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by

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# The Propagating Solutions and Far-field Patterns for Acoustic Harmonic Waves in a Finite Depth Ocean

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## 1. Introduction

In the theoretical study of the sound field produced by a source in an ocean, the model of a point source in an ocean of constant depth has been investigated very thoroughly (cf.[1]). Probably the reason for this is that it approximates a real sound source in a real ocean. In the past, this has been a reasonable assumption because the real sources considered were often small. However, it is anticipated that in the future the attention will be turned to quieter and dispersed sources of sound in the ocean. This means that one must be closer to the sources in order to be able to detect them.

In the past, observations were made, moreover, so far away that one could assume that the sound source was simply a point source. In the future, one must assess the effects of distributed sources of sound, that is, one must consider the shapes of the sound sources.

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This motivates us to investigate the two kinds of problems:

(1) *The direct propagation problem,* i.e., if we know the shape of an object, how is the sound field produced or scattered by it? What can we expect to detect in a reasonable distance? This problem is to find an operator **T** mapping the boundary of the obstacle and the incident field onto the far field patterns.

(2) *The inverse scattering problem,* i.e. if we have detected the scattered data at a distance, can we say anything about the shape of the obstacle, that is, can we invert the operator **T** to reconstruct the shape of the scattering obstacle?

Recently, much progress has been made on the inverse scattering problen for time-harmonic acoustic waves in the whole space (cf. [3],[4]). However, in a finite depth ocean, the interaction of the surface and bottom of the ocean on the sound waves causes a pronounced effect in the far field pattern. These surfaces, for example, permit only a finite number of modes to propagate, the other waves being evanescent.

For the direct problem, much investigation has been made. However, for the three dimensional wave propagation in an ocean, "the exact theory of wave scattering from rough surfaces has not yet been developed."([5], also [2]) In particular, when using integral equation method, there may be real values of k at which uniqueness fails.

In this paper, we will discuss problem (1). The main contribution of this paper is to prove that for any real value of k, our problem is uniquely solvable. We will obtain a representation in terms of single and double layer potentials for the solutions, establish an *existence theorem*, and furthermore, find a representation for the far field pattern. Based on these results, we can characterize the far-field pattern and discuss the inverse scattering problem. (cf. [6],[7],[8],[13])

## 2 The Representation of Propagating Solutions

Let $\mathbf{R}_b^3 = \{(\mathbf{x}, z) \in \mathbf{R}^3; \mathbf{x} = (x_1, x_2) \in \mathbf{R}^2, 0 \leq z \leq h\}$, where $h$ is a positive constant; moreover, let $\Omega$ be a bounded domain with a $C^2$ boundary having an outward unit normal $\nu$, such that $\overline{\Omega} \subset \mathbf{R}_b^3$. The direct propagation problem in a homogeneous ocean, of constant depth, with a pressure release surface, and a rigid bottom may be modeled as (cf.[1]):

$$\Delta_3 \, u + k^2 u = 0, \ in \ \mathbf{R}_b^3 \setminus \overline{\Omega}, \tag{2.1}$$

$$u = 0, \quad at \ z = 0, \tag{2.2}$$

$$\frac{\partial u}{\partial z} = 0, \quad at \ z = h, \tag{2.3}$$

$$\lim_{r \to \infty} r^{\frac{1}{2}} \left( \frac{\partial u_n^s}{\partial r} - ik a_n u_n^s \right) = 0, \ r = | \mathbf{x} |, \ n = 0, 1, ..., \infty. \tag{2.4}$$

$$u = g(\mathbf{x}, z), \quad for \ (\mathbf{x}, z) \in \partial\Omega. \tag{2.5}$$

Here $\Delta_3$ is the three dimensional Laplace operator, $k$ a real number, $g$ a given function, and $u_n$ the $n^{th}$ normal propagation mode. i.e. if

$$a_n = [1 - \frac{(2n+1)^2 \pi^2}{4k^2 h^2}]^{\frac{1}{2}} \tag{2.6}$$

$$\phi_n(z) = sin[k(1 - a_n^2)^{\frac{1}{2}} z] \tag{2.7}$$

then $u$ may have a normal mode representation

$$u^s = \sum_{n=0}^{\infty} \phi_n(z) u_n^s(\mathbf{x}). \tag{2.8}$$

$u_n(\mathbf{x})$ is the $n^{th}$ normal mode.[1]

Let $G(z, z_0, r)$ be the Green's function in $\mathbf{R}_b^3$ satisfying the radiating condition (2.4). Then $G(z, z_0, r)$ can be written as [1]:

$$G(z, z_0, r) = \frac{i}{2h} \sum_{n=0}^{\infty} \phi_n(z)\phi_n(z_0) H_0^{(1)}(k a_n r), \tag{2.9}$$

where $H_n^{(1)}(r)$ is Hankel function of first kind of order n.

If $u$ is a solution of (2.1) $\sim$ (2.4) in $C^2(\mathbf{R}_b^3 \setminus \overline{\Omega}) \cap C(\mathbf{R}_b^3 \setminus \Omega)$ such that the normal derivative on the boundary exists in the sense that the limit

$$\frac{\partial u}{\partial \nu}(\mathbf{x}, z) = \lim_{h \to 0}(\nu(\mathbf{x}, z), grad\ u((\mathbf{x}, z) - h\nu(\mathbf{x}, z))),\ (\mathbf{x}, z) \in \partial\Omega$$

exists uniformly on $\partial\Omega$. Then by Green's formula,

$$\int_{\Gamma_R} \{G(z, \zeta, |\ \mathbf{x} - \xi\ |) \frac{\partial u(\xi, \zeta)}{\partial \nu} - \frac{\partial G}{\partial \nu}(z, \zeta, |\ \mathbf{x} - \xi\ |) u(\xi, \zeta)\} d\sigma$$

$$- \int_{\partial\Omega} \{G(z, \zeta, |\ \mathbf{x} - \xi\ |) \frac{\partial u(\xi, \zeta)}{\partial \nu} - \frac{\partial G}{\partial \nu}(z, \zeta, |\ \mathbf{x} - \xi\ |) u(\xi, \zeta)\} d\sigma$$

$$= \begin{cases} 0 \quad \text{if } (\mathbf{x}, z) \in \Omega \\ -\int_{\Gamma_\epsilon} \{G(z, \zeta, |\ \mathbf{x} - \xi\ |) \frac{\partial u(\xi,\zeta)}{\partial \nu} - \frac{\partial G}{\partial \nu}(z, \zeta, |\ \mathbf{x} - \xi\ |) u(\xi, \zeta)\} d\sigma \\ \qquad \text{if } (\mathbf{x}, z) \in \mathbf{R}_b^3 \setminus \overline{\Omega} \end{cases} \tag{2.10}$$

where

$$\Omega_\epsilon := \{(\mathbf{y}, \zeta); [|\ \mathbf{y} - \mathbf{x}\ |^2 + (\zeta - z)^2]^{1/2} < \epsilon\},\ \ \Gamma_\epsilon := \partial\Omega_\epsilon;$$

$$\Omega_R := \{(\mathbf{y}, \zeta) \in \mathbf{R}_b^3; |\ \mathbf{y}\ | < R\},\ \ \Gamma_R := \partial\Omega_R;$$

and $\epsilon, R$ are positive numbers such that $\Omega_\epsilon \subset \mathbf{R}_b^3 \setminus \overline{\Omega}, \overline{\Omega} \subset \Omega_R$.

Since $G(z,\zeta,|\,\mathbf{x}-\xi\,|)$ can be written as [1]

$$G(z,\zeta,|\,\mathbf{x}-\xi\,|) = \frac{e^{ik\sqrt{|\mathbf{x}-\xi|^2+(z-\zeta)^2}}}{4\pi\sqrt{|\,\mathbf{x}-\xi\,|^2+(z-\zeta)^2}} + \Phi_1(z,\zeta,|\,\mathbf{x}-\xi\,|), \quad (2.11)$$

where

$$\Phi_1(z,\zeta,|\,\mathbf{x}-\xi\,|) = \frac{1}{4\pi}\sum_{n=-\infty,n\neq 0}^{\infty}\{\frac{e^{ik\sqrt{|\mathbf{x}-\xi|^2+(z-\zeta-2nh)^2}}}{\sqrt{|\,\mathbf{x}-\xi\,|^2+(z-\zeta-2nh)^2}}$$

$$-\frac{e^{ik\sqrt{|\mathbf{x}-\xi|^2+(z+\zeta-2nh)^2}}}{\sqrt{|\,\mathbf{x}-\xi\,|^2+(z+\zeta-2nh)^2}}\} - \frac{e^{ik\sqrt{|\mathbf{x}-\xi|^2+(z+\zeta)^2}}}{4\pi\sqrt{|\,\mathbf{x}-\xi\,|^2+(z+\zeta)^2}},$$

it may be seen that $\Phi_1(z,\zeta,|\,\mathbf{x}-\xi\,|)$ is bounded and continuous at $z = \zeta, \mathbf{x}=\xi$. Therefore

$$\int_{\Gamma_\epsilon} G\frac{\partial u}{\partial \nu}d\sigma \to 0,$$

and

$$\int_{\Gamma_\epsilon} u\frac{\partial G}{\partial \nu}d\sigma \to u(\mathbf{x},z), \quad as\ \epsilon \to 0.$$

From (2.10) we have

$$\int_{\Gamma_R}\{G\frac{\partial u}{\partial \nu} - \frac{\partial G}{\partial \nu}u\}d\sigma - \int_{\partial\Omega}\{G\frac{\partial u}{\partial \nu} - \frac{\partial G}{\partial \nu}u\}d\sigma$$

$$= \begin{cases} 0 & \text{if } (\mathbf{x},z) \in \Omega, \\ u(\mathbf{x},z) & \text{if } (\mathbf{x},z) \in \mathbf{R}_b^3 \setminus \overline{\Omega}. \end{cases} \quad (2.12)$$

Let

$$\hat{G}(z,z_0,r) = \frac{i}{2h}\sum_{n=0}^{\infty} a_n\phi_n(z)\phi_n(z_0)H_0^{(1)}(ka_n r),$$

and denote

$$G(z,z_0,r) = \sum_{n\leq N} + \sum_{n>N} =: G_N + G'_N,$$

$$\hat{G}(z,z_0,r) = \hat{\sum_{n\leq N}} + \hat{\sum_{n>N}} =: \hat{G}_N + \hat{G}'_N.$$

where $N = [(2kh - \pi)/2\pi]$. As usual, here $[a]$ is used to denote the integer part of a. From (2.6) we know that $a_n \geq 0$ if $n \leq N$, $ia_n < 0$ if $n > N$. In view of asymptotic behavior of $H_0^1(r)$, we know that

$$G'_N = 0(\frac{1}{r^{3/2}}),\ \ \hat{G}'_N = 0(\frac{1}{r^{3/2}}),$$

$$\frac{\partial}{\partial r}G'_N(ka_n r) = 0(\frac{1}{r^{3/2}}),\ \ \frac{\partial}{\partial r}\hat{G}'_N(ka_n r) = 0(\frac{1}{r^{3/2}}),$$

$$\frac{\partial}{\partial r}G_N - ik\hat{G}_N == \frac{i}{2h}\sum_{n\leq N}\phi_n(z)\phi_n(z_0)[\frac{\partial}{\partial r}H_0^{(1)}(ka_n r)$$

$$-ika_n H_0^{(1)}(ka_n r)] = 0(\frac{1}{r^{3/2}}),\ \ as\ r \to \infty.$$

Consequently, if we can prove

$$\int_{\Gamma_R} |\ u\ |^2\, d\sigma = 0(1),\ \ as\ R \to \infty, \tag{2.13}$$

then

$$\int_{\Gamma_R}\left\{G\frac{\partial u}{\partial r} - \frac{\partial G}{\partial r}u\right\}d\sigma = \int_{\Gamma_R}\left\{(G\frac{\partial u}{\partial r} - ik\hat{G}u) - (\frac{\partial G}{\partial r} - ik\hat{G})u\right\}d\sigma, \tag{2.14}$$

$$\left|\int_{\Gamma_R}\{\frac{\partial G}{\partial r} - ik\hat{G}u\}d\sigma\right| \leq (\int_{\Gamma_R} |\ \frac{\partial G}{\partial r} - ik\hat{G}\ |^2\, d\sigma)^{1/2}(\int_{\Gamma_R} |\ u\ |^2\, d\sigma)^{1/2}$$

$$\leq C\left[\int_0^h\int_0^{2\pi}\frac{R}{R^3}d\theta d\zeta\right]^{1/2} \to 0,\ \ as\ R \to \infty. \tag{2.15}$$

$$\int_{\Gamma_R}\left\{G\frac{\partial u}{\partial r} - ik\hat{G}u\right\}d\sigma$$

$$= \int_0^h\int_0^{2\pi}\left\{\left[\frac{i}{2h}\sum_{n=0}^{\infty}\phi_n(z)\phi_n(\zeta)H_0^{(1)}(ka_n\ |\ \mathbf{x} - \xi\ |)\right]\left[\sum_{n=0}^{\infty}\frac{\partial u_n}{\partial r}\phi_n(\zeta)\right]\right.$$

$$\left. - \left[\frac{i}{2h}\sum_{n=0}^{\infty}ika_n\phi_n(z)\phi_n(\zeta)H_0^{(1)}(ka_n\ |\ \mathbf{x} - \xi\ |)\right]\left[\sum_{n=0}^{\infty}u_n\phi_n(\zeta)\right]\right\}Rd\theta d\zeta$$

$$= \frac{i}{4}\sum_{n=0}^{\infty}\int_0^{2\pi}\left\{\phi_n(z)H_0^{(1)}(ka_n \mid \mathbf{x}-\xi \mid)\left[\frac{\partial u_n}{\partial r} - ika_n u_n\right] R\right\} d\theta.$$

By the radiating condition (2.4) and $H_0^{(1)}(ka_n \mid \mathbf{x}-\xi \mid) = 0(\frac{1}{r^{1/2}})$, as $r \to \infty$, we can conclude that

$$\int_{\Gamma_R}\left\{G\frac{\partial u}{\partial r} - ik\hat{G}u\right\} d\sigma \to 0, \quad as\ R \to \infty. \tag{2.16}$$

In order to prove (2.13), we note that for $0 \le n \le N,\ a_n > 0$,

$$0 = \lim_{R\to\infty}\int_{\Gamma_R}\left|\frac{\partial u}{\partial r} - ika_n u_n\right|^2 d\sigma$$

$$= \lim_{R\to\infty}\int_{\Gamma_R}\left\{\left|\frac{\partial u_n}{\partial r}\right|^2 + k^2a_n^2|u_n^2 + 2Im(ka_n u_n\frac{\partial \overline{u_n}}{\partial r})\right\} d\sigma. \tag{2.17}$$

Since

$$ka_n\int_{\Gamma_R} u_n\frac{\partial \overline{u_n}}{\partial r}d\sigma = ka_n\int_{\partial\Omega} u_n\frac{\partial \overline{u_n}}{\partial \nu}d\sigma - k^3a_n\int_{R_b^3\setminus\Omega}|u_n|^2 d\mathbf{X}$$

$$+ka_n\int_{R_b^3\setminus\Omega}|grad\ u_n|^2 d\mathbf{X},$$

upon substituting the imaginary part of the last equation into (2.17),

$$\lim_{R\to\infty}\int_{\Gamma_R}\sum_{n=0}^{N}\left\{\left|\frac{\partial u}{\partial u_n}\right|^2 + k^2|u_n|^2\right\} d\sigma = -2\sum_{n=0}^{N} Im(k\int_{\partial\Omega} a_n u_n\frac{\partial \overline{u_n}}{\partial \nu}d\sigma),$$

it follows that

$$\sum_{n=0}^{N}\int_{\Gamma_R}|u_n|^2 d\sigma = 0(1). \tag{2.18}$$

For $n > N$, we have $a_n = i|a_n|$; hence, by the asymptotic properties of the Hankel functions

$$u_n \sim H_0^{(1)}(ka_n r) = 0\left(\frac{e^{ika_n r}}{r^{1/2}}\right),$$

so that

$$\sum_{n=N+1}^{\infty}\int_{\Gamma_R}|u_n|^2 d\sigma = 0(1). \tag{2.19}$$

This proves (2.13).

In view of the above discussion, we can state:

**Theorem 2.1**: let $u \in C^2(R_b^3 \setminus \overline{\Omega}) \cap C(R_b^3 \setminus \Omega)$ such that the normal derivative exists in a strong sense on $\partial\Omega$ be a solution to the problem (2.1)-(2.4). Then

$$\int_{\partial\Omega} \left\{ u\frac{\partial G}{\partial \nu} - G\frac{\partial u}{\partial \nu} \right\} d\sigma = \begin{cases} 0, & \text{if } (\mathbf{x}, z) \in \Omega \\ u(\mathbf{x}, z), & \text{if } (\mathbf{x}, z) \in R_b^3 \setminus \overline{\Omega}. \end{cases} \tag{2.20}$$

*Remark*: Since $G(z, \zeta, |\mathbf{x} - \xi|)$ satisfy the radiating condition (2.4), it is easy to see that $u(\mathbf{x}, z)$ in (2.20) satisfies the same radiating condition.

# 3 The Uniqueness Theorem

We want to show that if $u \in C^2(\mathbf{R}_b^3 \setminus \overline{\Omega}) \cap C(\mathbf{R}_b^3 \setminus \Omega)$ is a solution of (2.1) $\sim$ (2.5) where $g = 0$,then $u \equiv 0$ in $\mathbf{R}_b^3 \setminus \Omega$. We will prove this in several steps.

**Lemma 3.1**: if $u \in C^2(\mathbf{R}_b^3 \setminus \overline{\Omega}) \cap C(\mathbf{R}_b^3 \setminus \Omega)$ is a solution of (2.1) $\sim$ (2.5) with homogeneous boundary data $g = 0$, then for any $R > 0$ such that $\Omega_R \supset \Omega$

$$\sum_{n=0}^{\infty} Im \int_0^{2\pi} [u_n \frac{\partial \overline{u_n}}{\partial r}]_{r=R} R d\theta = 0,$$

where $u_n$ is the $n^{th}$ normal mode of the solution.

Proof: By Green's formula, we have

$$\int_{\partial\Omega \cup \Gamma_R} (u\frac{\partial \overline{u}}{\partial r} - \overline{u}\frac{\partial u}{\partial \nu}) d\sigma = \int_{\Omega_R \setminus \Omega} (u \bigtriangleup \overline{u} - \overline{u} \bigtriangleup u) d\mathbf{X}$$
$$= \int_{\Omega_R \setminus \Omega} (\overline{k}^2 |u|^2 - k^2 |u|^2) d\mathbf{X} = 0.$$

Hence,

$$Im \int_{\Gamma_R} u\frac{\partial \overline{u}}{\partial \nu} d\sigma = \frac{1}{2i} Im \int_{\Gamma_R} (u\frac{\partial \overline{u}}{\partial r} - \overline{u}\frac{\partial u}{\partial r}) d\sigma = 0,$$

which follows from u being zero on $\partial\Omega$. Now expanding $u(\mathbf{x}, z)$ as

$$u(\mathbf{x}, z) = \sum_{n=0}^{\infty} \phi_n(z) u_n(\mathbf{x}),$$

we have

$$Im \int_{\Gamma_R} u \frac{\partial \overline{u}}{\partial r} d\sigma$$

$$= Im \int_0^{2\pi} \int_0^h \left[ \sum_{n=0}^{\infty} \phi_n(z) u_n(r, \theta) \right] \left[ \sum_{m=0}^{\infty} \phi_m(z) \frac{\partial \overline{u_m}}{\partial r} \right] |_{r=R} \, R dz d\theta$$

$$= \frac{h}{2} Im \sum_{n=0}^{\infty} \int_0^{2\pi} u_n(r, \theta) \frac{\partial \overline{u_m}}{\partial r} |_{r=R} \, R d\theta,$$

which lemma 3.1 follows.

**Lemma 3.2** Under the assumptions of Lemma 3.1, we may conclude that

$$u = 0\left(\frac{e^{-ka_{N+1} r}}{r^{1/2}}\right), \quad as \ r \to \infty. \tag{3.1}$$

Proof: In view of the radiating condition (2.4), for $0 \leq n \leq N, \ a_n > 0$

$$0 = \lim_{R \to \infty} \sum_{n=0}^{N} \frac{1}{a_n} \int_0^{2\pi} \left| \frac{\partial u}{\partial r} - ika_n u_n \right|^2_{r=R} R d\theta$$

$$= \lim_{R \to \infty} \sum_{n=0}^{N} \frac{1}{a_n} \int_0^{2\pi} \left\{ \left| \frac{\partial u_n}{\partial r} \right|^2 + k^2 a_n^2 |u_n|^2 + 2Im(ka_n u_n \frac{\partial \overline{u_n}}{\partial r}) \right\}_{r=R} R d\theta$$

$$= \lim_{R \to \infty} \sum_{n=0}^{N} \frac{1}{a_n} \int_0^{2\pi} \left\{ \left| \frac{\partial u_n}{\partial r} \right|^2 + k^2 a_n^2 |u_n|^2 \right\}_{r=R} R d\theta$$

$$+ \lim_{R \to \infty} 2kIm \sum_{n=0}^{\infty} \int_0^{2\pi} \left\{ u_n \frac{\partial \overline{u_n}}{\partial r} \right\}_{r=R} R d\theta$$

$$- \lim_{R \to \infty} 2kIm \sum_{n=N+1}^{\infty} \int_0^{2\pi} \left\{ u_n \frac{\partial \overline{u_n}}{\partial r} \right\}_{r=R} R d\theta \tag{3.2}$$

By Lemma 3.1

$$2kIm \sum_{n=0}^{\infty} \int_0^{2\pi} \left\{ u_n \frac{\partial \overline{u_n}}{\partial r} \right\}_{r=R} R d\theta = 0.$$

For $n \geq N+1$, $u_n = 0(e^{-k|a_n|R})$, as $R \to \infty$; hence,

$$\lim_{R\to\infty} 2kIm \sum_{n=N+1}^{\infty} \int_0^{2\pi} \left\{ u_n \frac{\partial \overline{u_n}}{\partial r} \right\}_{r=R} Rd\theta = 0$$

(3.2) becomes

$$\lim_{R\to\infty} \sum_{n=0}^{N} \frac{1}{a_n} \int_0^{2\pi} \left\{ \left| \frac{\partial u_n}{\partial r} \right|^2 + k^2 a_n^2 |u_n|^2 \right\}_{r=R} Rd\theta = 0. \tag{3.3}$$

In view of the $a_n$ being positive for $n = 0, 1, ..., N$, (3.3) implies that as $R \to \infty$ we must have

$$\int_0^{2\pi} |u_n|^2 Rd\theta = o(1), \; n = 0, 1, ..., N. \tag{3.4}$$

Since for sufficient large R, any solution of (2.1) $\sim$ (2.5) can be written in the form of

$$u(\mathbf{x}, z) = \sum_{n=0}^{\infty} \phi_n(z) u_n(\mathbf{x}) = \sum_{n=0}^{\infty} \left[ \sum_{j=\infty}^{\infty} C_{nj} H_j^{(1)}(ka_n r) e^{ij\theta} \right] \phi_n(z),$$

from (3.4),

$$\int_0^{2\pi} |u_n(R,\theta)|^2 Rd\theta = \sum_{j=\infty}^{\infty} |C_{nj}|^2 |H_j^{(1)}(ka_n R)|^2 2\pi R = o(1).$$

We must have

$$|C_{nj}|^2 |H_j^{(1)}(ka_n R)|^2 2\pi R = o(1) \;\; as \; R \to \infty,$$

$$n = 0, 1, ..., N; j = 0, \pm 1, \pm 2, ...$$

However,

$$H_j^{(1)}(ka_n R) = 0(\frac{1}{R^{1/2}}) \;\; as \; R \to \infty, \; n = 0, 1, ..., N. \tag{3.5}$$

So

$$C_{nj} = 0, \;\; for \; n = 0, 1, ..., N, \; j = 0, \pm 1, \pm 2, ...$$

that is

$$u_n = 0 \ \ for \ n = 0, 1, ..., N.$$

Hence,

$$u(\mathbf{x}, z) = \sum_{n=N+1}^{\infty} \phi_n(z) u_n(\mathbf{x}) = \sum_{n=N+1}^{\infty} \left[ \sum_{j=\infty}^{\infty} C_{nj} H_j^{(1)}(ka_n r) e^{ij\theta} \right] \phi_n(z)$$

$$= 0(\frac{e^{-ka_{N+1}r}}{r^{1/2}}), \ as \ r \to \infty.$$

Now let $\psi$ be a function in $C^2(\mathbf{R}_b^3 \setminus \overline{\Omega}) \cap C^1(\mathbf{R}_b^3 \setminus \Omega)$; then by the second Green's formula,

$$\int_{\Omega_R \setminus \Omega} [u\psi \triangle (\overline{u\psi}) - \overline{u\psi} \triangle (u\psi)] d\mathbf{X}$$

$$= \int_{\partial\Omega \cup \Gamma_R} [u\psi \frac{\partial}{\partial \nu} (\overline{u\psi}) - \overline{u\psi} \frac{\partial}{\partial \nu} (u\psi)] d\sigma \tag{3.6}$$

If $\psi = \psi(r)$, then the left-hand side of (3.6) becomes

$$\int_{\Omega_R \setminus \Omega} [|\psi|^2 (u \triangle \overline{u} - \overline{u} \triangle u) + |u|^2 (\psi \triangle \overline{\psi} - \overline{\psi} \triangle \psi)$$

$$+ 2(u \frac{\partial \overline{u}}{\partial r} \psi \frac{\partial \overline{\psi}}{\partial r} - \overline{u} \frac{\partial u}{\partial r} \overline{\psi} \frac{\partial \psi}{\partial r})] d\mathbf{X}.$$

Since $k$ is real, and $u|_{\partial\Omega} = 0$, $u|_{z=0} = 0$, $u_z|_{z=h} = 0$, $\psi_z = 0$, it follows that

$$\int_{\Omega_R \setminus \Omega} [|u|^2 (\psi \triangle \overline{\psi} - \overline{\psi} \triangle \psi) + 2(u \frac{\partial \overline{u}}{\partial r} \psi \frac{\partial \overline{\psi}}{\partial r} - \overline{u} \frac{\partial u}{\partial r} \overline{\psi} \frac{\partial \psi}{\partial r})] d\mathbf{X}$$

$$= \int_{\Gamma_R} [u\psi \frac{\partial}{\partial \nu} (\overline{u\psi}) - \overline{u\psi} \frac{\partial}{\partial \nu} (u\psi)] d\sigma. \tag{3.7}$$

Let us introduce the following notation:

$$Z(r, \theta) = \{z \in [0, h] | (r, \theta, z) \in \mathbf{R}_b^3 \setminus \Omega\}$$

$$A(r) := \int_0^{2\pi} \int_{Z(r,\theta)} |u|^2 dz d\theta,$$

$$B(r) := \int_0^{2\pi} \int_{Z(r,\theta)} \overline{u} u_r dz d\theta,$$

$$b(r) := \frac{B(r)}{A(r)}.$$

then

$$\int_{\Omega_R \backslash \Omega} |u|^2 d\mathbf{X} = \int_0^R A(r) r dr,$$

$$\int_{\Omega_R \backslash \Omega} \overline{u} u_r d\mathbf{X} = \int_0^R B(r) r dr.$$

**Lemma 3.3**: Im $B(r) = 0$ for $r \in [0, \infty)$, and $2B(r) = A'(r)$.

Proof: Let $\psi = \psi(r)$ be a real $C^2$ function in (3.7), such that $\psi(r)$ remains bounded as $r \to \infty$. By letting $R \to \infty$, the right-hand-side of (3.7) vanishes by (3.1). Then (3.7) follows

$$0 = 2 \int_{R_b^3 \backslash \Omega} \psi \frac{\partial \psi}{\partial r} \left[ u \frac{\partial \overline{u}}{\partial r} - \overline{u} \frac{\partial u}{\partial r} \right] d\mathbf{X} = 4i \int_0^\infty \psi \frac{\partial \psi}{\partial r} Im(B(r)) r dr. \tag{3.8}$$

Let us denote

$$g^+(r) := max\{Im[B(r)], 0\},$$

$$g^-(r) := max\{-Im[B(r)], 0\},$$

$$\psi^+(r) := \left( 2 \int_0^r g^+(t) dt \right)^{1/2},$$

then

$$\psi^+ \frac{\partial \psi^+}{\partial r} = \frac{1}{2} \frac{\partial}{\partial r} (\psi^{+2}) = g^+(r).$$

Since

$$B(r) \sim 0(\frac{e^{-2k|a_{n+1}|r}}{r}), \quad as\ r \to \infty,$$

$\psi^+(r)$ is bounded as $r \to \infty$. Now taking $\psi_n(r) \in C^2[0, \infty)$ such that

$$\|\psi_n(r) - \psi^+(r)\|_{L^2} \to 0 \quad as\ n \to \infty$$

and substituting $\psi_n$ into (3.7), we conclude that

$$4i\int_0^\infty [g^+(r)]^2 r dr = 0.$$

Hence, $g^+(r) = 0$. In the same way we can prove $g^-(r) = 0$; consequently,

$$Im[B(r)] = 0 \;\; for \; r \in [0, \infty), \tag{3.9}$$

and

$$A'(r) = \int_0^{2\pi}\int_Z (u\overline{u}_r + \overline{u}u_r) dz d\theta = 2B(r). \tag{3.10}$$

Now we are in a position to prove:

**Theorem 3.1**: Under the assumptions of Lemma 3.1, it follows that $u \equiv 0$ in $R_b^3 \setminus \Omega$.

Proof: Rewriting (3.7) as

$$\int_0^R [A(r)(\overline{\psi}_{rr} + \frac{1}{r}\overline{\psi}_r) + 2B(r)\frac{\partial\overline{\psi}}{\partial r}]\psi r dr$$

$$-\int_0^R [A(r)(\psi_{rr} + \frac{1}{r}\psi_r) + 2B(r)\psi_r]\overline{\psi} r dr$$

$$= \int_{\partial\Omega_R} [|u|^2(\psi\frac{\partial\overline{\psi}}{\partial r} - \overline{\psi}\frac{\partial\psi}{\partial r}) + |\psi|^2(u\frac{\partial\overline{u}}{\partial r} - \overline{u}\frac{\partial u}{\partial r})] d\sigma. \tag{3.11}$$

Without loss of generality, we can suppose that $A(r) \neq 0$. In fact, if $A(r)$ has zero points $r_n$, then

$$A(r_n) = \int_0^{2\pi}\int_Z |u|^2 dz d\theta = 0,$$

hence $u(r_0, \theta, z) = 0$ on $\{(r_n, \theta, z) \in R_b^3,\ 0 \le \theta \le 2\pi,\ z \in Z\}$, which follows from the continuity of u. Let $\hat{r} = \sup_n\{r_n\}$, if $\hat{r} = \infty$, then there is a $r_n$ such that $\Omega \subset \Omega_{r_n}$. By separation of the variables in $R_b^3 \setminus \Omega_{r_n}$ it is easy to see that $u \equiv 0$ in $R_b^3 \setminus \Omega_{r_n}$ and hence, by analyticity of the solutions to the Helmhotz equation, $u \equiv 0$ in $R^3_b \setminus \Omega$. If $\hat{r} < \infty$, we can consider the uniqueness problem in $R^3_b \setminus \Omega_{\hat{r}}$ instead of $R^3_b \setminus \Omega$. It is easy to see that $A(r) \neq 0$ in $R^3_b \setminus \Omega_{\hat{r}}$.

If $\psi = \psi(r) \equiv 0$ is a solution of

$$\begin{cases} \psi_{rr} + \frac{1}{r}\psi_r + 2b(r)\psi_r + [\alpha(r) + i\beta(r)]\psi = 0, & 0 < r < \infty \\ \psi = 0(r^{-\delta}e^{k|a_{n+1}|r}), & \text{as } r \to \infty \end{cases} \tag{3.12}$$

where $\alpha(r)$, $\beta(r)$ are properly chosen real functions, $\beta(r) > 0$, $\delta = const. > 1$, then as $R \to \infty$, the right-hand-side of (3.11) vanishes and (3.11) becomes

$$\int_0^\infty A(r)\beta(r)|\psi|^2 r dr = \int_{R^3_b\setminus\Omega} |u|^2\beta(r)|\psi(r)|^2 d\mathbf{X} = 0. \tag{3.13}$$

We may conclude that

$$u \equiv 0 \;\; in \; R^3_b \setminus \Omega.$$

Now we prove that (3.12) has a solution for some choice of $\alpha(r)$ and $\beta(r)$. Let

$$\alpha(r) = [b(r)]^2 + \frac{b(r)}{r} + b'(r),$$

$$\beta(r) = \begin{cases} \beta_0^2(6 - 8r + 3r^2), & 0 \le r \le 1 \\ \frac{\beta_0^2}{r^2}, & r \ge 1 \end{cases}$$

where $\beta_0$ is a constant to be determine, then $\beta(r) \in C^2[0,\infty)$ such that $\beta(r) > 0$. Let

$$\psi = \phi e^{-\int_0^r b(\tau)d\tau} \tag{3.14}$$

(3.12) becomes

$$\phi_{rr} + \frac{1}{r}\phi_r + i\beta(r)\phi = 0 \;\; r \in [0,\infty). \tag{3.15}$$

It is easy to see that (3.15) is solvable, such that for $r > 1$,

$$\phi(r) = r^{-\frac{\sqrt{2}}{2}\beta_0 + i\frac{\sqrt{2}}{2}\beta_0} \tag{3.16}$$

$$\psi(r) = r^{-\frac{\sqrt{2}}{2}\beta_0 + i\frac{\sqrt{2}}{2}\beta_0} e^{-\int_0^r b(\tau)d\tau}. \qquad (3.17)$$

By assumption, $A(r) \neq 0$,

$$|\psi(r)| \leq r^{-\frac{\sqrt{2}}{2}\beta_0} e^{-\int_0^r \frac{A'(\tau)}{2A(\tau)}d\tau}$$

$$\leq M r^{-\frac{\sqrt{2}}{2}\beta_0} e^{-1/2 \log A(r)}$$

$$= \frac{M r^{-\frac{\sqrt{2}}{2}\beta_0}}{[A(r)]^{1/2}} \sim 0\left(\frac{r^{1/2} e^{k|a_{n+1}|r}}{r^{\frac{\sqrt{2}}{2}\beta_0}}\right).$$

We can choose $\beta_0$ such that $\frac{\sqrt{2}}{2}\beta_0 - \frac{1}{2} > 1$ , then $\psi$ satisfies (3.12); consequently, Theorem 3.1 is proved.

# 4 An Existence Theorem

We are going to combine the double- and single-layer integral representations to prove the existence of a solution to (2.1) $\sim$ (2.5). We do this by seeking a solution of the exterior Dirichlet Problem (2.1) $\sim$ (2.5) in the form of a combined double- and single-layer potential:

$$u(\mathbf{x}, z) = \int_{\partial\Omega} \left\{ \frac{\partial G(z, \zeta, |\mathbf{x} - \xi|)}{\partial r_\xi} - i\eta G(z, \zeta, |\mathbf{x} - \xi|) \right\} \psi(\xi, \zeta) d\sigma_\xi, \qquad (4.1)$$

where we use the subscript $\xi$ to denote in which variables we compute the normal derivative and perform the integrator. The source point is at $(\mathbf{x}, z)$,and $\eta \neq 0$ is an arbitrary real parameter.

We recall that $G(z, \zeta, |\mathbf{x} - \xi|)$ has a representation of the form (2.11). If we define the operators $\hat{S}$, $\hat{K}$, $\hat{K}'$ from $C(\partial\Omega)$ to $C(\partial\Omega)$ by

$$\hat{S}\phi := 2\int_{\partial\Omega} G(z, \xi, |\mathbf{x} - \xi|)\phi(\xi, \zeta) d\sigma_\xi, \ (\mathbf{x}, z) \in \partial\Omega,$$

$$\hat{K}\phi := 2\int_{\partial\Omega} \frac{\partial G(z, \xi, |\mathbf{x} - \xi|)}{\partial \nu_\xi} \phi(\xi, \zeta) d\sigma_\xi, \ (\mathbf{x}, z) \in \partial\Omega.$$

$$\hat{K}'\phi := 2\int_{\partial\Omega} \frac{\partial G(z,\xi,|\mathbf{x}-\xi|)}{\partial \nu_x}\phi(\xi,\zeta)d\sigma_\xi,\ (\mathbf{x},z)\in\partial\Omega,$$

and $S,\ K,\ K'$ are the operator defined by

$$S\phi := 2\int_{\partial\Omega} \Phi(\mathbf{X},\mathbf{Y})\phi(\mathbf{Y})d\sigma_\xi,\ (\mathbf{x},z)\in\partial\Omega,$$

$$K\phi := 2\int_{\partial\Omega} \frac{\partial}{\partial \nu_\xi}\Phi(\mathbf{X},\mathbf{Y})\phi(\mathbf{Y},\zeta)d\sigma_\xi,\ (\mathbf{x},z)\in\partial\Omega,$$

$$K'\phi := 2\int_{\partial\Omega} \frac{\partial}{\partial \nu_x}\Phi(\mathbf{X},\mathbf{Y})\phi(\mathbf{Y},\zeta)d\sigma_\xi,\ (\mathbf{x},z)\in\partial\Omega,$$

where

$$\Phi(\mathbf{X},\mathbf{Y}) = \frac{e^{ik|\mathbf{X}-\mathbf{Y}|}}{4\pi|\mathbf{X}-\mathbf{Y}|},\ \ \mathbf{X}=(\mathbf{x},z),\ \mathbf{Y}=(\xi,\zeta),$$

then we have the decompositions

$$\hat{S}\phi = S\phi + S_1\phi,$$

$$\hat{K}\phi = K\phi + K_1\phi,$$

$$\hat{K}'\phi = K'\phi + K_1'\phi.$$

Note that $S_1,\ K_1,\ K_1'$ are integral operators with the continuous kernels $\Phi_1(z,\zeta,|\mathbf{x}-\xi|)$, $\frac{\partial}{\partial\nu_\xi}\Phi_1(z,\zeta,|\mathbf{x}-\xi|)$, and $\frac{\partial}{\partial\nu_x}\Phi_1(z,\zeta,|\mathbf{x}-\xi|)$, respectively. Based on the well known boundary properties for these operators $S,\ K,\ and\ K'$, we can conclude that $\hat{S},\ \hat{K},\ \hat{K}'$ satisfy the same jump condition as $S,\ K,\ K'$ respectively.

**Theorem 4.1** The composite double- single- layer potential $u(\mathbf{x},z)$ defined by (4.1) is a solution of (2.1) $\sim$ (2.5), provided that the density $\psi\in C(\partial\Omega)$ is a solution of the integral equation

$$\psi + \hat{K}\psi - i\eta\hat{S}\psi = 2\phi. \qquad (4.2)$$

Proof: The combined double- single- layer potential $u(\mathbf{x}, z)$ obviously satisfies Helmholtz's equation (2.1) in $\mathbf{R}_b^3 \setminus \Omega$ , the radiating condition (2.4), and the surface and bottom conditions (2.2),(2.3). Let $(\mathbf{x}, z) \in \mathbf{R}_b^3 \setminus \Omega$ and let $(\mathbf{x}, z) \to (\mathbf{x}_0, z_0) \in \partial\Omega$, then

$$u(\mathbf{x}, z) = \int_{\partial\Omega} (\frac{\partial \Phi}{\partial \nu_\xi} - i\eta\Phi)\psi d\sigma_\xi + \int_{\partial\Omega} (\frac{\partial \Phi_1}{\partial \nu_\xi} - i\eta\Phi_1)\psi d\sigma_\xi$$

$$\to (\frac{1}{2}K\psi + \psi - i\frac{\eta}{2}S\psi) + \frac{1}{2}(\frac{1}{2}K_1\psi - i\frac{\eta}{2}S_1\psi)$$

$$= \frac{1}{2}\hat{K}\psi + \psi - i\frac{\eta}{2}\hat{S}\psi$$

So if $\psi$ satisfies (4.2), then

$$\lim_{(\mathbf{x}, z) \to (\mathbf{x}_0, z_0)} u(\mathbf{x}, z) = \frac{1}{2}\hat{K}\psi + \psi - i\frac{\eta}{2}\hat{S}\psi = \phi(\mathbf{x}_0, z_0).$$

**Theorem 4.2**: The integral equation (4.2) is uniquely solvable.

Proof: Since $\hat{K} - i\eta\hat{S}$ is a compact operatorm we only need to prove that the homogeneous form of equation (4.2) has only the trivial solution $\psi = 0$.

Let $\psi \in C(\partial\Omega)$ be a solution to the homogeneous equation $\psi + \hat{K}\psi - i\eta\hat{S}\psi = 0$, then u as defined by (4.1) solves the equation (2.1) $\sim$ (2.5) with $g = 0$. Therefore, by the uniqueness theorem, $u = 0$ in $\mathbf{R}_b^3 \setminus \Omega$.

Let $(\mathbf{x}, z) \in \Omega$, $(\mathbf{x}_0, z_0) \in \partial\Omega$,

$$u_- := \lim_{(\mathbf{x}, z) \to (\mathbf{x}_0, z_0)} u(\mathbf{x}, z), \quad \frac{\partial u_-}{\partial \nu_x} := \lim_{(\mathbf{x}, z) \to (\mathbf{x}_0, z_0)} \frac{\partial u(\mathbf{x}, z)}{\partial \nu_x},$$

from the jump relations we have

$$-u_- = \psi, \quad -\frac{\partial u_-}{\partial \nu_x} = i\eta\psi, \quad on\ \partial\Omega.$$

The first Green's formula implies that

$$i\eta \int_{\partial\Omega} |\psi|^2 d\sigma = \int_{\partial\Omega} \overline{u} \frac{\partial u_-}{\partial \nu_x} d\sigma_x = \int_\Omega (|\nabla u|^2 - k^2|u|^2) d\mathbf{X}.$$

Since k is real, it must be

$$\int_{\partial\Omega} |\psi|^2 d\sigma = 0$$

and $\psi = 0$ on $\partial\Omega$.

Combining Theorem 4.1 and 4.2, we then obtain the desired conclusion.

**Theorem 4.3** The problem (2.1) $\sim$ (2.5) is uniquely solvable.

# 5 Propagating Far Field Patterns

We have known that in a constant depth ocean there are only finite many propagating modes of wave. the others will evanesce soon. Therefore, the far- field pattern in a constant depth ocean will contain only the information from the propagating modes. One wonders how much information has been lost in the propagating process. In this section, we will present a representation of far-field patterns in terms of the combined double- single-layer potential , and we will discuss some of the properties of far-field pattern. A further discussion can be found in [6],[10].

Let $R = |\mathbf{x} - \xi|$, $r = |\mathbf{x}|$, $r' = |\xi|$, $\mathbf{x} \sim (r, \theta)$, $\xi \sim (r', \theta')$. $R^2 = r^2 + r'^2 - 2rr'cos(\theta - \theta')$. We can expand the Hankel function $H_0^{(1)}(kR)$ as

$$H_0^{(1)}(kR) = \sum_{n=0}^{\infty} \epsilon_n H_n^{(1)}(kr) J_n(kr') cosn(\theta - \theta'), \quad r > r', \tag{5.1}$$

where $\epsilon_0 = 1$, and $\epsilon_n = 2$ for $n \geq 1$.

In view of (5.1),

$$\frac{\partial G(z,\zeta,|\mathbf{x}-\xi|)}{\partial \nu_\xi} - iG(z,\zeta,|\mathbf{x}-\xi|)$$

$$= \frac{i}{2h}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\epsilon_m \Big\{ \phi_n(z)H_m^{(1)}(ka_nr)cosm\theta[\frac{\partial}{\partial \nu_\xi}(J_m(ka_nr')\phi_n(\zeta)cosm\theta')$$

$$-i(J_m(ka_nr')\phi_n(\zeta)cosm\theta')]$$

$$+\phi_n(z)H_m^{(1)}(ka_nr)sinm\theta[\frac{\partial}{\partial \nu_\xi}(J_m(ka_nr')\phi_n(\zeta)sinm\theta')$$

$$-i(J_m(ka_nr')\phi_n(\zeta)sinm\theta')]\}$$

$$=: \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\alpha_{mn}(z,\theta,r',\zeta,\theta')H_m^{(1)}(ka_nr), \tag{5.2}$$

where

$$\alpha_{mn}(z,\theta,r',\zeta,\theta') = \frac{i\epsilon_m}{2h}\phi_n(z)\Big\{ cosm\theta[\frac{\partial}{\partial \nu_\xi}(J_m(ka_nr')\phi_n(\zeta)cosm\theta')$$

$$-i(J_m(ka_nr')\phi_n(\zeta)cosm\theta')] + sinm\theta[\frac{\partial}{\partial \nu_\xi}(J_m(ka_nr')\phi_n(\zeta)sinm\theta')$$

$$-i(J_m(ka_nr')\phi_n(\zeta)sinm\theta')]\Big\}.$$

Since for $n > N$, $a_n = i|a_n|$,

$$H_m^{(1)}(ka_nr) \sim \sqrt{\frac{2}{\pi ka_nr}}exp[-k|a_n|r - i(m+1/2)\frac{\pi}{2}]$$

$$0(\frac{e^{-k|a_n|r}}{r^{1/2}}) = 0(\frac{1}{r^{3/2}}), \quad as\ r \to \infty.$$

Hence,

$$\frac{\partial G(z,\zeta,|\mathbf{x}-\xi|)}{\partial \nu_\xi} - iG(z,\zeta,|\mathbf{x}-\xi|) = \sum_{n=0}^{N}\sqrt{\frac{2}{\pi ka_nr}}exp(ika_nr)$$

$$\Big\{\sum_{m=0}^{\infty} exp[i-(m+1/2)\frac{\pi}{2}]\alpha_{mn}(z,\theta,r',\zeta,\theta')\Big\} + 0(\frac{1}{r^{3/2}}), \tag{5.3}$$

$$as \quad r \to \infty.$$

Now from (4.1) (set $\eta = 1$),

$$u(\mathbf{x}, z) = \sum_{n=0}^{N} \sqrt{\frac{2}{\pi k a_n r}} exp(ika_n r)$$

$$\int_{\partial\Omega} \left\{ \sum_{m=0}^{\infty} exp[i - (m + 1/2)\frac{\pi}{2}]\alpha_{mn}(z, \theta, r', \zeta, \theta') \right\} \psi(\mathbf{x}, z) d\theta_\xi + 0(\frac{1}{r^{3/2}})$$

$$=: \sum_{n=0}^{N} \frac{1}{\sqrt{ka_n r}} exp(ika_n r) f_n(z, \theta) + 0(\frac{1}{r^{3/2}}), \quad as\ r \to \infty \qquad (5.4)$$

where

$$f_n(z, \theta) = \sqrt{\frac{2}{\pi}} \int_{\partial\Omega} \left\{ \sum_{m=0}^{\infty} exp[i - (m + 1/2)\frac{\pi}{2}]\alpha_{mn}(z, \theta, r', \zeta, \theta') \right\} \psi(\mathbf{x}, z) d\theta_\xi,$$

$$n = 0, 1, ..., N;$$

play the role of a far-field pattern, and $\psi \in C(\partial\Omega)$ is the unique solution of integral equation (4.2).

We call the function

$$F(z, \theta) := \sum_{n=0}^{N} f_n(z, \theta) \qquad (5.5)$$

the "propagating far-field pattern". From the representation of $\alpha_{mn}(z, \theta, r', \zeta, \theta')$.

$$F(z, \theta) = \frac{i}{2h\pi} \sum_{n=0}^{N} \phi_n(z) \left\{ \sum_{m=0}^{\infty} \epsilon_m cos m\theta \int_{\partial\Omega} [\frac{\partial}{\partial \nu_\xi}(J_m(ka_n r')\phi_n(\zeta) cos m\theta') \right.$$

$$-i(J_m(ka_n r')\phi_n(\zeta) cos m\theta')]\psi(\mathbf{x}, \zeta) d\sigma_\xi$$

$$+ \sum_{m=0}^{\infty} \epsilon_m sin m\theta \int_{\partial\Omega} [\frac{\partial}{\partial \nu_\xi}(J_m(ka_n r')\phi_n(\zeta) sin m\theta')$$

$$\left. -i(J_m(ka_n r')\phi_n(\zeta) sin m\theta')]\psi(\mathbf{x}, \zeta) d\sigma_\xi \right\},$$

$$(\theta, z) \in [0, 2\pi] \times [0, h]. \qquad (5.6)$$

Let $F$ be the set of far-field patterns. In the case of $\mathbf{R}^3$ and $\mathbf{R}^2$, we know that there is an one-to-one correspondence between $F$ and $C(\partial\Omega)$

(cf. [3]). In particular, if $u$ is a solution to the Helmholtz equation in the exterior region $\mathbf{R}^3 \setminus \Omega$ satisfying the radiating condition, if its far-field pattern vanishes identically, then $u = 0$ in $\mathbf{R}^3 \setminus \Omega$. Unfortunately, this is not true in the finite depth ocean case. Following is a typical example showing that it is possible that the far-field pattern of all scattered waves are identical to zero.

In fact, for $0 < k < \frac{\pi}{2h}$,

$$a_n = [1 - \frac{(2h+1)^2\pi^2}{4k^2h^2}]^{1/2} = i|a_n|,$$

for all $n = 0, 1, ..., \infty$. By the representation theorem, we can conclude that any solution to the problem (2.1) $\sim$ (2.5) with $0 < k < \frac{\pi}{2h}$ has the asymptotic property:

$$u(\mathbf{x}, z) = 0(\frac{1}{r^{3/2}}), \quad as\ r \to \infty;$$

that is, its far-field pattern is identical to zero.

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