chosen to be small enough that the exact location between these two grid points of the true value is not a great concern.

Table 1: Iterative finite difference and nonlinear least squares results

	FD	NLLS
R_A ($000s)	5.51	5.63
R_I ($000s)	4.31	4.20
R_X ($000s)	1.09	1.00
Solution Time (minutes)	41.6	0.38

The table shows that the finite difference and nonlinear least squares results are quite similar. It's important to remember that the nonlinear least squares results are themselves an approximation derived from a numerical routine, so the true error of the finite difference routine could be a bit more or less than the table suggests. Depending on the purposes of the analysis, it may be advisable to increase the density of state variable and time grids, but for our own work, we are quite satisfied with this degree of accuracy. The most striking thing about the table is the hundred-fold difference in solution times. However, the solution time for the nonlinear least squares routine reported above does not include the search for a good set of initial values, which in our experience can often eat up much of the time savings from using this method.

Figure 2: The iterative finite difference algorithm

Initialize
$k = 1, Count = 0$
$V_I^k = 0;\ R_I^k = 0$

$PAYOFF_A^k = \max\left(\dfrac{R}{\rho - \alpha} - \dfrac{C + L}{\rho} - \Omega + V_I^k,\ 0\right)$

$k=k+1$

V_A^k, R_A^k

$PAYOFF_I^k = \max\left(\Omega - \left\{\dfrac{R}{\rho - \alpha} - \dfrac{C + L}{\rho}\right\} + V_A^k + V_X^k,\ 0\right)$

V_I^k, R_I^k

$PAYOFF_X^k = \max\left(-\dfrac{L}{\rho} - V_A^k,\ 0\right)$

V_x^k, R_X^k

If $R_A^k = R_A^{k-1}$
and $R_I^k = R_I^{k-1}$
and $R_X^k = R_X^{k-1}$
then $Count = Count+1$
else $Count = 0$

No

$Count >$
Criterion?

Yes

END

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Assessment and conclusions

The iterative finite difference method described here meets the need for a solution technique for real options models when two conditions hold jointly: 1) the decision problem includes multiple interacting options and 2) the stochastic processes representing state variables do not permit closed-form representations of the PDEs that represent the value of each management option. When both these conditions are present, neither traditional finite difference methods nor methods based on solving systems of nonlinear equations can be used.

In this paper, we have developed the iterative finite difference technique in the context of an example that does in fact permit solution by nonlinear systems (here, with nonlinear least squares), in order to enable direct comparison of our method with the nonlinear least squares approach. Properly parameterized, the iterative finite difference method can yield solutions that are very comparable to the nonlinear systems methods (which are themselves approximations of the true solutions, being based on a Levenberg-Marquardt algorithm). However, the real point of having the iterative finite difference solver available is to allow solution of problems characterized by more general stochastic processes that do not allow application of the nonlinear systems method.

Our own interest in solving this class of models relates to limited-entry fisheries in which state variables such as average revenue appear to be mean-reverting, but the approach is applicable to a wide variety of real options analyses in which simpler solution techniques are not available. While the technique can be computationally demanding, the added flexibility provided by the iterative finite difference solver can enable analysts to develop and solve more realistic models of project valuation, option valuation, and dynamically optimal discrete choice under uncertainty.

Literature Cited

Bosetti, V., and D. Tomberlin. 2004. Real Options Analysis of Fishing Fleet Dynamics: A Test. FEEM Natural Resource Management Working Paper 102. Milan: Fondazione Eni Enrico Mattei.

Dixit, Avinash, and Robert Pindyck. 1994. Investment Under Uncertainty. Princeton: Princeton University Press.

Tomberlin, D. and V. Bosetti. 2004. Solving Real Options Models of Fisheries Investment When Salvage Value Is Difficult to Estimate. In *Proceedings of the 2004 IIFET Convention*. Corvallis, OR: International Institute for Fisheries Economics and Trade.

Appendix: Matlab Code for A Limited-Entry Permit Problem

```
% SOLVES ENTRY-EXIT-MOTHBALL PROBLEM UNDER GBM WITH FINITE DIFFERENCES

%%%%%%%%% PARAMETERS %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
RHO     = 0.05;         % CONTINUOUS DISCOUNT RATE
L       = 0.410;        % LICENSE FEE (IN $000s)
C       = 4.342;        % VARIABLE OPERATING COSTS (IN $000s)
SALVAGE = 10;           % EXIT SALVAGE VALUE (IN $000s)
ALPHA   = -0.0093;      % DRIFT RATE
SIGMA   = 0.32;         % VOLATILITY
DELTA   = RHO - ALPHA;  % CONVENIENCE PARAMETER
t       = 5000;         % NUMBER OF TIME POINTS IN MESH
r       = 600;          % NUMBER OF REVENUE LEVELS IN MESH
R_MAX   = 45;           % MAXIMUM REVENUE LEVEL IN MESH ($000S)
R_MIN   = 0;            % MINIMUM REVENUE LEVEL IN MESH ($000S)
T       = 50;           % TERMINAL TIME (IN YEARS)
T_COST  = .1       ;    % COST OF SWITCHING BETWEEN MOTHBALL AND ACTIVE
dt      = (T/t);
dR      = (R_MAX - R_MIN)/r;
R       = dR*(0:r) + R_MIN;
% partial difference coefficients
an = + 0.5*(RHO-DELTA)*(1:r-1)*dt - 0.5*(SIGMA*SIGMA*(1:r-1).*(1:r-1))*dt;
bn = 1 + SIGMA*SIGMA*(1:r-1).*(1:r-1)*dt + RHO*dt;
cn =  - 0.5*(RHO-DELTA)*(1:r-1)*dt - 0.5*(SIGMA*SIGMA*(1:r-1).*(1:r-1))*dt;
% coefficient matrix
A = diag(bn) + diag(an(2:r-1), -1) + diag(cn(1:r-2), 1);

Ventry = zeros(1, r+1);
Vexit = zeros(1, r+1);
Vmoth = zeros(1, r+1);


for volta=1:7

    Moth
        PAYOFFmoth = max(SALVAGE + Ventry + Vexit - L/RHO -(R/(RHO-
                        ALPHA)-C/RHO-L/RHO )- T_COST, zeros(1,r+1));
        V = zeros(t+1,r+1);
        V(t+1,:) = PAYOFFmoth;         % at terminal time you get payoff
        V(:,1)   = PAYOFFmoth(1,1);  % at min revenue, quitting gets you
                                        cost savings + salvage
        V(:,r+1) = PAYOFFmoth(1,end); % at max revenue, exit option is
                                        worthless
        for i = t:-1:1
            y = V(i+1, 2:r)';
            y(1) = y(1)-an(1)*V(i+1,1);         %boundary adjustment
            y(end) = y(end)-cn(end)*V(i+1,end); %boundary adjustment
            V(i, 2:r) = [A \ y]';
            V(i, :) = max(PAYOFFmoth, V(i,:));
        end
        Vmoth = V(1,:) ;
        PAYOFFmothrecord(volta,:)=PAYOFFmoth;
        Vmothrecord(volta,:)=Vmoth;
        Rmothrecord(volta,:)=Vmoth-PAYOFFmoth;
```

Entry

```
    PAYOFFentry = max(R/(RHO-ALPHA)-C/RHO-L/RHO + Vmoth -(SALVAGE-
                      L/RHO+Vexit)-T_COST , zeros(1,r+1));
    V = zeros(t+1,r+1);
    V(t+1,:) = PAYOFFentry ;        % at terminal time, you get payoff
    V(:,1)   = PAYOFFentry(1,1);    % at min revenue, you get
                                       min PAYOFFentry
    V(:,r+1) = PAYOFFentry(1,end); % at max revenue, option to enter
                                       is surely exercised
    for i = t:-1:1
        y = V(i+1, 2:r)';
        y(1) = y(1)-an(1)*V(i+1,1);          %boundary adjustment
        y(end) = y(end)-cn(end)*V(i+1,end); %boundary adjustment
        V(i, 2:r) = [A \ y]';
        V(i, :) = max(PAYOFFentry, V(i,:));
    end

    Ventry = V(1,:) ;

    PAYOFFentryrecord(volta,:)=PAYOFFentry;
    Ventryrecord(volta,:)=Ventry;
    Rentryrecord(volta,:)=Ventry-PAYOFFentry;
```

Exit

```
    PAYOFFexit = max(L/RHO - Ventry, zeros(1,r+1));
    V = zeros(t+1,r+1);
    V(t+1,:) = PAYOFFexit ;         % at terminal time, you get payoff
    V(:,1)   = PAYOFFexit(1,1);     % at min revenue, quitting gets you
                                       min PAYOFFexit
    V(:,r+1) = PAYOFFexit(1,end); % at max revenue, exit option
                                       gets you max PAYOFFexit
    for i = t:-1:1
    y = V(i+1, 2:r)';
    y(1) = y(1)-an(1)*V(i+1,1);          %boundary adjustment
    y(end) = y(end)-cn(end)*V(i+1,end); %boundary adjustment
    V(i, 2:r) = [A \ y]';
    V(i, :) = max(PAYOFFexit, V(i,:));
    end
    Vexit = V(1,:) ;
    PAYOFFexitrecord(volta,:)=PAYOFFexit;
    Vexitrecord(volta,:)=Vexit;
    Rexitrecord(volta,:)=Vexit-PAYOFFexit;
```

end

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