Supplemental Material

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Journal of Climate; DOI:10.1175/JCLI-D-18-0611.1

Physical Understanding of Human-Induced Changes in U.S. Hot Droughts Using Equilibrium Climate Simulations

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Conditional distribution functions and the inverse forms of five candidate copulas are given as following:

1. Gaussian (Normal) Copula

The bivariate Gaussian (Normal) copula has distribution function:

$$C(u,v;\ \theta) = \Phi_{\theta}(\Phi^{-1}(u), \Phi^{-1}(v))$$

where u and v are the marginal distribution functions of the random variables X and Y in the range [0, 1], Φ is the standard normal distribution $N(0,1)$ with mean zero and unit variance, Φ^{-1} is its inverse, and Φ_{θ} is the bivariate standard normal distribution with correlation θ.

The corresponding density function is:

$$c(u,v;\ \theta) = \frac{1}{\sqrt{1-\theta^2}}\exp[-\frac{\theta^2(u^2+v^2)-2\theta uv}{2(1-\theta^2)}].$$

The h and h^{-1} functions for Gaussian (Normal) copulas are:

$$h(u,v;\ \theta) = \Phi[\frac{\Phi^{-1}(u)-\theta\Phi^{-1}(v)}{\sqrt{1-\theta^2}}], \text{ and}$$

$$h^{-1}(u,v;\ \theta) = \Phi[\Phi^{-1}(u)\sqrt{1-\theta^2} + \theta\Phi^{-1}(v)]$$

The parameter space for the dependence parameter of Normal copulas is $\theta \in (-1,1)$.

2. Student-t Copula

The bivariate Student-t copula has distribution function:

$$C(u,v;\ \rho,\delta) = t_{\rho,\delta}[t_{\delta}^{-1}(u), t_{\delta}^{-1}(v)]$$

where $t_{\rho,\delta}$ is the bivariate Student-t distribution function with correlation parameter ρ and δ degrees of freedom, and t^{-1} denotes the inverse univariate Student-t distribution function with δ degrees of freedom.

The corresponding density function is:

$$c(u,v;\ \rho,\delta) = \frac{\Gamma(\frac{\delta+2}{2})\Gamma(\frac{\delta}{2})}{\sqrt{1-\rho^2}\ [\ \Gamma\left(\frac{\delta+1}{2}\right)]^2} \times \frac{\{[1+\frac{(t_{\delta}^{-1}(u))^2}{\delta}][1+\frac{(t_{\delta}^{-1}(v))^2}{\delta}]\}^{\frac{\delta+1}{2}}}{\{1+\frac{\left[t_{\delta}^{-1}(u)\right]^2+\left[t_{\delta}^{-1}(v)\right]^2-2\rho t_{\delta}^{-1}(v)}{\delta(1-\rho^2)}\}^{\frac{\delta+2}{2}}}$$

The h and h^{-1} functions for Student-t copulas are:

$h(u,v;\ \rho,\delta) = \ t_{\delta+1}\{\frac{t_{\delta}^{-1}(u)-\rho t_{\delta}^{-1}(v)}{\sqrt{\frac{\left[\delta+\left(t_{\delta}^{-1}(v)\right)^{2}\right](1-\rho^{2})}{\delta+1}}}\}$, and

$$h^{-1}(u,v;\ \rho,\delta) = t_{\delta}\{t_{\delta+1}^{-1}(u)\sqrt{\frac{(\delta+\left(t_{\delta}^{-1}(v)\right)^{2}(1-\rho^{2})}{\delta+1}}+\rho t_{\delta}^{-1}(v)\}$$

The parameter space for the correlation parameter is $\rho \in (-1,1)$, and for degrees of freedom parameter is $\delta > 2$.

3. Frank Copula

The bivariate Frank copula has distribution function:

$$C(u,v;\ \theta) = \ -\theta^{-1}\log(\left[1-e^{-\theta}-\left(1-e^{\theta u}\right)\left(1-e^{\theta v}\right)\right]/(\,1-e^{-\theta}))$$

The corresponding density function is:

$c(u,v;\ \theta) = \ \frac{\theta(1-e^{-\theta})e^{-\theta(u+v)}}{[(1-e^{-\theta})-(1-e^{\theta u})(1-e^{\theta v})]^{2}}$.

The h and h^{-1} functions for Frank copulas are:

$h(u,v;\ \theta) = \ \frac{e^{-\theta v}}{\frac{1-e^{-\theta}}{1-e^{-\theta u}}+e^{-\theta v}-1}$, and

$$h^{-1}(u,v;\ \theta) = \ -\log\left\{1-\frac{1-e^{-\theta}}{(u^{-1}-1)e^{-\theta v}+1}\right\}/\theta$$

The parameter space for θ is $0 \le \theta < \infty$.

4. Clayton Copula

The bivariate Clayton copula has distribution function:

$$C(u,v;\ \theta) = (u^{-\theta}+v^{-\theta}-1)^{-1/\theta}$$

The corresponding density function is:

$c(u,v;\ \theta) = (1+\theta)(uv)^{-\theta-1}(u^{-\theta}+v^{-\theta}-1)^{-2-1/\theta}$.

The h and h^{-1} functions for Clayton copulas are:

$h(u,v;\ \theta) = \ v^{-\theta-1}(u^{-\theta} + v^{-\theta} - 1)^{-1-1/\theta}$, and

$h^{-1}(u,v;\ \theta) = [\left(uv^{\theta+1}\right)^{-\theta(1+\theta)} + 1 - v^{-\theta}]^{-1/\theta}$

The parameter space for θ is $0 \leq \theta < \infty$.

5. Gumbel Copula

The bivariate Gumbel copula has distribution function:

$$C(u,v;\ \theta) = \ \exp\{-[(-\log u)^{\theta} + (-\log v)^{\theta}]^{\frac{1}{\theta}}\}$$

The corresponding density function is:

$$c(u,v;\ \theta) = \ C(u,v;\ \theta)(uv)^{-1} \times$$

$$\frac{[(\log u)\ (\log v)]^{\theta-1}}{[(-\log u)^{\theta} + (-\log v)^{\theta}]^{2-\frac{1}{\theta}}}\{[(-\log u)^{\theta} + (-\log v)^{\theta}]^{\frac{1}{\theta}} + \theta - 1\}$$

where the dependence is controlled by $\theta \geq 1$. Perfect dependence is obtained when $\theta \to \infty$, and $\theta = 1$ implies independence.

The h function for Gumbel copulas is:

$$h(u,v;\ \theta) = \ v^{-1}\exp\left\{-[(-\log u)^{\theta} + (-\log v)^{\theta}]^{\frac{1}{\theta}}\right\}[1 + \left(\frac{\log u}{\log v}\right)^{\theta}]^{-1+1/\theta}$$

There is no closed form of h^{-1} function for Gumbel copulas. Therefore, a numerical routine, i.e. Newton-Raphson method, is used to invert it.