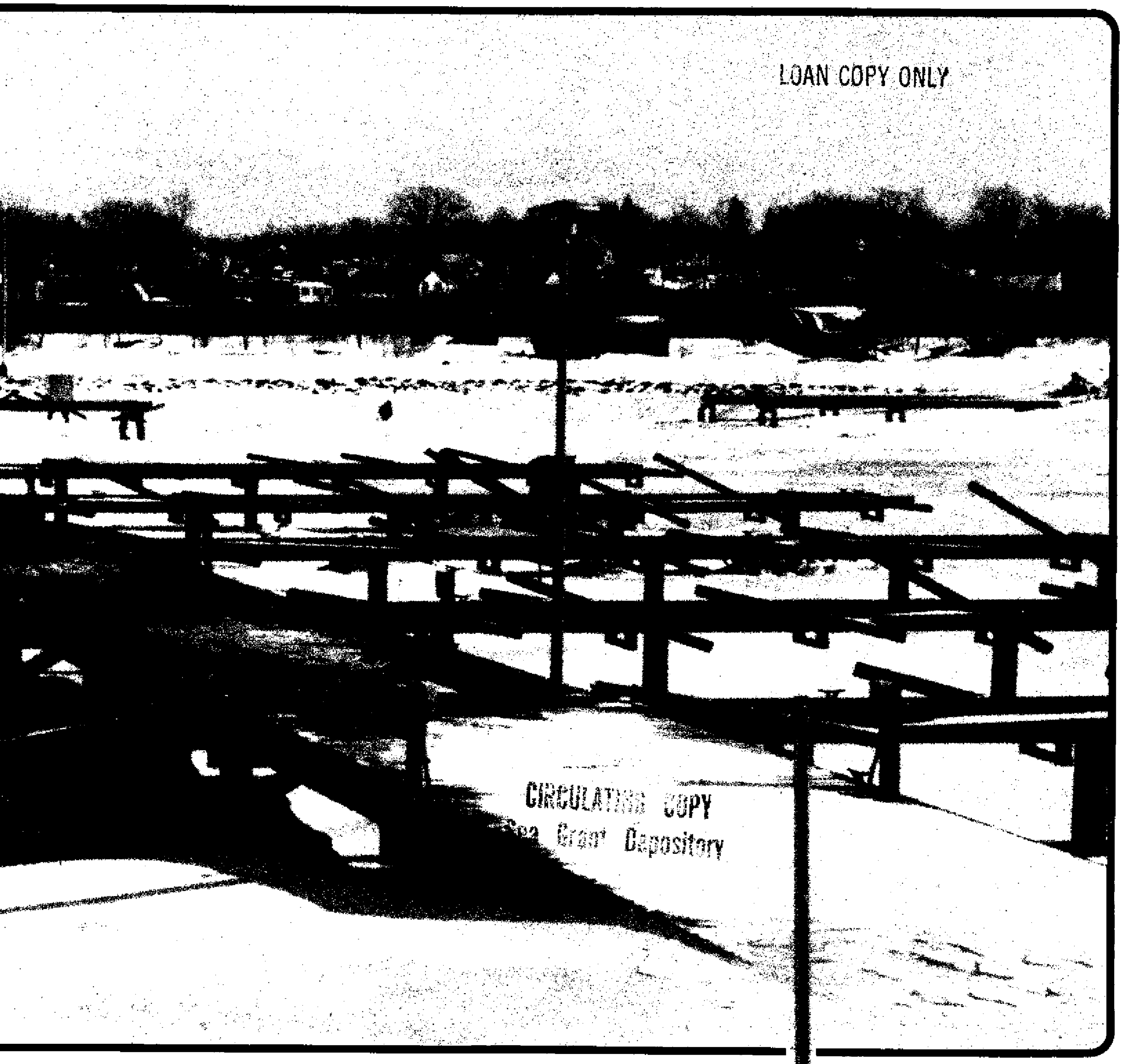

Ice Engineering Guide for Design and Construction of Small Craft Harbors

C. Allen Wortley

ICE ENGINEERING GUIDE

FOR

DESIGN AND CONSTRUCTION

OF

SMALL CRAFT HARBORS

C. Allen Wortley

Department of Engineering and Applied Science
University of Wisconsin-Extension
Madison, Wisconsin

University of Wisconsin Sea Grant College Program

This work was funded (in part) by the University of Wisconsin Sea Grant College Program under a grant from the Office of Sea Grant, National Oceanic and Atmospheric Administration, U.S. Department of Commerce and by the State of Wisconsin.

Advisory Report #WIS-SG-78-417
May 1978

University of Wisconsin
Sea Grant Communications Office
1800 University Avenue
Madison, WI 53706

608/263-3259

ABSTRACT

This ice engineering guide is an aid to designers and builders of small craft harbors in northern climates. In these areas ice does great damage to structures. The guide has been prepared from a review of the technical literature, consultation with knowledgeable engineers, contractors, and scientists, and field observations. It is applicable to lake ice in protected harbors. Information on the behavior and properties of ice and ice covers is presented. Descriptions of ice pressures of thermal origin and estimated thrusts on structures are given. The operation and design of compressed air suppression systems are explained. Safe bearing loads on ice sheets are estimated, as well as ice uplift loads for design. The information presented answers many questions about ice behavior and serves as a basis for innovative and economical small craft harbor design.

CONTENTS

PREFACE

More than a hundred years ago Dumble (1858) stated,

> "I may add, that the ignorance, or want of proper appreciation, of the properties of ice, evinced in the construction of numerous wharves, piers, and bridges on the inland lakes and rivers of Canada and the northern States, has proved a source of infinite annoyance and of immense expense".

Dumble was an engineer with the Cobourg and Peterborough Railway. Today, most designers of small craft harbors in northern climates have an appreciation of the behavior of ice. However, the level of relative ignorance about ice behavior is still high.

The successful design and construction of small craft harbor structures (such as docks, piers, and piles) subjected to ice forces and motions continues to be an uncertain science. The four methods currently being used are: (1) Predict the ice forces and motions, both horizontal and vertical, and design the marina structures to withstand them, (2) Predict the ice prevention or suppression capabilities of various systems to remove the adverse effects of ice or render them ineffective, (3) Estimate the relative scope of structure required by local comparative studies of successful and unsuccessful constructions and build accordingly, and (4) Remove the dockage by either beaching floating docks or retracting fixed piers.

Designers and builders should continue innovations and improvements in the state of the art of small craft harbor construction in northern climates. This publication endeavors to aid them and serve as a guide. It is a first step. It is incomplete. And likely will never be totally complete due to the uncertainty existing in the field of ice engineering. It is hoped the U.S. Cold Regions Research and Engineering Laboratory (US CRREL) will subsequently expand and update ice information necessary to construct small craft harbors in ice conditions.

Madison, Wisconsin
January, 1978

C. Allen Wortley
Principal Investigator

ACKNOWLEDGEMENTS

This work was funded by the University of Wisconsin Sea Grant College Program under a grant from the Office of Sea Grant, National Oceanic and Atmospheric Administration, U.S. Department of Commerce and by the State of Wisconsin.

The U.S. Government is authorized to produce and distribute reprints for governmental purposes notwithstanding any copyright notation that may appear hereon.

The project has been undertaken with the assistance of an Advisory Committee. Members were

Name	Affiliation
Guenther E. Frankenstein	U.S. Army Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire
Theodore Green	Departments of Civil and Environmental Engineering and Meteorology, University of Wisconsin-Madison
John P. Klus	Department of Engineering, University of Wisconsin-Extension, Madison
Raymond G. Lawrence	Waterways Division, Michigan Department of Natural Resources, Lansing, Michigan
J. Robert McCullough	Aeration Division, Schramm, Inc. West Chester, Pennsylvania
James E. Muschell	United Marine Associates, Cheboygan, Michigan
Harry Nelson	Madeline Island Ferry Lines, Inc. La Pointe, Wisconsin

Raymond F. Pesch	Roen Salvage Company Sturgeon Bay, Wisconsin
Max Pollack	Architectural-Engineering Division Milwaukee County Department of Public Works, Milwaukee, Wisconsin
A. R. Striegl	Civil Engineer Milwaukee, Wisconsin
John P. Zarling	Mechanical Engineering Department, University of Alaska College, Alaska
R. C. Zirbel	Edward E. Gillen Company Marine Engineers and Contractors, Milwaukee, Wisconsin

The project was commenced in 1973. In addition to the present principal investigator the following co-investigators also contributed to the work: Profs. John P. Klus and Lawrence A. Soltis of the University of Wisconsin-Extension Department of Engineering, and Prof. John P. Zarling, formerly with the University of Wisconsin-Parkside and currently with the Mechanical Engineering Department at the University of Alaska.

The principal investigator is especially grateful for the suggestions and assistance received from Messrs. G. D. Ashton, G. E. Frankenstein, and D. E. Nevel, all of the US CRREL.

INTRODUCTION

This publication is a guide for designers and builders of small craft harbors in northern climates, where ice forces and motions do great damage to man's works.

It has been prepared from a review of the literature, consultation with knowledgeable engineers, contractors and scientists, and the experiences and observations of the principal investigator and members of an Advisory Committee.

Zarling (1974) prepared an annotated bibliography entitled "Ice Engineering in Small Craft Marinas". This work served as the starting point for reviewing the literature. Additional and more recent publications were examined as well as many of those noted in the bibliography. There is a large amount of information in the overall field of ice engineering and ice physics. However, very little is easily applied by the small craft harbor designer or builder because of its research-technical nature. Also much of the information doesn't relate directly to the problems of designers and builders, or is of such orders of magnitude as to render it useless for safe and economical design.

However it is from the more useful, more current, and more applied information in the literature that much of this guide has been prepared. The literature has been freely cited. No additional laboratory experimental work has been undertaken. Field observations and measurements have been made and have influenced the recommendations derived from the information in the literature.

Ice for purposes of this guide is primarily stationary lake ice. River ice, ice floes and sea ice are not specifically dealt with. They can and do present different problems. Small craft harbors are customarily built in sheltered areas, away from dynamic ice forces.

The guide is divided into five parts. Part I presents information on the characteristics, behavior and properties of ice and ice covers. It presents as much specific quantitative information as is available. The other information presented is of a general nature but useful in understanding ice and its behavior.

Part II deals with ice pressures of thermal origin. It is derived largely from current laboratory research on thrusts on hydraulic structures. Certain information however is applicable to small craft harbors.

Part III presents recommendations for design and construction of compressed air ice suppression systems and other information on weakening ice. These suppression systems are effective in small craft harbors.

Part IV discusses the bearing capacity of ice and presents recommended maximum short term construction loadings.

Part V gives minimum and expected maximum values for ice uplift forces. Information about other ice forces is also presented.

As stated initially this is a guide for designers and builders of marinas in northern climates. It by no means answers all the pertinent questions, nor does it present all the "how to's". This is beyond our present scope and our present abilities.

It is hoped that the reader will review all Parts of this guide and form an overall understanding of ice behavior. Successful, innovative and economical design of small craft harbor structures is our goal.

UNITS

Conversion Factors: U.S. Customary to SI Metric Units

Measurements in this guide are shown in customary U.S. units with SI metric equivalents. Specific quantities are converted accurately, but where the measurements are general the conversion is approximate only. As much of the source data was in the metric system and not the U.S. customary or the SI systems, some data appear to have "odd" values. For example, pressures in kilograms per square centimeter (kg/cm^2) have been converted to pounds per square inch (psi) and kilonewtons per square meter (kN/m^2). The SI unit for pressure is pascal (Pa). One Pa is equal to one N/m^2. Because we are dealing with forces, and forces per unit length, as well as pressures, we have elected to leave pressures in force-length units rather than the derived pressure unit Pa.

Some U.S. Customary to SI Metric conversion factors are given below. The ASTM E380-76 Standard for Metric Practice gives additional conversion factors.

To Convert	To	Multiply By
inches (in)	millimeters (mm)	25.40
inches (in)	centimeters (cm)	2.540
feet (ft)	meters (m)	0.3048
yards (yd)	meters (m)	0.9144
miles (mi)	kilometers (km)	1.609
inches per yard (in/yd)	millimeters per meter (mm/m)	27.78
square inches (sq in)	square centimeters (cm^2)	6.45
square feet (sq ft)	square meters (m^2)	0.093
cubic inches (cu in)	cubic centimeters (cm^3)	16.39
cubic feet (cu ft)	cubic meters (m^3)	0.0283
pounds (lb)	kilograms (kg)	0.453
tons (ton)	kilograms (kg)	907.185
tons (ton)	metric tons (mt)	0.907

To Convert	To	Multiply By
one pound force (lb)	newtons (N)	4.448
one kip (kip)	kilonewtons (kN)	4.448
one kilogram force (kg)	newtons (N)	9.806
kips per foot (kips/ft)	kilonewtons per meter (kN/m)	14.59
pounds per square foot (psf)	newtons per square meter (N/m^2)	47.88
pounds per square inch (psi)	kilonewtons per square meter (kN/m^2)	6.89
pounds per square inch (psi)	meganewtons per square meter (MN/m^2)	0.00689
pounds per square inch (psi)	kilograms per square centimeter (kg/cm^2)*	0.07031
kips per square inch (ksi)	meganewtons per square meter (MN/m^2)	6.89
pounds per cubic foot (pcf)	kilograms per cubic meter (kg/m^3)*	16.02
Degree Fahrenheit (F)	Degree Celsius (C)*	0.555
Temperature Fahrenheit (F)	Temperature Celsius (C)*	(F-32)/1.8
square feet per minute (ft^2/min)	square meters per second (m^2/sec)	0.001548
square feet per hour (ft^2/hr)	square centimeters per second (cm^2/hr)	929.03
Btu per hour per square foot (Btu hr^{-1} ft^{-2})	watts per square meter (W m^{-2})	3.152
Btu per pound per degree Fahrenheit (Btu lb^{-1} F^{-1})	joule per kilogram per degree Celsius (J kg^{-1} C^{-1})*	4186.8
Btu per hour per square foot per degree Fahrenheit (Btu hr^{-1} ft^{-2} F^{-1})	watts per square meter per degree Celsius (W m^{-2} C^{-1})	5.678
Btu inch per hour per square foot per degree Fahrenheit (Btu in hr^{-1} ft^{-2} F^{-1})	watts per meter per degree Celsius (W m^{-1} C^{-1})*	0.1442
miles per hour (mph)	meters per second	0.4470

*not an SI unit

PART I

ICE AND ICE COVERS

This section of the guide contains information on the crystal structure of ice, the polycrystalline forms of ice, the nucleation and growth of ice, the freeze-up and break-up processes on fresh-water lakes, the thicknesses and behavior of ice covers and some introductory material on the deformation and strength of polycrystalline ice, a viscoelastic material.

Some of the material is pertinent to present design techniques for small craft harbors and some is only of general background interest. The general information may be useful in developing new design techniques, explaining behavior of ice covers, understanding other ice research papers and monographs that readers may come in contact with, and in other ways in the future. The material should answer some of the readers' questions about ice and confirm some of their observations in the field.

Simply stated, ice is not well understood. At the present time straight-forward and proven design approaches for small craft harbors are not available. Therefore the best approach for design is to begin with an understanding of the present state of knowledge about ice, especially in the context of small craft harbors, and go from there.

Those wishing more detailed technical and theoretical information on the physics and mechanics of ice should consult references such as the two monographs by Glen (1974, 1975) and textbooks by Pounder (1965) and Hobbs (1974).

Crystal Structure of Ice

Ice is a solid consisting of a crystallographic arrangement of water molecules. Each molecule has an oxygen atom with chemical bonds with two hydrogen atoms.

Although several forms of ice can exist, we shall be concerned only with the ordinary form of ice. It is the form with which most people are familiar. This form is hexagonal in crystal symmetry. The symmetry is clearly demonstrated in the symmetry of many snow crystals.

Each crystal of ice is made up of water molecules arranged in a regular geometric pattern forming hexagon prisms. The axis of the prism is known as the optic axis and the surface perpendicular to this axis is called the basal plane. Crystallographic terminology labels planes and directions. In the hexagonal system, there are three directions in the basal plane that are crystallographically equivalent. These directions are called a, b and d (sometimes a_1 a_2, and a_3) and the axis perpendicular to the basal plane is called the c-axis (i.e. the optic axis).

Runnels (1966) suggests that at the molecular level the seemingly rigid perfection of the ice crystal is disrupted by an astonishingly busy traffic of molecules and migrating lattice faults. There is a good deal of empty space between molecules and hence ice has a rather open structure. Because of this fact, water, in contrast to almost all other substances, is less dense in the solid state than in the liquid state. This results in ice floating on water. The density of pure ice is given by Pounder (1965) as 57.2 pcf (916.8 kg/m^3).

Ice encountered in nature, such as on a lake, is in a high temperature state. It is floating on its own melt with its bottom surface at a temperature equal to its melting point and the upper surface varying with ambient conditions. Predicting the properties and behavior of a material in such a state is quite difficult.

Polycrystalline Ice

Most bulk ice in nature consists of polycrystalline ice, that is, ice composed of a large number of single crystals in different orientations. Polycrystalline ice behaves differently from single crystals of ice because of the existence of grain boundaries and the fact that single crystals are anisotropic. A single crystal of ice has physical properties that are different in directions parallel and perpendicular to its c-axis. If the constituent crystals of polycrystalline ice are randomly oriented, then the bulk properties will be some kind of average of the properties of the single crystals of which it is composed. For some properties such as elasticity, this averaging is fairly straightforward, but for others, such as plastic deformation, there is such a radical difference in the behavior of ice crystals in different directions that the physical properties of the polycrystal are quite different from that of the single crystal.

Moreover, the grain size as well as the crystal orientation of polycrystalline ice has an effect on the strength and behavior of an ice sheet. Fortunately, from a design point of view, those effects may be minor for cold polycrystalline ice, the type that normally would present the most severe forces on structures placed in ice. This may not always be true for all kinds of polycrystalline ice encountered in lakes and rivers. The kind of ice which forms depends upon its genesis.

Nucleation and Growth of Ice Crystals

Ice in the absence of very great supercoolings or supersaturations will only form if some nucleus is present on which it can grow. This can be ice itself, and if an ice seed is present, ice will normally develop and grow. It may also be a nucleus consisting of some other material.

Once an ice nucleus is present, ice can grow from the surrounding water provided the temperature remains favorable. How ice will grow will depend on whether the water surface is supercooled, and also on the amount of impurities it contains.

Michel and Ramseier (1969) have classified river and lake ice based upon its genesis, structure and texture. An ice sheet or ice cover is divided into three basic ice formations: primary ice, secondary ice, and snow ice.

Primary ice is the first type of ice of uniform structure and texture which forms on a water body. On a calm water body the primary ice is in the form of an ice skim which grows horizontally in the supercooled layer and is about a hundredth of an inch (a few tenths of a mm) thick. Usually when water freezes a little plate-like disc forms. It may rapidly develop dendritic extensions depending on the amount of supercooling and the rate of heat loss. Since the planar structure will naturally float with its plane parallel to the surface, the ice which forms when water freezes often has its surface crystals oriented with their c-axes vertical, i.e. perpendicular to the free surface. These initial crystals spread over the surface until they interact and form a layer over the whole surface of the water. The subsequent growth of this layer downwards into the water is not usually dendritic.

In rough and turbulent water the primary ice consists of congealed frazil slush which can be an inch (several centimeters) thick. Frazil slush is a floating agglomerate of loosely packed individual ice crystals having the form of small discoids or spicules which are formed in supercooled turbulent water. If the nucleation occurred by snow, the resulting congealed snow slush (i.e. a loosely packed agglomerate of floating snow particles) would also be part of the primary ice.

Secondary ice forms parallel to the heat flow which is perpendicular to the primary ice. As the ice grows down into the water under conditions of calm formation and growth the grain boundaries are almost perpendicular to the surface. The result is a columnar-grained structure with the c-axes vertical in the long direction of the grains. This type of secondary ice is referred to as type S1. The crystal size increases with depth, because unfavorably oriented crystals disappear. S1 ice is found in lakes, reservoirs and in rivers with low flow velocities.

S2 columnar ice is a secondary ice form with the c-axes horizontal. It forms under similar conditions to S1 ice but where the primary ice has a random or preferred vertical orientation superimposed on a random orientation. As the ice grows down into the water, the c-axes vertical preferred orientation gives way to one in which the c-axes are horizontal. Some crystals are wedged out at the expense of others. This type of ice is found in lakes, reservoirs, shores and rivers.

Another type of secondary ice, but graunular rather than columnar in form, is S4 Congealed Frazil Slush. Its crystal boundaries are irregular and the grains highly angular and randomly oriented. Frazil is found in rivers where it is taken downstream and deposited upwards under the secondary ice overlying more slowly moving water. S4 ice is found in rivers and reservoirs, and lakes fed by turbulent waters. An entire ice cover can consist of S4 ice or it can be found layered between columnar forms of ice.

An interesting ice form, anchor ice, results when frazil is being formed in supercooled water. The frazil particles stick and grow on bottom weeds and stones. After a night of frazil formation the morning sun can cause anchor ice on stones and other objects to float to the surface and be carried downstream.

Snow ice is the third basic formation of ice. This type of ice always forms on top of the primary ice due to snow deposits which lie on the ice cover. Snow ice may form due to variations in discharge of water, by melt or rain, or by the depression of the ice cover due to a heavy snow load. In general snow ice is formed from any imaginable kind of water source. T1 is the designation for Snow Ice and its grains are round to angular and randomly oriented. Being granular it is similar in appearance to the S4 Congealed Frazil Slush ice found usually under the secondary ice layer.

The crystal size of ice depends greatly on meteorological parameters. If the water surface is calm and the temperature gradient is not too steep extra large crystals will develop. Their diameters can be greater than 0.8 inches (20 mm). If the temperature gradient becomes larger then the crystal size will decrease. If snow causes the nucleation, crystals will be of the 0.2 inches (5 mm) size and smaller. Extra large and giant size crystals can form under extremely calm conditions such as in a pool during the formation of the primary ice layer. Crystal dimensions in yards (meters) are possible.

During the evolution of an ice cover the crystal size will change. In lake ice it is possible to go from fine sized to extra large sized crystals in a columnar layer if nucleation has occurred by snow.

Gow and Langston (1977) describe the growth history of lake ice in relation to its stratigraphic, crystalline and mechanical structure. They have successfully correlated specific increments of ice growth with major weather events. This was accomplished by diagnosing bubble layering observed in stratigraphic samples of lake ice. The ice covers of the two New Hampshire lakes used in their studies contained two major components: lake ice formed solely from the freezing of lake water and snow ice formed from the freezing of water-saturated snow. The lake ice component consisted of predominately fibrous-textured crystals with a vertical c-axis orientation. Columnar crystals elongated in the vertical but with horizontal c-axes fabrics, were observed only in the top 4 to 6 inches (10 to 15 cm) of ice that fringed the shores of both lakes. These fabrics had reverted entirely to a vertical c-axis condition at 6 inches (15 cm) depth. The reason for this reversal in orientation texture has not clearly been established. However, indications are that the temperature gradient in the water may be a significant factor in initially controlling the growth textures, since continued growth of ice near shore resulted in rapid elimination of the horizontal c-axis in favor of crystals with vertical c-axes.

Several variants of snow ice were distinguished, depending upon the source of infiltrating water. Both ice types were readily distinguished on the basis of differences in their stratigraphic and crystalline structure.

Michel and Ramseier (1969) have also found that studying ice profiles from lakes and rivers enables one to determine what some of the meteorological conditions were upon formation of the primary ice. Similar analyses can be obtained from snow ice and overlying snow. They believe that in the future it may even be possible to predict directly the texture at a particular site from meteorological and hydrodynamic data.

The predominant inclusion in ice is air. During solidification the air is rejected at the ice-water interface and is supplemented by gases arising from biochemical processes which take place in a natural body of water. Air can also be incorporated from the atmosphere through cracks, drained snow ice or an agitated water surface. If the growth is slow most of the air will be rejected and the ice will be transparent like glass. This transparent ice is sometimes called black ice. (The term blue ice is sometimes used in connection with glacial ice, but not transparent lake ice.)

If the solidification process proceeds rapidly air in the form of bubbles will be distributed throughout some parts of the ice cover. If air is being emitted continuously, the bubbles take on a cylindrical shape. It is the air entrapped in ice that causes it to appear white. This type of ice is sometimes called white ice. Snow ice is an example of white ice because of entrapped air. Grain size does not determine the color of ice. Frozen frazil ice can appear transparent in spite of its granular texture.

Freeze-Up and Break-Up of Fresh Water Lakes

Water, in the temperature range of 39°F to 32°F (4°C to 0°C), increases in volume with decreasing temperature. This is contrary to the behavior of most substances. In this temperature range the coefficient of expansion of water is negative, that is, water expands as it becomes cooler. Since the volume of a given mass of water is smallest at 39°F (4°C), it has its maximum density at this temperature. The negative coefficient of expansion of water in this range is one of nature's anomalies. This explains why lakes first freeze at their surfaces and why somewhat warmer (and denser) water is found near the bottom.

Williams (1966) has prepared a detailed and comprehensive description of how fresh water lakes freeze and thaw. Michel (1971) also describes the winter regime of rivers and lakes in a monograph. It too is comprehensive and contains information on rivers as well as lakes. For our purposes we will recite the lake ice processes described by Williams (1966).

His descriptions are applicable to sheet ice, the type that forms on relatively still lakes, slow-moving rivers, and sheltered harbors. Williams designates three distinct periods in the life of sheet ice: the freeze-up period, the ice-growth period, and the ice-melting period. These periods roughly correspond with the fall, winter and spring seasons.

During the fall, or early winter, smaller lakes freeze over. A solid ice sheet forms. In larger lakes the sheet may only cover harbors and the area near the shoreline, but is generally extensive. The Great Lakes rarely completely freeze over, at least for any extended period of time.

The formation of an ice cover on a lake is a function of the water's heat exchange with the atmosphere, the initial amount of heat stored in the water body, and the amount of inflow of warm water (heat) to the site. The amount of heat lost to the atmosphere is a function of the air temperature, the wind velocity, and solar radiation. The amount of heat that can be stored in a water body is a function of depth. Usually, the deeper the lake, the deeper the convective mixing and the slower the rate of water cooling for a given surface heat loss. (In a man-made harbor, depths are usually shallow and hence significant thermal reserves because of depth are not present.) Inflow of heat by currents from warmer water in rivers or from deep reservoirs can prevent or delay sheet ice formation at certain sites.

The cooling of a fresh water lake occurs in two stages: gradual cooling until all the water is at a temperature of about 39°F (4°C), and cooling of the surface water from the time the water is isothermal (i.e. the same temperature, top to bottom) at 39°F (4°C) until sheet ice forms.

First the warm lake surface water cools down (even steams or mists when giving up its heat to the cool atmosphere) and thereby contracts and becomes more dense. This water sinks in the less dense lake and forces up the lower less dense, warmer water to in turn be cooled. This process is repeated until the lake is isothermal at 39°F (4°C). The lake has then "turned over".

From this point on the lighter, cooler surface water continues to cool down until it freezes as sheet ice. (A thin layer of ice first forms along the edges of the lake. A convective air motion from land toward rising warm air over the lake cools the lake edges more rapidly than the center.) Since the density of ice is even less than that of freezing water, the ice floats on the water below it, and further freezing can only result from heat flow upward by conduction.

The mid-winter period, when the permanent ice cover on lakes gradually increases to a maximum thickness late in the winter, is a period of interest to harbor engineers concerned with ice pressures, ice uplifts, and bearing capacities.

When the ice cover is thin, and the air very cold, there will be a relatively large heat flow from the water, through the ice, to the atmosphere. This will cause a rapid increase in ice thickness at the interface between the ice and the water. As the ice cover thickens, an insulating effect occurs, and ice grows at a slower rate. A snow cover on the initial ice formation can effectively prevent any further ice growth.

Ice normally grows downward into the water, but not always. The upper surface of ice can be submerged below the water level due to the weight of a snow cover. Snow ice then forms and freezes on top of the ice sheet. Sometimes this causes rapid increases in the total thickness.

A frozen snow covered lake can be highly variable in its underlying stratigraphy. The snow cover can conceal slush and weak layers of ice, as well as strong sound ice.

The clearing of ice from bodies of water at break-up is affected by heat gain from the atmosphere, snow and ice conditions, wind and currents, and inflow or runoff of warm water to a site. The heat gained from the atmosphere weakens and melts the ice. The amount of solar radiation absorbed is determined by the reflectivity of the snow or ice surface. The properties of the ice cover determine the depth solar radiation will penetrate, causing internal melting. The thickness of the snow and ice is a measure of the amount of ice to be melted. The mechanical action of wind and currents is of great importance in breaking up the ice after it has been weakened by surface and internal melting.

At the beginning of break-up, snow melt runoff from surrounding land areas may flood onto part of the lake surface. At the same time the snow cover on the ice densifies and begins to melt. The runoff usually accumulates near the mouths of streams draining into the lakes and along the shoreline, producing dark-looking slush on the ice surface. At this stage the darkened ice cover surface absorbs a large proportion of the incoming solar radiation.

The ice around the shoreline tends to melt first, partly because of the darkened surface layers but also because the ice thickness is often less close to the shore, where it can only grow as thick as the depth of the water. Eventually the ice cover completely melts around the shoreline, leaving the main body of ice floating free. This body of ice can still have considerable strength.

During melting, drainage holes can develop in the ice sheet. In the early stages of melt these holes develop near the shore often where runoff flows into old thermal cracks. The flow is due to difference in level between the lake water supporting the ice and the melt water on the ice surface. Such drainage holes can enlarge rapidly, sometimes developing into holes one to two feet (0.3 to 0.6 m) in diameter.

Later on during melting, the main body of ice, floating free, melts at the surface. This melt water flows along the surface of the ice and eventually creates surface drainage patterns. It can drain to the open water adjacent to the shoreline, or down the holes that appear to develop preferentially along old thermal cracks. If the surface melt water can drain away, the surface of the ice becomes a porous, crumbly white structure which reflects solar radiation readily, thus retarding melting by solar radiation. As the melt season progresses, penetrating solar radiation causes internal melting in the ice sheet. At this time the ice sheet may consist of a shallow, porous surface layer several inches (several cm) thick, a layer of water-logged ice also several inches (several cm) thick, and then solid unmelted ice (whose lower surface is melting in the lake water).

In the final stages of break-up the underlying entrapped water layers result in darkened surface ice and most of the incoming solar radiation is absorbed. The ice is now ripe for breaking by wind and currents and is unsafe for over-ice transportation. The current created by strong winds will break up the ice cover and induce circulation that brings the warmer subsurface water to the surface. This can cause rapid melting of ice from a lake. Indeed the final disappearance of ice covers has occurred so quickly at times that early observers believed that the ice actually sank.

The term rotten ice is sometimes used to refer to disintegrating ice at break-up time. If the ice has a columnar structure it becomes candled and is referred to as candle ice. Melting is concentrated at the boundaries of the columnar prisms leaving a weak candle-like structure.

Williams (1966) and Michel (1971), as well as others, have proposed methods to estimate the date of freeze-up, maximum ice thickness, and the date of break-up of fresh water lake ice. The method of doing this is approached in two ways: develop formulas based on physical principles, and make an analysis of past records to give statistical limits within which maximum ice thickness or freeze-up or break-up can be expected to occur.

The operational use of formulas is usually limited because of difficulties in making dependable forcasts of such variables as air temperature and snow depth. It is also difficult to allow for natural variability, as the rate at which ice forms, grows, and melts varies not only from lake to lake but on an individual lake.

The statistical climatological approach also has limitations. The required long-term records are only available for some lakes and it is difficult to use the records collected to estimate freeze-up or break-up for other lakes with different sites and thermal regimes. In addition, even for lakes where some long-term records are available, the statistical approach will only define upper and lower limits within which freeze-up or break-up will occur.

We believe direct on-site observations by the small craft harbor designer coupled with any information from local residents is the best approach to predicting freeze-up or break-up. Ice thickness determination can be handled in the same manner but of course modified by recommended design criteria for minimum thicknesses and strengths.

The U.S. Great Lakes Environmental Research Laboratories (GLERL) in cooperation with Canadian and other agencies are involved in efforts to understand and forcast behavior of ice formations on the Great Lakes. Samples and measurements of ice thicknesses from many places on the Great Lakes are underway. This information together with other published measurements and observations will be useful to marina designers.

Anecdotal Behavior of Natural Ice Covers

Many anecdotal accounts of the behavior of ice covers are contained in the literature. Some of the explanations about its behavior have since been proven wrong. Some have been substantiated with research and additional observations. During the nineteenth century Dumble (1858), who was an engineer with a railway company, published accounts of his observations on inland lakes and rivers in the northern states and Canada. A few of his observations are repeated below together with some of our parenthetical comments.

The most violent shoves of ice occur previous to rainstorms. (This would be associated with warm winter weather and thermal expansion.)

Contraction generally occurs at night, and is accompanied by sharp reports. (This would be associated with cooling nighttime weather and thermal contraction cracking.)

A coating of snow of any depth over six inches (15 cm) effectively prevents any motion of ice. (Snow is a very effective insulator and prevents thermal motion of ice.)

It is but reasonable to suppose that any solid, equally dense throughout its dimensions, and susceptible of expansion, would, when equally acted upon by the active agent or moving cause, expand from its center towards its circumference. This is the effect produced on any large field of ice of equal thickness and density, when acted upon uniformly by either the midday sun or warm winds. It is a fact, however, that it moves from other directions than from the center of the lake. Shoves are sometimes witnessed from the east and sometimes from the west, to the north and to the south.

Ice owing to the peculiar circumstances under which it sometimes forms, is not found to be equally pure or dense, neither is it of uniform thickness. This ice irregularly acted upon by warm winds, or by the slanting rays of the sun at different altitudes, shoves, or expands from various directions other than from the center of the lake.

Shoving is from the stronger and most susceptible ice toward the weaker and less expansive.

Ice on any large and irregular sheet of water studded with islands must naturally be of unequal thickness and density. There is therefore no doubt whatever, that the phenomenon of ice expanding and shoving from various directions is caused by the unequal thickness, density and glareness of the ice and likewise by the manner in which the heated atmosphere strikes it.

(Ice sheets and covers are variable in their composition and frequently expand, contract and move in erratic ways.)

It is observed, that when a large extent of field ice expands towards the shore it does not shove into deep bays but fractures from point to point in a zig-zag manner, across the chord at the mouth. A thrust of the main field must find less resistance across this chord than around the area of the bay. (Crack patterns can now easily be observed from the air and indicate where ice has failed as it moved into or by some shore configuration such as a bay.)

Much more recently Striegl (1952) described the formation of ice in the Great Lakes, and particularly in Lake Michigan.

Ice normally starts to form along the shores of the southern part of Lake Michigan about the middle of December. By the early part of January the initial ice sheet may reach a thickness of 6 to 8 inches (15 to 20 cm) and extend off shore distance of one to five miles (1.6 km to 8.0 km) depending on the continuity of cold weather and the strength of the winds. Frequently during this initial ice period the thin ice cover may be broken up by the wind and wave action. If the wind is toward shore, one of a series of windrows may develop where the drifting cake of ice will pile up on or under the sheet of ice until it may reach a depth of many feet (meters) above and below water, and may ultimately pack to the bottom where shallower water is reached. If the ice along the shore is weak the windrow may be pushed ashore until a large part of it is above the water level. If the wind is off shore the fields of floating ice are driven out into the lake where they may drift for long periods or pile up in ice windrows or along shore in another part of the lake, depending on the continuity and direction of the wind.

Engineering structures built in the lakes for harbor protection or other purposes must be designed to support enormous ice loads due to the ice build up on the structure by continued freezing of wave wash and spray or by loose ice piled on top of the structure by ice jams and then consolidated by continued freezing action. Those ice loads may pile up to 20 to 25 feet (6 to 8 m) above the top of the structures. Usually this load, does little damage to overloading individual structural members because of the extra support and bridging effect of the ice itself. However, it may cause considerable damage in other ways. Stones or concrete may be broken, through freezing and thawing action to such sizes that they will later be displaced by wave action. Also large breakwater stones, up to several tons each, may be carried bodily from the structure when encased in large ice blocks. Such stones have been carried by ice floes miles from structures and dropped in navigable channels where they have become obstructions to navigation. Smaller rocks have been found in the channels of harbors which were picked up by the ice on the beach or from the structures and floated with the ice to areas where they were dropped on melting or breaking of the ice.

In Chicago in January 1948, a strong easterly wind storm, accompanied by sleet, snow and low temperatures, piled ice in a solid mass along the entire Illinois shoreline. The ice was as much as 15 feet (5 m) above the water surface and extended 50 to 100 feet (15 to 30 m) behind sea walls and bulkheads in Chicago, and from 100 to 300 feet (30 to 90 m) in front of the beaches and natural shoreline and other points.

The section of this guide dealing with pressures of thermal origin exerted by ice sheets on structures will give additional and current information on the behavior of ice covers.

Thickesses of Great Lakes Ice

Striegl (1952), Aune, Beaudin, and Borrowman (1957), and others have reported observations on natural ice thicknesses on the Great Lakes. Because of the wide geographic extent of the lakes area, as well as the differences in the characteristics of individual lakes, the ice season varies considerably. Climatic conditions obviously are the major factors in the amount and rapidity of ice formation and disintegration.

The following typical cases indicate the range in temperatures and the differences in length of the cold season which are primary factors in connection with the formation of ice. As these figures are based on normal daily mean temperatures published by the U.S. Weather Bureau it is obvious that there will be wide variations from these means depending on the general character of the season. Please refer to Table 1.

Locality	Normal Length of Freezing Season From	To	Total Days	Ave. Temp. During Freezing Season	Freezing Index* F°-(C°-) Degree Days of Freezing
Duluth	Nov. 12	Apr. 2	142	16.73°F (-8.48°C)	2168 (1204)
Sault Ste. Marie	Nov. 16	Apr. 4	140	18.76°F (-7.36°C)	1854 (1030)
Milwaukee	Nov. 28	Mar. 15	108	24.22°F (-4.32°C)	840 (467)
Port Huron	Dec. 1	Mar. 20	110	24.98°F (-3.90°C)	772 (429)
Buffalo	Dec. 8	Mar. 19	102	26.35°F (-3.14°C)	576 (320)

Table 1
(Striegl--1952)

It will be noted that these figures indicate 3.76 times as much of the air temperature condition which, when in contact with undisturbed water, causes the formation of ice at Duluth as at Buffalo. Naturally one would expect the formation of more ice at Duluth, and the northerly areas of the Great Lakes, than in the southern areas. Other conditions such as sunshine, winds, volume of the body of water, etc., of course, modify the relationship between temperature and ice formation so there is no direct relationship that may be determined for natural conditions.

*Freezing Index is the total number of Degree Days using the freezing temperature as the reference. A Degree Day is defined as the departure of the daily mean temperature from the freezing temperature.

Aune, Beaudin, and Borrowman (1957) indicate maximum thicknesses of solid lake ice observed in locations from 1899 through 1951 and average daily temperatures for the month of January for a period of 20 years. They described solid lake ice as occurring in protected harbors, bays, and channels of low velocity current. Ice of this type begins to form in November and continues to increase in thickness throughout the winter occasionally obtaining a depth of three and one half feet (1.1 meters) with depths of two feet (0.6 meters) being quite ordinary in northern localities. Solid lake ice conditions usually prevail to the later part of March or early April at which time the ice begins to honeycomb and break up. Their data is presented in Table 2.

Location	North Latitude	Average Daily January Temperature	Maximum Thickness of Ice
Duluth, Minnesota	47°	10°F (-12°C)	38 in (97 cm)
Marquette, Michigan	46°	15°F (- 9°C)	27 in (69 cm)
Escanaba, Michigan	46°	15°F (- 9°C)	35 in (89 cm)
Green Bay, Wisconsin	44°	17°F (- 8°C)	36 in (91 cm)
Milwaukee, Wisconsin	43°	22°F (- 6°C)	-------------
Chicago, Illinois	42°	25°F (- 4°C)	-------------
Mackinaw City, Michigan	46°	17°F (- 8°C)	-------------
Saginaw Bay, Michigan	43°	22°F (- 6°C)	35 in (89 cm)
Detroit, Michigan	42°	26°F (- 3°C)	-------------
Cleveland, Ohio	41°	27°F (- 3°C)	17 in (43 cm)
Buffalo, New York	43°	25°F (- 4°C)	24 in (61 cm)
Oswego, New York	43°	23°F (- 5°C)	25 in (64 cm)
Ogdensburg, New York	45°	17°F (- 8°C)	30 in (76 cm)
Kingston, Ontario	44°	23°F (- 5°C)	25 in (64 cm)
Sault Ste. Marie, Michigan	46°	-------------	27 in (69 cm)
Port Arthur, Ontario	48°	10°F (-12°C)	41 in (104 cm)

Table 2
(Aune, Beaudin and Borrowman--1957)

Freshwater lakes and rivers lying between 35°F (2°C) and 30°F (-1°C) January isothermals will freeze on rare occasions of extremely cold temperatures (about 0°F or -18°C) which are usually of only several day's duration.

Between 30°F (-1°C) and 25°F (-4°C) isothermals some ice normally less than 4 inches (10 cm) in thickness can be expected almost every winter. Northward of the 25°F (-4°C) isothermal ice can be expected every winter. The thickness will vary from 6 inches (15 cm) to 12 inches (30 cm) in the vicinity of 25°F (-4°C) isothermal to 36 inches (91 cm) near the 10°F (-12°C) isothermal.

During January and February 1977 we made ice thickness measurements in harbors on Lakes Superior, Michigan and Huron. This was an unusually cold winter in the western Great Lakes area. Many thicknesses in the range of 30 to 39 inches (75 to 100 cm) were observed between 42° and 47° North latitudes. At 42° North latitude on Lake Michigan, 35 inches (90 cm) was measured. There seems little doubt but that three feet (one meter) of ice is probable in the western Great Lakes. Also, as was previously mentioned, the U.S. Great Lakes Environmental Research Laboratories is measuring and recording ice thickness in the Great Lakes.

Deformation and Strength Behavior of Polycrystalline Ice

Polycrystalline ice is a viscoelastic material. A comprehensive description of a complete linear viscoelastic model representing the stress-strain behavior is given by Nevel (1976).

For our purposes we will review deformation behavior by describing a laboratory unconfined uniaxial compressive strength test of ice. The test is run at a constant stress, i.e. a single load is placed on the sample. As it remains on the sample, the sample will compress or deform. The amount of deformation divided by the sample length is the strain.

The longer the load is left on the sample, the more the sample will deform. Strain is therefore a function of time. This is termed creep behavior. (Strain is also a function of other variables, e.g. the magnitude of the stress.)

The total creep occurring before the sample breaks is described by three creep periods. These periods are termed primary, secondary, and tertiary.

When the load is first placed on the ice, the ice has an instantaneous elastic response which can be thought of as an elastic compression in a spring. If the load is too great, the sample may break before creep can take place. Assuming however that the load is moderate, the ice will next exhibit a delayed elastic response. This response can be thought of as the travel in a dashpot, i.e. a plunger moving in a viscous fluid. Together these two responses make up the creep period called primary creep.

During secondary creep the strain increases with time and if the ice behaves as a linear viscoelastic material this strain increase is linear.

At some point in time the strain will accelerate. This increasing straining is the tertiary creep. The sample compresses at a faster and faster rate until it fails. At lower stresses ice may not exhibit tertiary creep, but may fail during secondary creep.

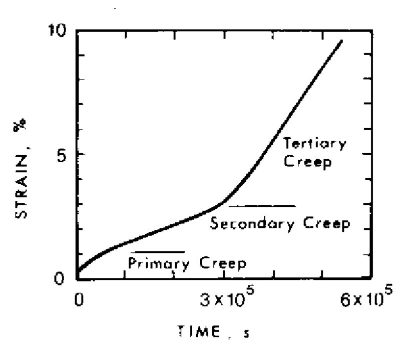

Figure 1
Creep Curve for Granular Ice
(Gold, 1973)

Figure 1 is a creep curve for granular ice reported by Gold (1973). The ice was stressed at 145 psi (1 MN/m^2). Its temperature was 14°F (-10°C). The primary creep represents about one percent strain. Secondary creep occurred until three and one-half days (3×10^5 s) had elapsed. The strain increased to an increasing rate and at the end of about six days had deformed ten percent of its original length.

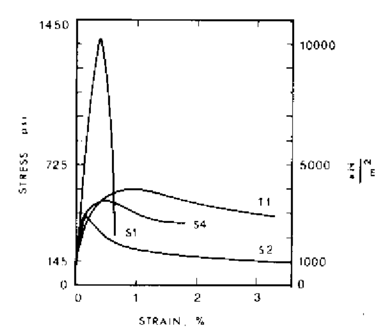

Figure 2
Strain Dependence on Stress
(Gold, 1973)

If now instead of applying a constant stress to the sample we apply a constant rate of strain we can obtain stress-strain curves like those in Figure 2 by Gold (1973). Some ice engineering problems involve conditions approaching constant strain. S1 and S2 are columnar ice and were tested with the load perpendicular to the long direction of the grains. T1 is granular ice and S4 is frazil ice. The temperature of the ice was 15°F (-9.4°C). The strain rate $1.67 \times 10^{-5}\ s^{-1}$, i.e. 0.0000167 per second. (The time to reach one percent strain was ten minutes and the time to reach three percent was one-half hour.) For the given strain rate and ice temperature the influence of ice's structure is clearly seen: columnar-grained, type S-1 ice, is appreciably more brittle in its behavior than the other three, and granular, type T1 ice, more ductile.

To adequately describe the deformation and strength behavior of ice many things must be considered. These include the type of ice, the temperature, the type of load (compression, tension, etc.), load direction with respect to ice structure, the strain rate, and the stress level. Furthermore, to analyze ice in an engineering context it is necessary to establish failure criteria in either ductile or brittle modes of behavior under a variety of boundary conditions. There is much that is not presently understood about ice behavior.

Use of published laboratory strength test data from ice specimens (or simulated ice specimens) must be done with care as the values determined by such testing may not be representative of conditions in the field. For example, uniaxial laboratory strength tests may not represent biaxial and triaxial conditions encountered in the field, because of influence of shear stress in the deformation and failure behavior of ice. Also, small samples generally do not adequately represent conditions in large, sometimes irregular, cracked sheets of ice containing various impurities.

Appended to this guide is an abbreviated summary of a comprehensive monograph entitled, "Deformation and Strength of Ice", by Lavrov (1969). For the reader wishing to delve more into ice strength and behavior it will be of interest. Also the previously mentioned viscoelastic ice model by Nevel (1976) will be very useful.

In subsequent sections of this guide we will introduce strength parameters and design criteria we believe applicable to ice engineering problems in small craft harbors.

PART II

ICE PRESSURES OF THERMAL ORIGIN

Drouin and Michel (1974), working in the Laval University Ice Mechanics Laboratory have prepared a comprehensive work entitled, "Pressures of Thermal Origin Exerted by Ice Sheets Upon Hydraulic Structures". This work, undertaken by Drouin for a doctoral thesis, we believe is the most comprehensive and recent treatment of thermally induced ice forces on structures.

Prior to this work, Michel (1970) prepared a monograph entitled, "Ice Pressure on Engineering Structures". With respect to thermal forces, the later work should be used as it corrects some previous assumptions now proven erroneous.

The material that follows in this section of the guide is largely based on Drouin and Michel (1974). It is recognized that it was originally prepared in the context of reservoir ice forces that affect dams, and that it is a laboratory study. Nevertheless, understanding this work and the ranges of thermally produced loads that can be predicted for dams, will give the harbor designer a feel for the problem he faces. With flexible harbor structures, the forces will be less. Also with real field ice, the forces will be less because of faults, cracks, discontinuities, etc. Laboratory samples test stronger than field ice.

Introduction

Thermally induced ice pressures on dams and other hydraulic structures are significant, and may even be the controlling design load for some structures. In the past these thermal thrusts have been calculated for the condition of ice failing by crushing at stresses like 400 psi (2756 kN/m^2). This resulted in thermal loads now known to be too high, e.g. 50 kips per foot (730 kN/m). Now values of ¼ to ½ these are normally used. Flexible structures experience even smaller loads than rigid structures, but ways to accurately determine ice loadings on flexible structures have not been developed. The loads would however be expected to be less than those predicted in this section.

During the past fifty years, engineers and scientists, e.g. Royen, Brown and Clarke, Rose, Monfore, Lofquist, and Lindgren researched and estimated thermally induced ice pressures. None of them took into account the crystallographic characteristics of the types of ice and actual strain rates that exist in nature in the regime of thermal expansion of ice sheets. Also the initial temperature of the ice was not considered in some of the previous work, and some research was based on too few tests. The composite picture of past studies is one with ambiguities and many deficiencies. Therefore we won't review this previous work, but instead will present the recent work of Drouin and Michel (1974) who have, however, comprehensively analyzed the work of those before them.

To estimate ice pressures, we must first consider the thermal properties of ice, heat transfer in ice sheets, types of naturally occurring air temperature rises, and the deformation characteristics of several types of ice strained slowly. With a laboratory testing program supported by theoretical rheological mathematical models, Drouin and Michel (1974) have developed estimates for thermally induced ice thrusts.

Thermal Properties of Ice

Conductivity, specific heat, diffusivity and expansion are all properties of ice important to estimating thermal forces. Like most properties of ice their values are still imprecise. Drouin and Michel (1974) estimate the most probable values of thermal properties from the works of others. Table 3 presents their estimated values.

Temperature °F (°C)	Thermal Conductivity Btu in hr^{-1} ft^{-2} $°F^{-1}$ ($W\ m^{-1}\ °C^{-1}$)	Specific Heat Btu lb^{-1} $°F^{-1}$ ($J\ kg^{-1}\ °C^{-1}$)	Thermal Diffusivity $ft^2\ hr^{-1}$ ($cm^2\ hr^{-1}$)	Thermal Expansion $°F^{-1}$ ($°C^{-1}$)
32° (0°)	15.6 (2.25)	.506 (2117)	.0449 (41.7)	30.0×10^{-6} (54.0×10^{-6})
14° (-10°)	16.1 (2.32)	.487 (2039)	.0481 (44.7)	29.0×10^{-6} (52.2×10^{-6})
-4° (-20°)	16.8 (2.42)	.468 (1961)	.0522 (48.5)	28.0×10^{-6} (50.4×10^{-6})
-22° (-30°)	17.6 (2.54)	.450 (1883)	.0570 (53.0)	27.0×10^{-6} (48.6×10^{-6})
-40° (-40°)	18.6 (2.68)	.431 (1805)	.0629 (58.4)	26.0×10^{-6} (46.8×10^{-6})

Table 3 Thermal Properties
(after Drouin and Michel, 1974)

In the above table, the thermal diffusivity (equal to the conductivity divided by the factor density times the specific heat) was computed assuming an ice density of 57.2 pcf (916 kg/m^3). Ice containing air bubbles may easily have a lower density, say as low as 49 pcf (785 kg/m^3).

Heat transfer through a porous medium, like bubbly ice, will be less because the air cells offer greater resistance to heat transfer than does the solid ice. The value for thermal conductivity becomes less as the amount of entrapped air increases.

Notwithstanding the inherent variations in the thermal properties of ice, we can adequately proceed, within the range of design accuracies, to make engineering estimates of ice thrusts.

Heat Transfer in Ice Sheets

The studies and solutions by Drouin and Michel (1974) do not take into account any thermal boundary layers (air-ice and air-snow) or any solar radiation on the ice. However, they believe their pressures calculated on the basis of heat transfer solely by conduction are of the proper order of magnitude, and perhaps greater than the pressure calculated with the air-ice boundary layer and absorption of solar radiation taken into consideration.

Air-ice or air-snow boundary layers can contribute to increasing the initial temperature of the surface of an ice sheet. Absorption of solar radiation by the ice adds to the heat transferred by conduction. The rates of temperature increase at various levels in the ice would then be higher than those computed on the basis of conduction alone. Increasing the initial surface temperature results in a decrease in the stresses throughout the ice sheet. Increasing the heat transferred by convection (because of solar radiation) results in an increase in stresses in the upper portions of the ice sheet and in shorter times to reach maximum stress levels which thereafter diminish rapidly. However, an increase in the average rate of temperature increase of the surface of an ice sheet implies a decrease in the maximum pressure. Hence their pressures are thought to be conservative, i.e. greater than necessary for design of conditions actually present in the field.

In the foregoing explanation it should be noted that an increase in the rate of temperature increase in the surface of the ice sheet results in a decrease in the maximum pressure. Drouin and Michel (1974) found, contrary to many other published theories and results, that the pressure in an ice sheet was highest for small rates of temperature increase at the surface of the ice. From a physical standpoint, they explain this by the deeper penetration into the ice sheet of the temperature variations at the time when the stress at the surface attains its maximum. It is in the top zone of an ice sheet where the stressed condition related to the maximum pressure exerted by thick ice sheets develops. Attenuation of temperature variation in the interior of an ice sheet is very rapid.

The stress in an ice sheet is determined by the temperature distribution in the ice sheet as a function of the variation of air temperature. Drouin and Michel (1974) use a sinusoidal air temperature variation, rather than a linear variation or step function. The use of sinusoidal variation produces larger stresses and fits well with natural temperature variations (in Canada).

The sinusoidal variation used is one where the temperature increases at an increasing rate until it has achieved one half of its total rise. This occurs when one half of the time to achieve its total rise has passed. The temperature continues to rise but at a decreasing rate until the total temperature rise has occurred for the period.

Other factors that affect the temperature variation in an ice sheet include the presence of snow on the ice sheet, the thickness of the ice sheet, increase of thickness in the ice sheet as a function of time, solar radiation absorbed, and variable thermal properties of ice. (These vary primarily with temperature.)

Analytic methods are available to describe temperature distribution in a mass as a function of the variation in temperature of an ambient medium. To solve the equations developed by these analyses requires simplifying assumptions. For example, Drouin and Michel (1974) had to assume that the thickness of the ice remained constant during the period of ambient temperature increase. This would not be the case for thin ice but would be reasonable for sheets thicker than 16 inches (40 cm).

Climatological Data

In order to compute the pressures from thermal ice expansion we need to know the total temperature rise and the time required to obtain that rise. Also we should know if the temperature rise is linear or varies somehow, such as sinusoidally. Drouin and Michel (1974) present climatological data for Quebec City from 1944 to 1967, inclusive. Quebec City is at 46°48'N latitude and 71°23'W longitude. Climatological data from Quebec City will not be applicable at most other sites, but by presenting them here we will be able to subsequently demonstrate a design methodology. The designer of a small craft harbor at another location would of course investigate climatological conditions there.

Table 4 presents a statistical analysis of data for 23 winters at Quebec City. For that city it was calculated that 90 percent of the air temperature increases had durations of less than 20 hours. About two-thirds of the time the duration was less than 10 hours. The most frequent temperature spread was 17°F (9.5°C) with a duration of 7 hours, which is an average rate of air temperature increase of 2.4 F/hr (1.3 C/hr). Sinusoidal variation in temperature rise was generally present.

Period of Recurrence (years)	Duration of Temperature Increase (hrs) 5	10	20
12.5	4.4 (2.4)	3.1 (1.7)	1.6 (0.9)
25	4.7 (2.6)	3.3 (1.8)	1.7 (0.9)
50	5.2 (2.9)	3.5 (1.9)	1.8 (1.0)
100	5.6 (3.1)	3.7 (2.1)	1.9 (1.1)

Table 4 Rates of Increase, F/hr (C/hr), for Linear Air Temperature Increases as a Function of the Period of Recurrence and Duration of the Increase (Quebec, December 1, 1944--March 31, 1967)

(after Drouin and Michel, 1974)

Rheological Aspects

When an ice specimen, mechanically fixed at its ends, is heated it will try to expand. However, since the ends are fixed a thermal stress will be produced. The strain from this thermal stress offsets that due to the temperature rise. Thus, even though the specimen undergoes no apparent change in length, it is subjected to mechanical strain equal to the thermal deformation that would have resulted if it had been unrestrained. Maintaining a constant length and measuring the forces (stresses) that occur at different temperatures is one way of carrying out laboratory tests on ice.

The laboratory procedure used by Drouin and Michel (1974) used a somewhat different approach, one that avoids errors caused by the metal parts of the testing machine expanding. The tests were performed, at different temperatures, on ice specimens at constant strain rates. The tests determined by means of the experimental curves obtained, the stresses and thrusts induced by the effects of temperature variation.

Ice in nature responds to thermal changes in a ductile manner. The strain rates are quite small and hence ice exhibits ductile behavior. The small strain rates accompanying thermal behavior can be shown with the following calculation.

Assume a temperature rise of 5 F/hr (2.8 C/hr) and a coefficient of thermal expansion of $28 \times 10^{-6}\ F^{-1}$ ($50.4 \times 10^{-6}\ C^{-1}$)

Strain Rate = Coefficient of Expansion x Rate of Temperature Rise

$$= \frac{(28 \times 10^{-6}\ F^{-1})\ (5\ F/hr)}{(3600\ s/hr)}$$

$$= 3.89 \times 10^{-8}\ s^{-1}$$

The thermal strain is also quite small.

Assume a rise in temperature from $-30^\circ F$ ($-34^\circ C$) to $32^\circ F$ ($0^\circ C$) at the above rate. The strain rate is $3.89 \times 10^{-8}\ s^{-1}$ for a coefficient of thermal expansion of $28 \times 10^{-6}\ F^{-1}$ ($50.4 \times 10^{-6}\ C^{-1}$)

$$\text{Strain} = \frac{(62\ F)\ (3600\ s/hr)\ (3.89 \times 10^{-8}\ s^{-1})}{(5\ F/hr)}$$

$$= 0.00174$$

$$= 0.174\%$$

The total strain is less than one quarter of one percent. However, if for this strain the ice were unrestrained, it would expand nearly 10 feet (3 m) in a mile (1.6 km)

To deform laboratory specimens by a factor of one percent at constant rates of 10^{-7}, 10^{-8}, and 10^{-9} per second, test durations of 28, 280, and 2800 hours are necessary. These long periods explain why few tests in the range of strain rates that accompany thermal expansion in the field are cited in the literature. At very low strain rates even small temperature variations in the ambient medium impair the validity of a test. Also, the deformation of a measuring element of a load cell, is not negligible in comparison with the actual deformation of the specimen. The laboratory testing program undertaken by Drouin and Michel (1974) overcame many difficulties of this nature.

Laboratory Test Results, Ice Thrusts

Two types of ice were tested, S1 columnar ice loaded perpendicular to the c-axis (and parallel to the basal plane) and T1 snowpack ice. Samples were generally columns 1 inch (2.54 cm) in diameter and 3 inches (7.62 cm) long. As they were compressed, at strain rates ranging between 10^{-7}, and 10^{-8} per second and at different temperatures ranging between freezing and about -20°F (-29°C), the deformations and loads were measured.

In the snowpack ice the maximum loads occurred after the stress had increased at a constant rate for some time. The stress, after having attained its maximum, usually remained constant or nearly so. The average deformation to reach maximum stress was 0.15 percent.

In the columnar ice the stress increased at a constant rate for a long time. Following that, the rate of increase of stress gradually diminished until it was nil. The maximum stress was reached when the specimen had been deformed between 0.08 and 0.20 percent. The stress, after reaching its maximum value, fell very rapidly and then stabilized at a near constant value.

At the low strain rates the sample behavior is plastic, i.e. the sample doesn't break or shatter, but yields as time proceeds.

The maximum thrusts for columnar and snowpack ice are given in Table 5. They have been computed from the laboratory test data and a sinusoidal varied ambient temperature rise from an initial linear steady temperature state in the ice (i.e. the initial ice temperature profile varies linearly from the ambient temperature at the surface to the melting point at the water-ice interface). With time the ice warms up and thereby exerts thermal stresses. These stresses reach maximums and the stress in the ice at various depths can be summed up to obtain the maximum ice thrust per unit length for a given ice thickness, total temperature rise, and duration of temperature rise.

Ice thrusts for ice thinner than indicated in the table will be less than the tabulated values. Also, the analysis of stresses indicates that the stresses in an ice sheet are particularly high in the first 8 to 12 inches (20 to 30 cm) of the thickness of a thermally stressed ice sheet. The bottom region of a thick ice sheet may consist of a different type of ice without appreciably altering the stressed conditions, the heat transfer being the same.

Ice Thickness and Duration of Temperature Increase (hrs)	Ice Surface Temperature, F (C) and Classification of Ice: S1 = Columnar Ice and T1 = Snowpack Ice					
	14°F (-10°C)		-4°F (-20°C)		-22°F (-30°C)	
	S1	T1	S1	T1	S1	T1
20 in (50 cm)						
t = 5 hrs	5 (73)	4 (58)	11 (160)	8 (117)	19 (277)	13 (190)
t = 10 hrs	6 (88)	5 (73)	14 (204)	11 (160)	23 (336)	16 (233)
t = 20 hrs	9 (131)	7 (102)	18 (263)	13 (190)	27 (394)	18 (263)
30 in (75 cm)						
t = 5 hrs	5 (73)	4 (58)	11 (160)	8 (117)	20 (292)	14 (204)
t = 10 hrs	7 (102)	5 (73)	15 (219)	11 (160)	24 (350)	18 (263)
t = 20 hrs	9 (131)	8 (117)	20 (292)	15 (219)	30 (438)	20 (292)
40 in (100 cm)						
t = 5 hrs	5 (73)	4 (58)	11 (160)	8 (117)	20 (292)	15 (219)
t = 10 hrs	7 (102)	5 (73)	15 (219)	12 (175)	25 (365)	19 (277)
t = 20 hrs	9 (131)	8 (117)	20 (292)	16 (233)	32 (467)	22 (321)

Table 5 Pressures of Thermal Origin by an Ice Sheet Restrained in One Direction
kips/ft (kN/m)

(after Drouin and Michel, 1974)

Use of the table is illustrated by the following example. Given a 30 inch (75 cm) ice sheet with a surface temperature of -4°F (-20°C). The thrust exerted by a rise in temperature to 32°F (0°C) over a period of 10 hours (i.e. at a rate of 3.6 F /hr or 2.0 C /hr) would be 11 to 15 kips per foot (160 to 219 kN/m) depending on whether the ice was granular or columnar (loaded perpendicular to the c-axis).

Except for complete biaxial restraint, most factors encountered in nature tend to decrease calculated ice thrusts. Among the factors which reduce stresses of thermal origin are the insulating qualities of a snow cover on the ice and cracks in the surface of the ice.

Biaxial Restraint

Determining rheological behavior of ice deformed in two directions is a subject of current research. Drouin and Michel (1974) performed some biaxial tests on snowpack ice and concluded that snowpack ice deforms in the manner of an almost perfectly plastic material when the maximum biaxial stress of thermal origin is obtained. This maximum biaxial stress is about twice the maximum stress previously obtained in the uniaxial tests. Also they concluded that maximum stresses for S1 and S2 columnar ice having pressures of thermal origin and biaxially restrained can be adequately estimated by using the uniaxial test results for S1 ice.

They give no ice thrusts for the biaxial condition as such, but suggest using uniaxial thrusts for a dam and reservoir with the following example. Assume an ice sheet is totally restrained in a direction perpendicular to a dam crest. Stresses of great magnitude develop in the first 8 inches (20 cm) of the ice thickness. Under natural conditions, the crystallographic orientation of the grains of columnar ice in the top layer of an ice sheet can be considered random. In such case, the stresses developed are considerably lower than if all the columns have, for instance, a vertical crystallographic orientation. Experimentally the biaxial restraint of S1 or S2 columnar ice having some grains randomly oriented gives substantially the same maximum stress values as are obtained from S1 ice specimens restrained in only a single direction (deformed perpendicular to the c-axis). Also a reservoir with rocky vertical walls around the entire perimeter would be unusual. It would correspond to an ice sheet in a swimming pool (or a small-craft harbor completely surrounded by vertical sheet piling walls).

Within the present state of knowledge, Drouin and Michel (1974) recommend using thrusts in reservoirs on laboratory testing of ice under uniaxial loading.

Effects of Cracks and Snow on Ice Thrusts

Snow covered ice or cracked ice or both significantly reduce the thrust of an ice sheet.

The effect of cracks (either from thermal contractions or water level changes) is to mitigate the effect of a temperature rise or spread.

Metge (1976) has reported on a five year study and field observation on thermal cracks at Kingston, Ontario. The frequency of cracking tends to decrease as an ice cover becomes thicker. He categorizes thermal cracks into three groups. The first are dry cracks which absorb a significant amount of thermal ice movement. They are common and consist of cracks that extend from the top of the sheet down one-half to two-thirds of the ice thickness. These dry cracks open and close according to the ice temperature. The second and third groups are respectively, wet cracks which are relatively rare, and wide wet cracks which are just too wide to easily refreeze into sound ice. Wet cracks are narrow enough to refreeze rapidly and in so doing add material to the ice sheet. Since wet cracks are not prevalent, we are mainly concerned with the beneficial effects of the dry thermal cracks. The following example illustrates their effect on thermal thrusts.

Assume a coefficient of thermal expansion of 28×10^{-6} F^{-1} (50.4×10^{-6} C^{-1}) and a total summation of crack widths of 0.02 inches per yard (0.56 mm/m)

$$\text{Strain (\%)} = (\text{Coef. of Expansion})\ (\text{Temp. Rise})\ (100)$$

$$\frac{(0.02\ \text{in/yd})\ (100)}{(36\ \text{in/yd})} = (28 \times 10^{-6}\ F^{-1})\ (100)\ (\text{Temp. Rise}\ F)$$

$$\text{Temp. Rise} = 20\ F\ \ (11\ C) = \text{Equivalent Temperature Spread}$$

In this example, the first 20 F (11 C) of a temperature rise would be taken up by the cracks. The remainder of the rise would produce thermal thrusts.

Table 6 gives temperature spreads for cracked ice.

Summation of Crack Widths per Unit of Length of an Ice Sheet		Total Deformation or Strain of the Ice	Equivalent Temperature Spread	
in/yd	mm/m	%	F	C
0.009	0.25	0.025	9	5
0.018	0.5	0.05	18	10
0.036	1.0	0.10	36	20
0.054	1.5	0.15	54	30
0.072	2.0	0.20	72	40

Table 6 Summation of Crack Widths Per Unit of Length of an Ice Sheet Expressed as Percent Thermal Deformation of the Ice and Equivalent Temperature Spreads (Coef. of Expansion = 28×10^{-6} F^{-1} or 50.4×10^{-6} C^{-1})

(after Drouin and Michel, 1974)

Snow accumulated on an ice sheet has a drastic effect on the temperature variations in the ice, and as a result the pressures of thermal origin are much less. A uniform fresh snow cover can drown an ice sheet causing no thrust until the ice reforms on the top. This can occur when a snow fall roughly equal to the ice thickness occurs.

Snow, a complex material which undergoes a continuous process of metamorphosis, can be converted to an equivalent thermal thickness of ice by proportioning through the coefficients of thermal conductivity for snow and for ice. However, the thermal conductivity of snow varies with density (as well as other things). Mellor (1964) gives snow conductivities as a function of density. Drouin and Michel (1974) report the average density of snow 20 to 40 inches (50 to 100 cm) thick on the ground to be 25 to 28 pcf (400 to 450 kg/m^3). Snow at this density will barely show a man's foot print. For snow layers of small thicknesses the density is much less--lower than 12 pcf (200 kg/m^3).

Table 7 gives equivalent ice thicknesses for snow densities.

Snow Density pcf (kg/m^3)	Ratio of Thermal Equivalent Thicknesses of Ice to the Actual Thicknesses of Snow
6.2 (100)	24.0
9.4 (150)	17.1
12.5 (200)	13.2
15.6 (250)	10.6
18.7 (300)	8.7
21.8 (350)	7.2
25.0 (400)	6.1
28.1 (450)	5.0

Table 7 Ratios of Thermally Equivalent Thickness of Ice to Actual Thicknesses of Snow (Drouin and Michel, 1974)

To illustrate the use of the above table, assume we have 6 inches (15 cm) of snow weighing 21.8 pcf (350 kg/m^3) on top of a two foot (0.6 m) thick ice sheet. The ratio of equivalent thickness of ice to actual thickness of snow is 7.2; therefore, thermally the snow simulates an additional layer of 43.2 (110 cm) thick. The fictitious sheet is now 67 inches (170 cm) thick. The temperature variations to be taken into account in calculating the stresses of thermal origin are those which prevail 43 inches (110 cm) below the surface of the fictitious sheet. At this depth into an ice sheet the attentuation of temperature is complete and thus the pressure of thermal origin is very small or nil.

Ice Thrust on Individual Piers

Petrunichev (1954), cited by Korzhavin (1971), considers static horizontal pressure of ice on individual piers. The question is raised as to whether one should determine the pressure from an ice field by the width equalling half the sum of the adjacent spans or be limited to considering only the pier width.

The deformation of the part of the ice field lying between the piers would not occur separately. It exerts a partial effect on the amount of pressure on a pier. It would be increased. Petrunichev (1954) suggests that the total force on a pier should be equal to the thrust (force per unit length) times the sum of the width of the pier and one-third the half sum of the spans contiguous with the pier.

The suggestion for somehow increasing forces for 'bridging' action would be applicable for small craft harbors. For example, consider the case of a marina pier constructed with horizontal elements spanned across intermittent cribs. The total lateral force would exceed that computed from ice contact with the crib alone. In part, the ice would 'bridge' over and produce a larger horizontal force.

Closure

In this section we have reviewed the thermal ice pressures proposed by Drouin and Michel (1974) and presented some estimates of thrusts. Flexible marina structures and conditions existing in nature should result in forces smaller than those given.

The forces were computed for assumed temperature rises equal to the difference in the cold ice sheet surface temperatures and the melting point. Temperature rise is a function of the site's climatology. The highest pressure occurs for the smaller rates of temperature rise. Also the attenuation of temperature rise occurs very rapidly and it is the top zone of the ice sheet that experiences the thermal stresses. (Query, can one mitigate these stresses in this area somehow. If the ice were weakened or broken, or if made thin or warmer, or if separated from the structure with a pliable material, the force on the structure would be reduced.)

Strain rates in the regime of thermal expansion of ice sheets are very low (less than 10^{-7} s^{-1}) and the ice behaves in a ductile manner. Also forces produced from uniaxial testing can adequately represent some level of biaxial conditions that might be present at a particular site.

Each site where thermal pressures will exist should be separately studied. The climatology, characteristics of the ice sheet, bank configuration, and type of structure are all important.

PART III

ICE SUPPRESSION AND WEAKENING

Ashton (1974) has prepared a monograph on air bubbler systems to suppress ice. Analytical methods were used to develop a procedure for predicting the effectiveness of using bubbler systems to suppress ice formation under various field conditions. We believe air bubbling is an effective method to suppress ice. The material contained in this section is based largely on the procedures suggested by Ashton but simplified somewhat to fit our concern, namely, melting ice in a small-craft harbor (rather than in a lake shipping channel or a deep-water port). The resulting suppression predictions agree well with our field observations and experiences of air bubbler manufacturers.

Those wishing to examine air bubbler systems in more detail than what is presented here can refer to the original Ashton (1974) monograph, Ashton (1975) and Ashton (1977). The 1975 work is a laboratory experimental report on heat transfer coefficients associated with flow induced by a line source bubbler system. The 1977 work is a refinement of the monograph and a numerical simulation for a system. US CRREL has developed a computer program in BASIC language that simulates analyses of diffuser lines, plumes and heat transfer, ice melting, and thermal reserves. The simulation agrees satisfactorily with one example case for a winter in Duluth, Minnesota.

We will begin with a description of how a compressed air bubbler deicing system works. This will be followed with a review of the suggested design method and practical considerations to achieve a successful installation. Also included are a few notes on other ways to suppress or weaken ice.

Principles of Bubbler System Operation

Figure 3 is a cross sectional view along the axis of an air diffuser pipe on a lake bottom. It is also representative of the action of a point source diffuser.

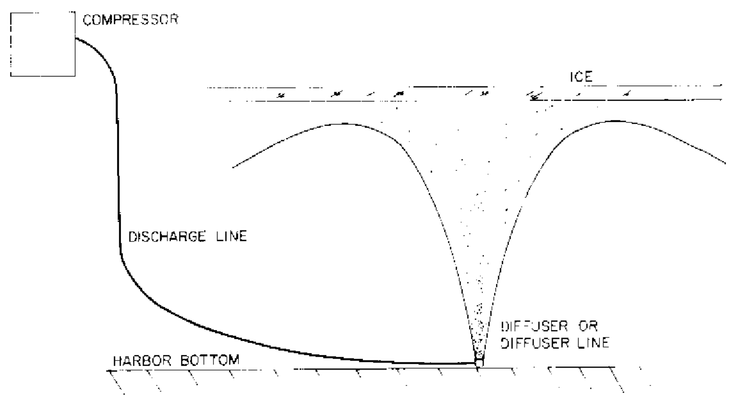

Figure 3
Bubbler System Schematic
(after Ashton, 1974)

Air is compressed by a compressor either located indoors or suitably protected and mounted on a marina pier. In an overall ice suppression system, several compressors may be used and interconnected through a discharge manifold piping system. Air travels through the discharge line to the diffuser or diffuser line on the lake bottom. The air has been warmed during compression but normally is cooled down to the ambient lake temperature by the time it reaches the diffuser. Here the air is discharged through an orifice. It must have enough pressure at the diffuser to overcome the depth of water hydrostatic pressure.

The discharging air is cooled as it expands going through the orifice. If it expands too quickly an icing condition can occur at the orifice. With proper air pressures this does not happen. The momentum of the air jet sets the bubbles in motion. The momentum quickly dissipates and bubble buoyancy takes over. As the bubbles rise they entrain water into the rising plume from the lateral direction. Many bubble sizes are present, but the tinier ones are more efficient as they move more water for the same volume of air. In fact, this principle explains why large air bubbles, like those produced by belcher devices, aren't as effective as a continuous stream of tiny bubbles. The ice melting that occurs on the underside of the sheet is the result of both temperature and volume of water being moved upwards from the warmer bottom water by the bubble plume.

At some point near the surface the buoyant plume spreads as it either impinges on the ice cover or encounters open water. If a free water surface is present, the bubbles will escape directly to the atmosphere. This results in a certain amount of heat being wasted. If a cover exists the bubbles will move laterally along the underside of the ice. As they do, melting primarily by convection occurs. This heat loss results in cooling of the flow whose velocity has also decreased as it spread out. Rapid decay in the heat transfer rate is the result.

Finally, the plume imposes a net circulation on the water and this allows more warm water to be drawn into the area from distant lateral directions. A bubbler system would not work in a swimming pool for example, because the amount of warm water is limited by the pool's volume and eventually it would all be cooled down. The pool would freeze to the bottom. Design of bubblers in sea water is complicated because salinity may vary with depth and theoretically the maximum density occurs at the freezing point. (There may be no "warm" dense bottom water.) Also rivers with currents tend to be isothermal and even super-cooled, and hence do not have enough heat. What heat there may be is already impinging on the river cover.

In a fresh water or brackish water marina, however, a bubbler system can work if lateral warm water recharge is present and it is designed and maintained properly.

Ice Suppression Bubbler Systems Design

The following design procedure is a trial-and-error procedure based upon Ashton (1974). For a given site and conditions the quantity of air required Q_a is estimated for a tolerated ice equilibrium thickness n_e. This thickness is a steady state thickness of an ice cover that will persist with a continual flow of heat to the cover's lower surface and a cooler atmosphere forming ice. At the equilibrium thickness, the ice is melting as fast as it forms.

The design of an ice suppression system requires the selection of this thickness. To reduce it to zero or near zero requires disproportionate amounts of heat, i.e. the air flows required become large. The ice itself is an insulator and a free water surface is heat wasteful. (However, the designer may wish to choose to obtain free water in order to better monitor the bubbler's operation throughout the winter.) In selecting a thickness to be tolerated, factors like the following must be considered: first cost and operating costs, resistance available to ice uplift through embedment of piles being protected, magnitude of lateral forces from thicknesses of ice, availability of manpower to chop ice during severe cold periods, temperature extremes existing at the site, and the amount of damage to be tolerated.

The quantity of air required Q_a is estimated from experience and local conditions. Table 8 gives heat transfer coefficients as a function of water depth H and air flow rates Q_a. (Q_a is in terms of volume per time per length of diffuser which reduces to area per time.) The depth of water used is from the underside of the ice (nominally the top surface for suppressed ice) to the diffuser level (on the bottom generally).

Water Depth	Air Flow Rate, Q_a, ft^2 min^{-1} (m^2 s^{-1})						
H, ft (m)	.01 (1.55)*	.02 (3.10)	.03 (4.64)	.04 (6.19)	.05 (7.74)	.06 (9.29)	.07 (10.8)
6 (1.8)	151 (857)	169 (960)	180 (1022)	189 (1073)	195 (1107)	201 (1141)	206 (1170)
8 (2.4)	142 (806)	158 (897)	169 (960)	177 (1005)	184 (1045)	189 (1073)	194 (1102)
10 (3.0)	134 (761)	150 (852)	159 (903)	167 (948)	173 (982)	178 (1011)	183 (1039)
12 (3.7)	127 (721)	142 (806)	151 (857)	158 (897)	164 (931)	169 (960)	174 (988)
14 (4.3)	121 (687)	135 (767)	144 (818)	151 (857)	157 (891)	162 (920)	166 (943)
16 (4.9)	116 (659)	130 (738)	138 (784)	145 (823)	150 (852)	155 (880)	159 (903)

*Multiply values by 10^{-5}

Table 8 HEAT TRANSFER COEFFICIENTS, h_b
Btu hr^{-1} ft^{-2} F^{-1} (W m^{-2} C^{-1})
(after Ashton, 1974)

The heat transfer rate q_w is obtained from

$$q_w = h_b (T_w - T_m)$$

where T_w is the water temperature

and T_m is the melting point temperature of ice

The water temperature can be measured at a given site or conservative estimates made for its value. We have made water temperature measurements in many western Great Lakes small craft harbors and found them to be isothermal and only slightly above the freezing temperature (½ F, 0.3 C, or cooler). Others have also reported very cold water. It is reasonable to assume that the water temperature is constant throughout its depth because being shallow and agitated by rising bubbles are conditions tending to destroy any thermal stratification.

Figure 4 gives the equilibrium thicknesses n_e for the calculated heat transfer rate q_w as functions of the ambient air temperatures T_a. It has been assumed that no snow cover is present and that the average wind speed is 10 mph (4.5 m s^{-1}). If snow is present the equilibrium thicknesses become smaller for a given q_w. Conversely, if windier conditions prevail the equilibrium thicknesses increase.

The average daily temperature during the period of the winter under consideration can, for most sites, be used for T_a. Ice once formed grows rather slowly during cold spells. During the day, warmer temperatures counteract cooler evening temperatures.

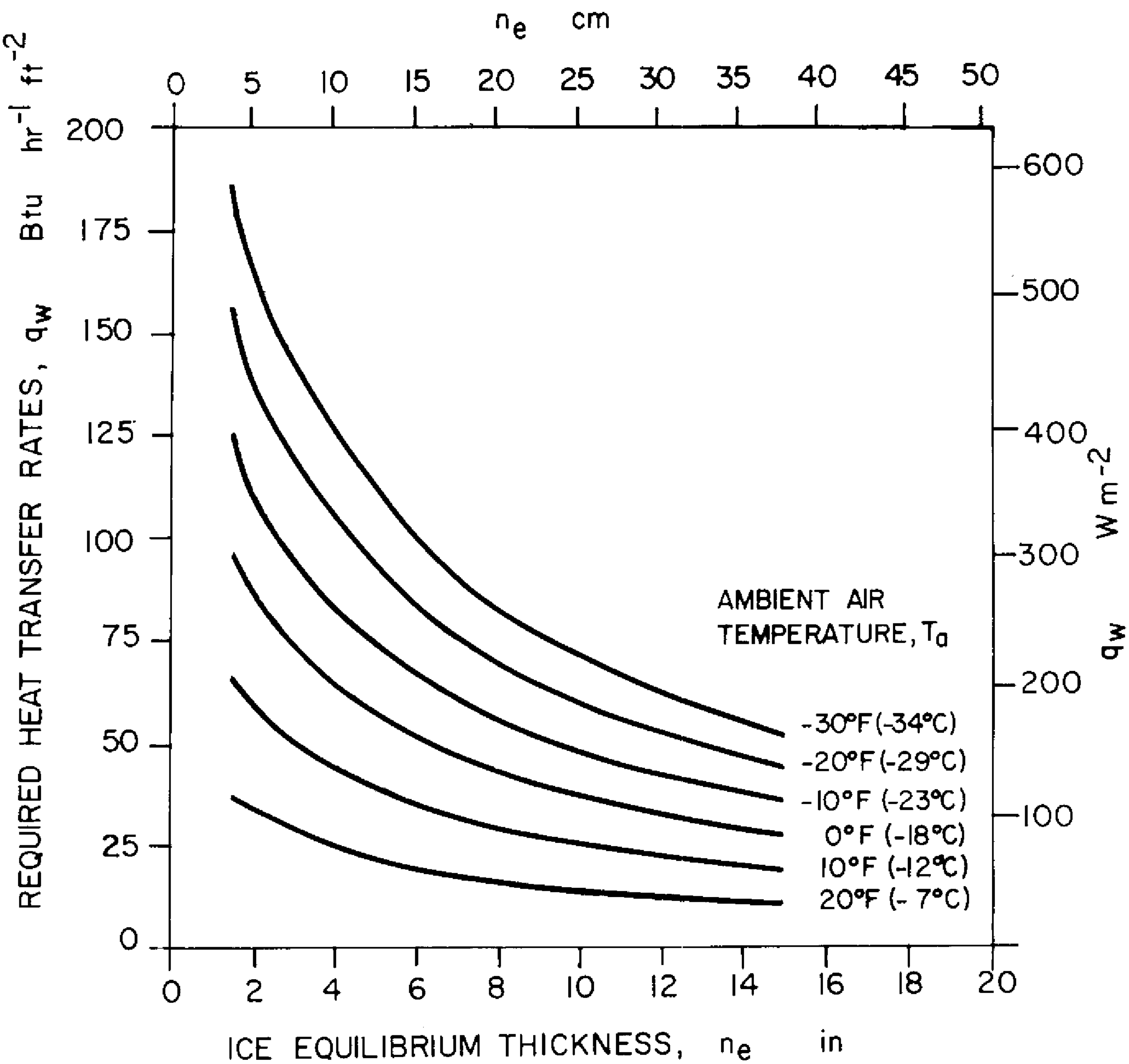

Figure 4. REQUIRED HEAT TRANSFER RATES TO ATTAIN EQUILIBRIUM THICKNESS AS FUNCTION OF AMBIENT AIR TEMPERATURE (after Ashton, 1974)

The use of Table 8 and Figure 4 is illustrated by the following example.

Assume Q_a = 0.06 ft^2 min^{-1} (9.29 x 10^{-5} m^2 s^{-1}) of air

H = 10 ft (3 m) water depth

T_w = 32.5°F (0.3°C) water temperature

T_a = -10°F (-23°C) ambient air temperature

From Table 8 at Q_a = 0.06 and H = 10 find the heat transfer coefficient h_b

h_b = 178 Btu hr^{-1} ft^{-2} F^{-1}
(1011 W m^{-2} C^{-1})

Find q_w, the heat transfer rate from

$$q_w = h_b\ (T_w - T_m)$$

$$= 178 \text{ Btu hr}^{-1} \text{ ft}^{-2}\ \text{F}^{-1}\ (32.5^\circ\text{F} - 32.0^\circ\text{F})$$

$$= 89 \text{ Btu hr}^{-1} \text{ ft}^{-2}\ (281 \text{ W m}^{-2})$$

From Figure 4 at T_a = -10°F and q_w = 89 find

n_e = 4 inches (10 cm)

Should the ambient temperature (approximately the average daily temperature) fall below -10°F (-23°C) say to -30°F (-34°C) for a few days, then the ice equilibrium thickness would increase to 7 inches (18 cm). Whether 7 inches (18 cm) of ice would be tolerable would depend on the site and what is being protected.

Now assume instead of Q_a = 0.06 ft^2 min^{-1} (9.29 x 10^{-5} m^2 s^{-1}) we use one-half as much air. From Table 8 at Q_a = 0.03 ft^2 min^{-1} (4.64 x 10^{-5} m^2 s^{-1}) h_b = 159 Btu hr^{-1} ft^{-2} F^{-1} (903 W m^{-2} C^{-1}). For ½ F (0.3 C), q_w = 80 Btu hr^{-1} ft^{-2} (252 W m^{-2}). From Figure 4 , at an air temperature of -10°F (-23°C) we find n_e = 5 inches (13 cm) and at -30°F (-34°C), n_e = 9 inches (23 cm). Changing the quantity of air does not have a linear effect on the ice thickness. If however, the water temperature is cooler (or warmer) than estimated, large changes in n_e occur.

Assume the water temperature is 32¼°F (0.1°C) instead of 32½°F (0.3°C). For Q_a = 0.06 ft^2 min^{-1} (9.29 x 10^{-5} m^2 s^{-1}) h_b = 178 Btu hr^{-1} ft^{-2} F^{-1} (1011 W m^2 C^{-1}). The required heat transfer rate is (¼) (178) = 45 Btu hr^{-1} ft^{-2} (142 W m^{-2}). Corresponding equilibrium thicknesses at -10°F (-23°C) and -30°F (-34°C) would be 11 inches (28 cm) and more than 15 inches (38 cm). These ice conditions would probably be intolerable.

The above analysis is for a line source diffuser system. A solution to a point source diffuser is not presently available. It is estimated that a series of point sources (such as would exist in a bubbler system layout protecting individual piles spaced throughout a marina) would approximate the line source condition. This of course would be a function of how far apart the point sources become. But in general, adding up the air discharges at each point source and dividing by the distribution diffuser length would be a reasonable approximation for Q_a in a typical small craft harbor.

We can estimate the time required to melt out an ice cover if one is already in place. For thicker ice there is little tendency for it to thicken. The simplest analysis occurs when we assume there is no conduction through the cover and that it is melted by the warm water impinging on its lower surface.

For these conditions the melting rate can be calculated as follows:

$$\text{Melting Rate} = \frac{q_w}{\rho_i \lambda}$$

where q_w is the heat transfer rate in Btu hr^{-1} ft^{-2} (W m^{-2})

ρ_i is the mass density of ice and equal to 57.2 lbs ft^{-3} (916 kg m^{-3})

λ is the heat fusion of ice and equal to 143.7 Btu lb^{-1} (3.34 x 10^5 J kg^{-1})

Find q_w to melt one inch (2.5 cm) in a day.

$$1 \text{ in da}^{-1} = \frac{(q_w \text{ Btu hr}^{-1} \text{ ft}^{-2})(12 \text{ in ft}^{-1})(24 \text{ hr da}^{-1})}{(57.2 \text{ lbs ft}^{-3})(143.7 \text{ Btu lb}^{-1})}$$

$$q_w = 28.5 \text{ Btu hr}^{-1} \text{ ft}^{-2} \ (89.8 \text{ W m}^{-2})$$

Therefore with 100 Btu hr^{-1} ft^{-2} (315 W m^{-2}) we can melt 3½ inches (9 cm) of ice per day.

We can also make an estimate of the exhaustion of the thermal reserve in a closed body of water. The following computations show the absolute necessity of having a warm water recharge at any small craft harbor using air diffusers to suppress the ice.

We can compute the volume of warm water necessary to melt a given volume of ice as follows. Assume we have ice at the freezing point and warm water at 33°F (0.55°C). On a cubic foot (0.0283 m^3) basis then:

$$\text{Ratio of Volumes} = \frac{\text{Heat Required to Melt Ice}}{\text{Heat Content of Water}}$$

$$= \frac{\rho_i \lambda}{\rho_w c_p (T_w - T_m)}$$

where ρ_i and λ are defined above

ρ_w is the mass density of water and equal to 62.4 lbs ft^{-3} (1000 kg m^{-3})

c_p is the specific heat capacity of water and equal to 1.0 Btu lb^{-1} F^{-1} (4187 J kg^{-1} C^{-1})

$(T_w - T_m)$ is the difference in the temperature of the water and the melting point of ice

$$= \frac{(57.2 \text{ lbs ft}^{-3})\ (143.7 \text{ Btu lb}^{-1})}{(62.4 \text{ lbs ft}^{-3})\ (1.0 \text{ Btu lb}^{-1}\ F^{-1})\ (1^\circ F)}$$

$$= 132$$

It takes 132 cubic feet (3.74 m^3) of water at 33°F (0.55°C) to melt 1 cubic foot (0.0283 m^3) of ice at the freezing point.

Assume we have an air diffuser system installed in a closed body of water that is at a temperature of 33°F (0.55°C). Also assume a depth of water of 10 feet (3 m), diffuser lines spaced 30 feet (9 m) on centers, and that we want to use the diffuser lines to melt ice in a band 3 feet (1 m) wide.

$$\text{Suppressed Thickness} = \frac{(30 \text{ ft})\ (10 \text{ ft})\ (1 \text{ ft})}{(132)\ (3 \text{ ft})\ (1 \text{ ft})}$$

$$= 0.76 \text{ ft or } 9 \text{ in } (23 \text{ cm})$$

Under these closed conditions the thermal reserve would be exhausted in a few days. If an ice cover grows rapidly a large amount of heat can be sealed in. However it would still not be enough to operate a suppression system throughout a winter. There must be a source of warm water. Also operating these systems in shallow water, say six feet (2 m) or less, is not recommended because of lack of enough warm water to melt an ice cover.

The detailed design and layout of an ice suppression bubbler system is outside the scope of this ice engineering guide. A few comments on the design of systems that are presently installed and are working are in order however.

Having determined the required amount of air (a function of water depth, water temperature, water body thermal reserves, atmospheric conditions, and thickness of ice to be tolerated) an air manifold distribution system together with branch diffuser lines would be designed in accordance with standard air flow design procedures. It is important that manifold and diffuser lines are large enough to reduce pressure losses and to make sure the air supply is not exhausted before the last diffuser is reached. Balancing air flows and pressures is quite important for a successful compression system.

The total air pressure required will be a function of the line and orfice losses and the hydrostatic head to be overcome at the diffuser. Where the harbor bottom is sloping, it is best to distribute the air in an uphill direction from the deep end; otherwise if the air is first emitted in shallow water, the deeper diffusers may be starved.

Spacing of orifices would normally be between 1/2 and 1/3 the water depth along a line diffuser. Pre-slit vinyl weighted bubbler hoses are available from component manufacturers. Fixed opening orifices are usually a nominal 3/64 inch (1 mm) size. The orifice diameter (or head differential) can be determined from a standard discharge equation given below.

$$Q_o = sQ_a = C_d \frac{\pi d^2}{4} \sqrt{2 \Delta p/p_a}$$

where Q_o = discharge from a single orifice

s = spacing of orifice

Q_a = air discharge per unit length

C_d = orifice loss coefficient (from fluid mechanics texts, etc.)

d = orifice diameter

Δp = pressure difference across orifice

p_a = mass density of air

Usually an air bubbler system is a low pressure, high volume design. This suggests the use of rotary blowers instead of piston compressors used previously. A disadvantage of a piston compressor is the presence of finely atomized oil in the air diffuser lines which has passed by the pistons. This causes clogging of the lines and orifices. Noise is a disadvantage of the rotary compressor system.

Standby power or other alternative procedures should be available at a marina in the event that power is lost during a storm or for other reasons.

A well designed air bubbler ice suppression system will rapidly prove inadequate if little attention is paid to its pre-season cleaning and balancing. It is essential to begin an icing season with clean, balanced and operable system components.

Other Ways to Suppress or Weaken Ice

At this time it appears that compressed air ice suppression systems are best for the small craft harbor. A few other methods are given below and they may have some applicability.

Velocity or propeller systems drive a propeller that churns warm water to the surface. A lot of water is moved with only a few percent of the volume contributing to melting ice where suppression is desired. Methods to analyze melting ice with these systems have not been published. The velocity system's chief applicability appears to be in keeping limited areas open, e.g. around a boat left in the ice.

US CRREL personnel have observed a field experiment where 57°F (14°C) warm water was emitted from a diffuser 19 feet (6 m) below the ice surface. The warm water rose until it cooled to 39°F (4°C); at which time it descended as it was denser than the surrounding water. It did not reach the surface to melt the cover. Warm water diffusers might work with shallow water, higher pressures for momentum and insulated manifold and diffuser lines.

Insulating ice is very effective in reducing its thickness and retarding its growth. A few trials have been made in river ice. They effectively reduced a normal cover of 12 inches (30 cm) to zero. Problems involved are installing, maintaining and removing the insulation material.

Snow is a good insulator and snow thrown by a snow-blower is reworked and tends to set up. In this form, a snow cover might provide satisfactory insulation during severe periods of a winter.

Dark substances will absorb heat and cause melting. However coal dusting and other techniques are one-time temporary measures. Such procedures as well as chemical applications for melting are not acceptable today.

Electrical resistance heating is more expensive than other suppression methods but is feasible and perhaps applicable. Frazil ice, which is quite sticky can be electrically melted from intake structures. Lake ice could also be melted with electrical heat. As the area of contact between the structure to be protected and the ice diminishes, the more favorable resistance heating becomes. (Query, could pilings whose diameters were made small at the ice line, an area not usually requiring a large pile section modulus, be incorporated in a dock system design with electric heating to suppress the ice.)

Finally, ice can be weakened and cracked by impacting, chopping, wave generation schemes and other mechanical means. These do not appear feasible for a small craft harbor however.

PART IV

BEARING CAPACITY OF ICE

This section of the guide contains information on the capacity of ice to support loads, especially construction loads resulting from using ice as a work platform for building small craft harbors. Winter construction has proven very successful in the Great Lakes. The ice easily handles men and small equipment with few accidents. Heavier loads have been moved and supported by ice also, including driving piling with light pile drivers and excavation dredging with regular construction cranes and draglines. Predicting safe allowable loads necessarily must be conservative. Ice strength must be assumed to be weaker than strengths which will be assigned in the next section for estimating the forces and uplifts due to ice.

This section begins with a discussion of ice parameters and bearing capacity theory. A figure is presented for maximum safe loads. The section concludes with some comments on thickening and strengthening ice.

Flexural Rigidity Length

Nevel (1976) introduced a term, flexural rigidity length, for a characteristic feature of all ice plate problems. The term in fact has been called characteristic length, or sometimes action radius. To derive the term, let us first consider the differential equation for a beam.

$$EI \frac{d^4w}{dx^4} = q$$

where w = deflection

x = distance

q = load

E = modulus of elasticity

I = moment of inertia

The term EI is a measure of how stiff the beam is. If now the beam rests on water the deflection increases the water pressure which reduces the load.

$$EI \frac{d^4w}{dx^4} = q - kw$$

where k = weight density of water

$$EI \frac{d^4w}{dx^4} + kw = q$$

$$\frac{EI}{k} \frac{d^4w}{dx^4} + w = \frac{q}{k}$$

The term $\frac{EI}{k}$ on a unit width basis has the dimensions of the length to the fourth power, so the fourth root of $\frac{EI}{k}$ is a length, i.e.

$$\frac{EI}{k} = \frac{FL^{-2}\ L^3}{F\ L^{-3}} = L^4$$

where F and L represent force and length units

$$\frac{EI}{k} = \frac{(E)\ (1/12)\ (h^3)}{k} = \frac{Eh^3}{12k}$$

where h is the plate thickness

Now for a plate instead of a beam the term $(1 - v^2)$ is introduced through strain compatibility relationships, where v is Poisson's Ratio. The flexural rigidity length l for a plate is defined as follows:

$$l = \left[\frac{Eh^3}{12\ (1 - v^2)} \right]^{1/4}$$

The value of l is rather insensitive to E and v, and sensitive to h, the plate thickness. This means that we need not know E and v precisely, but do need to know the ice thickness to make good engineering estimates. The flexural rigidity lengths for various E's and h's for a Poisson's Ratio of 1/3 are listed in Table 9. The values will be used in solving bearing capacity and other plate problems.

Ice Thickness, h inches (cm)	Modulus of Elasticity, E kips/in^2 (MN/m^2)				
	500 (3445)	750 (5168)	1000 (6890)	1250 (8613)	1500 (10335)
6 (15)	10.8 (3.3)	11.9 (3.6)	12.8 (3.9)	13.6 (4.1)	14.2 (4.3)
12 (30)	18.1 (5.5)	20.1 (6.1)	21.6 (6.6)	22.8 (6.9)	23.9 (7.3)
18 (46)	24.6 (7.5)	27.2 (8.3)	29.2 (8.9)	30.9 (9.4)	32.3 (9.8)
24 (61)	30.5 (9.3)	33.8 (10.3)	36.3 (11.1)	38.3 (11.7)	40.1 (12.2)
30 (76)	36.1 (11.0)	39.9 (12.2)	42.9 (13.1)	45.3 (13.8)	47.5 (14.5)
36 (91)	41.3 (12.6)	45.7 (13.9)	49.2 (15.0)	52.0 (15.8)	54.4 (16.6)
42 (107)	46.4 (14.1)	52.4 (16.0)	55.2 (16.8)	58.4 (17.8)	61.1 (18.6)

TABLE 9 FLEXURAL RIGIDITY (CHARACTERISTIC) LENGTHS, l, Feet (m)
(Poisson's Ratio = 1/3)

Bearing Capacity of Ice Sheets

Kerr (1976) has presented a critical survey of the literature on bearing capacity of floating ice plates subjected to static or quasi-static loads. The work contains discussions of general questions, analytical solutions, and field and laboratory results.

When using an ice sheet for supporting loads one question of obvious interest is what is the maximum load or the "breakthrough" load. Also the time at which this breakthrough occurs is of interest. Simple proven methods have not been developed to tell us the breakthrough load and when it will occur. It is a viscoelastic problem.

Field and laboratory observations indicate that indeed an ice sheet continues to carry a load after the first crack occurs. To solve analytically any plate problem requires limiting the problem, choosing an idealized model and formulating the proper equations from mechanics (statics and dynamics). Having formulated the problem, exact or approximate solutions are sought that agree with field and laboratory observations. However, if we limit the plate problem, for ease of analysis and conservatism, to an uncracked plate we then have a different problem after the first crack appears. More complex and different problem formulations are then necessary. We will however use the "first crack" analysis to determine our safe loads for construction on the ice.

If we place a uniform load on a circular area and increase the load, the ice will crack. There will be one or more radial cracks that extend out from the load. If we continue to add load a circumferential crack will form somewhat concentrically around the load but at a distance out from the load at the ends of the radial cracks. This forms wedges which usually number from four to eight or more. When this circumferential crack forms breakthrough can be expected shortly thereafter. On thick ice, even more load may be added due to side interaction between the wedges. However this possible support cannot be counted upon. In fact, it is reasonable to assume that when radial cracking begins the ice is sustaining a load that is no longer a safe load.

Nevel (1976) has formulated a mathematical creep model for ice which includes primary, secondary and tertiary creep. His equations show that at the load, the deflection increases with time while the stresses decrease, or relax. This means that the maximum tensile stress (which is usually the critical stress in an ice plate bearing problem) occurs at the moment the plate is loaded. A usual failure criterion is to limit the maximum tensile stress. Using this criterion means the sheet should fail at once or not at all as the stresses relax with increasing deflection. This however is contrary to observations on ice sheets under sustained loads. A possible explanation is that the tensile strength is somehow affected by the creep process.

Frankenstein (1966) describes the results of a number of large scale breakthrough field tests, for both concentrated loads and distributed loads placed on an Arctic lake cover. In the distributed load tests a

15-ft (4.6-m) diameter aluminum tank, with a height adjustable to 20 ft (6.1 m), was placed directly on the ice surface. Lake water was pumped into the tank at a more or less constant rate to load the ice. The concentrated load tests were conducted in the same manner as the distributed tests except that the tank was placed on a platform balanced on a 24 inch (61 cm) diameter wooden block.

For distributed load tests the first circumferential crack did not produce breakthrough. Additional cracking of the wedges occurred parallel to the first circumferential but closer to the load. Some side wedge interaction probably occurred before final breakthrough on the closer circumferential cracks, numbering one or two in addition to the first crack connecting the far ends of the radials.

The crack phenomenon for the concentrated tests differed greatly from that of the distributed tests. The distributed tests always yielded a circumferential crack but in some of the concentrated tests, no circumferential cracks were visible either during loading or after failure. The failure hole diameter for the concentrated tests was very close to the diameter of the bearing block.

Tables 10 and 11 present some of Frankenstein's load test data. Temperatures taken in the ice ranged between 30°F and 15°F (-1°C and -9 °C). The load contact pressures were computed on a nominal 15-ft (4.6-m) tank diameter and a 2-ft (0.6-m) block diameter. The punching shears for the concentrated load tests data were computed using the nominal block diameter, the ice thickness, and the collapse load.

Ice Thickness inches (cm)	Maximum Loads pounds (kN)	Time of Failure minutes	Maximum Deflection feet (m)	Ratio of Max. Load and Square of Ice Thickness lbs/in^2 (N/cm^2)	Load Contact Pressure lbs/sf (kN/m^2)	Pressure per Ice Thickness lbs/sf/in (kN/m^2/cm)
6.3 (16.0)	23955 (107)	11.7	1.06 (0.32)	604 (416)	136 (6.5)	22 (0.41)
8.2 (20.8)	28550 (127)	12.2	0.77 (0.23)	424 (292)	162 (7.8)	20 (0.38)
9.5 (24.1)	36980 (164)	14.6	0.87 (0.27)	410 (283)	209 (10.0)	22 (0.41)
12.8 (32.5)	44330 (197)	30.5	1.56 (0.48)	271 (187)	251 (12.0)	20 (0.38)
12.4 (31.5)	48303 (215)	21.6	0.98 (0.30)	314 (216)	273 (13.1)	22 (0.41)
15.6 (39.6)	77945 (347)	30.9	1.26 (0.38)	320 (221)	441 (21.1)	28 (0.53)

Table 10
Distributed Load Test Data
(Frankenstein 1966)

Ice Thickness inches (cm)	Maximum Load pounds (kN)	Time of Failure minutes	Maximum Deflection feet (m)	Ratio of Max. Load and Square of Ice Thickness lbs/in^2 (N/cm^2)	Load Contact Pressure lbs/sf (kN/m^2)	Punching shear lbs/in^2 (kN/m^2)
11.4 (29.0)	30,968 (138)	9.1	0.76 (0.23)	238 (164)	9,860 (472)	36 (248)
11.0 (27.9)	28,970 (129)	8.6	0.45 (0.14)	240 (165)	9,220 (441)	35 (241)
13.5 (34.3)	34,810 (155)	9.7	0.65 (0.20)	191 (132)	11,080 (531)	34 (234)
15.4 (39.1)	51,339 (228)	16.8	0.71 (0.22)	216 (149)	16,340 (782)	44 (303)
17.8 (45.2)	61,603 (274)	21.2	0.76 (0.23)	194 (134)	19,610 (939)	46 (317)

Table 11
Concentrated Load Test Data
(Frankenstein 1966)

Frankenstein's distributed load tests indicate a fairly narrow range of collapse load contact pressures per unit thickness of ice, i.e. 20 to 28 lbs/sf/in thickness (0.38 to 0.53 $kN/m^2/cm$ thickness). The concentrated load tests also give a narrow range for punching shear, i.e. 34 to 46 lbs/sq in (236 to 317 kN/m^2). The ratios of collapse loads to the squares of the ice thicknesses varied in the range of 320 to 604 lbs/in^2 (221 to 416 N/cm^2) for distributed loads and 191 to 240 lbs/in^2 (132 to 165 N/cm^2) for concentrated loads.

In the next section we will consider safe loadings for ice sheets that could be developed during wintertime construction of a small craft harbor. Quick intense loads causing breakthrough and long term viscoelastic loadings are not dealt with here. (Refer to Kerr, 1976 and Nevel, 1976.)

Safe Loads on Ice Sheets

Nevel (1977) has developed a pocket calculator program to predict safe loads based upon an elastic first crack analysis. His analysis and the program are included in Appendix III. Circular and rectangular loads or combinations of loads can be handled.

For our purposes we have solved his general equations for the simple case of one concentrated load. The computed maximum "safe load" can be used for estimating single construction loads on ice sheets; provided the ice is sound and free of discontinuities. The load cannot be placed near free edges of the sheet or in the vicinity of walls or shores where cracks behave as a free edge condition. To be out of the direct influence of these edge conditions the load should be away a distance equal to two or more flexural rigidity lengths. The expression for the maximum safe load P with Poisson's Ratio assumed to be 1/3 is:

$$P = 1.396\, \sigma h^2 \frac{(a/l)}{kei'(a/l)}$$

where σ is the allowable stress

h is the ice thickness

a is the radius of the load distribution

l is the flexural rigidity length

kei' is the first derivative of the Kelvin function kei

The expression for the maximum deflection w which will occur directly under the load is

$$w = \frac{(P)\left[(a/l)ker'(a/l) + 1\right]}{(a/l)^2\, k\, l^2 \pi}$$

where ker' is the first derivative of the Kelvin function ker

k is the weight density of water

Kelvin functions and their derivatives are tabulated in mathematical handbooks. They are difficult to work with for small values of argument (a/l) due to the sensitivity of the functions' values with changes in (a/l). Included in Appendix IV are tables of Kelvin functions and derivatives from Nevel (1959) for arguments ranging between 0.000 and 0.500. Also the appended pocket calculator program has a Kelvin function subroutine that can be used for these or other arguments.

For small values of (a/l) the following approximation for the maximum deflection w is useful.

$$w = \frac{P}{8kl^2}$$

Figure 5 is a plot of the maximum safe load for ice with a flexural strength σ of 100 psi (689 kN/m^2) and a modulus of elasticity E of 750 ksi (5168 MN/m^2). The maximum safe load is linearly proportional to the flexural strength. For this type of ice loading problem, design flexural strengths would range between 50 and 100 psi (345 and 689 kN/m^2) for fair to good ice. Loadings beyond this stress level are not recommended for usual construction procedures. Of course care is necessary in placing loads on any thin ice or placing very large loads regardless of the ice thickness or the analysis. Figure 5 simply gives recommended maximum values that should not be exceeded and that should be used with caution.

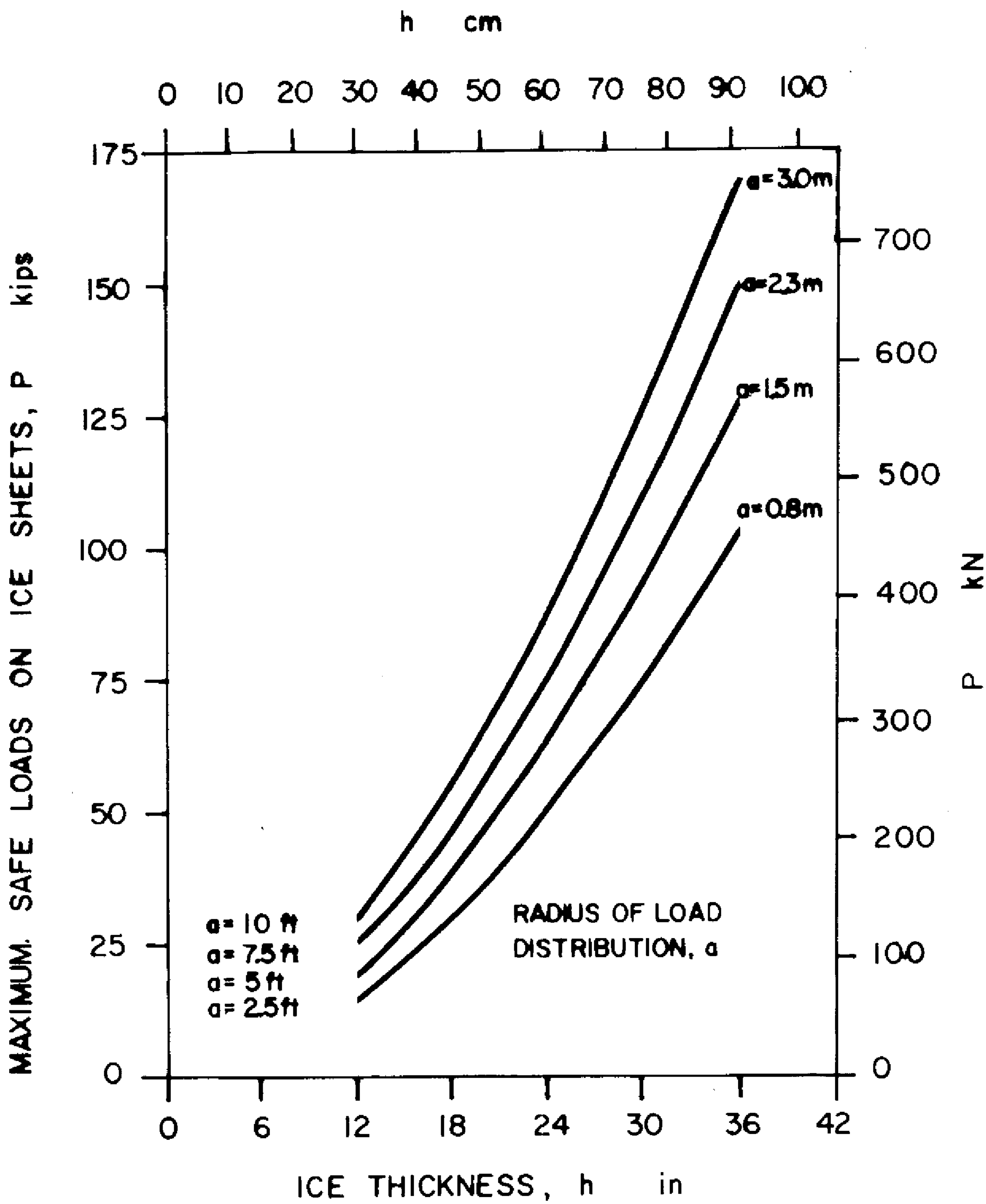

Figure 5. MAXIMUM SAFE LOADS ON ICE SHEETS (after Nevel, 1977)

To illustrate the use of Figure 5 and the equations let us assume we want to place a 20 kip (89 kN) load on a 5 ft by 8 ft (1.5 m by 2.4 m) area. We have 15 inches (38 cm) of sound ice. Through area ratios we will convert the rectangular area to an equivalent circular area.

$$a = \left[\frac{5\ \text{ft} \times 8\ \text{ft}}{\pi}\right]^{\frac{1}{2}} = 3.6\ \text{ft}\ (1.1\ \text{m})$$

From Figure 5 at 15 inches (38 cm) and 3.6 feet (1.1 m) we find the maximum safe load to be 23 kips (102 kN) which is greater than the specified 20 kips (89 kN).

If we have weaker ice, because of warm temperatures or previous damage from construction we would want to reduce the load. The flexural strength may be only 50 psi (345 kN/m^2) and for this case the safe load would be halved. In this example we should also consider the contact pressure and the deflection.

$$\text{contact pressure} = \frac{20{,}000\ \text{lbs}}{5\ \text{ft} \times 8\ \text{ft}} = 500\ \text{psf}\ (23.9\ \text{kN/m}^2)$$

This pressure is high for this relatively small loading area and it would be well to use a larger area for the 20 kip (89 kN) load. But assuming the given loading condition we can compute the deflection of the ice sheet from the formula

$$w = \frac{(P)\left[(a/l)\text{ker}'(a/l) + 1\right]}{(a/l)^2\ k\ l^2\ \pi}$$

where $a/l = 3.6/27.0 = 0.1333$
(l interpolated from Table 9)

$k = 62.4\ \text{lbs/ft}^3\ (1000\ \text{kg/m}^3)$

$$w = \frac{(20{,}000\ \text{lbs})\ \left[(0.1333)\ (-7.4501) + 1\right]}{(0.133)^2\ (62.4\ \text{lbs ft}^{-3})\ (27.0^2\ \text{ft}^2)\ \pi}$$

$$= .054\ \text{ft or } 0.65\ \text{in}\ (1.6\ \text{cm})$$

or $$w = \frac{P}{8kl^2} = \frac{(20{,}000\ \text{lbs})}{(8)\ (62.4\ \text{lbs ft}^3)\ (27.0^2\ \text{ft}^2)} = 0.55\ \text{ft}\ (1.7\ \text{cm})\ \text{approx.}$$

One criterion for allowable deflection is that the deflection be less than 0.08 times the ice thickness, i.e. the ice sheet will not be submerged. For our 15 inch (38 cm) sheet the allowable deflection criterion is equal to 1.2 inches (3 cm).

If the freeboard of the ice is exceeded water can flood the surface through cracks and other openings. This warms the ice and weakens it and also adds further loads. Loadings on ice less than 18 inches (46 cm) in thickness should be checked for possible submergence. Ice thicker than this will not submerge under the recommended maximum loads.

Figure 5 should be of assistance in estimating construction loads on small craft harbor ice covers. Special precautions are necessary however if moving loads across frozen bodies of water is planned. Moving loads create dynamic effects in the ice cover and water. Unfortunately the natural critical velocity is easily achieved on many lakes. At this velocity, the deflections and stresses are amplified. It is necessary to go at a slow speed or very fast. The technical literature should be reviewed by those planning moving loads across frozen lakes.

Thickening and Strengthening Ice Covers

In using ice for bearing capacity, actual thicknesses should be measured. Dates when a natural ice sheet will be thick enough to support a specified load are of interest and can be estimated from past observations and rules of thumb. One rule is that the thickness is directly proportional to the square root of the Freezing Index (previously defined in Part I). Using α as a proportionality factor we can write the following rule of thumb expression:

$$h = \alpha\sqrt{FI}$$

where h is ice thickness in inches

FI is Freezing Index, Fahrenheit

α is a locality factor ranging between 0.5 for snow covered lakes and 1.0 for lakes without snow.

Assume we have had 45 days of weather where it has generally been below freezing. The Freezing Index would be calculated by summing algebraically the plus and minus departures of the daily mean temperatures from the freezing temperature. The Freezing Index in our example will be assumed to be 750 Fahrenheit degree-days and our locality factor equal to 0.8.

$$h = 0.8\sqrt{750}$$

$$= 22 \text{ inches (56 cm)}$$

The locality factor α can be historically established by several years' measurements of ambient temperatures and ice thicknesses. It can be used for estimating when sufficient thicknesses of ice might naturally become available. Ice can also be intentionally thickened and strengthened to aid nature.

Hoffman (1967) describes surface flooding techniques for improving natural ice areas. These techniques have been developed by the U.S. Naval Civil Engineering Laboratory. Among their recommendations are the following items.

The maximum mean daily temperature at which ice flooding can be performed satisfactorily is considered to be about +15°F (-9°C). At temperatures much above this point, freezing rates are slow and long periods are required between applications.

The depth of water applied to any point should not be greater than that which will freeze through in 24 hours. At temperatures from 0°F to -10°F (-18°C to -23°C) with little wind, this is about 4 inches (10 cm) of water.

A cooling period equal to the freezing period should be allowed before an ice area is reflooded. Such cooling is necessary for restoration of ice temperature and recovery of ice strength and resistance to creep.

Additional water should not be applied until all areas of the previous flood have frozen solid. The premature reflooding of an unfrozen area is very undesirable since the freezing time increases exponentially with depth.

Air bubbles which form in the flooded ice surface should be broken before reflooding.

Duff (1958) describes in general terms the thinking of logging operators on the strengthening of ice sheets. Ice landings are constructed and maintained by flooding on top of existing ice and rolling any snow that falls.

Where snow is lying on an ice sheet it is, if at all possible, compacted by rolling before flooding. Slush always seems to form after a snowfall. The weight of the snow cover depresses the ice sheet and free water comes up through cracks or the insulating blanket of snow allows a current, probably a thermal in lakes, to weaken the ice sheet from below. Rolling immediately after a snowfall, or during a storm if it is a severe one, improves frost penetration and lessens the slush problem.

Rose and Silversides (1958) describe the merits of surface flooding for increasing the speed at which ice thickens. Starting with a layer of ice 12 inches (30 cm) thick and -20°F (-29°C) weather, if 5 inches (13 cm) of water are added on the surface it will take approximately 15 hours for it to freeze and give a total thickness of 17 inches (43 cm) of ice. If the ice surface is bared and ice is to be formed on the under-surface of the 12 inch (30 cm) ice sheet, it will take 60 hours to add an additional 5 inches (13 cm). This is 4 times as long.

Ohstrom and DenHartog (1976) have reported on a series of cantilever beam tests designed to determine the efficacy of adding reinforcement to an ice cover. Tests were run using 1 inch (2.5 cm) diameter tree branches, 3/16 inch (5 mm) diameter wire rope and 9/16 inch (14 mm) half-round wood dowels as reinforcement. A definite advantage was noted from using reinforcement, even when poorly placed. The reinforced ice carries a load even after it cracks. Thus, after the initial cracks there is time to remove people and equipment before final breakthrough. Disadvantages to the reinforcement are that the darker types absorb radiation and thereby cause weakening of the ice cover. Also, in many cases, the time and effort required to place reinforcement may exceed those required to achieve equal strength by additional thickening of the ice sheet.

The reinforcing process consists of laying reinforcement material on the ice, then flooding the area and allowing the reinforcement to freeze into the ice. For optimum strength, the reinforcement should be added to the side of the ice that carries the tensile forces. Initial cracking of the ice sheet is caused by tensile stresses near the bottom surface of the ice, but final breakthrough is caused by tensile stresses near the top surface.

PART V

VERTICAL ICE FORCES AND OTHER FORCES

One of the most significant and damaging ice forces results from changes in water levels. These changes cause the ice sheet to move up and down tearing and pulling structures built in small craft harbors.

In Part IV we considered the capacity of an ice sheet to support construction loads. Necessarily we used a conservative value for the strength of the ice. In estimating the uplift forces from ice we will assume stronger ice as it will probably exist sometime during the winter. Computing uplift loads is complicated because we do not know which of several failure criteria best represents the uplift force. We have selected a "first crack" analysis to predict the minimum uplift for which we should design. The maximum uplift force for design still remains a matter of engineering judgment and site circumstances.

This part begins with a discussion of lake level changes. Ranges of uplift loads are then presented together with some ways to attenuate these loads. The section concludes with some comments on buckling of ice sheets and other loads.

Water Level Fluctuations

Water levels, and hence ice levels, in the Great Lakes vary seasonally. The small craft harbor designer must anticipate and estimate these seasonal variations when planning fixed harbor structures, such as docks and piers, launching ramps and bulkheads. To a lesser extent, water level variations affect floating structures. They are of concern, however, e.g. in connection with anchorage and gangways.

To illustrate the importance of water levels on design, consider a fixed height dock specified to be a desired freeboard above normal summer water levels. Should levels prove otherwise, due to larger natural variations or inaccurate estimates, such a dock can end up well above the water or too close to the water. It becomes barely usable by the intended boats. Also, during wind storms over water, severe water level changes can occur. At Buffalo, New York the lake has risen to as much as 8 feet (2.4 m) above the normal water level.

Problems can result with winter ice levels also. Individual pilings supporting a fixed dock may be protected (deiced) with a compressed air bubbler system. Normally the number of pilings would be kept to a minimum to reduce the amount of air required. By doing this, dock members tend to have greater depths because of their longer spans. Should the ice become higher than estimated, the lower elements of the dock members (which are not deiced) may come in contact with the ice sheet or be shoved by broken ice pieces that are frequently found on top of an ice cover.

Local experience can aid the designer in estimating water and ice levels. If severe water level changes can be anticipated, a floating or adjustable height design would be called for. The following publications will also help the designer.

Monthly Bulletin of Lake Levels for the Great Lakes, Department of the Army, Detroit District, Corps of Engineers, P. O. Box 1027, Detroit, Michigan 48231. (Bulletin lists maximum, minimum, and expected levels.)

Hydrograph of Monthly Mean Levels of the Great Lakes, U. S. Department of Commerce, NOAA, National Ocean Survey, C3314, 6001 Executive Boulevard, Rockville, Maryland 20852. (Lists monthly mean lake levels from 1860 to date.)

In addition to seasonal lake level variations, there is an unusual water level phenomenon known as a seiche (pronounced sāsh). It is a standing wave oscillation of an enclosed or semi-enclosed water body that continues, pendulum fashion, after the cessation of the originating force. It is a short-term rise and fall of the water level and is caused by either persistent, strong winds piling up the water at one end of a basin, or changes in barometric pressure over the lake, and sometimes a combination of both. Although seiches are unnoticeable on small lakes, they are quite noticeable on the Great Lakes and in bays and harbors. The period of a seiche is a few minutes in a bay or harbor and about ten hours for a Great Lake.

Hodek and Doud (1975) measured an almost constant fluctuation of the winter water level in the Ontonagon, Michigan harbor. The amplitude had an observed maximum of 0.8 feet (0.24 m). The major period varied from 5 minutes to more than 10 hours. Higher frequency water oscillations also were observed and a change in the water level of 3 inches (8 cm) in 10 minutes was observed.

Seiches are easily observed in a marina in the summertime by noting a drop in the water surface with respect to some fixed object like a sloping launching ramp or a piling. In fact, severe seiches have been observed where the water level fluctuated three feet (1 m) and more in a few hours.

As a result of seiche action, an ice plate in a harbor moves up and down exerting large forces on structures penetrating the ice sheet. In the next section we will estimate the minimum level of these forces.

Minimum Ice Sheet Uplift Loads

In a manner similar to that used in computing safe loads on ice sheets, we can estimate minimum uplift loads to be expected. The analysis is an elastic one and is compatible with the rapidity with which water levels change under seiche action. The analysis is also a thin plate analysis which is not strictly correct for ice sheets thicker than the embedded pile diameter. For this case the stresses computed are too high in the vicinity of the load. However this is acceptable because it is conservative.

We will assume a round pile frozen into (rather than setting upon) an ice plate and determine the force when the plate first cracks. This will be the minimum force we can expect when a pile or structure is frozen into the ice.

The differential equation formulating this problem has been solved. We have evaluated the solution for the following boundary conditions. At a distance from the center of the concentrated load equal to the radius of the load (distance a), the slope of the deflected ice plate is zero; and at distance a the total load is carried by the stress on the failure surface which is the thickness of the ice plate times $2\pi a$. The crack that forms at failure is a circumferential crack located a distance a out from the center of the load. This crack is consistent with what is seen in the field when an ice sheet pulls upward on a strong well-embedded piling and eventually cracks. The distance a is not the radius of the piling, but is larger. This is due to the failure occurring by cracking in the ice rather than slippage at the ice-pile interface, and because immediately adjacent to the pile the ice is thicker than the rest of the sheet. (An ice collar forms in this area.) This thickening phenomenon results from heat transfer through metal piling (and to some extent through wood piling) increasing the ice thickness in this area. Also as the winter goes on and the crack has formed and refrozen many times, the piling experiences a kind of dipping action in the cold water, (somewhat like dipping a candle wick in wax to form a candle).

The equation for the minimum ice sheet uplift load, based upon this first circumferential crack is as follows:

$$P = \frac{(1/3)\,\pi\,\sigma\,h^2\,(a/l)}{\left[\dfrac{\text{kei}\,(a/l)\,\text{kei}'\,(a/l) + \text{ker}\,(a/l)\,\text{ker}'\,(a/l)}{\text{kei}'\,(a/l)\,\text{kei}'\,(a/l) + \text{ker}'\,(a/l)\,\text{ker}'\,(a/l)}\right]}$$

where σ is the allowable stress

h is the ice thickness

a is the radius of the load distribution

l is the flexural rigidity length

kei and ker are Kelvin functions

kei' and ker' are first derivatives

Figure 6 is a plot of the minimum ice sheet uplift load for ice with a flexural strength σ of 200 psi (1378 kN/m^2) and a modulus of elasticity E of 750 ksi (5168 MN/m^2). The minimum uplift load is linearly proportioned to the flexural strength, which could be greater than the assumed value of 200 psi (1378 kN/m^2). Also, as we will show later, making different assumptions for the failure criterion will result in larger loads.

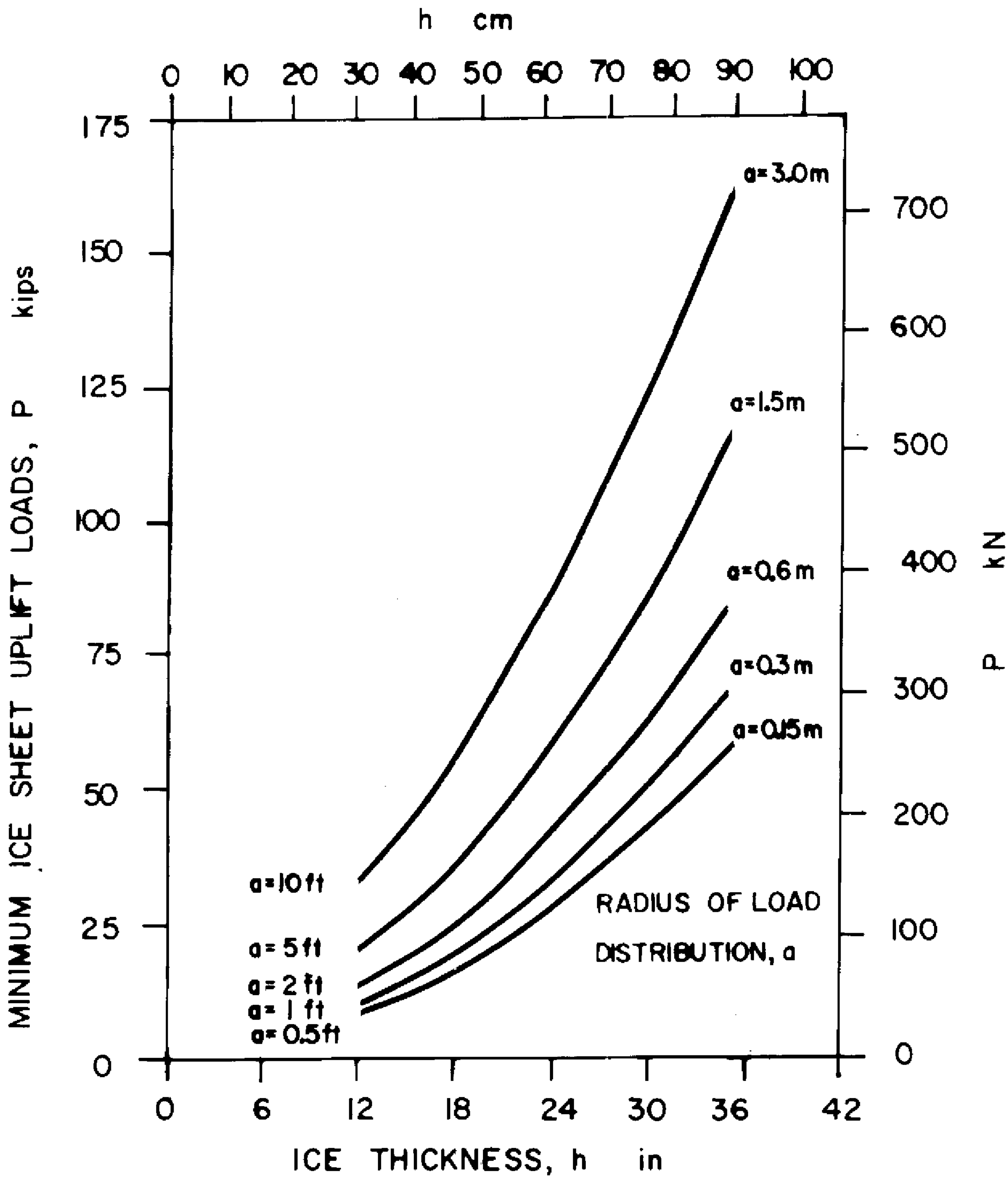

Figure 6. MINIMUM ICE SHEET UPLIFT LOADS

To illustrate the use of Figure 6, let us compute the minimum uplift for a 12 inch (30 cm) round steel pile in 24 inches (61 cm) of ice. As stated above the radius a will be greater than one half pile diameter. For the usual steel pile shapes, the radius of the ice collar will be about six inches (15 cm) greater than the pile radius. (For wood, a value of about half this much has been observed.) From Figure 6 at h = 24 inches (61 cm) and a = 1 foot (0.3 m), we find the minimum uplift load is 33 kips (147 kN). This estimated load is the least that we should design for with a 12 inch (30 cm) pile in 24 inches (61 cm) of sound cold ice (σ assumed = 200 psi, 1378 kN/m^2). We do not know precisely what the maximum load could be, but it could be several times as much. The ice does not fail on the ice-pile interface, but fails in the strength of the ice sheet itself.

The uplift on a square construction could be reasonably approximated by an inscribed circle of radius a equal to ½ of the length of the side. This would be conservative since the angles of a square would tend to cause stress-concentrations in the ice.

Nevel (1972) has formulated a more severe failure criterion for the ice pile uplift problem. It is a criterion which gives a near, if not the maximum, upper bound to the problem. It is assumed that radial cracks form and then a circumferential crack forms connecting the ends of the radials. This cracking pattern forms a series of truncated wedges whose tips are supporting the load and whose bases are failure planes when the circumferential crack develops. If we neglect any side interaction between the wedges, and assume that six wedges have formed the Nevel (1972) solution for the maximum load becomes:

$$P = 1.154\,\sigma\,h^2 \left[1.05 + 2.00\,(a/l) + 0.50\,(a/l)^3\right]$$

where the bracketed term is an approximation to an exact solution.

For our example we can compute the maximum load P as follows:

$a = 1.0$ feet (30 cm)

$l = 33.8$ ft (10.3 m) at $\underline{E} = 750$ ksi (5168 kN/m^2)

$a/l = .0296$

$[\;] = 1.109$

$P = (1.154)\,(200 \text{ lbs in}^{-2})\,(24^2 \text{ in}^2)\,(1.109)$

$P = 147{,}400$ lbs or 147 kips (654 kN)

This truncated wedges load estimate is $4\frac{1}{2}$ times the estimate based upon the first crack analysis. For smaller values of a the loads are 6 times as great and for larger values of a, 3 times as great.

The proper values to use for design are matters of engineering judgment and experience. The values should lie within the ranges presented and an uplift of not less than the minimums indicated on Figure 6 should be used for design with appropriate safety factors. If for example a minimum load is estimated to be 25 kips (111 kN), then the ultimate resistance of the soil foundation system preventing uplift, or the dead weight of the structure, should be greater than 25 kips (111 kN) by perhaps a factor of 2 or 3. In selecting a design criteria for a particular small craft harbor, other factors will enter in; such as whether some damage will be tolerable, the ease with which pullout resistance can be developed, or the location of the pile and its relationships with other piles or dock members.

Ice can also exert down-drag loads when the water level drops and leaves the ice hanging on the dock piles. The ice can easily span across between piles and develop the full dead weight of the ice sheet. Large drops in levels have occurred and it is believed that the ice sheet is capable of losing all its buoyancy. The down load situation is aggravated by a tendency to have thicker ice under shady dock areas than in the more exposed aisle or fairway areas.

There is limited field data on forces on full size piles but several direct observations have been made. Hodek and Doud (1975) and Hodek (1976) have measured vertical loads of 18 kips (80 kN) compression and $11\frac{1}{2}$ kips (51 kN) tension or uplift on a 15 inch (38 cm) instrumented sleeve placed around a dock pile. The thickness of the ice was about 16 inches (41 cm). The loads were those actually felt by force transducers on the instrumented pile sleeve.

Muschell (1976) jacked against an ice cover to pull frozen-in piles. Loads up to 25 kips (111 kN) were recorded. The loading rate was 0.5 psi per second (3.4 $kN/m^2/s$). Ice adhesion to steel was computed by dividing the pullout load by the actual ice thickness and area of ice contact to steel on HP 8 x 36 and HP 10 x 42 piles. The ice adhesion values ranged between 12 and 24 psi (83 and 165 kN/m^2).

In January 1977, Muschell (unpublished) recorded a value of 68 psi (469 kN/m^2) for ice adhesion to an 8-inch (20-cm) pipe pile in a comparable field test.

The lifting force per unit of length of circumference of a structure frozen into an ice sheet becomes less as the size of the structure increases. The lifting force per unit of length will approach the values for long straight walls. Lofquist (1951) and Michel (1970) have estimated the vertical forces exerted by ice rigidly attached to walls. The estimates are based upon an elastic analysis and rapid rises in water levels.

$$p = \frac{\sqrt{2}\,\sigma\,h^2}{6\,l}$$

$$\Delta = \frac{p}{\sqrt{2}\,k\,l}$$

where p is the lifting force per unit length

Δ is the rise in water level

Assume we have 24 inches (61 cm) of ice with a strength of 200 psi (1378 kN/m^2) and a modulus E = 750 ksi (5168 MN/m^2). The uplift force per unit length and the water level rise are computed as follows:

$$p = \frac{(\sqrt{2})\,(200\text{ lb in}^{-2})\,(24^2\text{ in}^2)}{(6)\,(33.8\text{ ft})}$$

$$= 803\text{ lbs/ft }(11.7\text{ kN/m})$$

$$\Delta = \frac{(803\text{ lbs ft}^{-1})}{(\sqrt{2})\,(62.4\text{ lbs ft}^{-3})\,(33.8\text{ ft})}$$

$$= .27\text{ ft or }3\tfrac{1}{4}\text{ inches }(8\tfrac{1}{4}\text{ cm})$$

Assuming a different strength value for the ice will change the uplift load in a linear manner. However the loads computed at σ equal 200 psi (1378 kN/m^2) are believed to be adequate since the method of analysis is conservative. There normally is cracking parallel to a wall or long crib structure and this reduces the uplift from that assumed for rigid attachment. Walls and long cribs seem to experience little damage due to ice uplift. Occasionally, however, the top of a crib will be pulled off because it is inadequately secured to the rest of the crib.

The next section presents some suggestions for reducing and eliminating uplift forces.

Reducing Ice Sheet Uplift Loads

In Part III we discussed ways to suppress ice and reduce its thickness. This is the most positive way at this time to reduce ice uplift on small craft harbor structures. A number of other techniques have been tried and are being tested now. The projections are optimistic that reliable and economical ways will be developed to reduce ice uplift forces.

Ice adheres tenaciously to most construction materials. There is no comprehensive treatise on ice adhesion; only technical and research papers.

Representative adhesion values for ice to various materials and coatings determined by Freiberger and Lacks (1961) are presented in Table 12. The results are based on experiments for artificial ice loaded at the rate of 5 psi/s (34 $kN/m^2/s$), and depend strongly on the rate.

	Ice Adhesion Range of Values	
	psi	kN/m^2
Metals	85-120	586-827
Woods	45-80	310-551
Plastics	25-40	172-276
Glass	20-150	138-1034
Paints	80-100	551-689
Resin Films	30-130	207-896

Table 12 Ice Adhesion
(after Freiberger and Lacks, 1961)

Jellinek (1957) performed shear experiments with snow-ice sandwiched between two stainless steel plates. The results of his work on small samples indicated the adhesive strength to be independent of thickness of the ice layer and cross-sectional area. He found the adhesive strength to be a linear function of the temperature until it becomes larger than the cohesive strength of ice at about 9°F (-13°C), where a very sharp transition from adhesive to cohesive breaks takes place. Representative values are as follows: 28°F (-2°C)--30.4 psi (209 kN/m^2), 7°F (-14°C)--236 psi (1626 kN/m^2), and -4°F (-20°C)--233 psi (1605 kN/m^2).

Michel (1970) has observed that there is very little available data on strength of adhesion of ice to structures and, there is a lot of scatter in the experimental results because of insufficient control of the uniformity of the structure of the ice and other relevant factors. Michel notes that in most cases the strength of adhesion of ice to construction materials is as high as the shear strength of the river or lake ice itself, which is usually taken to be from 80 to 150 psi (551 to 1034 kN/m^2).

And as previously stated, Muschell (1976 and unpublished) found values ranging up to 68 psi (469 kN/m^2).

For our purposes we can say that slippage between ice and the piling material will probably not naturally occur at loads less than the calculated minimums. Therefore in order to reduce uplift effects we must cause the ice to slip at loads less than these minimums.

Field and laboratory experiments have shown that wrapping a pile with various smooth materials doesn't work very well because the materials become unattached after a number of cycles of uplift. Materials that have been tried include tetrafluorethylene polymer wrap (teflon-like) and thin polyethylene sheets. Both materials were torn loose and worked their way up and out of the ice. A three ply wrap with polyethylene sheets was unsuccessful too. The three ply system was tested to see whether the outer layers of polyethylene, which would be assumed stuck to the ice and to the pile, would slide on the inner sheet sandwiched between the two. All layers worked their way out of the ice.

Laboratory studies and field tests are in progress now (1977) to evaluate protective jackets and coatings that could be installed or applied on existing piles or specified for new piles. US CRREL is undertaking this work and reports on their results are expected in 1978. One system of jacketing uses existing fiberglass products and methods for repairing damaged underwater piling. Preliminary laboratory test results show about a 4 fold reduction in the maximum force recorded on a test pile frozen into an ice sheet and displaced up and down in a simulated cyclic seiche action. Work is also underway to evaluate navigation lock wall epoxy and co-polymer coating systems described by Frankenstein et al (1976). Large reductions in adhesion values have been demonstrated but work on coating life and methods of application are needed.

Ice uplift loads can be reduced by causing the ice to crack so that large loads are not transmitted to structures. Group action is frequently effective in accomplishing this cracking. For example, several piles lashed together to act as a single dolphin, are effective in resisting uplift. A series of piles spaced 5 feet (1.5 m) on centers under a pier supporting a boat hoist machine have resisted uplift because of the group action. Also a T-head pier with pairs of pilings spaced about 25 feet (8 m) on centers has been found to act as a unit. A crack forms following the outline of the pier and completely encircling it. This crack demonstrates the group action of the entire piling system.

The piles toward the center of a marina and the piles at the ends of finger piers are more prone to pulling than those closer to shore. This is apparently due to the plate action and boundary conditions on the plate subjected to hydrostatic uplift. The ice sheet may also be frozen fast to the shoreline. This phenomenon suggests designing the outer piles for large uplift forces and thereby cause the ice to fail about them. If this happens the piles closer in should feel little if any uplift force.

Another system that has been successfully used to eliminate uplift forces on marina pilings is the sleeved pile system (U.S. Patent No. 3,543,523). It allows vertical motion of a pile. It consists of a round hollow steel pile driven (with a follower) so that the top of the pile is 2 feet (0.6 m) below the bottom surface of the ice. A slightly larger diameter hollow round steel pile piece (the sleeve) fits over the top of the driven pile. The driven pile has a cap plate, and welded inside the sleeve about 4 ft (1.2 m) from its lower end is a round bearing plate. When the sleeve is dropped over the pile, the round plate bears on the cap plate. On top of the sleeve is another cap plate for bearing of dock beams. As ice freezes to the sleeve, the sleeve moves up and down over the lower, stationary pile (analogous to a stationary piston in a moving cylinder). The sleeve works well, allowing vertical motion. However horizontal ice forces may still be acting on the piles. Also dockage systems using sleeved piles are more complicated and costly to build because differential vertical motions must be provided for in the dock members and supported utilities that may be on the docks.

Other Ice Forces and Actions

In Part II we reviewed thermally induced thrusts, and in this part have considered uplift. Ice exerts other forces. For example, ice is very abrasive and causes damage to structures, particularly wood structures whose life in ice may be only twenty years, Striegl (1952). Cladding wood structures is necessary to achieve longer service lives. Unprotected concrete is subject to abrasion and freeze-thaw spalling.

Sheltered small craft harbors can experience damage from chunks of ice being driven into the harbor and against the docks during Spring breakup, and during periods when ice goes out of a harbor during Winter storms.

In addition to thermally induced horizontal forces, wind and currents can produce dynamic forces. Specific design recommendations are not now available but in general, marina dockage should be located in protected areas away from dynamic ice forces. However, those wishing information on dynamic effects should refer to the Neill (1976) comprehensive review and assessment of dynamic ice forces on piers and piles. Analytical approaches, design formulas, and full-scale and small-scale laboratory investigations are considered.

Ice under horizontal loadings may buckle before it crushes. Sodhi and Hamza (1977), using the finite element method, have performed a buckling analysis of a semi-infinite plate on an elastic foundation. The analysis determines the effective pressure which will cause the plate to buckle. If the crushing strength of ice is greater than this effective pressure, the plate will buckle; if less, a crushing failure will occur before buckling. Nevel (unpublished) has developed a useful expression based upon the Sodhi and Hamza (1977) finite element analysis.

$$P = k\,l^3 \left[b/l + \frac{3.32}{1.0 + 0.25\ b/l} \right]$$

where P is the buckling load

b is the structure width

l is the flexural rigidity length

k is the weight density of water

To illustrate, let us assume we have 6 inches (15 cm) of ice pushing against a wall 100 feet (30 m) long. The flexural rigidly length l, assuming E equal 750 ksi (5168 MN/m^2), is 11.9 feet (3.6 m). The ratio b/l is 100 feet/11.9 feet or 8.40.

$$P = (62.4 \text{ lbs ft}^{-3})\ (11.9^3 \text{ ft}^3) \left[8.40 + \frac{3.32}{1.0 + 0.25\ (8.40)} \right]$$

$$= 996{,}000 \text{ lbs or } 996 \text{ kips } (4430 \text{ kN})$$

We next compute the effective pressure as follows:

$$\text{Effective Pressure} = \frac{P}{b\ h}$$

$$= \frac{996{,}000 \text{ lbs}}{(100 \text{ ft})\ (0.5 \text{ ft})\ (144 \text{ in}^2 \text{ ft}^{-2})}$$

$$= 138 \text{ lbs in}^{-2}\ (951 \text{ kN/m}^2)$$

If the crushing strength of ice is greater than this pressure, buckling will occur. The maximum crushing strength may be on the order of 300 psi (2067 kN/m^2). In general, for small structures the ice crushes before buckling, except in the presence of thick ice where crushing always takes place.

There is little or no experimental data, expecially from full scale structures, to assist in the design of piles for horizontal focus. The American Association of State Highway and Transportation Officials (AASHTO) recommends that highway bridges be designed with the following provisions:

> All piers and other portions of structures which are subject to the force of flowing water, floating ice, or drift shall be designed to resist the maximum stresses induced thereby.
>
> The pressure of ice on piers shall be calculated at 400 psi (2756 kN/m^2). The thickness of ice and height at which it applies shall be determined by investigation at the site of the structure.

The shortcoming of the AASHTO provision for 400 psi (2756 kN/m^2) ice pressure is that it is representative of an impact loading for strong ice failing in compression. The effects of pile shape and orientation, e.g. sloping piles where ice would fail in bending, are not considered. The maximum measured effective ice pressure on a bridge pier to date has been less than 200 psi (1378 kN/m^2).

In a marina where Spring ice movements occur the effective ice pressure is probably less than 200 psi (1378 kN/m^2). Spring ice in a moving field has very low internal strength.

There is a tendency of ice to squeeze objects placed in it. How this happens is not understood and no methods of analyses are known. When a pontoon is placed in water (ice) it will either be squeezed up and finally rest on the ice, or be squeezed down and drawn further into the ice. The result of this action may be damage to the dock framing members connecting the pontoon units together; or sometimes rupturing of the pontoon material itself.

One hypothesis of the squeezing action is that it is the result of refreezing water. A small amount of ice adjacent to the pontoon melts during the day's warming sun. At night an ice cover forms over the melt water. This encloses the melt water which subsequently freezes in a partially confined space. This refreezing exerts a squeezing force which either ruptures or moves the pontoon.

A field study to investigate the action of pontoon units frozen in lake ice is scheduled for the winter of 1977-1978.

APPENDIX I--LITERATURE CITED

Ashton, G. D. 1974. "Air Bubbler Systems to Suppress Ice". Special Report 210. Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

Ashton, G. D. 1975. "Experimental Evaluation of Bubbler Induced Heat Transfer Coefficients". Proceedings of Third International Symposium on Ice Problems. International Association of Hydraulic Research, Hanover, New Hampshire. 133-142.

Ashton, G. D. 1977. "Numerical Simulation of Air Bubbler Systems". Paper presented at the Third Canadian Hydrotechnical Conference. Quebec, P.Q., Canada. May 26-27, 1977.

Aune, C. A., Beaudin, L. A., and Borrowman, J. K. 1957. "Effects of Ice on Inland Navigation". Section 1, Communication 3, Permanent International Association of Navigation Congresses, XIX Congress, London: pp. 71-94.

Drouin, M. and Michel, B. 1974. "Pressure of Thermal Origin Exerted by Ice Sheets upon Hydraulic Structures". CRREL Draft Translation 427. Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

Duff, C. H. 1956. "Ice Landings". Conference on the Bearing Strength of Ice, NRC, Ottawa. Transactions of the Engineering Institute of Canada. 2:3:99-100.

Dumble, J. H. 1858. "Ice Phenomena, from Observations on Rice Lake". Canadian Journal of Industry, Science and Art, New Series, 3:414-422.

Frankenstein, G. E. 1966. "Strength of Ice Sheets". Proceedings of Ice Pressures Against Structures Conference, Laval University, Quebec, Nov. 1966. Technical Memorandum No. 92, National Research Council, NRC No. 9851, Ottawa, March 1968, 79-87.

Frankenstein, G. E. et al. 1976. "Ice Removal from the Walls of Navigation Locks". Proceedings of the Symposium on Inland Waters for Navigation, Flood Control & Water Diversions, Colorado State University, Aug. 10-12, 1976. American Society of Civil Engineers, New York.

Freiberger, A. and Lacks, H. 1961. "Ice-phobic Coatings for Deicing Naval Vessels". Proceedings of the Fifth Naval Sciences Symposium. p. 234-237.

Glen, J. W. 1974. "The Physics of Ice". Cold Regions Science and Engineering Monograph II-C2a. Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

Glen, J. W. 1975. "The Mechanics of Ice". Cold Regions Science and Engineering Monograph II-C2b. Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

Gold, L. W. 1973. "Ice--A Challenge to the Engineer". Proceedings of Fourth Canadian Congress of Applied Mechanics, Montreal, 1973, pp. G19-G36. National Research Council of Canada, Technical Paper No. 395, NRC 13436.

Gow, A. J. and Langston, D. 1977. "Growth History of Lake Ice in Relation to its Stratigraphic, Crystalline and Mechanical Structure". CRREL Report 77-1. Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

Hobbs, P. V. 1974. Ice Physics. Clarendon Press, Oxford.

Hodek, R. J. and Doud, J. O. 1975. "Instrumented Piles for the Measurement of Ice-Uplift Forces". Proceedings of Third International Symposium on Ice Problems, International Association of Hydraulic Research, Hanover, New Hampshire. 409-417.

Hodek, R. J. 1976. "A Field Study of Ice-Pile Interaction 1975-1976 Season". Unpublished research report submitted to US Cold Regions Research and Engineering Laboratory under Contract No. DACA 89-76-C-0007. December 1976.

Hoffman, C. R. 1967. "Ice Construction--Method of Surface Flooding". Naval Facilities Engineering Command Technical Report R511. U.S. Naval Engineering Laboratory, Port Hueneme, California.

Jellinek, H. H. G. 1957. "Adhesive Properties of Ice". Research Report 38. Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

Kerr, A. D. 1976. "The Bearing Capacity of Floating Ice Plates Subjected to Static or Quasi-Static Loads". Journal of Glaciology. 17:76:229-268.

Korzhavin, K. N. 1971. "Action of Ice on Engineering Structures". CRREL Draft Translation 260. Original 1962. Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

Lavrov, V. V. 1969. "Deformation and Strength of Ice". Main Administration of the Hydrometeorological Service of the Council of Ministers of the USSR, Arctic and Antarctic Scientific Research Institute, Leningrad. Translated from Russian, Israel Program for Scientific Translations, Jerusalem, 1971.

Lofquist, B. 1951. "Lifting Force and Bearing Capacity of an Ice Sheet". National Research Council of Canada Technical Translation No. 164. Original 1944.

Mellor, M. 1964. "Properties of Snow". Cold Regions Science and Engineering Monograph III-A1. Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

Metge, M. 1976. "Thermal Cracks in Ice". Doctoral Thesis. Department of Civil Engineering. Queen's University, Kingston, Ontario.

Michel, B. and Ramseier, R. O. 1969. "Classification of River and Lake Ice Based Upon Its Genesis, Structure and Texture". Report S-15 Departement deGenie Civil, Section Mecanique des Glaces, Universite Laval.

Michel, B. 1970. "Ice Pressure on Engineering Structures". Cold Regions Science and Engineering Monograph III-B1b. Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

Michel, B. 1971. "Winter Regime of Rivers and Lakes". Cold Regions Science and Engineering Monograph III-B1a. Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

Muschell, J. E. 1976. "Design, Specifications, and Construction of Pilings for Small Craft Harbors". Unpublished paper presented at Docks and Marinas Institute, University of Wisconsin-Extension, Department of Engineering, May 4-5, 1976. James E. Muschell, President, United Marine Associates, 111 North Main, Cheboygan, Michigan 49721.

Neill, C. R. 1976. "Dynamic Ice Forces on Piers and Piles. An Assessment of Design Guidelines in the Light of Recent Research". Canadian Journal of Civil Engineering. Vol. 3:305-341.

Nevel, D. E. 1959. "Tables of Kelvin Functions and Their Derivatives". U.S. Army Snow, Ice and Permafrost Research Establishment (SIPRE) Technical Report 67. SIPRE is now the U.S. Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

Nevel, D. E. 1972. "The Ultimate Failure of a Floating Ice Sheet". Proceedings of the Ice Symposium, 1972. International Association of Hydraulic Research, Leningrad. 1-5.

Nevel, D. E. 1976. "Creep Theory for a Floating Ice Sheet". CRREL Special Report 76-4. Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

Nevel, D. E. 1977. "Safe Ice Loads Computed with a Pocket Calculator". Unpublished. U.S. Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

Ohstrom, E. G. and DenHartog, S. L. 1976. "Cantilever Beam Tests on Reinforced Ice". CRREL Report 76-7. Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

Petrunichev, N. N. 1954. "On the Static Pressure of Ice", (O staticheskom davlenii l'da). Ice Thermal Problems in Hydraulic Engineering. Leningrad.

Pounder, E. R. 1965. Physics of Ice. Pergamon Press, Ltd., London.

Rose, L. B. and Silversides, C. R. 1958. "The Preparation of Ice Landings by Pulp and Paper Companies in Eastern Canada". Conference on the Bearing Strength of Ice, NRC, Ottawa. Transactions of the Engineering Institute of Canada. 2:3:101-105.

Runnels, L. K. 1966. "Ice". Scientific American, 215:118-124.

Sodhi, D. S. and Hamza, H. E. 1977. "Buckling Analysis of a Semi-Infinite Ice Sheet". Fourth International Conference on Port and Ocean Engineering Under Arctic Conditions. St. John's, Newfoundland. September 26-30, 1977.

Striegl, A. R. 1952. "Ice on the Great Lakes and Its Engineering Problems". Paper presented at Joint Meeting of American Meteorological Society and Geophysical Union. September 11, 1952, Chicago.

Williams, G. P. 1966. "Freeze-Up and Break-Up of Fresh-Water Lakes". Proceedings of Ice Pressures Against Structures Conference, Laval University, Quebec, November 1966. Technical Memorandum No. 92, National Research Council, NRC No. 9851, Ottawa, March 1968, 203-215.

Zarling, J. P. 1974. "Ice Engineering in Small Craft Marinas: An Annotated Bibliography". Advisory Report No. 8. University of Wisconsin Sea Grant College Program, University of Wisconsin-Madison.

APPENDIX II--DEFORMATION AND STRENGTH OF ICE, LAVROV

This appendix contains an abbreviated summary of a monograph prepared by Lavrov (1969). It is based on experimental studies on structurally simulated ice whose small-scale crystal structure is similar to that of an ice cover. This very important requirement is not fulfilled in small samples cut from a natural ice cover. The laws of deformation and strength presented are for forces acting on ice for short periods, i.e. from a few seconds to a few minutes. Problems with very long-term strength and creep of ice are not dealt with.

Three peculiarities distinguish ice from other materials. First, it exists at a temperature within a few degrees (a difference of tens of Celsius degrees being rare) of its melting point. Ice therefore usually contains some liquid water between crystals. The water acts like a lubricant, reducing the sliding resistance between crystals. Secondly, ice does not react chemically with impurities and forms no solid solutions with them. The impurities accumulate gradually and form many pores, cavities, etc. in the ice. The third peculiarity is the comparatively large size of ice crystals. Their transverse dimensions vary between about a tenth of an inch (a few mm) and an inch (several cm). In metals these dimensions come out to small fractions of a few hundredths of an inch (a mm). These peculiarities cause ice to be a material which in its totality appears to consist of plastic hinges, so that it sharply reacts to static loads.

The Ratio of Transverse and Longitudinal Strains (Poisson's Ratio)

Poisson's ratio plays an important role in the laws governing the behavior of ice under load. Poisson's ratio for ice is usually determined dynamically from the difference between the propagation velocities of the transverse and longitudinal elastic waves. Its average value at temperatures below 32°F (0°C) is 0.33. Experimental values range from 0 to more than 2.5. Poisson's ratio depends upon stress, in some cases exceeds the theoretically possible maximum, and is different for tension and for compression. The results obtained can be explained only on the basis of the actual behavior of the ice under load, i.e., by allowing for its plasticity.

Behavior of Ice Under Load

Whatever the type of static load acting on ice (compression, tension, bending, shear) slip plays a decisive role in the overall deformation pattern. Slip in ice begins simultaneously with the load application. It is most intensive at the beginning, and then decreases according to a power law. The widely held view that at the beginning the deformation is purely elastic, and that the above-mentioned processes take place only after some time, is erroneous. The elastic limit of ice is small, less than 7 psi ($50 kN/m^2$).

The fraction of the elastic deformation in the total deformation at the beginning of a short-time load action is less than later, i.e., ice is not a linearly elastic material and does not obey Hooke's law. This basic peculiarity of ice distinguishes it from other materials, in particular metals.

The linearity of some parts of the deformation diagrams for compression and bending do not indicate that ice obeys Hooke's law in its usual meaning, since Young's modulus E in the expression "stress equals E times strain" must be replaced by another proportionality factor. This is the strain modulus which is not constant but depends on the duration of the load application.

The failure mechanism of ice varies with the test conditions (temperature, duration of load action, and direction of force), and is in the final analysis determined by the extent to which slip takes place both inside and between the crystals.

None of the existing models simulating the mechanical properties of a body is fully applicable to ice, because one of its constituent elements, the "dashpot" (considered as plunger moving in a liquid according to Newton's law of viscosity) must be replaced by a more complicated structure, by means of which not only the permanent set but also the elastic lag is allowed for. Studies carried out during recent years have shown that ice does not obey the law of viscosity since the relationship between the stress and the strain rate is not linear for it. It is at present impossible to express the stress-strain relationship analytically by allowing for all the laws mentioned above, which govern the behavior of ice under load.

The Dependence of the Properties of Ice on the Conditions of its Formation and Growth

It is well known that the quality of ice (including the mechanical properties of the crystals) depends on the conditions of its growth. The mechanical strength of the crystals vary directly with the time taken for the ice to grow. A number of defects in the crystal structure is small when the ice grows slowly, since the build-up of new lattice layers proceeds in small steps; these new layers are connected more regularly to the already finished planes. Furthermore, impurities are able to diffuse from the growing faces into the solution.

Other conditions being equal, the upper layers grow faster than the lower ones, so that the quality of the upper layers is worse.

The quality of ice depends on the rate of its growth, so the strength of an ice cover formed at low temperatures is less than that of an ice cover of equal thickness, formed during milder weather.

The presence of snow on ice retards the removal of the latent heat of crystallization from the lower boundary of the ice, so that its growth rate decreases. The thickness and strength of natural ice cover therefore fluctuates.

The Modulus of Elasticity and the Shear Modulus

Natural ice, particularly when its crystals are small, has various orientations of the crystal axes; the crystals are separated by intercrystalline layers containing impurities and liquid phase. In addition, air bubbles are distributed over the bulk of the ice. The moduli of elasticity for the ice cover are therefore smaller than the elastic constants of monocrystals, and vary with the structure of the ice.

The elastic properties of ice are most accurately determined by means of dynamical methods, i.e., by measuring the propagation velocity of elastic (sound) waves in ice. Ultrasonic waves and the method of resonance vibrations yield the local values of the moduli of elasticity, while seismic methods yield averaged values for an area.

General shortcoming of all dynamical methods of determining the elastic properties is the approximateness of the theory, which results from considering the ice as an isotropic body.

Allowing for the peculiarities of the various dynamical methods of determining the moduli of elasticity, it appears that the latter differ insignificantly for fresh-water ice of different origin and from different water bodies.

Young's modulus E and the shear modulus G vary with temperature and method by which determined. Approximate average values are 1200 ksi (8300 MN/m^2) and 400 ksi (2800 MN/m^2) respectively.

The Strain Moduli (Uniaxial Compression, Tension, Bending, and Slip and Shear)

In studying the behavior of ice under static load it is useful to introduce the concept of the strain modulus. The reason is that the stress-strain diagrams have considerable linear parts.

In contrast to the modulus of elasticity the strain modulus is computed from the total, i.e., the elastic, the recovered, and the remanent, strain. It therefore depends not only on the temperature and structure of the ice but also on the magnitude of the load and the time during which it acts. The strain modulus may therefore have many values for each kind of deformation.

If only the properties of the samples are considered, all values of the strain modulus should be assumed to be equally probable, taking into account the dependence of the strain modulus on the above-mentioned factors. The scatter of its values indicates only that the test conditions or the structure of the ice in some tests differed from those in others. The data found in the various sources on the mechanical properties of ice should, therefore, be grouped according to their similar features. This, however, is not always possible due to the lack of information on the test conditions. The strain modulus must in any case be selected according to the test conditions. The less time the load acts on the ice, the larger should be the value of the strain modulus.

When information on the properties of the ice cover is required, and the question arises of which value of the strain modulus should be given preference, there is, strictly speaking, no possibility of solving this problem by using the results of tests of samples having an arbitrary structure. Far more accurate data on the mechanical properties will then be provided by samples of structurally simulated ice.

The results of strain moduli in uniaxial compression, tension, bending, and slip and shear are presented in Table II-1.

	Strain Modulus Range	
	ksi	MN/m^2
Uniaxial Compression	71- 768	490- 5296
Tension	17- 71	118- 490
Bending	135-1707	932-11768
Slip and Shear	6- 36	41- 245

Table II-1 Strain Moduli
Structurally Simulated Ice
(after Lavrov, 1969)

The strain moduli in compression and tension increase under repeated load applications.

The large differences in the strain modulus in bending are due not only to differences in the ice structure and temperature but also to the influence of the sample dimensions. This last factor was not considered important until recently, although it plays an important role in bending tests. The clear dependence of the strain modulus in bending on the ratio of the distance between supports to the sample height is due to the proximity of the ice temperature to the melting point and to the appearance of stress concentrations and microcracks at the corners of cavities. This causes irreversible and slowly reversible deformations which take place in the ice practically as soon as the load is applied.

The strain modulus in bending decreases when the load is increased and increases when a load is applied again unless the stress is very high.

The strain modulus in shear depends on the temperature, the concentration of impurities, the stress system, the dimensions of the shear surface, and the duration of the load application.

Ultimate Compressive Strength

The ultimate compressive strength of ice is, like that of other materials, usually established from the results of tests of samples (cubes) subjected to forces acting on one side. The sample is under these conditions deformed in the direction of compression but is free to expand in the transverse direction. The quality of work done in preparing the sample has a considerable influence on the test results.

Carefully prepared samples of structurally simulated fresh-water ice gave the following range of results: ultimate compressive strength with the direction of force perpendicular to the ice surface 730-1220 psi (5031-8414 kN/m^2) and for the direction of the force parallel to the ice surface 156-614 psi (1079-4236 kN/m^2).

Tensile Strength

The tensile strength of ice has been little investigated, like its shear strength. This may be due to the lack of examples in which ice is under real conditions subjected to tension.

Lavrov reports on the research of several other people. They found that tensile strengths fell in the range of 71 to 199 psi (490 to 1373 kN/m^2).

Lavrov's tests on structurally simulated ice show lower strengths than natural ice. This may be due to the different dimensions of the crystals and to differences in the development of the intercrystalline layers.

The Differences in the Properties of Ice Under Tension and Compression

As noted, the tensile strength and strain modulus in tension are considerably less than the corresponding magnitudes in compression. Lavrov carried out special investigations on homogeneous structurally simulated ice. He found the compressive strength of this type of ice to be 9 times higher than its tensile strength. This is due to the structure of the ice. Part of the vertical cross-sectional area of ice consists of intercrystalline layers and cavities filled with impurities and air. During tension the force is counteracted by the cohesive strength of the material connecting adjacent grains. Only part of the cross-section thus takes up the force. The crystal layers bo dering the cavities can move freely in opposite directions, thus altering the shape of the cavities and increasing their size.

During compression the cavities become filled and the entire cross-section of the sample thereafter takes up the force.

On the basis of the experimentally established differences in the ultimate strength (1:9) it may be assumed that in fresh-water ice not subjected to any external force about 90% of the cross-sectional area is taken up by intercrystalline layers and cavities, with only 10% of it formed by direct bonds between adjacent crystals. However, this area ratio is not always observed since the quality of the ice (including the extent to which intercrystalline layers have developed in it) depends on the conditions of its formation and its growth rate. Furthermore, the microstructure of the ice changes with the lapse of time, due to recrystallization and other factors. There are thus cases in practice where the differences in the tensile and compressive strength of ice are less than 1:9. On the other hand, in thawing ice these differences are greater than 1:9.

Shear Strength

The results of Lavrov's tests carried out on structurally simulated ice gave a shear strength range of 68 to 422 psi (471 to 2913 kN/m^2). The shear strength depends on the sample size (scale factor), the duration of the load action, and the temperature when shear occurs along a circular surface in the center of an ice sheet. The direction of the force during the tests was perpendicular to the ice surface (along the vertical axes of the crystallites).

Bending Strength

Lavrov determined an average bending strength of 316 psi (2177 kN/m^2) for structurally simulated ice. The direction of the force was perpendicular to the ice surface and the duration of the load action was 4 seconds.

The bending strength was computed by means of the usual formulas of the strength of materials, which are suitable for determining the stresses only when the elastic strains are small. In the case of failure the strains are comparatively large and irreversible. Furthermore, the behavior of the ice at the beginning of the load action considerably differs from that assumed in the theory of elasticity. Hence, the bending strength, determined from the formula for a freely supported beam, cannot be considered as the true maximum tensile stress, but is an arbitrary magnitude. The computed bending strength therefore exceeds the computed tensile strength.

Lavrov determined an average bending strength of 161 psi (1108 kN/m^2) for structurally simulated ice in cantilever bending. The tensile strength and bending strength are of similar magnitude when cantilever beams are tested.

The Freezing Strength of Ice

Lavrov reports the research results of several others. In one case the average value of the normal freezing strength of ice to different surfaces at 27°F (-3°C) was about 284 psi (1961 kN/m^2). Surfaces included were ice to copper, ice to iron, and ice to glass. Tests on ice to polystyrene gave values about one-tenth as much.

Researchers have concluded that the strength with which water and other liquids freeze to solid surfaces does not depend upon whether the latter are wetted by the liquid considered or not. They state further: "The strength with which ice freezes to plastics and varnishes is approximately one-tenth of that with which it freezes to glass or metals".

Other tests have given freezing strengths of fresh-water ice lower values, in the range of 71 to 142 psi (490 to 980 kN/m^2).

The Friction Coefficient

The friction coefficient was determined as the ratio of the towing force to the weight of the sample together with an additional load. The static friction coefficient was determined by starting the sample, while the dynamic friction coefficient was determined during steady motion of the sample.

An ice surface almost always carries a thin liquid film consisting of water molecules. The frictional force thus is the sum of two components, namely dry friction and boundary friction with water lubrication. This is supported by the increase in the friction coefficient when the temperature of the ice surface is lowered.

It was found by experiment that steel skids are better than wooden ones at temperatures down to -22°F (-30°C), since the coefficient of boundary friction of steel is less than that of wood. Skids made of a material which is a poor heat conductor are better at temperatures below -31°F (-35°C), since part of the heat developed by friction is lost to the surroundings via heat-conducting skids; this interferes with the formation of a watery film, so that friction increases.

Researchers found that the static friction coefficient of ice on non-rusty steel is 0.15 to 0.25 and 0.30 to 0.35 on painted red lead.

Tests on Lake Ladoga were performed in an area cleared of snow. The ice surface was smooth and without cracks, but was uniformly covered by several hundreths to a tenth of an inch (1 to 3 mm) prominences. The lower surface of the ice was smooth.

The average friction coefficients of ice on steel and steel on ice were at 36°F (2°C) as follows: static friction coefficient ranged from 0.48 to 0.91 and the dynamic friction coefficient ranged from 0.08 to 0.15. The field results differed considerably from those obtained in the laboratory, perhaps due to scale effects.

Summary of Strength Properties Reported by Lavrov

A summary of strength properties reported by Lavrov is presented in Table II-2. The summarized values are qualified on one or more variables such as strain rate, temperature or sample size. They are applicable only for forces acting on ice for short periods, i.e., from a few seconds to a few minutes.

Property	Average Value or Range of Values
Poisson's Ratio	0.33
Elastic Limit	less than 7 psi (50 kN/m^2)
Young's Modulus	1200 ksi (8300 MN/m^2)
Shear Modulus	400 ksi (2800 MN/m^2)
Uniaxial Compression Strain Modulus	71-768 ksi (490-5296 MN/m^2)
Tension Strain Modulus	17-71 ksi (118-490 MN/m^2)
Bending Strain Modulus	135-1707 ksi (932-11768 MN/m^2)
Slip and Shear Strain Modulus	6-36 ksi (41-245 MN/m^2)
Ultimate Compressive Strength (Force perpendicular to surface)	730-1220 psi (5031-8414 kN/m^2)
Ultimate Compressive Strength (Force parallel to surface)	156-614 psi (1079-4236 kN/m^2)
Tensile Strength	71-199 psi (490 to 1373 kN/m^2)
Shear Strength	68-422 psi (471 to 2913 kN/m^2)
Freely Supported Beam Bending Strength (Force perpendicular to surface)	316 psi (2177 kN/m^2)
Cantilever Beam Bending Strength (Force perpendicular to surface and applied from above)	161 psi (1108 kN/m^2)
Freezing Strength to Metals	71-284 psi (490-1961 kN^2/m^2)
Freezing Strength to Plastics and Varnishes	about one-tenth that of metals

Table II-2 Strength Properties
(after Lavrov, 1969)

APPENDIX III

SAFE ICE LOADS COMPUTED WITH A POCKET CALCULATOR

by

D. E. Nevel

U.S. Army Cold Regions Research and Engineering Laboratory
Hanover, NH 03755

April 1977

INTRODUCTION

When a load is placed on a floating ice sheet, the ice sheet is bent downward creating stresses within the ice. The downward deflection increases the water pressure on the bottom of the ice which in turn supports the load. If the maximum tensile stress within the ice exceeds the tensile strength of the ice, the ice will crack. Although the ice sheet can still carry more load before breakthrough occurs, the first crack is generally used to predict safe bearing capacity. In some cases flooding of the ice sheet may create operational difficulties, and a limiting deflection is used rather than first crack.

Previously the computations for the deflection and stresses of a floating ice sheet have been obtained by computers or programmable desk-top calculators. Recently programmable pocket calculators have become powerful enough to perform these computations. Although they are slower, they do provide a means for computation in the field which has been non-existent before.

The purpose of this report is to provide a program to calculate the deflection and stresses for a floating ice sheet on the Hewlett-Packard model 67 pocket calculator. CRREL does not necessarily endorse the HP-67 since there are other programmable pocket calculators that may also perform the same computations.

The user of the program must select appropriate values for the mechanical properties of the ice in order to compute reliable deflection and stresses. Engineering judgement must be used in selecting the allowable ice strength and when dealing with non-ideal situations.

REQUIRED INFORMATION

A rectangular coordinate system must be defined on the ice sheet for locating the loads relative to each other. For each load, the coordinates of the center of the load, the magnitude of the load, and the distribution of the load must be known. The load may be assumed to be uniformily distributed over a circular area or a rectangular area whose dimensions and orientation are known. The load from a pneumatic tire can be assumed to be uniformly distributed over a circular area whose radius can be determined from the air pressure in the tire.

Young's modulus, Poisson's ratio, and the ice thickness must be specified for the ice sheet, as well as the density of supporting water.

The coordinates where the deflection and stresses are to be computed must be specified. For safe bearing capacity prediction, one should choose the coordinates that will give the largest stress within the ice sheet. When the loads are sufficiently far apart, the largest stress will occur directly under the center of the heaviest load. When two loads of equal magnitude are extremely close together the largest stress may occur between the loads. A few trial computations may be necessary to find where the stresses are the largest.

THEORETICAL APPROXIMATIONS AND LIMITATIONS

The calculator program uses the elastic thin plate theory to represent a floating ice sheet of uniform thickness, and considers a load applied uniformly to a circular or rectangular area. Previously the rectangular area solution has been expressed by a Fourier series which frequently is slowly convergent. This program uses a new solution expressed by a power series which is more rapidly convergent. In order to further reduce the space and time requirements, the deflection is calculated from four concentrated loads, one at each corner of the rectangular area. This is a good approximation because the deflection does not depend significantly upon the distribution of the load.

Even with these simplifications, the rectangular area solution requires much more time than the circular area solution. Further time-savings can be obtained by noting that the stresses becomes less dependent on the load's distribution as the calculation point moves farther away from the load. Hence, at a sufficiently large distance from the load, an equivalent circular area or concentrated load may be substituted for the rectangular area. For distances greater than five times the flexural rigidity length ℓ, the deflection and stresses are nearly zero for either the circular or rectangular area solution and the calculation may be omitted.

The elastic thin plate theory is adequate except for calculating the stresses in the vicinity of a relatively concentrated load. For this situation, the three-dimensional theory of elasticity should be used. For this theory solutions exist for both rectangular and circular load distributions, but they are expressed inconveniently for numerical computation. In practice, loads distributed over a circular area (e.g. a pneumatic tire) tend to be more concentrated than loads distributed

over a rectangular area (e.g. a crawler track). The program uses a method proposed by Westergaard when the stresses are evaluated directly under the center of a circular area. The method replaces the radius a of the load distribution circle with $\sqrt{1.6\ a^2 + h^2} - 0.675\ h$, when the radius a is less than 1.742 times the ice thickness h. Using this new radius in the thin plate theory gives the same stress as the three-dimensional theory of elasticity.

EQUATIONS

The most convenient form of the solution for a load uniformly distributed over a circular area has been given by Max Wyman. The symbols are defined as follows:

- E is Young's modulus of the ice,
- ν is Poisson's ratio of the ice,
- h is the thickness of the ice,
- k is the weight density of the water,
- ℓ is $[Eh^3/12k(1-\nu^2)]^{1/4}$, the flexural rigidity length,
- P is the total load,
- r and θ are polar coordinates from the center of the load,
- R is r/ℓ, the non-dimensional radial coordinate,
- a is the radius of the load's distribution,
- A is a/ℓ, the non-dimensional load radius,
- w is the deflection of the ice sheet,
- σ_r is the radial stress at the bottom of the ice sheet.
- σ_θ is the tangential stress at the bottom of the ice sheet.

When R > A, the equations are:

$$\frac{(\sigma_r-\sigma_\theta)\, h^2\, \pi}{2P(1-\nu)\ 3} = \frac{2}{R}\ [-D\ \mathrm{kei}'R + C\ \mathrm{ker}'R] + \frac{(\sigma_r+\sigma_\theta)h^2\ \pi}{2P(1+\nu)\ 3},$$

$$\frac{(\sigma_r+\sigma_\theta)h^2\ \pi}{2P(1+\nu)\ 3} = C\ \mathrm{kei}\ R + D\ \mathrm{ker}\ R,$$

$$\frac{wk\ell^2\ \pi}{P} = -\ D\ \mathrm{kei}\ R + C\ \mathrm{ker}\ R,$$

where $C = \frac{\mathrm{ber}'A}{A}$ and $D = \frac{\mathrm{bei}'A}{A}$. When $A = 0$, $C = 0$ and $D = 1/2$.

When A > R, the equations are:

$$\frac{(\sigma_r-\sigma_\theta)h^2\ \pi}{2P(1-\nu)3} = \frac{2}{R}\ [-D\ \mathrm{bei}'R + C\ \mathrm{ber}'R] + \frac{(\sigma_r+\sigma_\theta)h^2\ \pi}{2P(1+\nu)3},$$

$$\frac{(\sigma_r+\sigma_\theta)h^2\ \pi}{2P(1+\nu)3} = C\ \mathrm{bei}\ R + D\ \mathrm{ber}\ R,$$

$$\frac{w\ k\ \ell^2\ \pi}{P} = -D\ \mathrm{bei}\ R + C\ \mathrm{ber}\ R + \frac{1}{A^2},$$

where $C = \frac{\mathrm{ker}'A}{A}$ and $D = \frac{\mathrm{kei}'A}{A}$. When $R = 0$, $\sigma_r - \sigma_\theta = 0$.

The stresses must be determined relative to the x, y coordinate system in order to add to the stresses from other loads. The equations for this are

$$\frac{\sigma_x+\sigma_y}{2} = \frac{\sigma_r+\sigma_\theta}{2}$$

$$\frac{\sigma_x-\sigma_y}{2} = \frac{\sigma_r-\sigma_\theta}{2}\ \cos 2\theta$$

$$\sigma_{xy} = \frac{\sigma_r-\sigma_\theta}{2}\ \sin 2\theta$$

Consider a load uniformly distributed over a rectangular area as shown in Figure 1. The symbols are defined by:

x and y are coordinates which define the center of the load,

x_o and y_o are coordinates where the stresses are evaluated,

x_r and y_r are coordinates where the stresses are evaluated in a coordinate system whose origin is at the center of the load and is parallel to the sides of the rectangle,

θ is the angle of the x_r axis measured from the x axis in a counterclockwise direction,

X is x_r/ℓ and Y is y_r/ℓ, non-dimensional coordinates,

a is one half the length of the load in the x_r direction,

b is one half the length of the load in the y_r direction,

A is a/ℓ and B is b/ℓ, non-dimensional length of the load, and

E, ν, h, k, ℓ, P and w are defined as before.

The equations to determine the x_r and y_r coordinates are:

$$x_r = (x_o-x)\cos\theta + (y_o-y)\sin\theta$$
$$y_r = -(x_o-x)\sin\theta + (y_o-y)\cos\theta$$

The equations to determine the stresses at the bottom of the ice sheet relative to the X, Y coordinate system are:

$$\frac{(\sigma_X \pm \sigma_Y)h^2}{2P(1 \pm \nu)} = \frac{3}{4\pi AB}\left[I(X+A,\ Y+B) - I(X+A,\ Y-B) + I(X-A,\ Y-B) - I(X-A, Y+B)\right]$$

$$\pm \frac{3}{4\pi AB}\left[I(Y+B, X+A) - I(Y-B, X+A) + I(Y-B, X-A) - I(Y+B, X-A)\right]$$

$$\frac{\sigma_{XY}\, h^2}{P(1-\nu)} = \frac{3}{4\pi AB}\left[\mathrm{kei}\sqrt{(X+A)^2+(Y+B)^2} - \mathrm{kei}\sqrt{(X+A)^2+(Y-B)^2}\right.$$

$$\left. + \mathrm{kei}\sqrt{(X-A)^2+(Y-B)^2} - \mathrm{kei}\sqrt{(X-A)^2+(Y+B)^2}\right]$$

where

$$I(x,y) = \left(\frac{x}{2}\right)^2 \left(\frac{y}{x}\right) \sum_{n=1,3}^{\infty} \frac{(-1)^{\frac{n+1}{2}} (x/2)^{2n}}{n!\ (n+1)!} \left[-\frac{\pi}{4}\right] \sum_{k=0,1}^{n} \frac{n!\ (y/x)^{2k}}{k!\ (n-k)!} \frac{1}{2k+1}$$

$$+\left(\frac{x}{2}\right)^2 \left(\frac{y}{x}\right) \sum_{n=0,2}^{\infty} \frac{(-1)^{n/2} (x/2)^{2n}}{n!\ (n+1)!} \left[-\gamma - \mathrm{Ln}\frac{\sqrt{x^2+y^2}}{2} + \phi(n) + \frac{\cdot 2}{n+1}\right] \sum_{k=0,1}^{n} \frac{n!(y/x)^{2k}}{k!(n-k)} \frac{1}{2k+1}$$

$$+\left(\frac{x}{2}\right)^2 \left(\frac{y}{x}\right) \sum_{n=0,2}^{\infty} \frac{(-1)^{n/2} (x/2)^{2n}}{n!\ (n+1)!} \sum_{k=0,1}^{n} \frac{n!(-1)^k}{k!(n-k)!} \frac{1}{2k+1} \left[-\frac{\arctan(y/x)}{(y/x)}\right.$$

$$\left. + \sum_{r=0,1}^{k} \frac{(-1)^r (y/x)^{2r}}{2r+1}\right]$$

where γ is Euler's number and $\phi(n) = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{n}$. The equations to determine the stresses relative to the x, y coordinate system are:

7

$$\frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_X + \sigma_Y}{2}$$

$$\frac{\sigma_x - \sigma_y}{2} = \left(\frac{\sigma_X - \sigma_Y}{2}\right) \cos 2\theta - \sigma_{XY} \cdot \sin 2\theta$$

$$\sigma_{xy} = \left(\frac{\sigma_X - \sigma_Y}{2}\right) \sin 2\theta + \sigma_{XY} \cos 2\theta$$

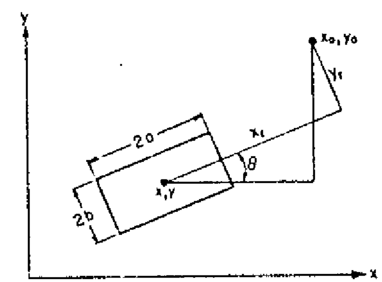

Figure 1. Rectangular load distribution.

GENERAL PROCEDURE

Turn on the calculator and set the other switch to run. The program to compute the deflection and the stresses for a load uniformly distributed over a circular area is divided into two parts which are labeled 1 and 2. The program to compute the deflection and stresses for a load uniformly distributed over a rectangular area is divided into two parts which are labeled $\boxed{3}$ and $\boxed{4}$. The results of either program 2 or $\boxed{4}$ are:

8

REG 0	Identification #		
REG 6	deflection	w	(inches)
REG 7	stress sum	$(\sigma_x+\sigma_y)/2$	(psi)
REG 8	stress difference	$(\sigma_x-\sigma_y)/2$	(psi)
REG 9	shearing stress	σ_{xy}	(psi)

Program 5 adds the deflection and stresses from either program 2 or $\boxed{4}$ to the respective sums which are in registers 1 through 4. From these new sums it determines and stores the maximum stress in register A and the angle from the x axis of the crack in register B. Between computation of each load, the sums are saved on a magnetic card referred to as the sum card. Side 2 of the program 5 magnetic card is not used, and can used for the sum card.

INPUT DATA

1. For a load uniformly distributed over a rectangular area (Fig. 1), the following data should be keyed into memory. Neither the length a or b of the rectangle may be equal to zero.

REGISTER	DATA (units)	DEFINITION
0	ID#	a load identification number
1	x (inches)	coordinate of the load's center
2	y (inches)	coordinate of the load's center
3	P (pounds)	magnitude of the load
4	a (inches)	rectangle half-length along x_r
5	b (inches)	rectangle half-length along y_r
6	θ (degrees)	the angle of x_r from x
7	E (psi)	Young's modulus of ice
8	ν	Poisson's ratio of ice
9	h (inches)	thickness of ice
A	k (lb/in^3)	weight density of water
B	x_o (inches)	coordinate of computations
C	y_o (inches)	coordinate of computations

2. For a load uniformly distributed over a circular area, the input is the same except for the distribution of the load in registers 4, 5 and 6. Registers 5 and 6 are not used, while register 4 contains the radius a (inches) of the circular area. This radius may be equal to zero.

3. To store data into a register, key the data into the display register, press STO followed by the register symbol. For example to store 1 for the ID#, press 1, STO, 0. To display data from a register in order to check the input, press RCL followed by the register symbol.

4. The operator may find it time consuming to repeatedly key in all the data for each load. For each load the ice and water data as well as the place of computation do not change. The operator may find it convenient to record the input data for the first load on a magnetic card. Then for succeeding loads, this card may be read into memory and only data which are different need be keyed into memory.

When the computation is to be repeated for a different ice thickness or other parameter change, the operator may find it convenient to record the input for each load on a magnetic card. Then for succeeding computations, each card can be read into memory, and only the appropriate parameter need be keyed.

If the input data is to be recorded on a magnetic card, the secondary registers should be clear since this allows memory to be recorded on one side of a magnetic card rather than on two sides. Consult the HP manual for ways to clear these registers.

To record the memory registers on a magnetic card, press f, W/DATA and insert the card. To read the registers from a magnetic card into memory, just insert the card.

5. The units of the input data are pounds, inches and degrees. A different force and length unit may be used for input which will give the deflection and stress expressed in these same units.

Programs 1 and 2

Program 1 is for a load uniformly distributed over a circular area. Program 2 is a continuation of program 1 and assumes that the results of program 1 are still in memory. The operating instructions are:

1. Read program 1 by inserting side 1 of the 1 magnetic card (side 2 is not used).

2. Store input data in memory.

3. Press A to start the program. After about 5 seconds, the calculator stops and displays 2, the next required program.

4. Read program 2 by inserting side 1 of the 2 magnetic card followed by side 2.

5. Press A to start the program. After about 30 seconds the program stops and displays 5, the next required program.

6. Proceed to program 5.

Program $\boxed{3}$ and $\boxed{4}$

Program $\boxed{3}$ is for a load uniformly distributed over a rectangular area. Program $\boxed{4}$ is a continuation of program $\boxed{3}$ and assumes that the results of program $\boxed{3}$ are still in memory. The operating instructions are:

1. Read program $\boxed{3}$ by inserting side 1 of the $\boxed{3}$ magnetic card followed by side 2.

2. Store input data in memory.

3. Press A to start the program. After about 60 seconds, the calculator stops and displays 4, the next required program.

4. Read program $\boxed{4}$ by inserting side 1 of the $\boxed{4}$ magnetic card followed by side 2.

5. Press A to start the program. After about 300 seconds, the calculator stops and displays 5, the next required program.

6. Proceed to program 5.

Program 5

Program 5 is a continuation of program 2 or $\boxed{4}$. Program 5 assumes that the results of program 2 or $\boxed{4}$ are still in memory. The operating instructions are:

1. Read program 5 into memory by inserting side 1 of the 5 magnetic card. Side 2 is not used by program 5.

3. For a succeeding load (any load except the beginning load of a computation), press A to start the program. The program pauses for about one second and displays the ID#. During this pause, insert the sum card, and the sums will be read. If the sum card is not inserted during the pause, the program will repeat until it is inserted.

For the BEGINNING load of a computation ONLY, press B to start the program. This zeros the sum and eliminates the reading of the sum card.

3. When the program stops, CRD is displayed. Insert the sum card, and memory will be recorded.

4. The angle of the crack is stored in register B and the maximum stress is stored in register A as well as being displayed. If this was not the last load of a computation, process any additional load by either program 1 or $\boxed{3}$.

Subroutines

The subroutines Kelvin (z) and I (x,y) may be of use in other programs. The following is a brief description of these subroutines that will allow their use for this purpose.

The Kelvin (z) subroutine is stored in steps 102 through 213 of program 2. This subroutine uses registers A, B, I, and 0 through 9. The subroutine also uses Euler's number γ = 0.5772156649 which is assumed to be in register E. Therefore, be sure that register E contains γ. To execute the subroutine, place the argument z in the x register and

press E. The subroutine will compute ker (z), ber (z), ker'(z), ber'(z) kei (z), bei (z), kei'(z), bei'(z) in registers 1 through 8 respectively. The argument z will be in register A. The method of computation is from Don Nevel, Tables of Kelvin functions and their derivatives, CRREL Technical Report 67, June 1959. The series is truncated when the last term becomes less than 10^{-6}. The exponent 6 is stored in step 159 and may be changed to obtain other accuracies.

The subroutine I (x,y) is stored in steps 61 through 220 of program $\boxed{4}$. This subroutine uses registers A, B, I, and 10 through 19. The subroutine also uses Euler's number γ = 0.5772156649 which is assumed to be in register E. Therefore, be sure register E contains γ. To execute the subroutine, place the argument y in register y, the argument x in register x, and press E. The subroutine will give I (x,y) in register x. The series is truncated when the last term becomes less than 10^{-6}. The exponent 6 is stored in step 135 and may be changed to obtain other accuracies.

Examples

The following numerical examples are provided for the purpose of checking and debugging the programs. For the Kelvin (z) subroutine with the accuracy set at 10^{-6}, the results for z=1 are:

REG 1	2.867062088 -01	REG 5	-4.949946366 -01
REG 2	9.843817812 -01	REG 6	2.495660400 -01
REG 3	-6.946038910 -01	REG 7	3.523699135 -01
REG 4	-6.244575218 -02	REG 8	4.973965115 -01

For programs 1 and 2 there are various options which should be checked. For R>A, A may or may not be zero. For A>R, R may or may not be zero. For A>R and R=0, Westergaard's substitution may or may not be made. The following examples cover all cases.

The input data for program 1 are:

REG	R>A=0	A>R=0	Westergaard A>R=0
1	100	100	0
2	100	100	0
3	10000	10000	10000
4	0	20	5
7	10^6	10^6	10^6
8	1/3	1/3	1/3
9	10	10	10
A	62.4/1728	62.4/1728	62.4/1728
B	170	100	0
C	170	100	0

The output data from program 2 are:

REG	R>A=0	A>R=0	Westergaard A>R=0
6	5.982837411 -01	6.761493320 -01	6.790823603 -01
7	6.216841289 +01	1.935510530 +02	2.807218953 +02
8	0	0	0
9	-1.534340994 +01	0	0

The next two options can also be used to check program 5. The input data are:

REG	R>A≠0	A>R≠0
0	1	2
1	0	63
2	0	63
3	10000	10000
4	20	20
7	10^6	10^6
8	1/3	1/3
9	10	10
A	62.4/1728	62.4/1728
B	70	70
C	70	70

The output data from program 2 are:

REG	R>A≠0	A>R≠0
6	5.974546370 -01	6.748975099 -01
7	6.221160230 +01	1.857763777 +02
8	0	0
9	-1.501987756 +01	-1.943671630 +00

The results of program 5 are:

REG		
1	1.272352147 +00	ΣW
2	2.479879800 +02	$\Sigma(\sigma_x+\sigma_y)/2$
3	0	$\Sigma(\sigma_x-\sigma_y)/2$
4	- 1.696354919 +01	$\Sigma\sigma_{xy}$
A	2.649515292 +02	maximum stress
B	4.500000000 +01	crack angle (degrees)

For the I (x,y) subroutine with the accuracy set at 10^{-6}, the result is I (1,.5) = 8.407935381 -02.

The example to check program $\boxed{3}$ and $\boxed{4}$ is:

REG	Input	Output
1	0	
2	0	
3	10000	
4	30	
5	5	
6	30	6.008578047 -01
7	10^6	6.566463308 +01
8	1/3	7.014794041 +00
9	10	-1.337575308 +01
A	62.4/1728	
B	50	
C	60	

MEMORY REGISTER USE

EXCEPT FOR INPUT AND OUTPUT

REG	1 and 2	Kelvin (z)	$\boxed{3}$ and $\boxed{4}$	I(x,y)
0	ID#		ID#	
1	R		x_r, X+A	
2	2θ		y_r, Y+B	
3			$3P(1+\nu)/4\pi ABh^2$	
4	A		X - A	
5			Y - B	
6	$P/\pi k\ell^2$			
7	$3P(1+\nu)/\pi h^2$		$\pi k\ell^2/P$	
8	$3P(1-\nu)/\pi h^2$			
9				
A		z/2 - z		$(x/2)^2$
B		PT		$(y/x)^2$
C	C		2θ	
D	D		$3P(1-\nu)/4\pi ABh^2$	
E	Y	Y	Y	Y
I	ℓ	Index	ℓ	k
10		n		n
11		ker z		Σ
12		ber z		P
13		ker'z		Q
14		ber'z		Σ_x
15		kei z		P_k
16		bei z		r
17		kei'z		Σ_r
18		bei'z		P_r
19		φ		φ

PROGRAM (1)

1	LBL A	
2	RCL 7	E
3	RCL 9	h
4	STO X 9	h^2
5	RCL 9	h^2
6	X	h^3
7	X	Eh^3
8	RCL A	k
9	1/X	1/k
10	STO 6	1/k
11	X	Eh^3/k
12	1	1
13	RCL 8	ν
14	X^2	ν^2
15	-	$1 - \nu^2$
16	÷	$Eh^3/k(1-\nu^2)$
17	1	
18	2	12
19	÷	ℓ^4
20	$\sqrt{\ }$	ℓ^2
21	STO ÷ 6	$1/k\ell^2$
22	$\sqrt{\ }$	ℓ
23	STI	ℓ
24	RCL C	y_o
25	RCL 2	y
26	-	y_o-y
27	RCL B	x_o
28	RCL 1	x
29	-	x_o-x
30	→ P	polar
31	RCI	ℓ
32	÷	R
33	STO 1	R
34	R ↓	θ
35	2	2
36	X	2θ
37	STO 2	2θ
38	RCL 3	P
39	π	π
40	÷	P/π
41	STO X 6	$P/\pi k\ell^2$
42	RCL 9	h^2
43	÷	$P/\pi h^2$
44	3	3
45	X	$3P/\pi h^2$
46	STO 7	$3P/\pi h^2$
47	STO X 8	$3\nu P/\pi h^2$
48	RCL 8	$3\nu P/\pi h^2$
49	STO + 7	$3(1+\nu)P/\pi h^2$
50	-	$3(1-\nu)P/\pi h^2$
51	STO 8	$3(1-\nu)P/\pi h^2$
52	RCL 1	R
53	X ≠ 0	R ≠ 0
54	GTO 1	
55	1	
56	.	

19

PROGRAM (1)

57	7	
58	4	1.74
59	RCL 4	a
60	X^2	a^2
61	1	
62	.	
63	6	1.6
64	X	$1.6\ a^2$
65	RCL 9	h^2
66	÷	$1.6\ a^2/h^2$
67	1	1
68	+	$1.6\ a^2/h^2+1$
69	$\sqrt{\ }$	$\sqrt{1.6\ a^2/h^2+1}$
70	.	
71	6	
72	7	
73	5	.675
74	-	new a/h
75	X > Y	a/h > 1.74
76	GTO 1	
77	RCL 9	h^2
78	$\sqrt{\ }$	h
79	X	new a
80	STO 4	new a
81	LBL 1	
82	RCL 4	a
83	RCL	ℓ
84	÷	A
85	STO	A
86	.	
87	5	
88	7	
89	7	
90	2	
91	1	
92	5	
93	6	
94	6	
95	4	
96	9	Y
97	STO E	Y
98	2	2
99	RTN	

20

PROGRAM (2)

1	LBL A	
2	RCL 1	R
3	RCL 4	A
4	CF 0	IN
5	X < Y	A < R
6	SF 0	OUT
7	X = 0	A = 0
8	GTO 9	
9	P ⇄ S	
10	GSB E	KELVIN (A)
11	RCL 3	IN-ker'A
12	F? 0	
13	RCL 4	OUT-ber'A
14	RCL A	A
15	+	C
16	STO C	C
17	RCL 7	IN-kei'A
18	F? 0	
19	RCL 8	OUT-bei'A
20	RCL A	A
21	+	D
22	STO D	D
23	P ⇄ S	
24	LBL 7	
25	RCL 1	R
26	P ⇄ S	
27	GSB E	KELVIN (R)
28	8	IN-ber's
29	F? 0	
30	7	OUT-ker's
31	STI	Index
32	RCL (i)	bei'R or kei'R
33	RCL D	D
34	X	
35	STO 9	$\sigma_r-\sigma_\theta$
36	DSZ	
37	DSZ	
38	RCL (i)	bei R or kei R
39	RCL C	C
40	X	
41	STO B	$\sigma_r+\sigma_\theta$
42	RCL (i)	bei R or kei R
43	RCL D	D
44	X	
45	STO 0	w
46	DSZ	
47	DSZ	
48	RCL (i)	ber'R or ker'R
49	RCL C	C
50	X	
51	RCL 9	$\sigma_r-\sigma_\theta$
52	-	
53	2	2
54	X	
55	DSZ	
56	DSZ	

PROGRAM (2)

57	RCL (i)	ber R or ker R
58	RCL D	D
59	X	
60	RCL B	$\sigma_r+\sigma_\theta$
61	+	
62	STO B	$\sigma_r+\sigma_\theta$
63	RCL A	A
64	X	
65	+	
66	RCL A	A
67	X ≠ 0	A = 0
68	÷	
69	STO A	$\sigma_r-\sigma_\theta$
70	RCL (i)	ber R or ker R
71	RCL C	C
72	X	
73	RCL 0	w
74	-	
75	P ⇄ S	
76	F? 0	
77	GTO 8	
78	RCL 4	A
79	X^2	A^2
80	1/X	$1/A^2$
81	÷	
82	LBL 8	
83	STO X 6	w
84	RCL 2	2θ
85	RCL A	$\sigma_r-\sigma_\theta$
86	RCL 8	$3P(1-\nu)/\pi h^2$
87	X	
88	R ↓	rect
89	STO 8	$(\sigma_x-\sigma_y)/2$
90	R ↓	σ_{xy}
91	STO 9	σ_{xy}
92	RCL B	$\sigma_r+\sigma_\phi$
93	STO X 7	$(\sigma_x+\sigma_y)/2$
94	5	5
95	RTN	
96	LBL 9	
97	STO C	C
98	.	
99	5	.5
100	STO D	D
101	GTO 7	
102	LBL E	KELVIN (z)
103	2	2
104	÷	z/2
105	STO A	z/2
106	CLX	0
107	STO 0	n
108	STO 1	ker z
109	STO 3	ker'z
110	STO 4	ber'z
111	STO 5	kei z
112	STO 6	bei z

PROGRAM (2)

113	STO 7	kei'z
114	STO 8	bei'z
115	STO 9	ϕ
116	1	1
117	STO B	PT
118	STO 2	ber z
119	RCL A	z/2
120	X ≠ 0	z ≠ 0
121	GTO 2	
122	π	π
123	4	4
124	÷	π/4
125	STO - 5	kei z
126	RTN	
127	LBL 2	
128	8	8
129	STI	Index
130	GSB 3	n, ϕ, PT, Σ
131	GSB 4	PT, Σ
132	RCL B	PT
133	CHS	-PT
134	STO B	PT
135	GSB 3	n, ϕ, PT, Σ
136	GTO 4	PT, Σ
137	LBL 3	
138	1	1
139	STO + 0	n
140	RCL 0	n
141	÷	1/n
142	STO + 9	ϕ
143	LBL 4	
144	RCL A	z/2
145	RCL 0	n
146	÷	z/2n
147	RCL B	PT
148	X	PT z/2n
149	STO B	PT
150	STO + (i)	Σ
151	DSZ	
152	RCL 9	ϕ
153	X	
154	STO + (i)	Σ
155	DSZ	
156	RTN	
157	ABS	ABS (ϕ PT)
158	EEX	
159	6	
160	CHS	10^{-6}
161	X < Y	$10^{-6} < \lvert\phi\ PT\rvert$
162	GTO 2	
163	RCL A	z/2
164	LN	LN(z/2)
165	RCL E	γ
166	+	γ+LN(z/2)
167	STO B	γ + LN (z/2)
168	RCL 8	bei'z

PROGRAM (2)

169	X	
170	STO - 7	kei'z
171	RCL B	γ+LN(z/2)
172	RCL 6	bei z
173	X	
174	STO - 5	kei z
175	RCL B	γ+LN(z/2)
176	RCL 4	ber'z
177	X	
178	STO - 3	ker'z
179	RCL B	γ+LN(z/2)
180	RCL 2	ber z
181	X	
182	STO - 1	ker z
183	π	π
184	4	4
185	÷	π/4
186	STO B	π/4
187	RCL 4	ber'z
188	X	π/4 ber'z
189	STO - 7	kei'z
190	RCL B	π/4
191	RCL 2	ber z
192	X	π/4 ber z
193	STO - 5	kei z
194	RCL B	π/4
195	RCL 8	bei'z
196	X	π/4 bei'z
197	STO + 3	ker'z
198	RCL B	π/4
199	RCL 6	bei z
200	X	π/4 bei z
201	STO + 1	ker z
202	RCL 6	bei z
203	RCL A	z/2
204	2	2
205	X	z
206	STO A	z
207	÷	bei z/z
208	STO - 7	kei'z
209	RCL 2	ber z
210	RCL A	z
211	÷	ber z/z
212	STO - 3	ker'z
213	RTN	

PROGRAM $\boxed{3}$

1	LBL A	
2	RCL 7	E
3	RCL 9	h
4	STO X 9	h^2
5	RCL 9	h^2
6	X	h^3
7	X	Eh^3
8	RCL A	k
9	STO 7	k
10	÷	Eh^3/k
11	1	1
12	RCL 8	ν
13	x^2	ν^2
14	-	$1-\nu^2$
15	÷	$Eh^3/k(1-\nu^2)$
16	1	1
17	2	12
18	÷	ℓ^4
19	$\sqrt{\ }$	ℓ^2
20	STO X 7	$k\ell^2$
21	$\sqrt{\ }$	ℓ
22	STI	ℓ
23	RCL B	x_o
24	STO - 1	$x - x_o$
25	RCL C	y_o
26	STO - 2	$y - y_o$
27	RCL 6	θ
28	RCL 2	$-(y_o - y)$
29	RCL 6	θ
30	RCL 1	$-(x_o-x)$
31	R ↓	$-(x_o-x)\cos\theta$
32	CHS	$(x_o-x)\cos\theta$
33	STO 1	x_r
34	R ↓	$-(x_o-x)\sin\theta$
35	STO 2	y_r
36	R ↓	$-(y_o-y)$
37	R ↓	$-(y_o-y)\cos\theta$
38	STO - 2	y_r
39	R ↓	$-(y_o-y)\sin\theta$
40	STO - 1	x_r
41	RCL 6	θ
42	2	2
43	X	2θ
44	STO C	2θ
45	RCI	ℓ
46	STO ÷ 1	X
47	STO ÷ 2	Y
48	STO ÷ 4	A
49	STO ÷ 5	B
50	RCL 3	P
51	π	π
52	÷	P/π
53	STO ÷ 7	$\pi k\ell^2/P$
54	RCL 4	A
55	RCL 5	B
56	X	AB

PROGRAM $\boxed{3}$

57	RCL 9	h^2
58	X	ABh^2
59	÷	$P/\pi ABh^2$
60	.	
61	7	
62	5	3/4
63	X	$3P/4\pi ABh^2$
64	STO 3	$3P/4\pi ABh^2$
65	STO X 3	$3P\nu/4\pi ABh^2$
66	RCL 8	$3P\nu/4\pi ABh^2$
67	STO + 3	$3P(1+\nu)/4\pi ABh^2$
68	-	$3P(1-\nu)/4\pi ABh^2$
69	STO D	$3P(1-\nu)/4\pi ABh^2$
70	.	
71	5	
72	7	
73	7	
74	2	
75	1	
76	5	
77	6	
78	6	
79	4	
80	9	γ
81	STO E	γ
82	CLX	0
83	STO 6	w
84	RCL 1	X
85	RCL 4	A
86	STO + 1	X + A
87	-	X - A
88	STO 4	X - A
89	RCL 2	Y
90	RCL 5	B
91	STO + 2	Y + B
92	-	Y - B
93	STO 5	Y - B
94	RCL 4	X - A
95	GSB E	kei $\sqrt{X^2+Y^2}$
96	STO 9	σ_{XY}
97	RCL 2	Y + B
98	RCL 4	X - A
99	GSB E	kei $\sqrt{X^2+Y^2}$
100	STO - 9	σ_{XY}
101	RCL 5	Y - B
102	RCL 1	X + A
103	GSB E	kei $\sqrt{X^2+Y^2}$
104	STO - 9	σ_{XY}
105	RCL 2	Y + B
106	RCL 1	X + A
107	GSB E	kei $\sqrt{X^2+Y^2}$
108	STO + 9	σ_{XY}
109	RCL D	$3P(1-\nu)/4\pi ABh^2$
110	STO X 9	σ_{XY}
111	RCL 7	$\pi k\ell^2/P$
112	8	8

PROGRAM $\boxed{3}$

113	X	$8\pi k\ell^2/P$
114	STO ÷ 6	W
115	4	4
116	RTN	
117	LBL E	kei (z)
118	→ P	polar
119	P ⇄ S	
120	2	2
121	÷	z/2
122	STO A	z/2
123	CLX	0
124	STO 0	n
125	STO 1	ker z
126	STO 3	ker'z
127	STO 4	ber'z
128	STO 5	kei z
129	STO 6	bei z
130	STO 7	kei'z
131	STO 8	bei'z
132	STO 9	ϕ
133	1	1
134	STO B	PT
135	STO 2	ber z
136	RCL A	z/2
137	X ≠ 0	z ≠ 0
138	GTO 2	
139	π	π
140	4	4
141	÷	π/4
142	CHS	-π/4
143	GTO 5	
144	LBL 2	
145	8	8
146	STI	index
147	GSB 3	n,ϕ,PT,Σ
148	GSB 4	PT,Σ
149	RCL B	PT
150	CHS	-PT
151	STO B	PT
152	GSB 3	n,ϕ,PT,Σ
153	GTO 4	PT,Σ
154	LBL 3	
155	1	1
156	STO + 0	n
157	RCL 0	n
158	÷	1/n
159	STO + 9	ϕ
160	LBL 4	
161	RCL A	z/2
162	RCL 0	n
163	÷	z/2n
164	RCL B	PT
165	X	PT z/2n
166	STO B	PT
167	STO + (i)	bei'z
168	DSZ	

PROGRAM $\boxed{3}$

169	RCL 9	ϕ
170	X	ϕ PT
171	STO + (i)	kei'z
172	DSZ	
173	RTN	
174	ABS	ABS(ϕ PT)
175	EEX	
176	6	
177	CHS	10^{-6}
178	X < Y	$10^{-6} < \lvert\phi\ PT\rvert$
179	GTO 2	
180	RCL A	z/2
181	LN	LN(z/2)
182	RCL E	γ
183	+	γ + LN(z/2)
184	RCL 6	bei z
185	X	
186	STO - 5	kei z
187	π	π
188	4	4
189	÷	π/4
190	RCL 2	ber z
191	X	
192	STO - 5	kei z
193	RCL 5	kei z
194	LBL 5	
195	P ⇄ S	
196	STO - 6	W
197	RTN	

PROGRAM 4

1	LBL A	
2	RCL 5	Y-B
3	RCL 4	X-A
4	GSB E	I(x,y)
5	STO 7	$(\sigma_X+\sigma_Y)/2$
6	STO 8	$(\sigma_X-\sigma_Y)/2$
7	RCL 2	Y+B
8	RCL 4	X-A
9	GSB E	I(x,y)
10	STO - 7	$(\sigma_X+\sigma_Y)/2$
11	STO - 8	$(\sigma_X-\sigma_Y)/2$
12	RCL 4	X-A
13	RCL 5	Y-B
14	GSB E	I(x,y)
15	STO + 7	$(\sigma_X+\sigma_Y)/2$
16	STO - 8	$(\sigma_X-\sigma_Y)/2$
17	STO 4	X-A
18	RCL 2	Y+B
19	GSB E	I(x,y)
20	STO - 7	$(\sigma_X+\sigma_Y)/2$
21	STO + 8	$(\sigma_X-\sigma_Y)/2$
22	RCL 2	Y+B
23	RCL 1	X+A
24	GSB E	I(x,y)
25	STO + 7	$(\sigma_X+\sigma_Y)/2$
26	STO + 8	$(\sigma_X-\sigma_Y)/2$
27	RCL 5	Y-B
28	RCL 1	X+A
29	GSB E	I(x,y)
30	STO - 7	$(\sigma_X+\sigma_Y)/2$
31	STO - 8	$(\sigma_X-\sigma_Y)/2$
32	RCL 1	X+A
33	RCL 5	Y-B
34	GSB E	I(x,y)
35	STO - 7	$(\sigma_X+\sigma_Y)/2$
36	STO + 8	$(\sigma_X-\sigma_Y)/2$
37	RCL 1	X+A
38	RCL 2	Y+B
39	GSB E	I(x,y)
40	STO + 7	$(\sigma_X+\sigma_Y)/2$
41	STO - 8	$(\sigma_X-\sigma_Y)/2$
42	RCL 3	$3P(1+\nu)/4\pi ABh^2$
43	STO X 7	$(\sigma_X+\sigma_Y)/2$
44	RCL D	$3P(1-\nu)/4\pi ABh^2$
45	STO X 8	$(\sigma_X-\sigma_Y)/2$
46	RCL C	2θ
47	RCL 9	σ_{XY}
48	RCL C	2θ
49	RCL 8	$(\sigma_X-\sigma_Y)/2$
50	R ←	rect
51	STO 8	$(\sigma_x-\sigma_y)/2$
52	R ↓	
53	STO 9	σ_{xy}
54	R ↓	
55	R ←	rect
56	STO + 9	σ_{xy}

PROGRAM 4

57	R ↓	
58	STO - 8	$(\sigma_x-\sigma_y)/2$
59	5	5
60	RTN	
61	LBL E	I(x,y)
62	X = 0	x = 0
63	RTN	
64	÷	y/x
65	X = 0	y = 0
66	RTN	
67	P ⇄ S	
68	STO B	y/x
69	LST X	x
70	2	2
71	÷	x/2
72	x^2	$(x/2)^2$
73	STO A	$(x/2)^2$
74	X	$(x/2)^2(y/x)$
75	STO 2	P
76	1	1
77	RCL B	y/x
78	TAN^{-1}	$TAN^{-1}(y/x)$
79	→ R	radians
80	RCL B	y/x
81	÷	$TAN^{-1}(y/x)/(y/x)$
82	-	Q
83	STO 3	Q
84	RCL B	y/x
85	x^2	$(y/x)^2$
86	STO B	$(y/x)^2$
87	1	1
88	+	$1+(y/x)^2$
89	RCL A	$(x/2)^2$
90	X	$R^2/4$
91	√	R/2
92	LN	LN(R/2)
93	RCL E	γ
94	+	γ+LN(R/2)
95	STO 9	$-\phi_o$
96	-	$Q+\phi_o$
97	.	
98	5	.5
99	+	$Q+\phi_o+.5$
100	X	Σ_o
101	STO 1	Σ
102	CLX	0
103	STO 0	n
104	GTO 1	
105	LBL 2	
106	GSB 3	
107	.	
108	5	.5
109	RCL 0	n
110	1	1
111	+	n+1
112	÷	.5/(n+1)

PROGRAM 4

113	RCL 9	$-\phi$
114	-	$\phi+.5/(n+1)$
115	X	
116	STO + 1	Σ
117	SF 0	
118	GSB 4	
119	STO + 1	Σ
120	LBL 1	
121	RCL 2	P
122	CHS	-P
123	STO 2	P
124	CF 0	
125	GSB 3	
126	π	π
127	X	
128	4	4
129	÷	
130	STO - 1	Σ
131	RCL 1	Σ
132	÷	
133	ABS	
134	EEX	
135	6	
136	CHS	10^{-6}
137	X < Y	
138	GTO 2	
139	RCL 1	Σ
140	P ⇄ S	

141	RTN	
142	LBL 3	
143	RCL A	$(x/2)^2$
144	1	1
145	STO + 0	n
146	RCL 0	n
147	+	n+1
148	÷	$(x/2)^2/(n+1)$
149	RCL 0	n
150	1/X	1/n
151	STO - 9	$-\phi$
152	X	$(x/2)^2/(n+1)n$
153	STO x 2	P
154	LBL 4	
155	CLX	0
156	STI	k
157	1	1
158	STO 5	P_k
159	F? 0	
160	RCL 3	Q
161	STO 4	Σ_k
162	LBL 5	
163	RCL 0	n
164	RCI	k-1
165	-	n-k+1
166	ISZ	
167	RCI	k
168	÷	(n-k+1)/k

PROGRAM 4

169	F? 0	
170	GTO 7	
171	RCL B	$(y/x)^2$
172	X	
173	STO X 5	P_k
174	RCL 5	P_k
175	LBL 6	
176	RCI	k
177	2	2
178	X	2k
179	1	1
180	+	2k+1
181	÷	
182	STO + 4	Σk
183	RCI	k
184	RCL 0	n
185	X > Y	n > k
186	GTO 5	
187	RCL 4	Σ_k
188	RCL 2	P
189	X	$P\ \Sigma_k$
190	RTN	
191	LBL 7	
192	CHS	-(n-k+1)/k
193	STO X 5	P_k
194	CLX	0
195	STO 6	r
196	RCL 3	Q

197	STO 7	Σ_r
198	1	1
199	STO 8	P_r
200	LBL 8	
201	RCL B	$(y/x)^2$
202	CHS	$-(y/x)^2$
203	STO X 8	P_r
204	RCL 8	P_r
205	1	1
206	STO + 6	r
207	RCL 6	r
208	2	2
209	x	2r
210	+	2r+1
211	÷	$P_r/2r+1)$
212	STO + 7	Σ
213	RCL 6	r
214	RCI	k
215	X > Y	k > r
216	GTO 8	
217	RCL 7	Σ_r
218	RCL 5	P_k
219	X	$P_k\ \Sigma r$
220	GTO 6	

PROGRAM 5

1	LBL A	
2	4	4
3	STI	
4	CF 3	
5	RCL 0	
6	MERGE	
7	PAUSE	
8	F? 3	
9	GTO 1	
10	GTO A	
11	LBL 1	
12	STO 0	ID#
13	RCL 6	w
14	STO + 1	Σw
15	RCL 7	$(\sigma_x+\sigma_y)/2$
16	STO + 2	$\Sigma(\sigma_x+\sigma_y)/2$
17	RCL 8	$(\sigma_x-\sigma_y)/2$
18	STO + 3	$\Sigma(\sigma_x-\sigma_y)/2$
19	RCL 9	σ_{xy}
20	STO + 4	$\Sigma\ \sigma_{xy}$
21	P $\rightleftarrows$ S	
22	CL REG	
23	P $\rightleftarrows$ S	
24	3	
25	6	
26	0	360
27	RCL 4	$\Sigma\ \sigma_{xy}$
28	RCL 3	$\Sigma(\sigma_x-\sigma_y)/2$
29	→ P	polar
30	RCL 2	$\Sigma(\sigma_x+\sigma_y)/2$
31	+	σ
32	STO A	σ
33	R ↓	
34	X < 0	angle < 0
35	+	
36	2	2
37	÷	
38	9	
39	0	90
40	−	angle
41	STO B	angle
42	RCL A	σ
43	W/DATA	
44	RTN	
45	LBL B	Begin Load
46	CLX	0
47	STO 1	ΣW
48	STO 2	$\Sigma(\sigma_x+\sigma_y)/2$
49	STO 3	$\Sigma(\sigma_x-\sigma_y)/2$
50	STO 4	$\Sigma\ \sigma_{xy}$
51	STO 5	
52	RCL 0	ID#
53	GTO 1	

Technical Report 67

JUNE, 1959

Tables of Kelvin Functions and their Derivatives

by D. E. Nevel

U. S. ARMY SNOW ICE AND PERMAFROST
RESEARCH ESTABLISHMENT
Corps of Engineers
Wilmette, Illinois

TABLES OF KELVIN FUNCTIONS AND THEIR DERIVATIVES

by

D. E. Nevel

The bearing capacity of a floating ice sheet can be computed by considering it a plate on an elastic foundation. The differential equation describing this is

$$\nabla_{\chi}^{4} w + w = \frac{q}{k} \quad \text{where the operator}$$

$$\nabla_{\chi}^{2} = \frac{d^2}{d\chi^2} + \frac{1}{\chi^2}\frac{d^2}{d\theta^2} + \frac{1}{\chi}\frac{d}{d\chi} .$$

Assuming that the boundary conditions are symmetrical with respect to the origin, the differential equation becomes

$$\frac{d^4 w}{d\chi^4} + \frac{1}{\chi}\frac{d^3 w}{d\chi^3} - \frac{2}{\chi^2}\frac{d^2 w}{d\chi^2} + \frac{2}{\chi^3}\frac{dw}{d\chi} + w = \frac{q}{k}$$

where

q = uniform distributed load

k = foundation modulus

w = deflection

x, θ = polar coordinates

$\chi = x/\ell$

$\ell = \sqrt[4]{\frac{Eh^3}{12k(1-\mu^2)}}$ = characteristic length

E = modulus of elasticity

h = thickness of the plate

μ = Poisson's ratio.

The solution of this differential equation is given in terms of Kelvin functions as (Wyman, 1950):

$$w = A\,\mathrm{ber}(\chi) + B\,\mathrm{bei}(\chi) + C\,\mathrm{ker}(\chi) + D\,\mathrm{kei}(\chi) + \frac{q}{k} .$$

These Kelvin functions are also important in other physical problems, for instance in the fields of electrical engineering, heat flow, and shell analysis. Because of the wide application of Kelvin functions, there is a need for a detailed accurate table presently unavailable.

The formulas for these functions and their first derivatives are:

$$\mathrm{ber}(\chi) = \sum_{n=0,2,4,6}^{\infty} \frac{(-1)^{\frac{n}{2}} \left(\frac{\chi}{2}\right)^{2n}}{n!\; n!} \qquad \mathrm{ber}_{\phi}(\chi) = \sum_{n=2,4,6}^{\infty} \frac{(-1)^{\frac{n}{2}} \left(\frac{\chi}{2}\right)^{2n}}{n!\; n!}\, \phi(n)$$

$$\mathrm{ber}'(\chi) = \sum_{n=2,4,6}^{\infty} \frac{(-1)^{\frac{n}{2}} (\frac{\chi}{2})^{2n-1}}{(n-1)!\ n!} \qquad \mathrm{ber}_{\phi}'(\chi) = \sum_{n=2,4,6}^{\infty} \frac{(-1)^{\frac{n}{2}} (\frac{\chi}{2})^{2n-1}}{(n-1)!\ n!} \phi(n)$$

$$\mathrm{bei}(\chi) = \sum_{n=1,3,5}^{\infty} \frac{(-1)^{\frac{n-1}{2}} (\frac{\chi}{2})^{2n}}{n!\ n!} \qquad \mathrm{bei}_{\phi}(\chi) = \sum_{n=1,3,5}^{\infty} \frac{(-1)^{\frac{n-1}{2}} (\frac{\chi}{2})^{2n}}{n!\ n!} \phi(n)$$

$$\mathrm{bei}'(\chi) = \sum_{n=1,3,5}^{\infty} \frac{(-1)^{\frac{n-1}{2}} (\frac{\chi}{2})^{2n-1}}{(n-1)!\ n!} \qquad \mathrm{bei}_{\phi}'(\chi) = \sum_{n=1,3,5}^{\infty} \frac{(-1)^{\frac{n-1}{2}} (\frac{\chi}{2})^{2n-1}}{(n-1)!\ n!} \phi(n)$$

$$\mathrm{ker}(\chi) = \mathrm{ber}_{\phi}(\chi) - \mathrm{ber}(\chi) \ln \frac{\gamma\chi}{2} + \frac{\pi}{4} \mathrm{bei}(\chi)$$

$$\mathrm{ker}'(\chi) = \mathrm{ber}_{\phi}'(\chi) - \mathrm{ber}'(\chi) \ln \frac{\gamma\chi}{2} + \frac{\pi}{4} \mathrm{bei}'(\chi) - \frac{\mathrm{ber}(\chi)}{\chi}$$

$$\mathrm{kei}(\chi) = \mathrm{bei}_{\phi}(\chi) - \mathrm{bei}(\chi) \ln \frac{\gamma\chi}{2} - \frac{\pi}{4} \mathrm{ber}(\chi)$$

$$\mathrm{kei}'(\chi) = \mathrm{bei}_{\phi}'(\chi) - \mathrm{bei}'(\chi) \ln \frac{\gamma\chi}{2} - \frac{\pi}{4} \mathrm{ber}'(\chi) - \frac{\mathrm{bei}(\chi)}{\chi}$$

where $\phi(n) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

and $\ln \gamma = 0.577215664901533$ (Euler's constant)

Other important derivatives and integrals may be expressed in terms of the functions and first derivatives.

$$\mathrm{ber}''(\chi) = -\mathrm{bei}(\chi) - \frac{\mathrm{ber}'(\chi)}{\chi}$$

$$\mathrm{ber}'''(\chi) = -\mathrm{bei}'(\chi) + \frac{\mathrm{bei}(\chi)}{\chi} + \frac{2\,\mathrm{ber}'(\chi)}{\chi^2}$$

$$\int \chi\, \mathrm{ber}(\chi)\, d\chi = \chi\, \mathrm{bei}'(\chi)$$

$$\mathrm{bei}''(\chi) = \mathrm{ber}(\chi) - \frac{\mathrm{bei}'(\chi)}{\chi}$$

$$\mathrm{bei}'''(\chi) = \mathrm{ber}'(\chi) - \frac{\mathrm{ber}(\chi)}{\chi} + \frac{2\,\mathrm{bei}'(\chi)}{\chi^2}$$

$$\int \chi\, \mathrm{bei}(\chi)\, d\chi = -\chi\, \mathrm{ber}'(\chi)$$

$$\mathrm{ker}''(\chi) = -\mathrm{kei}(\chi) - \frac{\mathrm{ker}'(\chi)}{\chi}$$

$$\mathrm{ker}'''(\chi) = -\mathrm{kei}'(\chi) + \frac{\mathrm{kei}(\chi)}{\chi} + \frac{2\,\mathrm{ker}'(\chi)}{\chi^2}$$

$$\int \chi\, \mathrm{ker}(\chi)\, d\chi = \chi\, \mathrm{kei}'(\chi)$$

$$\mathrm{kei}''(\chi) = \mathrm{ker}(\chi) - \frac{\mathrm{kei}'(\chi)}{\chi}$$

$$\mathrm{kei}'''(\chi) = \mathrm{ker}'(\chi) - \frac{\mathrm{ker}(\chi)}{\chi} + \frac{2\,\mathrm{kei}'(\chi)}{\chi^2}$$

$$\int \chi\,\mathrm{kei}(\chi)\,d\chi = -\chi\,\mathrm{ker}'(\chi)$$

Table I gives the Kelvin functions and their first derivatives. Table II gives $\mathrm{ber}_\phi(\chi)$ and $\mathrm{bei}_\phi(\chi)$ and their first derivatives. These functions were computed by means of a Bendix G-15 D electronic computer.

The tabulated value consists of a sign, followed by two digits representing a floating point exponent p as used in Bendix G-15 D Intercom 1000 D (Bendix Computer Division, 1958), followed by a twelve digit number. The correct value of the function is

$$(\text{sign})\ (12 \text{ digit number})\ (10)^{p-50}$$

Example: $\mathrm{ker}'(0.050) = -52.199803991221$ in floating point
or -19.9803991221 in decimal notation.

The computed functions are accurate to the last digit; except that $\mathrm{ker}(\chi)$, $\mathrm{ker}'(\chi)$, $\mathrm{kei}(\chi)$, and $\mathrm{kei}'(\chi)$ are occasionally in error only in the last digit.

The values of the functions are plotted in Figures 1-3.

The appendix gives the method used in computing the functions.

The solution of the symmetrical plate problem is

$$w = A\,\mathrm{ber}(\chi) + B\,\mathrm{bei}(\chi) + C\,\mathrm{ker}(\chi) + D\,\mathrm{kei}(\chi) + \frac{q}{k}$$

$$\text{radial slope} = \frac{1}{\ell}\frac{dw}{d\chi}$$

$$\text{radial moment} = -k\ell^2\left[\frac{d^2w}{d\chi^2} + \frac{\mu}{\chi}\frac{dw}{d\chi}\right]$$

$$\text{radial shear} = -k\ell\left[-A\,\mathrm{bei}'(\chi) + B\,\mathrm{ber}'(\chi) - C\,\mathrm{kei}'(\chi) + D\,\mathrm{ker}'(\chi)\right]$$

$$\text{tangential slope} = 0$$

$$\text{tangential moment} = -k\ell^2\left[\frac{1}{\chi}\frac{dw}{d\chi} + \mu\frac{d^2w}{d\chi^2}\right]$$

$$\text{tangential shear} = 0$$

$$\text{displaced vol of water} = 2\pi\ell^2\int \chi w\,d\chi.$$

REFERENCES

Bendix Computer Division of Bendix Aviation Corp* (1958) Manual CB-078, Intercom 1000 programming system for the Bendix G-15 computer.

McLachlan, N. W. (1934) Bessel functions for engineers, Oxford University Press.

Wyman, M. (1950) Deflections of an infinite plate, Canadian Journal of Research, A28.293-302.

* 5630 Arbor Vitae Street, Los Angeles 45, California.

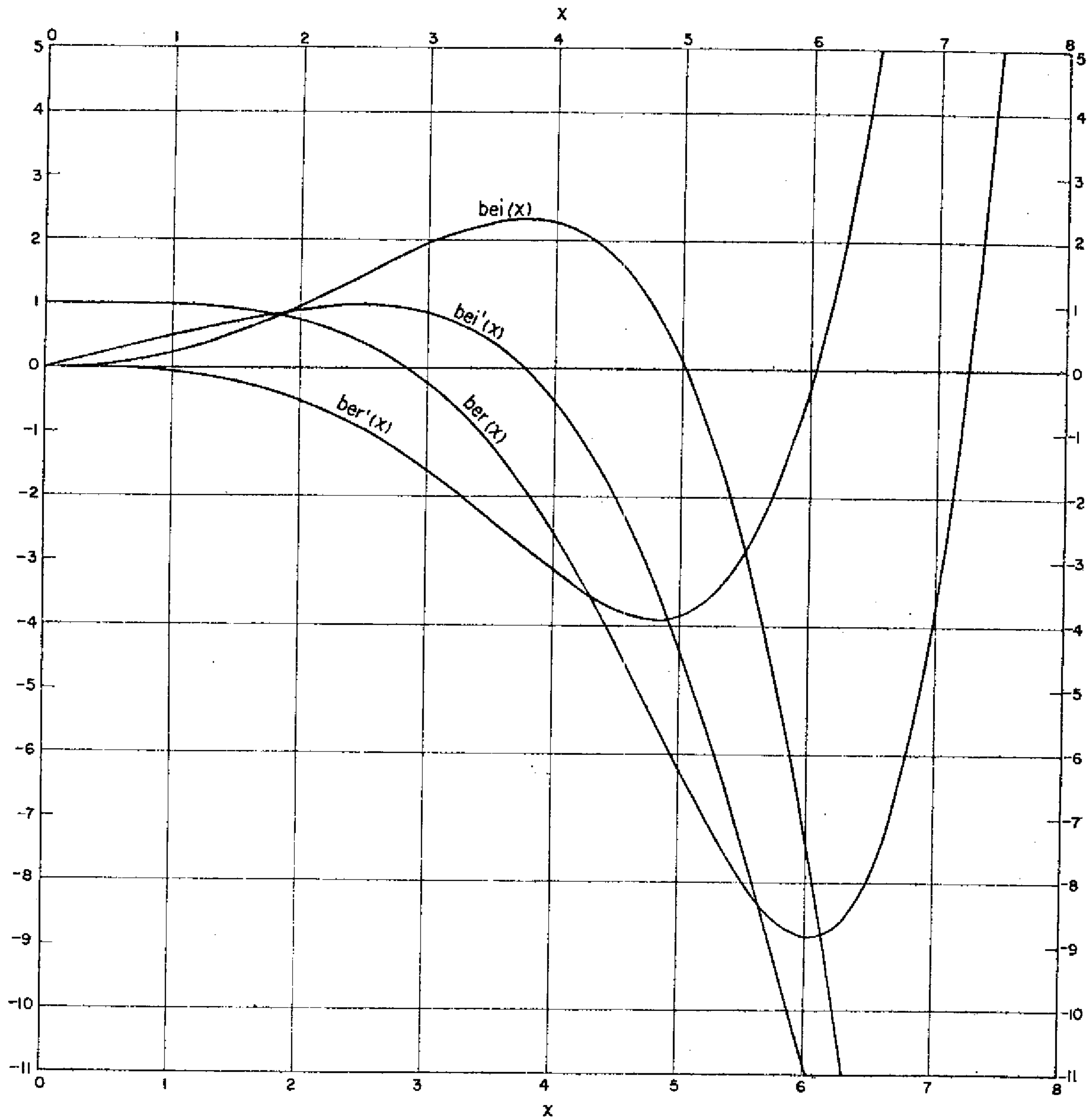

Figure 1.

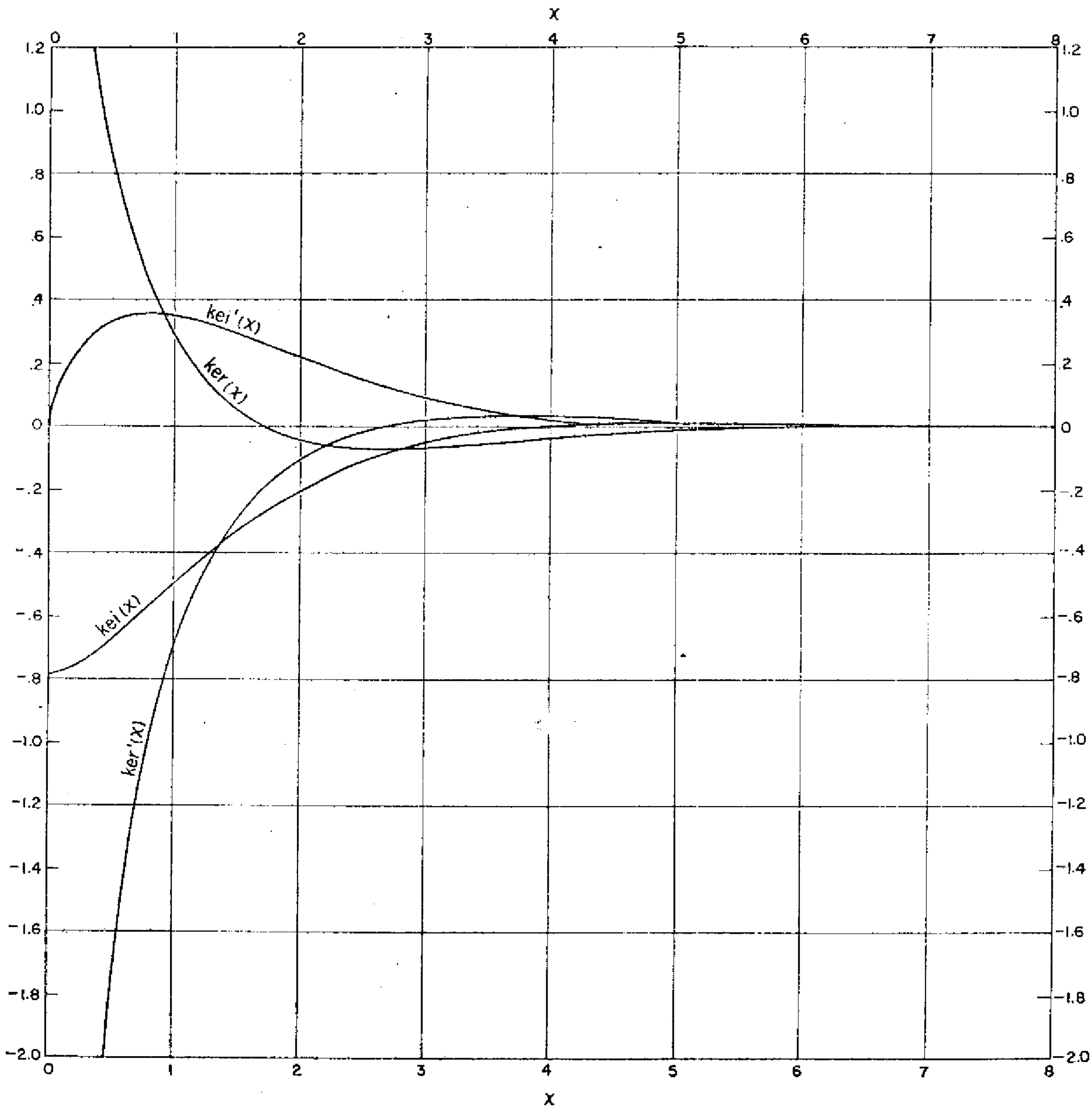
x
0
1
2
3
4
5
6
7
8
1.2
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
-1.2
-1.4
-1.6
-1.8
-2.0
kei'(x)
ker(x)
kei(x)
ker'(x)

Figure 2.

χ	ber(χ)	ber'(χ)	bei(χ)	bei'(χ)	ker(χ)	ker'(χ)	kei(χ)	kei'(χ)	χ
.000	51.100000000000	-00.000000000000	00.000000000000	00.000000000000	+ ∞	- ∞	-50.785398163397	00.000000000000	.000
.001	51.100000000000	-40.624999912935	44.249999999993	47.500000000000	51.702368699099	-53.999999607301	-50.785396157476	48.376184344641	.001
.002	51.100000000000	-41.499999999737	44.999999999987	48.100000000000	51.633054039948	-53.499999214606	-50.785390832858	48.683054000679	.002
.003	50.999999999999	-42.168749998697	45.224999999999	48.150000000000	51.592507627311	-53.333332155248	-50.785382581979	48.963761308432	.003
.004	50.999999999996	-42.399999999789	45.399999999999	48.200000000000	51.563739557509	-53.249998429231	-50.785371613825	49.122747880086	.004
.005	50.999999999990	-42.781249999415	45.624999999999	48.249999999999	51.541425379088	-53.199998036557	-50.785358074334	49.147856283414	.005
.006	50.999999999980	-43.134999999929	45.899999999997	48.299999999998	51.523193439386	-53.166664310560	-50.785342076036	49.171957925790	.006
.007	50.999999999962	-43.214374999973	46.122499999999	48.349999999996	51.507778626647	-53.142854394099	-50.785323710604	49.195222350962	.007
.008	50.999999999936	-43.319999999969	46.159999999999	48.399999999991	51.494425781892	-53.124996858606	-50.785303055423	49.217770061440	.008
.009	50.999999999897	-43.455624999948	46.202499999998	48.449999999985	51.482647812097	-53.111107577096	-50.785280177457	49.239691157616	.009
.010	50.999999999844	-43.624999999943	46.249999999996	48.499999999974	51.472112133563	-52.999960733824	-50.785255135732	49.261055575940	.010
.011	50.999999999771	-43.831874999985	46.302499999992	48.549999999958	51.462581527874	-52.909047717080	-50.785227983024	49.281919187028	.011
.012	50.999999999676	-44.107999999997	46.359999999987	48.599999999935	51.453880841723	-52.833286215695	-50.785198767058	49.302327656881	.012
.013	50.999999999554	-44.137312499992	46.422499999979	48.649999999903	51.445877061759	-52.769179726189	-50.785167531390	49.322319011807	.013
.014	50.999999999400	-44.171499999987	46.489999999967	48.699999999860	51.438466794600	-52.714230746078	-50.785134316082	49.341925409425	.014
.015	50.999999999209	-44.210937499986	46.562499999950	48.749999999802	51.431568076759	-52.666607773544	-50.785099158218	49.361174401096	.015
.016	50.999999998976	-44.255999999976	46.639999999927	48.799999999727	51.425114833200	-52.624937182230	-50.785062092317	49.380089856250	.016
.017	50.999999998695	-44.307062499974	46.722499999895	48.849999999630	51.419053018817	-52.588168551979	-50.785023150666	49.398692654746	.017
.018	50.999999998360	-44.364499999955	46.809999999852	48.899999999508	51.413337864476	-52.555484889343	-50.784982363591	49.417001215798	.018
.019	50.999999997964	-44.428687499950	46.902499999796	48.949999999355	51.407931868630	-52.526241199494	-50.784939759683	49.435031909003	.019
.020	50.999999997500	-44.499999999917	46.999999999722	48.999999999167	51.402803304708	-52.499921486573	-50.784895365982	49.452799378625	.020
.021	50.999999996961	-44.578812499901	47.110249999963	49.104999999894	51.397925093041	-52.476108039650	-50.784849208141	49.470316802859	.021
.022	50.999999996340	-44.665499999861	47.120999999951	49.109999999866	51.393273935458	-52.454459095238	-50.784801310558	49.487596103614	.022
.023	50.999999995627	-44.760437499812	47.132249999936	49.114999999832	51.388829642406	-52.434692326980	-50.784751696495	49.504648118083	.023
.024	50.999999994816	-44.863999999740	47.143999999917	49.119999999793	51.384574603392	-52.416572462913	-50.784700388180	49.521482740412	.024
.025	50.999999993896	-44.976562499660	47.156249999894	49.124999999746	51.380493365586	-52.399901874593	-50.784647406890	49.538109039734	.025
.026	50.999999992860	-45.109849999956	47.168999999866	49.129999999691	51.376572295131	-52.384513337951	-50.784592773037	49.554535359279	.026
.027	50.999999991696	-45.123018749943	47.182249999832	49.134999999626	51.372799302404	-52.370264402855	-50.784536506227	49.570769400220	.027
.028	50.999999990396	-45.137199999927	47.195999999791	49.139999999552	51.369163617265	-52.357032969197	-50.784478625328	49.586818293086	.028
.029	50.999999988949	-45.152431249905	47.210249999742	49.144999999466	51.365655603763	-52.344713778262	-50.784419148519	49.602688658954	.029
.030	50.999999987344	-45.168749999881	47.224999999684	49.149999999367	51.362266606273	-52.333215605833	-50.784358093341	49.618386662194	.030
.031	50.999999985570	-45.186193749849	47.240249999615	49.154999999254	51.358988820862	-52.322458998559	-50.784295476740	49.633918056178	.031
.032	50.999999983616	-45.204799999813	47.255999999534	49.159999999126	51.355815187090	-52.312374434761	-50.784231315104	49.649288223084	.032
.033	50.999999981470	-45.224606249768	47.272249999439	49.164999998981	51.352739296467	-52.302900819633	-50.784165624301	49.664502208714	.033
.034	50.999999979120	-45.245649999715	47.288999999329	49.169999998817	51.349755314575	-52.293984245990	-50.784098419710	49.679564753100	.034
.035	50.999999976553	-45.267968749650	47.306249999202	49.174999998632	51.346857914484	-52.285576967473	-50.784029716249	49.694480317494	.035
.036	50.999999973756	-45.291599999574	47.323999999055	49.179999998425	51.344042219554	-52.277636542875	-50.783959528405	49.709253108282	.036
.037	50.999999970716	-45.316581249485	47.342249998886	49.184999998194	51.341303754065	-52.270125119226	-50.783887870254	49.723887098230	.037
.038	50.999999967420	-45.342949999379	47.360999998693	49.189999997937	51.338638400446	-52.263008828082	-50.783814755487	49.738386045438	.038
.039	50.999999963852	-45.370743749254	47.380249998473	49.194999997650	51.336042362048	-52.256257274688	-50.783740197427	49.752753510295	.039
.040	50.999999960000	-45.399999999110	47.399999998222	49.199999997333	51.333512130640	-52.249843103762	-50.783664209054	49.766992870694	.040
.041	50.999999955847	-45.430756248942	47.420249997938	49.204999996983	51.331044457913	-52.243741628833	-50.783586803016	49.781107335728	.041
.042	50.999999951380	-45.463049998748	47.440999997618	49.209999996597	51.328636330425	-52.237930514524	-50.783507991649	49.795099958041	.042
.043	50.999999946581	-45.496918748525	47.462249997256	49.214999996172	51.326284947484	-52.232389503167	-50.783427786989	49.808973645010	.043
.044	50.999999941436	-45.532399998267	47.483999996851	49.219999995705	51.323987701587	-52.227100178701	-50.783346200792	49.822731168887	.044
.045	50.999999935928	-45.569531247973	47.506249996396	49.224999995195	51.321742161046	-52.222045762050	-50.783263244539	49.836375176007	.045
.046	50.999999930040	-45.608349997635	47.528999995888	49.229999994636	51.319546054531	-52.217210933189	-50.783178929456	49.849908195196	.046
.047	50.999999923755	-45.648893747251	47.552249995322	49.234999994027	51.317397257273	-52.212581675924	-50.783093266519	49.863332645426	.047
.048	50.999999917056	-45.691199996815	47.575999994692	49.239999993364	51.315293778718	-52.208145142079	-50.783006266468	49.876650842839	.048
.049	50.999999909925	-45.735306246320	47.600249993992	49.244999992644	51.313233751456	-52.203889532310	-50.782917939818	49.889865007169	.049
.050	50.999999902344	-45.781249995761	47.624999993218	49.249999991862	51.311215421270	-52.199803991221	-50.782828296864	49.902977267648	.050

x	ber (x)	ber'(x)	bei(x)	bei'(x)	ker(x)	ker'(x)	kei(x)	kei'(x)	x
.050	50.999999902344	-45.781249995761	47.624999993218	49.249999991862	51.311215421270	-52.199803991221	-50.782828296864	49.902977267648	.050
.051	50.999999894294	-45.829068745131	47.650249992363	49.254999991015	51.309237138168	-52.195878514818	-50.782737347693	49.915989668440	.051
.052	50.999999885756	-45.878799994422	47.675999991419	49.259999990099	51.307297348234	-52.192103868649	-50.782645102193	49.928904173643	.052
.053	50.999999876711	-45.930481243627	47.702249990380	49.264999989109	51.305394566556	-52.188471515201	-50.782551570058	49.941722671910	.053
.054	50.999999867140	-45.984149992735	47.728999989238	49.269999988043	51.303527470085	-52.184973549369	-50.782456760798	49.954446980724	.054
.055	50.999999857021	-46.103984374174	47.756249987986	49.274999986894	51.301694692102	-52.181602640968	-50.782360683744	49.967078850346	.055
.056	50.999999846336	-46.109759999063	47.783999986614	49.279999985658	51.299895016482	-52.178351983394	-50.782263348053	49.979619967490	.056
.057	50.999999835062	-46.115745623939	47.812249985114	49.284999984331	51.298127272743	-52.175215247705	-50.782164762721	49.992071958721	.057
.058	50.999999823180	-46.121944998802	47.840999983477	49.289999982907	51.296390351475	-52.172186541433	-50.782064936530	50.100443639363	.058
.059	50.999999810666	-46.128361873650	47.870249981692	49.294999981382	51.294683200167	-52.169260371606	-50.781963878310	50.101671478777	.059
.060	50.999999797500	-46.134999998481	47.899999979750	49.299999979750	51.293004819381	-52.166431611445	-50.781861596442	50.102890860541	.060
.061	50.999999783659	-46.141863123295	47.930249977639	49.304999978005	51.291354259242	-52.163695470357	-50.781758099361	50.104101926210	.061
.062	50.999999769120	-46.148954998089	47.960999975347	49.309999976142	51.289730616213	-52.161047466818	-50.781653395316	50.105304812709	.062
.063	50.999999753860	-46.156279372863	47.992249972863	49.314999974155	51.288133030133	-52.158483403849	-50.781547492418	50.106499652551	.063
.064	50.999999737856	-46.163839997614	48.102399997017	49.319999972038	51.286560681477	-52.155999346779	-50.781440398650	50.107686574053	.064
.065	50.999999721084	-46.171640622340	48.105624996727	49.324999969784	51.285012788844	-52.153591603073	-50.781332121868	50.108865701532	.065
.066	50.999999703520	-46.179684997040	48.108899996413	49.329999967387	51.283488606623	-52.151256703986	-50.781222669305	50.110037155465	.066
.067	50.999999685139	-46.187976871712	48.112224996074	49.334999964840	51.281987422842	-52.148991387863	-50.781112050075	50.111201052768	.067
.068	50.999999665916	-46.196519996352	48.115599995709	49.339999962137	51.280508557175	-52.146792584911	-50.781000270180	50.112357506758	.068
.069	50.999999645826	-46.205318120960	48.119024995316	49.344999959270	51.279051359091	-52.144657403297	-50.780887337506	50.113506627504	.069
.070	50.999999624844	-46.214374995532	48.122499994894	49.349999956232	51.277615206144	-52.142583116435	-50.780773259334	50.114648521872	.070
.071	50.999999602942	-46.223694370065	48.126024994440	49.354999953015	51.276199502373	-52.140567151351	-50.780658042837	50.115783293686	.071
.072	50.999999580096	-46.233279994558	48.129599993953	49.359999949612	51.274803676826	-52.138607078015	-50.780541695087	50.116911043851	.072
.073	50.999999556277	-46.243135619006	48.133224993432	49.364999946014	51.273427182177	-52.136700599547	-50.780424223057	50.118031870476	.073
.074	50.999999531460	-46.253264993407	48.136899992873	49.369999942213	51.272069493440	-52.134845543226	-50.780305633622	50.119145868992	.074
.075	50.999999505615	-46.263671867758	48.140624992275	49.374999938202	51.270730106773	-52.133039852209	-50.780185923564	50.120253132258	.075
.076	50.999999478716	-46.274359992054	48.144399991636	49.379999933971	51.269408538352	-52.131281577907	-50.780065129572	50.121353750663	.076
.077	50.999999450734	-46.285333116293	48.148224990954	49.384999929511	51.268104323328	-52.129568872958	-50.779943228248	50.122447812224	.077
.078	50.999999421640	-46.296594990470	48.152099990226	49.389999924813	51.266817014847	-52.127899984736	-50.779820236105	50.123535402683	.078
.079	50.999999391405	-46.308149364581	48.156024989449	49.394999919868	51.265546183133	-52.126273249358	-50.779696159572	50.124616605586	.079
.080	50.999999360000	-46.319999988622	48.159999988622	49.399999914667	51.264291414628	-52.124687086135	-50.779571004996	50.125691502375	.080
.081	50.999999327395	-46.332150612588	48.164024987742	49.404999909198	51.263052311186	-52.123139992430	-50.779444778643	50.126760172459	.081
.082	50.999999293560	-46.344604986475	48.168099986805	49.409999903453	51.261828489319	-52.121630538917	-50.779317486701	50.127822693298	.082
.083	50.999999258464	-46.357366860278	48.172224985810	49.414999897421	51.260619579483	-52.120157365148	-50.779189135281	50.128879140465	.083
.084	50.999999222076	-46.370439983990	48.176399984753	49.419999891091	51.259425225410	-52.118719175457	-50.779059730420	50.129929587721	.084
.085	50.999999184365	-46.383828107607	48.180624983631	49.424999884452	51.258245083482	-52.117314735154	-50.778929278081	50.130974107079	.085
.086	50.999999145300	-46.397534981124	48.184899982441	49.429999877493	51.257078822139	-52.115942866974	-50.778797784158	50.132012768861	.086
.087	50.999999104848	-46.411564354533	48.189224981179	49.434999870203	51.255926121318	-52.114602447782	-50.778665254473	50.133045641762	.087
.088	50.999999062976	-46.425919977828	48.193599979844	49.439999862570	51.254786671932	-52.113292405494	-50.778531694782	50.134072792904	.088
.089	50.999999019653	-46.440605601003	48.198024978430	49.444999854582	51.253660175370	-52.112011716208	-50.778397110773	50.135094287891	.089
.090	50.999998974844	-46.455624974051	48.202499976934	49.449999846227	51.252546343031	-52.110759401531	-50.778261508070	50.136110190856	.090
.091	50.999998928516	-46.470981846964	48.207024975353	49.454999837492	51.251444895880	-52.109534526077	-50.778124892234	50.137120564517	.091
.092	50.999998880636	-46.486679969735	48.211599973683	49.459999828364	51.250355564032	-52.108336195127	-50.777987268764	50.138125470217	.092
.093	50.999998831169	-46.502723092356	48.216224971919	49.464999818831	51.249278086354	-52.107163552448	-50.777848643096	50.139124967973	.093
.094	50.999998780080	-46.519114964818	48.220899970058	49.469999808879	51.248212210092	-52.106015778239	-50.777709020611	50.140119116516	.094
.095	50.999998727334	-46.535859337113	48.225624968095	49.474999798495	51.247157690515	-52.104892087220	-50.777568406627	50.141107973336	.095
.096	50.999998672896	-46.552959959231	48.230399966026	49.479999787663	51.246114290582	-52.103791726831	-50.777426806409	50.142091594716	.096
.097	50.999998616730	-46.570420581164	48.235224963847	49.484999776371	51.245081780618	-52.102713975543	-50.777284225164	50.143070035773	.097
.098	50.999998558800	-46.588244952901	48.240099961552	49.489999764604	51.244059938018	-52.101658141282	-50.777140668046	50.144043350494	.098
.099	50.999998499069	-46.606436824432	48.245024959137	49.494999752346	51.243048546953	-52.100623559932	-50.776996140155	50.145011591770	.099
.100	50.999998437500	-46.624999945746	48.249999956597	49.499999739583	51.242047398104	-51.996095939455	-50.776850646537	50.145974811429	.100

χ	ber(χ)	ber'(χ)	bei(χ)	bei'(χ)	ker(χ)	ker'(χ)	kei(χ)	kei'(χ)	χ
.100	50.999998437500	-46.624999945746	48.249999956597	49.499999739583	51.242047398104	-51.996095939455	-50.776850646537	50.145974811429	.100
.101	50.999998374056	-46.643938066833	48.255024953927	49.504999726299	51.241056288400	-51.986156310226	-50.776704192189	50.146933060267	.101
.102	50.999998308700	-46.663254937680	48.260099951121	49.509999712479	51.240075020770	-51.976410828763	-50.776556782056	50.147886388082	.102
.103	50.999998241393	-46.682954308275	48.265224948175	49.514999698106	51.239103403913	-51.966853840656	-50.776408421037	50.148834843699	.103
.104	50.999998172096	-46.703039928606	48.270399945082	49.519999683163	51.238141252072	-51.957479908976	-50.776259113977	50.149778475002	.104
.105	50.999998100772	-46.723515548660	48.275624941836	49.524999667635	51.237188384822	-51.948283803917	-50.776108865679	50.150717328958	.105
.106	50.999998027380	-46.744384918423	48.280899938432	49.529999651504	51.236244626868	-51.939260493025	-50.775957680896	50.151651451646	.106
.107	50.999997951881	-46.765651787881	48.286224934864	49.534999634752	51.235309807855	-51.930405131980	-50.775805564338	50.152580888281	.107
.108	50.999997874236	-46.787319907019	48.291599931125	49.539999617362	51.234383762178	-51.921713055880	-50.775652520667	50.153505683237	.108
.109	50.999997794404	-46.809393025822	48.297024927209	49.544999599317	51.233466328814	-51.913179771018	-50.775498554504	50.154425880072	.109
.110	50.999997712344	-46.831874894275	48.302499923109	49.549999580596	51.232557351151	-51.904800947093	-50.775343670425	50.155341521551	.110
.111	50.999997628015	-46.854769262361	48.308024918819	49.554999561183	51.231656676826	-51.896572409855	-50.775187872965	50.156252649665	.111
.112	50.999997541376	-46.878079880063	48.313599914331	49.559999541057	51.230764157579	-51.888490134136	-50.775031166617	50.157159305654	.112
.113	50.999997452385	-46.901810497363	48.319224909638	49.564999520199	51.229879649100	-51.880550237253	-50.774873555831	50.158061530027	.113
.114	50.999997361000	-46.925964864243	48.324899904732	49.569999498590	51.229003010895	-51.872748972761	-50.774715045021	50.158959362581	.114
.115	50.999997267178	-46.950546730685	48.330624899607	49.574999476209	51.228134106150	-51.865082724527	-50.774555638557	50.159852842419	.115
.116	50.999997170876	-46.975559846668	48.336399894254	49.579999453036	51.227272801606	-51.857548001113	-50.774395340774	50.160742007969	.116
.117	50.999997072051	-47.100100796217	48.342224888665	49.584999429050	51.226418967432	-51.850141430451	-50.774234155966	50.161626897003	.117
.118	50.999996970660	-47.102689482718	48.348099882832	49.589999404230	51.225572477114	-51.842859754781	-50.774072088393	50.162507546649	.118
.119	50.999996866658	-47.105322419166	48.354024876746	49.594999378554	51.224733207341	-51.835699825847	-50.773909142274	50.163383993414	.119
.120	50.999996760000	-47.107999980560	48.359999870400	49.599999352000	51.223901037896	-51.828658600341	-50.773745321795	50.164256273192	.120
.121	50.999996650643	-47.110722541897	48.366024863783	49.604999324546	51.223075851555	-51.821733135558	-50.773580631105	50.165124421288	.121
.122	50.999996538540	-47.113490478175	48.372099856888	49.609999296170	51.222257533987	-51.814920585280	-50.773415074318	50.165988472426	.122
.123	50.999996423647	-47.116304164392	48.378224849704	49.614999266848	51.221445973665	-51.808218195848	-50.773248655514	50.166848460763	.123
.124	50.999996305916	-47.119163975544	48.384399842222	49.619999236555	51.220641061765	-51.801623302434	-50.773081378740	50.167704419909	.124
.125	50.999996185303	-47.122070286630	48.390624834432	49.624999205271	51.219842692089	-51.795133325486	-50.772913248007	50.168556382935	.125
.126	50.999996061760	-47.125023472646	48.396899826324	49.629999172970	51.219050760975	-51.788745767345	-50.772744267295	50.169404382387	.126
.127	50.999995935240	-47.128023908590	48.403224817887	49.634999139626	51.218265167222	-51.782458209017	-50.772574440552	50.170248450298	.127
.128	50.999995805696	-47.131071969458	48.409599809113	49.639999105215	51.217485812006	-51.776268307106	-50.772403771694	50.171088618203	.128
.129	50.999995673080	-47.134168030248	48.416024799988	49.644999069712	51.216712598813	-51.770173790880	-50.772232264606	50.171924917147	.129
.130	50.999995537344	-47.137312465957	48.422499790503	49.649999033091	51.215945433363	-51.764172459478	-50.772059923140	50.172757377699	.130
.131	50.999995398439	-47.140505651581	48.429024780646	49.654998995326	51.215184223544	-51.758262179242	-50.771886751120	50.173586029961	.131
.132	50.999995256317	-47.143747962117	48.435599770406	49.659998956390	51.214428879345	-51.752440881170	-50.771712752339	50.174410903581	.132
.133	50.999995110927	-47.147039772562	48.442224759770	49.664998916255	51.213679312791	-51.746706558490	-50.771537930562	50.175232027761	.133
.134	50.999994962220	-47.150381457912	48.448899748727	49.669998874896	51.212935437885	-51.741057264334	-50.771362289524	50.176049431268	.134
.135	50.999994810147	-47.153773393163	48.455624737264	49.674998832283	51.212197170549	-51.735491109520	-50.771185832930	50.176863142444	.135
.136	50.999994654657	-47.157215953313	48.462399725368	49.679998788389	51.211464428565	-51.730006260432	-50.771008564460	50.177673189216	.136
.137	50.999994495698	-47.160709513356	48.469224713027	49.684998743184	51.210737131521	-51.724600936991	-50.770830487764	50.178479599104	.137
.138	50.999994333221	-47.164254448289	48.476099700227	49.689998696641	51.210015200759	-51.719273410717	-50.770651606465	50.179282399229	.138
.139	50.999994167172	-47.167851133108	48.483024686955	49.694998648728	51.209298559327	-51.714022002871	-50.770471924160	50.180081616325	.139
.140	50.999993997501	-47.171499942810	48.489999673197	49.699998599417	51.208587131925	-51.708845082680	-50.770291444418	50.180877276744	.140
.141	50.999993824155	-47.175201252388	48.497024658939	49.704998548676	51.207880844861	-51.703741065634	-50.770110170784	50.181669406466	.141
.142	50.999993647081	-47.178955436839	48.504099644166	49.709998496476	51.207179626006	-51.698708411861	-50.769928106774	50.182458031106	.142
.143	50.999993466226	-47.182762871159	48.511224628863	49.714998442784	51.206483404749	-51.693745624560	-50.769745255881	50.183243175923	.143
.144	50.999993281537	-47.186623930343	48.518399613016	49.719998387569	51.205792111954	-51.688851248513	-50.769561621574	50.184024865825	.144
.145	50.999993092960	-47.190538989385	48.525624596610	49.724998330799	51.205105679920	-51.684023868644	-50.769377207293	50.184803125380	.145
.146	50.999992900441	-47.194508423282	48.532899579627	49.729998272441	51.204424042344	-51.679262108650	-50.769192016458	50.185577978820	.146
.147	50.999992703925	-47.198532607027	48.540224562053	49.734998212462	51.203747134277	-51.674564629678	-50.769006052463	50.186349450048	.147
.148	50.999992503358	-47.202611915616	48.547599543871	49.739998150828	51.203074892092	-51.669930129062	-50.768819318678	50.187117562646	.148
.149	50.999992298683	-47.206746724043	48.555024525064	49.744998087506	51.202407253449	-51.665357339108	-50.768631818449	50.187882339882	.149
.150	50.999992089845	-47.210937407303	48.562499505615	49.749998022461	51.201744157255	-51.660845025929	-50.768443555102	50.188643804715	.150

χ	ber(χ)	ber'(χ)	bei(χ)	bei'(χ)	ker(χ)	ker'(χ)	kei(χ)	kei'(χ)	χ
.150	50.999992089845	-47.210937407303	48.562499505615	49.749998022461	51.201744157255	-51.660845025929	-50.768443555102	50.188643804715	.150
.151	50.999991876789	-47.215184340389	48.570024485507	49.754997955658	51.201085543638	-51.656391988323	-50.768254531937	50.189401979800	.151
.152	50.999991659458	-47.219487898297	48.577599464722	49.759997887062	51.200431353910	-51.651997056703	-50.768064752232	50.190156887498	.152
.153	50.999991437795	-47.223848456020	48.585224443242	49.764997816637	51.199781530536	-51.647659092058	-50.767874219243	50.190908549877	.153
.154	50.999991211742	-47.228266388552	48.592899421049	49.769997744347	51.199136017107	-51.643376984963	-50.767682936206	50.191656988723	.154
.155	50.999990981242	-47.232742070886	48.600624398123	49.774997670154	51.198494758308	-51.639149654626	-50.767490906333	50.192402225540	.155
.156	50.999990746238	-47.237275878017	48.608399374446	49.779997594022	51.197857699890	-51.634976047967	-50.767298132815	50.193144281560	.156
.157	50.999990506671	-47.241868184937	48.616224349997	49.784997515912	51.197224788646	-51.630855138739	-50.767104618823	50.193883177743	.157
.158	50.999990262482	-47.246519366640	48.624099324757	49.789997435787	51.196595972381	-51.626785926675	-50.766910367506	50.194618934791	.158
.159	50.999990013613	-47.251229798118	48.632024298706	49.794997353607	51.195971199886	-51.622767436674	-50.766715381993	50.195351573142	.159
.160	50.999989760003	-47.255999854364	48.639999271822	49.799997269334	51.195350420917	-51.618798718015	-50.766519665393	50.196081112983	.160
.161	50.999989501593	-47.260829910372	48.648024244085	49.804997182927	51.194733586169	-51.614878843595	-50.766323220794	50.196807574253	.161
.162	50.999989238323	-47.265720341133	48.656099215474	49.809997094347	51.194120647250	-51.611006909208	-50.766126051264	50.197530976644	.162
.163	50.999988970132	-47.270671521640	48.664224185965	49.814997003552	51.193511556665	-51.607182032834	-50.765928159853	50.198251339610	.163
.164	50.999988696960	-47.275683826885	48.672399155537	49.819996910502	51.192906267786	-51.603403353968	-50.765729549592	50.198968682370	.164
.165	50.999988418744	-47.280757631859	48.680624124167	49.824996815154	51.192304734838	-51.599670032969	-50.765530223489	50.199683023911	.165
.166	50.999988135424	-47.285893311555	48.688899091832	49.829996717467	51.191706912874	-51.595981250426	-50.765330184538	50.200394382993	.166
.167	50.999987846937	-47.291091240964	48.697224058509	49.834996617397	51.191112757757	-51.592336206557	-50.765129435711	50.201102778154	.167
.168	50.999987553220	-47.296351795076	48.705599024172	49.839996514901	51.190522226142	-51.588734120624	-50.764927979963	50.201808227714	.168
.169	50.999987254212	-47.301675348884	48.714023988799	49.844996409936	51.189935275451	-51.585174230368	-50.764725820231	50.202510749775	.169
.170	50.999986949848	-47.307062277377	48.722498952363	49.849996302457	51.189351863864	-51.581655791468	-50.764522959434	50.203210362232	.170
.171	50.999986640066	-47.312512955547	48.731023914839	49.854996192418	51.188771950294	-51.578178077013	-50.764319400471	50.203907082769	.171
.172	50.999986324801	-47.318027758383	48.739598876202	49.859996079776	51.188195494372	-51.574740376998	-50.764115146226	50.204600928870	.172
.173	50.999986003989	-47.323607060877	48.748223836426	49.864995964483	51.187622456433	-51.571341997837	-50.763910199566	50.205291917816	.173
.174	50.999985677565	-47.329251238018	48.756898795483	49.869995846493	51.187052797495	-51.567982261887	-50.763704563337	50.205980066693	.174
.175	50.999985345465	-47.334960664795	48.765623753347	49.874995725760	51.186486479249	-51.564660506998	-50.763498240373	50.206665392392	.175
.176	50.999985007622	-47.340735716198	48.774398709989	49.879995602235	51.185923464037	-51.561376086067	-50.763291233488	50.207347911615	.176
.177	50.999984663972	-47.346576767216	48.783223665382	49.884995475871	51.185363714843	-51.558128366616	-50.763083545480	50.208027640878	.177
.178	50.999984314447	-47.352484192838	48.792098619497	49.889995346618	51.184807195276	-51.554916730383	-50.762875179131	50.208704596513	.178
.179	50.999983958981	-47.358458368053	48.801023572304	49.894995214428	51.184253869556	-51.551740572919	-50.762666137206	50.209378794671	.179
.180	50.999983597507	-47.364499667849	48.809998523775	49.899995079251	51.183703702501	-51.548599303213	-50.762456422455	50.210050251326	.180
.181	50.999983229959	-47.370608467215	48.819023473879	49.904994941037	51.183156659513	-51.545492343312	-50.762246037612	50.210718982278	.181
.182	50.999982856268	-47.376785141138	48.828098422586	49.909994799734	51.182612706566	-51.542419127966	-50.762034985394	50.211385003153	.182
.183	50.999982476366	-47.383030064606	48.837223369863	49.914994655291	51.182071810194	-51.539379104284	-50.761823268504	50.212048329412	.183
.184	50.999982090185	-47.389343612607	48.846398315681	49.919994507656	51.181533937474	-51.536371731394	-50.761610889628	50.212708976346	.184
.185	50.999981697656	-47.395726160126	48.855623260006	49.924994356776	51.180999056024	-51.533396480116	-50.761397851439	50.213366959087	.185
.186	50.999981298709	-47.402178082152	48.864898202805	49.929994202599	51.180467133980	-51.530452832657	-50.761184156593	50.214022292602	.186
.187	50.999980893276	-47.408699753671	48.874223144047	49.934994045070	51.179938139993	-51.527540282295	-50.760969807732	50.214674991703	.187
.188	50.999980481287	-47.415291549668	48.883598083695	49.939993884135	51.179412043215	-51.524658333094	-50.760754807483	50.215325071045	.188
.189	50.999980062670	-47.421953845131	48.893023021718	49.944993719740	51.178888813288	-51.521806499613	-50.760539158459	50.215972545131	.189
.190	50.999979637355	-47.428687015044	48.902497958078	49.949993551828	51.178368420336	-51.518984306632	-50.760322863257	50.216617428312	.190
.191	50.999979205272	-47.435491434392	48.912022892742	49.954993380343	51.177850834950	-51.516191288883	-50.760105924461	50.217259734792	.191
.192	50.999978766349	-47.442367478162	48.921597825673	49.959993205230	51.177336028185	-51.513426990793	-50.759888344641	50.217899478628	.192
.193	50.999978320513	-47.449315521336	48.931222756835	49.964993026430	51.176823971544	-51.510690966232	-50.759670126353	50.218536673734	.193
.194	50.999977867693	-47.456335938901	48.940897686189	49.969992843886	51.176314636973	-51.507982778268	-50.759451272139	50.219171333883	.194
.195	50.999977407817	-47.463429105839	48.950622613700	49.974992657539	51.175807996851	-51.505301998934	-50.759231784526	50.219803472707	.195
.196	50.999976940811	-47.470595397135	48.960397539327	49.979992467330	51.175304023979	-51.502648208996	-50.759011666029	50.220433103704	.196
.197	50.999976466602	-47.477835187772	48.970222463033	49.984992273199	51.174802691574	-51.500020997734	-50.758790919150	50.221060240234	.197
.198	50.999975985116	-47.485148852732	48.980097384778	49.989992075086	51.174303973258	-51.497419962726	-50.758569546376	50.221684895525	.198
.199	50.999975496279	-47.492536767000	48.990022304521	49.994991872930	51.173807843055	-51.494844709641	-50.758347550182	50.222307082674	.199
.200	50.999975000017	-47.499999305556	48.999997222223	49.999991666670	51.173314275374	-51.492294852031	-50.758124933029	50.222926814650	.200

x	ber(x)	ber'(x)	bei(x)	bei'(x)	ker(x)	ker'(x)	kei(x)	kei'(x)	x
.200	50.999975000017	-47.499999305556	48.999997222223	49.999991666670	51.173314275374	-51.492294852031	-50.758124933029	50.222926814650	.200
.201	50.999974496256	-47.507536843382	49.101002213784	50.100499145624	51.172823245012	-51.489770011145	-50.757901697366	50.223544104294	.201
.202	50.999973984919	-47.515149755462	49.102009705133	50.100999124159	51.172334727137	-51.487269815728	-50.757677845630	50.224158964323	.202
.203	50.999973465931	-47.522838416774	49.103022196266	50.101499102264	51.171848697286	-51.484793901844	-50.757453380243	50.224771407328	.203
.204	50.999972939216	-47.530603202302	49.104039687177	50.101999079933	51.171365131358	-51.482341912691	-50.757228303617	50.225381445782	.204
.205	50.999972404699	-47.538444487024	49.105062177863	50.102499057160	51.170884005603	-51.479913498432	-50.757002618149	50.225989092035	.205
.206	50.999971862302	-47.546362645921	49.106089668319	50.102999033939	51.170405296617	-51.477508316023	-50.756776326226	50.226594358322	.206
.207	50.999971311948	-47.554358053973	49.107122158540	50.103499010262	51.169928981339	-51.475126029050	-50.756549430222	50.227197256761	.207
.208	50.999970753560	-47.562431086159	49.108159648523	50.103998986123	51.169455037038	-51.472766307566	-50.756321932498	50.227797799354	.208
.209	50.999970187060	-47.570582117457	49.109202138261	50.104498961516	51.168983441310	-51.470428827944	-50.756093835405	50.228395997991	.209
.210	50.999969612369	-47.578811522847	49.110249627751	50.104998936433	51.168514172074	-51.468113272718	-50.755865141279	50.228991864452	.210
.211	50.999969029410	-47.587119677306	49.111302116988	50.105498910867	51.168047207560	-51.465819330444	-50.755635852449	50.229585410405	.211
.212	50.999968438104	-47.595506955813	49.112359605967	50.105998884813	51.167582526310	-51.463546695553	-50.755405971228	50.230176647411	.212
.213	50.999967838370	-47.603973733343	49.113422094683	50.106498858262	51.167120107167	-51.461295068216	-50.755175499920	50.230765586926	.213
.214	50.999967230130	-47.612520384874	49.114489583131	50.106998831208	51.166659929269	-51.459064154208	-50.754944440816	50.231352240297	.214
.215	50.999966613302	-47.621147285383	49.115562071305	50.107498803643	51.166201972050	-51.456853664779	-50.754712796198	50.231936618772	.215
.216	50.999965987808	-47.629854809845	49.116639559202	50.107998775561	51.165746215226	-51.454663316528	-50.754480568334	50.232518733494	.216
.217	50.999965353566	-47.638643333235	49.117722046815	50.108498746953	51.165292638796	-51.452492831275	-50.754247759482	50.233098595505	.217
.218	50.999964710494	-47.647513230529	49.118809534139	50.108998717814	51.164841223034	-51.450341935949	-50.754014371890	50.233676215749	.218
.219	50.999964058512	-47.656464876701	49.119902021169	50.109498688135	51.164391948434	-51.448210362465	-50.753780407794	50.234251605072	.219
.220	50.999963397537	-47.665498646725	49.120999507900	50.109998657909	51.163944795956	-51.446097847613	-50.753545869420	50.234824774221	.220
.221	50.999962727487	-47.674614915574	49.122101994326	50.110498627129	51.163499746522	-51.444004132948	-50.753310758982	50.235395733851	.221
.222	50.999962048280	-47.683814058222	49.123209480441	50.110998595786	51.163056781508	-51.441928964681	-50.753075078685	50.235964494522	.222
.223	50.999961359831	-47.693096449642	49.124321966239	50.111498563873	51.162615882493	-51.439872093578	-50.752838830722	50.236531066698	.223
.224	50.999960662059	-47.702462464805	49.125439451716	50.111998531381	51.162177031303	-51.437833274854	-50.752602017278	50.237095460756	.224
.225	50.999959954879	-47.711912478683	49.126561936865	50.112498498307	51.161740210006	-51.435812268078	-50.752364640524	50.237657686981	.225
.226	50.999959238206	-47.721446866247	49.127689421680	50.112998464638	51.161305400909	-51.433808837073	-50.752126702623	50.238217755567	.226
.227	50.999958511957	-47.731066002467	49.128821906156	50.113498430368	51.160872586551	-51.431822749827	-50.751888205728	50.238775676623	.227
.228	50.999957776046	-47.740770262314	49.129959390286	50.113998395489	51.160441749704	-51.429853778395	-50.751649151982	50.239331460169	.228
.229	50.999957030388	-47.750560020757	49.131101874064	50.114498359992	51.160012873364	-51.427901698819	-50.751409543517	50.239885116141	.229
.230	50.999956274897	-47.760435652764	49.132249357483	50.114998323870	51.159585940749	-51.425966291032	-50.751169382456	50.240436654390	.230
.231	50.999955509487	-47.770397533306	49.133401840539	50.115498287114	51.159160935297	-51.424047338780	-50.750928670911	50.240986084682	.231
.232	50.999954734073	-47.780446037348	49.134559323224	50.115998249717	51.158737840657	-51.422144629539	-50.750687410986	50.241533417703	.232
.233	50.999953948566	-47.790581539859	49.135721805531	50.116498211668	51.158316640693	-51.420257954432	-50.750445604774	50.242078660056	.233
.234	50.999953152881	-47.800804415805	49.136889287455	50.116998172961	51.157897319472	-51.418387108154	-50.750203254359	50.242621824265	.234
.235	50.999952346928	-47.811115040152	49.138061768988	50.117498133587	51.157479861268	-51.416531888897	-50.749960361815	50.243162918775	.235
.236	50.999951530621	-47.821513787866	49.139239250124	50.117998093537	51.157064250552	-51.414692098269	-50.749716929208	50.243701952951	.236
.237	50.999950703871	-47.832001033912	49.140421730856	50.118498052802	51.156650471993	-51.412867541233	-50.749472958593	50.244238936082	.237
.238	50.999949866590	-47.842577153254	49.141609211178	50.118998011374	51.156238510455	-51.411058026025	-50.749228452016	50.244773877381	.238
.239	50.999949018687	-47.853242520856	49.142801691082	50.119497969243	51.155828350990	-51.409263364094	-50.748983411516	50.245306785985	.239
.240	50.999948160075	-47.863997511681	49.143999170560	50.119997926402	51.155419978838	-51.407483370031	-50.748737839119	50.245837670957	.240
.241	50.999947290662	-47.874842500691	49.145201649607	50.120497882840	51.155013379421	-51.405717861505	-50.748491736845	50.246366541288	.241
.242	50.999946410360	-47.885777862849	49.146409128215	50.120997838550	51.154608538346	-51.403966659200	-50.748245106705	50.246893405894	.242
.243	50.999945519076	-47.896803973116	49.147621606376	50.121497793521	51.154205441393	-51.402229586750	-50.747997950699	50.247418273620	.243
.244	50.999944616721	-47.907921206452	49.148839084083	50.121997747745	51.153804074520	-51.400506470683	-50.747750270820	50.247941153243	.244
.245	50.999943703203	-47.919129937817	49.150061561328	50.122497701213	51.153404423856	-51.398797140358	-50.747502069053	50.248462053466	.245
.246	50.999942778431	-47.930430542171	49.151289038104	50.122997653914	51.153006475700	-51.397101427911	-50.747253347370	50.248980982926	.246
.247	50.999941842311	-47.941823394471	49.152521514404	50.123497605840	51.152610216516	-51.395419168194	-50.747004107741	50.249497950191	.247
.248	50.999940894753	-47.953308869677	49.153758990219	50.123997556982	51.152215632934	-51.393750198726	-50.746754352121	50.250012963762	.248
.249	50.999939935663	-47.964887342745	49.155001465541	50.124497507328	51.151822711742	-51.392094359635	-50.746504082462	50.250526032072	.249
.250	50.999938964947	-47.976559188632	49.156248940363	50.124997456871	51.151431439890	-51.390451493607	-50.746253300703	50.251037163490	.250

x	ber(x)	ber'(x)	bei(x)	bei'(x)	ker(x)	ker'(x)	kei(x)	kei'(x)	x
.250	50.999938964947	-47.976559188632	49.156248940363	50.124997456871	51.151431439890	-51.390451493607	-50.746253300703	50.251037163490	.250
.251	50.999937982513	-47.988324782294	49.157501414676	50.125497405600	51.151041804482	-51.388821445837	-50.746002008777	50.251546366319	.251
.252	50.999936988266	-48.100018449869	49.158758888472	50.125997353505	51.150653792777	-51.387204063976	-50.745750208610	50.252053648799	.252
.253	50.999935982113	-48.101213871276	49.160021361743	50.126497300577	51.150267392183	-51.385599198085	-50.745497902117	50.252559019105	.253
.254	50.999934963957	-48.102418779947	49.161288834480	50.126997246806	51.149882590258	-51.384006700587	-50.745245091207	50.253062485351	.254
.255	50.999933933705	-48.103633213378	49.162561306676	50.127497192180	51.149499374707	-51.382426426220	-50.744991777779	50.253564055589	.255
.256	50.999932891261	-48.104857209063	49.163838778321	50.127997136692	51.149117733379	-51.380858231991	-50.744737963725	50.254063737808	.256
.257	50.999931836529	-48.106090804497	49.165121249407	50.128497080329	51.148737654263	-51.379301977134	-50.744483650930	50.254561539938	.257
.258	50.999930769413	-48.107334037176	49.166408719925	50.128997023082	51.148359125491	-51.377757523067	-50.744228841269	50.255057469849	.258
.259	50.999929689816	-48.108586944594	49.167701189866	50.129496964941	51.147982135329	-51.376224733346	-50.743973536611	50.255551535353	.259
.260	50.999928597642	-48.109849564247	49.168998659221	50.129996905895	51.147606672181	-51.374703473628	-50.743717738817	50.256043744201	.260
.261	50.999927492792	-48.111121933629	49.170301127981	50.130496845934	51.147232724582	-51.373193611630	-50.743461449739	50.256534104089	.261
.262	50.999926375170	-48.112404090235	49.171608596136	50.130996785046	51.146860281202	-51.371695017088	-50.743204671223	50.257022622654	.262
.263	50.999925244678	-48.113696071559	49.172921063678	50.131496723222	51.146489330835	-51.370207561719	-50.742947405105	50.257509307477	.263
.264	50.999924101216	-48.114997915097	49.174238530598	50.131996660451	51.146119862407	-51.368731119185	-50.742689653217	50.257994166085	.264
.265	50.999922944686	-48.116309658343	49.175560996884	50.132496596721	51.145751864967	-51.367265565056	-50.742431417379	50.258477205948	.265
.266	50.999921774990	-48.117631338791	49.176888462529	50.132996532022	51.145385327689	-51.365810776772	-50.742172699409	50.258958434482	.266
.267	50.999920592026	-48.118962993935	49.178220927521	50.133496466342	51.145020239865	-51.364366633609	-50.741913501112	50.259437859049	.267
.268	50.999919395696	-48.120304661270	49.179558391852	50.133996399671	51.144656590913	-51.362933016649	-50.741653824289	50.259915486957	.268
.269	50.999918185900	-48.121656378291	49.180900855511	50.134496331998	51.144294370363	-51.361509808737	-50.741393670734	50.260391325464	.269
.270	50.999916962535	-48.123018182490	49.182248318489	50.134996263311	51.143933567864	-51.360096894459	-50.741133042232	50.260865381773	.270
.271	50.999915725502	-48.124390111362	49.183600780774	50.135496193598	51.143574173180	-51.358694160102	-50.740871940562	50.261337663036	.271
.272	50.999914474699	-48.125772202401	49.184958242357	50.135996122849	51.143216176188	-51.357301493627	-50.740610367496	50.261808176355	.272
.273	50.999913210024	-48.127164493101	49.186320703228	50.136496051052	51.142859566874	-51.355918784635	-50.740348324797	50.262276928781	.273
.274	50.999911931375	-48.128567020956	49.187688163375	50.136995978196	51.142504335336	-51.354545924342	-50.740085814223	50.262743927316	.274
.275	50.999910638650	-48.129979823458	49.189060622788	50.137495904267	51.142150471778	-51.353182805544	-50.739822837524	50.263209178912	.275
.276	50.999909331744	-48.131402938102	49.190438081457	50.137995829256	51.141797966513	-51.351829322592	-50.739559396445	50.263672690471	.276
.277	50.999908010556	-48.132836402381	49.191820539370	50.138495753150	51.141446809956	-51.350485371364	-50.739295492721	50.264134468849	.277
.278	50.999906674982	-48.134280253789	49.193207996516	50.138995675936	51.141096992627	-51.349150849234	-50.739031128083	50.264594520853	.278
.279	50.999905324916	-48.135734529818	49.194600452885	50.139495597604	51.140748505148	-51.347825655050	-50.738766304253	50.265052853243	.279
.280	50.999903960256	-48.137199267962	49.195997908464	50.139995518141	51.140401338241	-51.346509689104	-50.738501022947	50.265509472733	.280
.281	50.999902580896	-48.138674505714	49.197400363244	50.140495437534	51.140055482727	-51.345202853109	-50.738235285876	50.265964385989	.281
.282	50.999901186731	-48.140160280567	49.198807817211	50.140995355771	51.139710929524	-51.343905050170	-50.737969094742	50.266417599633	.282
.283	50.999899777655	-48.141656630013	49.200220270355	50.141495272841	51.139367669647	-51.342616184765	-50.737702451241	50.266869120242	.283
.284	50.999898353563	-48.143163591545	49.201637722664	50.141995188730	51.139025694206	-51.341336162716	-50.737435357063	50.267318954346	.284
.285	50.999896914348	-48.144681202657	49.203060174126	50.142495103426	51.138684994405	-51.340064891166	-50.737167813892	50.267767108433	.285
.286	50.999895459903	-48.146209500840	49.204487624728	50.142995016916	51.138345561538	-51.338802278560	-50.736899823404	50.268213588947	.286
.287	50.999893990122	-48.147748523587	49.205920074460	50.143494929188	51.138007386992	-51.337548234616	-50.736631387270	50.268658402287	.287
.288	50.999892504897	-48.149298308391	49.207357523308	50.143994840229	51.137670462242	-51.336302670308	-50.736362507153	50.269101554811	.288
.289	50.999891004120	-48.150858892742	49.208799971260	50.144494750025	51.137334778854	-51.335065497842	-50.736093184712	50.269543052833	.289
.290	50.999889487683	-48.152430314135	49.210247418304	50.144994658565	51.137000328478	-51.333836630637	-50.735823421597	50.269982902625	.290
.291	50.999887955477	-48.154012610060	49.211699864427	50.145494565834	51.136667102852	-51.332615983300	-50.735553219454	50.270421110420	.291
.292	50.999886407394	-48.155605818010	49.213157309617	50.145994471820	51.136335093799	-51.331403471608	-50.735282579922	50.270857682406	.292
.293	50.999884843325	-48.157209975476	49.214619753860	50.146494376508	51.136004293225	-51.330199012491	-50.735011504632	50.271292624733	.293
.294	50.999883263158	-48.158825119950	49.216087197143	50.146994279887	51.135674693118	-51.329002524006	-50.734739995213	50.271725943510	.294
.295	50.999881666785	-48.160451288923	49.217559639453	50.147494181943	51.135346285547	-51.327813925324	-50.734468053285	50.272157644806	.295
.296	50.999880054096	-48.162088519887	49.219037080777	50.147994082661	51.135019062664	-51.326633136706	-50.734195680461	50.272587734649	.296
.297	50.999878424978	-48.163736850334	49.220519521102	50.148493982028	51.134693016697	-51.325460079490	-50.733922878350	50.273016219031	.297
.298	50.999876779322	-48.165396317753	49.222006960413	50.148993880031	51.134368139953	-51.324294676069	-50.733649648556	50.273443103903	.298
.299	50.999875117014	-48.167066959638	49.223499398698	50.149493776656	51.134044424819	-51.323136849873	-50.733375992674	50.273868395178	.299
.300	50.999873437945	-48.168748813478	49.224996835941	50.149993671888	51.133721863753	-51.321986525355	-50.733101912295	50.274292098732	.300

x	ber(x)	ber'(x)	bei(x)	bei'(x)	ker(x)	ker'(x)	kei(x)	kei'(x)	x
.300	50.999873437945	-48.168748813478	49.224996835941	50.149993671888	51.133721863753	-51.321986525355	-50.733101912295	50.274292098732	.300
.301	50.999871742001	-48.170441916764	49.226499272131	50.150493565715	51.133400449292	-51.320843627973	-50.732827409004	50.274714220401	.301
.302	50.999870029069	-48.172146306987	49.228006707251	50.150993458120	51.133080174046	-51.319708084169	-50.732552484379	50.275134765988	.302
.303	50.999868299037	-48.173862021637	49.229519141288	50.151493349092	51.132761030697	-51.318579821361	-50.732277139995	50.275553741254	.303
.304	50.999866551791	-48.175589098206	49.231036574228	50.151993238614	51.132443012000	-51.317458767918	-50.732001377418	50.275971151929	.304
.305	50.999864787217	-48.177327574183	49.232559006056	50.152493126673	51.132126110781	-51.316344853151	-50.731725198211	50.276387003702	.305
.306	50.999863005201	-48.179077487059	49.234086436757	50.152993013255	51.131810319937	-51.315238007296	-50.731448603929	50.276801302229	.306
.307	50.999861205629	-48.180838874323	49.235618866316	50.153492898344	51.131495632434	-51.314138161496	-50.731171596122	50.277214053130	.307
.308	50.999859388385	-48.182611773466	49.237156294718	50.153992781926	51.131182041304	-51.313045247790	-50.730894176337	50.277625261990	.308
.309	50.999857553355	-48.184396221977	49.238698721949	50.154492663986	51.130869539650	-51.311959199096	-50.730616346111	50.278034934359	.309
.310	50.999855700422	-48.186192257347	49.240246147993	50.154992544510	51.130558120639	-51.310879949200	-50.730338106978	50.278443075752	.310
.311	50.999853829471	-48.187999917064	49.241798572834	50.155492423482	51.130247777507	-51.309807432736	-50.730059460468	50.278849691653	.311
.312	50.999851940385	-48.189819238618	49.243355996457	50.155992300887	51.129938503551	-51.308741585180	-50.729780408102	50.279254787508	.312
.313	50.999850033047	-48.191650259498	49.244918418847	50.156492176711	51.129630292135	-51.307682342831	-50.729500951397	50.279658368731	.313
.314	50.999848107341	-48.193493017194	49.246485839986	50.156992050937	51.129323136684	-51.306629642801	-50.729221091867	50.280060440706	.314
.315	50.999846163148	-48.195347549194	49.248058259860	50.157491923551	51.129017030689	-51.305583422999	-50.728940831017	50.280461008779	.315
.316	50.999844200350	-48.197213892988	49.249635678452	50.157991794538	51.128711967699	-51.304543622120	-50.728660170349	50.280860078269	.316
.317	50.999842218830	-48.199092086063	49.251218095745	50.158491663880	51.128407941325	-51.303510179636	-50.728379111359	50.281257654457	.317
.318	50.999840218469	-48.200982165909	49.252805511724	50.158991531564	51.128104945240	-51.302483035778	-50.728097655536	50.281653742598	.318
.319	50.999838199147	-48.202884170014	49.254397926371	50.159491397573	51.127802973174	-51.301462131526	-50.727815804368	50.282048347911	.319
.320	50.999836160746	-48.204798135867	49.255995339670	50.159991261891	51.127502018916	-51.300447408600	-50.727533559333	50.282441475586	.320
.321	50.999834103144	-48.206724100955	49.257597751603	50.160491124502	51.127202076315	-51.299438809445	-50.727250921907	50.282833130781	.321
.322	50.999832026224	-48.208662102766	49.259205162154	50.160990985390	51.126903139275	-51.298436277223	-50.726967893560	50.283223318625	.322
.323	50.999829929862	-48.210612178788	49.260817571305	50.161490844540	51.126605201757	-51.297439755798	-50.726684475757	50.283612044214	.323
.324	50.999827813940	-48.212574366510	49.262434979039	50.161990701935	51.126308257779	-51.296449189728	-50.726400669958	50.283999312615	.324
.325	50.999825678334	-48.214548703417	49.264057385338	50.162490557558	51.126012301411	-51.295464524254	-50.726116477616	50.284385128867	.325
.326	50.999823522925	-48.216535226999	49.265684790184	50.162990411393	51.125717326781	-51.294485705288	-50.725831900182	50.284769497976	.326
.327	50.999821347589	-48.218533974741	49.267317193560	50.163490263424	51.125423328069	-51.293512679406	-50.725546939101	50.285152424922	.327
.328	50.999819152205	-48.220544984132	49.268954595447	50.163990113634	51.125130299509	-51.292545393833	-50.725261595812	50.285533914654	.328
.329	50.999816936648	-48.222568292657	49.270596995826	50.164489962006	51.124838235386	-51.291583796438	-50.724975871749	50.285913972093	.329
.330	50.999814700798	-48.224603937804	49.272244394681	50.164989808523	51.124547130037	-51.290627835722	-50.724689768343	50.286292602130	.330
.331	50.999812444528	-48.226651957060	49.273896791991	50.165489653168	51.124256977852	-51.289677460806	-50.724403287019	50.286669809630	.331
.332	50.999810167717	-48.228712387910	49.275554187738	50.165989495925	51.123967773271	-51.288732621427	-50.724116429197	50.287045599430	.332
.333	50.999807870239	-48.230785267840	49.277216581903	50.166489336776	51.123679510781	-51.287793267925	-50.723829196291	50.287419976336	.333
.334	50.999805551970	-48.232870634338	49.278883974467	50.166989175703	51.123392184922	-51.286859351235	-50.723541589713	50.287792945130	.334
.335	50.999803212785	-48.234968524889	49.280556365410	50.167489012690	51.123105790282	-51.285930822878	-50.723253610869	50.288164510566	.335
.336	50.999800852558	-48.237078976978	49.282233754714	50.167988847719	51.122820321497	-51.285007634952	-50.722965261158	50.288534677369	.336
.337	50.999798471163	-48.239202028092	49.283916142358	50.168488680772	51.122535773248	-51.284089740125	-50.722676541979	50.288903450240	.337
.338	50.999796068475	-48.241337715715	49.285603528323	50.168988511831	51.122252140267	-51.283177091623	-50.722387454721	50.289270833852	.338
.339	50.999793644367	-48.243486077332	49.287295912588	50.169488340880	51.121969417331	-51.282269643226	-50.722098000772	50.289636832851	.339
.340	50.999791198711	-48.245647150430	49.288993295134	50.169988167900	51.121687599263	-51.281367349257	-50.721808181515	50.290001451859	.340
.341	50.999788731381	-48.247820972492	49.290695675939	50.170487992872	51.121406680930	-51.280470164577	-50.721517998327	50.290364695471	.341
.342	50.999786242249	-48.250007581004	49.292403054984	50.170987815779	51.121126657246	-51.279578044572	-50.721227452581	50.290726568255	.342
.343	50.999783731187	-48.252207013449	49.294115432248	50.171487636603	51.120847523167	-51.278690945150	-50.720936545646	50.291087074756	.343
.344	50.999781198066	-48.254419307313	49.295832807709	50.171987455325	51.120569273696	-51.277808822732	-50.720645278886	50.291446219493	.344
.345	50.999778642758	-48.256644500079	49.297555181347	50.172487271927	51.120291903877	-51.276931634246	-50.720353653660	50.291804006959	.345
.346	50.999776065133	-48.258882629231	49.299282553141	50.172987086391	51.120015408797	-51.276059337114	-50.720061671323	50.292160441624	.346
.347	50.999773465062	-48.261133732253	49.301014923068	50.173486898696	51.119739783586	-51.275191889253	-50.719769333226	50.292515527933	.347
.348	50.999770842415	-48.263397846629	49.302752291107	50.173986708826	51.119465023416	-51.274329249061	-50.719476640715	50.292869270305	.348
.349	50.999768197061	-48.265675009842	49.304494657237	50.174486516761	51.119191123500	-51.273471375416	-50.719183595132	50.293221673137	.349
.350	50.999765528871	-48.267965259375	49.306242021435	50.174986322482	51.118918079090	-51.272618227663	-50.718890197814	50.293572740801	.350

x	ber(x)	ber'(x)	bei(x)	bei'(x)	ker(x)	ker'(x)	kei(x)	keri	x
.350	50.999765528871	-48.267965259375	49.306242021435	50.174986322482	51.118918079090	-51.272618227663	-50.718890197814	50.293572740801	.350
.351	50.999762837712	-48.270268632712	49.307994383679	50.175486125970	51.118645885482	-51.271769765612	-50.718596450094	50.293922477645	.351
.352	50.999760123454	-48.272585167334	49.309751743947	50.175985927205	51.118374538010	-51.270925949531	-50.718302353300	50.294270887996	.352
.353	50.999757385965	-48.274914900725	49.311514102216	50.176485726169	51.118104032048	-51.270086740136	-50.718007908758	50.294617976154	.353
.354	50.999754625112	-48.277257870367	49.313281458463	50.176985522842	51.117834363007	-51.269252098588	-50.717713117787	50.294963746398	.354
.355	50.999751840763	-48.279614113742	49.315053812665	50.177485317205	51.117565526341	-51.268421986487	-50.717417981703	50.295308202985	.355
.356	50.999749032786	-48.281983668332	49.316831164799	50.177985109237	51.117297517537	-51.267596365863	-50.717122501818	50.295651350146	.356
.357	50.999746201046	-48.284366571620	49.318613514842	50.178484898920	51.117030332124	-51.266775199173	-50.716826679438	50.295993192094	.357
.358	50.999743345410	-48.286762861086	49.320400862770	50.178984686232	51.116763965667	-51.265958449291	-50.716530515867	50.296333733015	.358
.359	50.999740465744	-48.289172574212	49.322193208559	50.179484471156	51.116498413766	-51.265146079508	-50.716234012404	50.296672977077	.359
.360	50.999737561913	-48.291595748479	49.323990552185	50.179984253669	51.116233672060	-51.264338053522	-50.715937170344	50.297010928423	.360
.361	50.999734633784	-48.294032421369	49.325792893624	50.180484033752	51.115969736223	-51.263534335432	-50.715639990977	50.297347591175	.361
.362	50.999731681220	-48.296482630361	49.327600232852	50.180983811385	51.115706601965	-51.262734889736	-50.715342475590	50.297682969435	.362
.363	50.999728704086	-48.298946412937	49.329412569843	50.181483586548	51.115444265031	-51.261939681321	-50.715044625465	50.298017067281	.363
.364	50.999725702246	-48.301423806577	49.331229904574	50.181983359219	51.115132721201	-51.261148675463	-50.714746441881	50.298349888772	.364
.365	50.999722675564	-48.303914848761	49.333052237019	50.182483129378	51.114921966290	-51.260361837815	-50.714447926112	50.298681437944	.365
.366	50.999719623903	-48.306419576968	49.334879567153	50.182982897005	51.114661996147	-51.259579134409	-50.714149079428	50.299011718814	.366
.367	50.999716547127	-48.308938028679	49.336711894951	50.183482662078	51.114402806655	-51.258800531646	-50.713849903095	50.299340735377	.367
.368	50.999713445097	-48.311470241373	49.338549220386	50.183982424577	51.114144393728	-51.258025996291	-50.713550398377	50.299668491607	.368
.369	50.999710317676	-48.314016252529	49.340391543434	50.184482184480	51.113886753317	-51.257255495471	-50.713250566531	50.299994991459	.369
.370	50.999707164726	-48.316576099626	49.342238864067	50.184981941766	51.113629881403	-51.256488996668	-50.712950408812	50.300320238868	.370
.371	50.999703986108	-48.319149820143	49.344091182260	50.185481696414	51.113373774001	-51.255726467715	-50.712649926470	50.300644237746	.371
.372	50.999700781683	-48.321737451558	49.345948497987	50.185981448403	51.113118427155	-51.254967876792	-50.712349120751	50.300966991989	.372
.373	50.999697551312	-48.324339031351	49.347810811220	50.186481197710	1.112863836945	-51.254213192418	-50.712047992899	50.301288505470	.373
.374	50.999694294856	-48.326954596997	49.349678121932	50.186980944315	51.112609999479	-51.253462383450	-50.711746544152	50.301608782045	.374
.375	50.999691012174	-48.329584185977	49.351550430097	50.187480688195	51.112356910897	-51.252715419079	-50.711444775746	50.301927825548	.375
.376	50.999687703125	-48.332227835767	49.353427735687	50.187980429328	51.112104567369	-51.251972268820	-50.711142688911	50.302245639795	.376
.377	50.999684367570	-48.334885583844	49.355310038674	50.188480167693	51.111852965098	-51.251232902516	-50.710840284875	50.302562228584	.377
.378	50.999681005366	-48.337557467686	49.357197339031	50.188979903267	51.111602100313	-51.250497290325	-50.710537564861	50.302877595692	.378
.379	50.999677616373	-48.340243524769	49.359089636730	50.189479636028	51.111351969275	-51.249765402725	-50.710234530090	50.303191744879	.379
.380	50.999674200449	-48.342943792571	49.360986931742	50.189979365954	51.111102568276	-51.249037210500	-50.709931181776	50.303504679884	.380
.381	50.999670757450	-48.345658308567	49.362889224040	50.190479093022	51.110853893632	-51.248312684743	-50.709627521133	50.303816404430	.381
.382	50.999667287235	-48.348387110234	49.364796513593	50.190978817209	51.110605941693	-51.247591796851	-50.709323549370	50.304126922220	.382
.383	50.999663789660	-48.351130235047	49.366708800374	50.191478538493	51.110358708835	-51.246874518517	-50.709019267690	50.304436236938	.383
.384	50.999660264582	-48.353887720482	49.368626084353	50.191978256851	51.110112191463	-51.246160821732	-50.708714677295	50.304744352253	.384
.385	50.999656711858	-48.356659604015	49.370548365501	50.192477972260	51.109866386007	-51.245450678777	-50.708409779384	50.305051271813	.385
.386	50.999653131342	-48.359445923120	49.372475643789	50.192977684697	51.109621288930	-51.244744062219	-50.708104575149	50.305356999249	.386
.387	50.999649522891	-48.362246715273	49.374407919185	50.193477394138	51.109376896716	-51.244040944910	-50.707799065782	50.305661538175	.387
.388	50.999645886359	-48.365062017947	49.376345191661	50.193977100561	51.109133205882	-51.243341299982	-50.707493252468	50.305964892187	.388
.389	50.999642221602	-48.367891868618	49.378287461186	50.194476803942	51.108890212968	-51.242645100844	-50.707187136391	50.306267064863	.389
.390	50.999638528473	-48.370736304759	49.380234727730	50.194976504256	51.108647914541	-51.241952321175	-50.706880718731	50.306568059765	.390
.391	50.999634806827	-48.373595363845	49.382186991261	50.195476201482	51.108406307194	-51.241262934927	-50.706574000663	50.306867880436	.391
.392	50.999631056517	-48.376469083348	49.384144251749	50.195975895594	51.108165387548	-51.240576916316	-50.706266983360	50.307166530404	.392
.393	50.999627277396	-48.379357500741	49.386106509163	50.196475586568	51.107925152248	-51.239894239820	-50.705959667991	50.307464013179	.393
.394	50.999623469318	-48.382260653499	49.388073763470	50.196975274382	51.107685597963	-51.239214880178	-50.705652055722	50.307760332254	.394
.395	50.999619632134	-48.385178579093	49.390046014640	50.197474959010	51.107446721390	-51.238538812384	-50.705344147713	50.308055491105	.395
.396	50.999615765697	-48.388111314996	49.392023262639	50.197974640428	51.107208519249	-51.237866011684	-50.705035945125	50.308349493194	.396
.397	50.999611869858	-48.391058898680	49.394005507437	50.198474318612	51.106970988286	-51.237196453575	-50.704727449111	50.308642341963	.397
.398	50.999607944469	-48.394021367617	49.395992749001	50.198973993537	51.106734125269	-51.236530113799	-50.704418660824	50.308934040840	.398
.399	50.999603989381	-48.396998759278	49.397984987297	50.199473665178	51.106497926994	-51.235866968342	-50.704109581412	50.309224593237	.399
.400	50.999600004444	-48.399991111135	49.399982222293	50.199973333511	51.106262390276	-51.235206993430	-50.703800212019	50.309514002548	.400

x	ber(x)	ber'(x)	bei(x)	bei'(x)	ker(x)	ker'(x)	kei(x)	kei'(x)	x
.400	50.999600004444	-48.399991111135	49.399982222293	50.199973333511	51.106262390276	-51.235206993430	-50.703300212019	50.309514002546	.400
.401	50.999595989509	[illegible]	49.401986453056	[illegible]	51.106027511958	-51.234550165526	-50.703490593787	50.309802272153	.401
.402	50.999591944425	-48.406020845319	49.403991632252	[illegible]	51.105793288904	-51.233896461328	-50.703180607853	50.310089405416	.402
.403	50.999587869042	-48.409058302583	49.406003907148	50.201472319411	51.105559718001	-51.233245557766	-50.702870375354	50.310375405684	.403
.404	50.999583763209	-48.412110869934	49.408021126609	50.201971973250	51.105326796162	-51.232598331996	-50.702559857419	50.310660276290	.404
.405	50.999579626774	-48.415178584829	49.410043346602	50.202471624675	51.105094520319	-51.231953861401	-50.702249055176	50.310944020551	.405
.406	50.999575459586	-48.418261484740	49.412070561091	50.202971272630	51.104862887428	-51.231312423589	-50.701937969752	50.311226641767	.406
.407	50.999571261494	-48.421359607137	49.414102772043	50.203470917099	51.104631894468	-51.230673996386	-50.701626602266	50.311508143226	.407
.408	50.999567032343	-48.424472989490	49.416139979421	50.203970558058	51.104401538439	-51.230038557834	-50.701314953837	50.311738528198	.408
.409	50.999562771983	-48.427601669267	49.418182183192	50.204470195479	51.104171816364	-51.229406086194	-50.701003025581	50.312067799939	.409
.410	50.999558480259	-48.430745683936	49.420229383319	50.204969829336	51.103942725285	-51.228776559935	-50.700690818608	50.312345961691	.410
.411	50.999554157018	-48.433905070964	49.422281579767	50.205469459604	51.103714262269	-51.228149957739	-50.700378334026	50.312623016680	.411
.412	50.999549802106	-48.437079867821	49.424338772499	50.205969086257	51.103486424402	-51.227526258493	-50.700065572942	50.312898968118	.412
.413	50.999545415369	-48.440270111972	49.426400961480	50.206468709267	51.103259208792	-51.226905441289	-50.699752536457	50.313173819202	.413
.414	50.999540996652	-48.443475840886	49.428468146672	50.206968328607	51.103032612566	-51.226287485421	-50.699439225669	50.313447573114	.414
.415	50.999536545801	-48.446697092028	49.430540328040	50.207467944253	51.102806632874	-51.225672370383	-50.699125641675	50.313720233023	.415
.416	50.999532062659	-48.449933902865	49.432617505545	50.207967556176	51.102581266885	-51.225060075867	-50.698811785567	50.313991802083	.416
.417	50.999527547070	-48.453186310864	49.434699679151	50.208467164349	51.102356511789	-51.224450581758	-50.698497658434	50.314262283434	.417
.418	50.999522998880	-48.456454353489	49.436786848819	50.208966768746	51.102132364794	-51.223843868136	-50.698183261362	50.314531680201	.418
.419	50.999518417931	-48.459738068206	49.438879014513	50.209466369339	51.101908823132	-51.223239915268	-50.697868595434	50.314799995496	.419
.420	50.999513804066	-48.463037492481	49.440976176193	50.209965966101	51.101685884050	-51.222638703612	-50.697553661730	50.315067232417	.420
.421	50.999509157129	-48.466352663778	49.443078333822	50.210465559004	51.101463544817	-51.222040213811	-50.697238461327	50.315333394048	.421
.422	50.999504476961	-48.469683619561	49.445185487360	50.210965148021	51.101241802722	-51.221444426691	-50.696922995299	50.315598483460	.422
.423	50.999499763404	-48.473030397295	49.447297636769	50.211464733124	51.101020655069	-51.220851323259	-50.696607264717	50.315862503709	.423
.424	50.999495016300	-48.476393034443	49.449414782010	50.211964314285	51.100800099187	-51.220260884702	-50.696291270647	50.316125457840	.424
.425	50.999490235490	-48.479771568468	49.451536923042	50.212463891476	51.100580132418	-51.219673092384	-50.695975014156	50.316387348881	.425
.426	50.999485420815	-48.483166036834	49.453664059826	50.212963464669	51.100360752126	-51.219087927845	-50.695658496303	50.316648179850	.426
.427	50.999480572116	-48.486576477002	49.455796192322	50.213463033836	51.100141955692	-51.218505372797	-50.695341718148	50.316907953749	.427
.428	50.999475689232	-48.490002926437	49.457933320489	50.213962598947	50.999237405167	-51.217925409122	-50.695024680747	50.317166673570	.428
.429	50.999470772004	-48.493445422598	49.460075444287	50.214462159976	50.997061040164	-51.217348018873	-50.694707385151	50.317424342290	.429
.430	50.999465820270	-48.496904002949	49.462222563675	50.214961716892	50.994890436271	-51.216773184268	-50.694389832412	50.317680962873	.430
.431	50.999460833870	-48.500378704950	49.464374678611	50.215461269668	50.992725568019	-51.216200887693	-50.694072023574	50.317936538271	.431
.432	50.999455812642	-48.503869566062	49.466531789054	50.215960818273	50.990566410115	-51.215631111694	-50.693753959683	50.318191071422	.432
.433	50.999450756425	-48.507376623745	49.468693894963	50.216460362680	50.988412937441	-51.215063838980	-50.693435641778	50.318444565252	.433
.434	50.999445665056	-48.510899915460	49.470860996294	50.216959902859	50.986265125048	-51.214499052420	-50.693117070898	50.318697022675	.434
.435	50.999440538372	-48.514439473667	49.473033093006	50.217459438780	50.984122948162	-51.213936735038	-50.692798248077	50.318948446593	.435
.436	50.999435376212	-48.517995350825	49.475210185055	50.217958970414	50.981986382173	-51.213376870017	-50.692479174348	50.319198839893	.436
.437	50.999430178411	-48.521567569392	49.477392272400	50.218458497731	50.979855402642	-51.212819440692	-50.692159850740	50.319448205452	.437
.438	50.999424944806	-48.525156171828	49.479579354995	50.218958020703	50.977729985295	-51.212264430551	-50.691840278279	50.319696546134	.438
.439	50.999419675233	-48.528761195590	49.481771432799	50.219457539297	50.975610106022	-51.211711823233	-50.691520457988	50.319943864791	.439
.440	50.999414369527	-48.532382678138	49.483968505767	50.219957053486	50.973495740875	-51.211161602523	-50.691200390889	50.320190164262	.440
.441	50.999409027524	-48.536020656927	49.486170573854	50.220456563238	50.971386866070	-51.210613752357	-50.690880077998	50.320435447376	.441
.442	50.999403649059	-48.539675169416	49.488377637017	50.220956068523	50.969283457980	-51.210068256815	-50.690559520332	50.320679716947	.442
.443	50.999398233965	-48.543346253061	49.490589695209	50.221455569311	50.967185493137	-51.209525100120	-50.690238718901	50.320922975781	.443
.444	50.999392782078	-48.547033945318	49.492806748388	50.221955065571	50.965092948233	-51.208984266637	-50.689917674716	50.321165226669	.444
.445	50.999387293231	-48.550738283643	49.495028796506	50.222454557272	50.963005800112	-51.208445740873	-50.689596388783	50.321406472392	.445
.446	50.999381767257	-48.554459305492	49.497255839518	50.222954044384	50.960924025775	-51.207909507473	-50.689274862106	50.321646715719	.446
.447	50.999376203989	-48.558197048320	49.499487877378	50.223453526875	50.958847602373	-51.207375551221	-50.688953095685	50.321885959407	.447
.448	50.999370603260	-48.561951549581	49.501724910040	50.223953004715	50.956776507210	-51.206843857034	-50.688631090519	50.322124206202	.448
.449	50.999364964902	-48.565722846730	49.503966937457	50.224452477872	50.954710717742	-51.206314409967	-50.688308847604	50.322361458839	.449
.450	50.999359288747	-48.569510977221	49.506213959581	50.224951946314	50.952650211570	-51.205787195204	-50.687986367932	50.322597720041	.450

x	ber(x)	ber'(x)	bei(x)	bei'(x)	ker(x)	ker'(x)	kei(x)	kei'(x)	x
.450	50.999359288747	-48.569510977221	49.506213959581	50.224951946314	50.952650211570	-51.205787195204	-50.687986367932	50.322597720041	.450
.451	50.999353574626	-48.573315978508	49.508465976367	50.225451410010	50.950594966446	-51.205262198064	-50.687663652493	50.322832992520	.451
.452	50.999347822371	-48.577137888042	49.510722987766	50.225950868929	50.948544960265	-51.204739403995	-50.687340702275	50.323067278977	.452
.453	50.999342031812	-48.580976743278	49.512984993730	50.226450323039	50.946500171071	-51.204218798571	-50.687017518263	50.323300582101	.453
.454	50.999336202780	-48.584832581667	49.515251994210	50.226949772307	50.944460577046	-51.203700367497	-50.686694101438	50.323532904571	.454
.455	50.999330335104	-48.588705440661	49.517523989160	50.227449216701	50.942426156520	-51.203184096601	-50.686370452780	50.323764249056	.455
.456	50.999324428614	-48.592595357711	49.519800978528	50.227948656189	50.940396887961	-51.202669971836	-50.686046573265	50.323994618212	.456
.457	50.999318483140	-48.596502370269	49.522082962267	50.228448090739	50.938372749976	-51.202157979280	-50.685722463868	50.324224014685	.457
.458	50.999312498510	-48.600426515784	49.524369940326	50.228947520319	50.936353721314	-51.201648105129	-50.685398125559	50.324452441111	.458
.459	50.999306474552	-48.604367831708	49.526661912657	50.229446944894	50.934339780858	-51.201140335703	-50.685073559308	50.324679900114	.459
.460	50.999300411096	-48.608326355489	49.528958879207	50.229946364434	50.932330907630	-51.200634657438	-50.684748766080	50.324906394309	.460
.461	50.999294307968	-48.612302124577	49.531260839928	50.230445778904	50.930327080784	-51.200131056890	-50.684423746840	50.325131926298	.461
.462	50.999288164996	-48.616295176421	49.533567794768	50.230945188271	50.928328279611	-51.199629520730	-50.684098502548	50.325356498677	.462
.463	50.999281982006	-48.620305548469	49.535879743677	50.231444592503	50.926334483532	-51.199130035745	-50.683773034162	50.325580114026	.463
.464	50.999275758827	-48.624333278168	49.538196686601	50.231943991565	50.924345672102	-51.198632588835	-50.683447342638	50.325802774918	.464
.465	50.999269495283	-48.628378402967	49.540518623491	50.232443385425	50.922361825005	-51.198137167014	-50.683121428929	50.326024483917	.465
.466	50.999263191201	-48.632440960312	49.542845554292	50.232942774048	50.920382922055	-51.197643757407	-50.682795293986	50.326245243573	.466
.467	50.999256846405	-48.636520987649	49.545177478954	50.233442157401	50.918408943192	-51.197152347249	-50.682468938758	50.326465056428	.467
.468	50.999250460722	-48.640618522426	49.547514397423	50.233941535449	50.916439868487	-51.196662923885	-50.682142364188	50.326683925015	.468
.469	50.999244033976	-48.644733602086	49.549856309645	50.234440908159	50.914475678134	-51.196175474766	-50.681815571222	50.326901851854	.469
.470	50.999237565992	-48.648866264076	49.552203215568	50.234940275497	50.912516352452	-51.195689987452	-50.681488560798	50.327118839459	.470
.471	50.999231056592	-48.653016545840	49.554555115137	50.235439637427	50.910561871887	-51.195206449609	-50.681161333855	50.327334890330	.471
.472	50.999224505602	-48.657184484823	49.556912008298	50.235938993915	50.908612217003	-51.194724849005	-50.680833891328	50.327550006960	.472
.473	50.999217912844	-48.661370118467	49.559273894997	50.236438344927	50.906667368489	-51.194245173513	-50.680506234152	50.327764191831	.473
.474	50.999211278141	-48.665573484217	49.561640775179	50.236937690427	50.904727307155	-51.193767411109	-50.680178363255	50.327977447417	.474
.475	50.999204601315	-48.669794619514	49.564012648787	50.237437030381	50.902792013930	-51.193291549870	-50.679850279566	50.328189776179	.475
.476	50.999197882189	-48.674033561801	49.566389515768	50.237936364753	50.900861469860	-51.192817577971	-50.679521984010	50.328401180572	.476
.477	50.999191120584	-48.678290348521	49.568771376064	50.238435693508	50.898935656111	-51.192345483691	-50.679193477512	50.328611663038	.477
.478	50.999184316322	-48.682565017113	49.571158229619	50.238935016611	50.897014553966	-51.191875255401	-50.678864760991	50.328821226014	.478
.479	50.999177469224	-48.686857605020	49.573550076377	50.239434334026	50.895098144822	-51.191406881575	-50.678535835366	50.329029871924	.479
.480	50.999170579110	-48.691168149680	49.575946916280	50.239933645717	50.893186410191	-51.190940350779	-50.678206701552	50.329237603183	.480
.481	50.999163645801	-48.695496688535	49.578348749272	50.240432951649	50.891279331700	-51.190475651677	-50.677877360463	50.329444422198	.481
.482	50.999156669116	-48.699843259022	49.580755575294	50.240932251784	50.889376891089	-51.190012773023	-50.677547813011	50.329650331366	.482
.483	50.999149648876	-48.704207898582	49.583167394288	50.241431546088	50.887479070208	-51.189551703670	-50.677218060103	50.329855333076	.483
.484	50.999142584898	-48.708590644652	49.585584206196	50.241930834523	50.885585851021	-51.189092432558	-50.676888102646	50.330059429706	.484
.485	50.999135477002	-48.712991534670	49.588006010959	50.242430117053	50.883697215599	-51.188634948721	-50.676557941545	50.330262623626	.485
.486	50.999128325007	-48.717410606073	49.590432808517	50.242929393642	50.881813146125	-51.188179241282	-50.676227577699	50.330464917196	.486
.487	50.999121128729	-48.721847896298	49.592864598812	50.243428664252	50.879933624888	-51.187725299454	-50.675897012009	50.330666312769	.487
.488	50.999113887988	-48.726303442782	49.595301381782	50.243927928847	50.878058634286	-51.187273112539	-50.675566245372	50.330866812688	.488
.489	50.999106602600	-48.730777282959	49.597743157369	50.244427187389	50.876188156822	-51.186822669926	-50.675235278682	50.331066419287	.489
.490	50.999099272381	-48.735269454264	49.600189925510	50.244926439842	50.874322175108	-51.186373961090	-50.674904112831	50.331265134891	.490
.491	50.999091897149	-48.739779994134	49.602641686145	50.245425686167	50.872460671856	-51.185926975593	-50.674572748708	50.331462961817	.491
.492	50.999084476720	-48.744308940000	49.605098439213	50.245924926327	50.870603629886	-51.185481703081	-50.674241187202	50.331659902372	.492
.493	50.999077010909	-48.748856329298	49.607560184651	50.246424160285	50.868751032119	-51.185038133283	-50.673909429198	50.331855958856	.493
.494	50.999069499532	-48.753422199460	49.610026922398	50.246923388003	50.866902861579	-51.184596256015	-50.673577475579	50.332051133560	.494
.495	50.999061942403	-48.758006587918	49.612498652390	50.247422609442	50.865059101391	-51.184156061171	-50.673245327224	50.332245428765	.495
.496	50.999054339338	-48.762609532104	49.614975374565	50.247921824564	50.863219734781	-51.183717538728	-50.672912985014	50.332438846746	.496
.497	50.999046690151	-48.767231069450	49.617457088860	50.248421033331	50.861384745075	-51.183280678746	-50.672580449822	50.332631389768	.497
.498	50.999038994655	-48.771871237387	49.619943795211	50.248920235705	50.859554115698	-51.182845471361	-50.672247722525	50.332823060087	.498
.499	50.999031252664	-48.776530073344	49.622435493553	50.249419431646	50.857727830173	-51.182411906791	-50.671914803992	50.333013859952	.499
.500	50.999023463991	-48.781207614751	49.624932183822	50.249918621116	50.855905872120	-51.181979975332	-50.671581695094	50.333203791604	.500