University of Delaware

SEA GRANT

DEL-SG-06-89

The Propagation Problem and Far-field

Patterns in a Stratified Finite Depth Ocean

by

R.P. Gilbert
Dept. of Mathematical Sciences
University of Delaware
Newark, DE 19716, USA

Yongzhi Xu
Dept. of Mathematical Sciences
University of Delaware
Newark, DE 19716, USA

February 1989

University of Delaware Sea Grant College Program
Newark, DE 19716

This publication is the result of research sponsored by NOAA Office of Sea Grant, Department of Commerce, under Grant No. NA86AA-D-SG040 (Project No. R/OE-2). The U.S. Government is authorized to produce and distribute reprints for governmental purposes, notwithstanding any copyright notation that may appear hereon.

The University of Delaware Sea Grant College Program is supported cooperatively by the National Oceanic and Atmospheric Administration, U.S. Department of Commerce, and by the State of Delaware.

The Propagation Problem and Far-field Patterns in a Stratified Finite Depth Ocean

R. P. Gilbert
Yongzhi Xu
Dept. of Math. Sciences ,
University of Delaware ,
Newark , DE 19716 . [1]

1. Introduction

In this paper we investigate the direct, or propagation problem associated with the scattering of an "plane wave" off of a submerged body Ω. We assume that the body is contained in an ocean of finite, constant depth, which we refer to as R_b^3. The index of refraction, moreover, is assumed to be dependent only on the depth. Consequently, we are seeking solutions to the Helmholtz equation

$$\Delta_3 u + k^2 n^2(z) u = 0 \tag{1.1}$$

in the region $R_b^3 \setminus \Omega$, where moreover, $u(\mathbf{x}, z)$ must satisfy certain boundary conditions on the ocean surface and ocean bottom. For the purposes of this paper we assume that the ocean surface (z=0) is sound-soft; whereas, the ocean bottom ($z = z_b$) is sound-hard, i.e.

$$u(x, y, 0) = 0, \; and \; \frac{\partial u}{\partial z}(x, y, z_b) = 0. \tag{1.2}$$

[1] This work is supported in part by Sea Grant NA86AA-D-SG040.

If the solution is decomposed into sum of the incident wave $u^i(\mathbf{x}, z)$, and the scattered wave $u^s(\mathbf{x}, z)$, then on the object surface we have the additional boundary condition

$$u^s(\mathbf{x}, z) = -u^i(\mathbf{x}, z) \;\; on \; \partial\Omega. \tag{1.3}$$

This problem was already considerd by Gilbert and Xu [8],[9] for the case of an ocean with a constant index of refraction. It is clear that the above problem leads to a kind of exterior Dirichlet problem in a slab R_b^3 . In order to make sure of the uniqeness of the problem, we should post a radiating condition in addition.

It is known, for instance, see Ahluwalia and Keller [1] that the scattered wave u^s has the modal representation

$$u^s = \sum_{n=0}^{\infty} \phi_n(z) u_n^s(\mathbf{x}), \tag{1.4}$$

where $\phi_n(z)$ is the solution to the eigenvalue problem

$$\phi_{zz} + k^2 n^2(z)\phi = k^2 a^2 \phi, \tag{1.5}$$

$$\phi(0) = 0, \;\; \phi_z(z_b) = 0.$$

The problem (1.4) has an infinite number of simple, real eigenvalues [1]

$$a_0^2 > a_1^2 > a_2^2 > ...,$$

and corresponding eigenfunctions

$$\phi_0(z), \; \phi_1(z), \; \phi_2(z), \; ...,$$

which form a complete orthogonal set. An algorethm to construct the modal solution $\phi_n(z)$ analytically by the method of transmutations has been developed in Gilbert-Wood [7] and Dustin-Gilbert-Verma-Wood [6].

In view of the discussions in [1], [11] ,we know a proper radiating condition for our problem is that the n^{th} mode of u^s, u_n^s, satisfying the condition

$$\lim_{r\to\infty} r^{1/2}(\frac{\partial u_n^s}{\partial r} - ika_n u_n^s) = 0,\ r = \mid \mathbf{x} \mid,\ n = 0, 1, ..., \infty. \tag{1.6}$$

We refer the problem (1.1) $\sim$ (1.3), (1.6) as problem (D).

In the section 2 we prove the problem is uniqely solvable; further, we prove a dense property of the farfield pattern.

2 The Uniqueness and Existence Theorem

Let $\mathbf{R}_b^3 = \{(\mathbf{x}, z) \in \mathbf{R}^3, 0 \leq z \leq h\}$ where $\mathbf{x} = (x_1, x_2)$, and h is a positive constant. Let Ω be a bounded, connected domain in $\mathbf{R}_b^3$ with C^2 boundary $\partial\Omega$ having an outward unit normal ν. Moreover, define the relative complement of Ω as $\Omega_e := \mathbf{R}_b^3 \setminus \overline{\Omega}$.

As showed in [1], the Green's function $G(z, z_0, r)$ has the normal mode representation:

$$G(z, z_0, r) = \frac{i}{4}\sum_{n=0}^{\infty} \frac{\phi_n(z)\phi_n(z_0)}{\int_0^h \phi_n^2(s)ds} H_0^{(1)}(ka_n r), \tag{2.1}$$

which satisfies

$$\triangle G + k^2 n^2(z) G = -\frac{1}{2\pi r}\delta(z - z_0)\delta(r), \tag{2.2}$$

and (1.2) as well as the radiating condition (1.6). Hence

$$\int_{\partial\Omega} \left(u\frac{\partial G}{\partial \nu} - G\frac{\partial u}{\partial \nu} \right) d\sigma = -\int_{\Omega_e} (u \triangle G - G \triangle u)\, dX$$

$$= \begin{cases} u(\mathbf{x}, z) & (\mathbf{x}, z) \in \Omega_e \\ 0 & (\mathbf{x}, z) \in \Omega \end{cases} \tag{2.3}$$

where ν is the outward normal.

A uniqueness theorem for the constant index case has been proved in an earlier paper [11]. We can go through the proof in the present variable index case provided we use $\phi_n(z)$ as the solutions of (1.4) instead of $\phi_n(z) = sin[k(1-a_n^2)^{1/2}z]$ for the constant index case. We are led to:

Theorem 2.1 *If $u \in C^2(R_b^3 \setminus \overline{\Omega}) \cap C(R_b^3 \setminus \Omega)$ is a solution to the problem (D) with homogeneous boundary data, then $u \equiv 0$ in $R_b^3 \setminus \Omega$.*

To the existence problem, we seek a solution in the form of a combined double- and single-layer potential:

$$u(\mathbf{x}, z) := \int_{\partial\Omega} (\frac{\partial}{\partial \nu_\xi} + i\eta) G(z, \zeta, \mid \mathbf{x} - \xi \mid) g(r', \zeta', \theta') d\sigma_\xi, \qquad (2.4)$$

where we use the subscript ξ to denote in which variables we compute the normal derivative and perform the integrator. $G(z, \zeta, \mid \mathbf{x} - \xi \mid)$ is the Green's function and the source point is at $(\mathbf{x}, z)$, and $\eta \neq 0$ is an arbitrary real parameter.

We now need some information concerning the nature of the Green's function's singularity. To this end we recall that each Green's function is also a *Levi function.* With regard to Levi functions it is known that the generalized potentials act pretty much the same way as Newtonian potentials. More precisely let us form a single-layer potential with the Levi function L($\mathbf{X}$,$\mathbf{Y}$) as

$$v(\mathbf{X}) = \int_{\partial D} L(\mathbf{X}, \mathbf{Y}) \zeta(\mathbf{Y}) d\sigma_Y, \qquad (2.5)$$

and let $[f] := f^+(\mathbf{X}) - f^-(\mathbf{X})$ be the jump in the function $f(\mathbf{X})$ as the point $\mathbf{X}$ passes over the surface ∂D. Then the derivative of the single layer potential is discontinuous across the surface ∂D. The jump discontinuity obeys the rule (see for example Miranda [10] pg 35.)

$$\left(\frac{\partial v}{\partial \nu}\right)^{\pm} = \mp\frac{\zeta(\mathbf{X})}{2} + \int_{\partial D} \frac{\partial L(\mathbf{X},\mathbf{Y})}{\partial \nu_X}\zeta(\mathbf{Y})d\sigma_Y. \tag{2.6}$$

A double-layer potential using the Levi function may be constructed in the form

$$w(\mathbf{X}) = \int_{\partial D} \frac{\partial L(\mathbf{X},\mathbf{Y})}{\partial \nu_Y}\rho(\mathbf{Y})d\sigma_Y. \tag{2.7}$$

The double-layer potential has a jump across the surface ∂D given by [10]

$$w^+(\mathbf{X}) - w^-(\mathbf{X}) = \rho(\mathbf{X}). \tag{2.8}$$

Using these fact concerning single and double layer potentials we may conclude that the potential defined by (2.4) satisfies the following two jump conditions across the surface $\partial\Omega$:

$$u^+ - u^- = 2g \;\; on \;\; \partial\Omega, \tag{2.9}$$

and

$$\left(\frac{\partial u}{\partial \nu}\right)^{+} - \left(\frac{\partial u}{\partial \nu}\right)^{-} = -i2\eta g \;\; on \; \partial\Omega. \tag{2.10}$$

It is now convenient for us to introduce certain singular operators $\mathbf{S}$,$\mathbf{K}$, $\mathbf{K}'$ as was done in Gilbert-Xu[9]:

$$\mathbf{S}\phi := 2\int_{\partial\Omega} G(z,\zeta,|\,\mathbf{x}-\xi\,|)\phi(\xi,\zeta)d\sigma_\xi,\; (\mathbf{x},z)\in\partial\Omega, \tag{2.11}$$

$$\mathbf{K}\phi := 2\int_{\partial\Omega} \frac{\partial}{\partial \nu_\xi} G(z,\zeta,|\,\mathbf{x}-\xi\,|)\phi(\xi,\zeta)d\sigma_\xi,\; (\mathbf{x},z)\in\partial\Omega, \tag{2.12}$$

and

$$\mathbf{K}'\phi := 2\int_{\partial\Omega} \frac{\partial}{\partial \nu_x} G(z,\zeta,|\,\mathbf{x}-\xi\,|)\phi(\xi,\zeta)d\sigma_\xi,\; (\mathbf{x},z)\in\partial\Omega. \tag{2.13}$$

It is clear that $\mathbf{K}'$ is the adjoint to $\mathbf{K}$ with respect to the pairing

$$< \phi, \psi >:= \int_{\partial\Omega} \phi\psi d\sigma. \tag{2.14}$$

Now we are in the position to prove the existence theorem:

Theorem 2.1: *There is a unique solution to the exterior Dirichlet problem (D). The solution can be represented in the form of combined double and single layer potential (2.4) in which $g \in C(\partial\Omega)$ satisfies integral equation*

$$g + Kg - i\eta Sg = -2u^i. \tag{2.15}$$

Proof: It is easy to see that $u(\mathbf{X})$ defined by (2.4) satisfies (1.1),(1.2), and (1.6). If g is a solution to the integral equation (2.15), then (cf. [11])

$$u(\mathbf{X}) \to \frac{1}{2}(g + Kg - i\eta Sg) = -u^i, \ as\ \mathbf{X} \to \mathbf{X_0} \in \partial\Omega.$$

To show that the equation (2.15) has a solution, we need only to prove its respective homogeneous equation has only trivial solution since $K - i\eta S$ is a compact operator in $C(\partial\Omega)$.

Let $g \in C(\partial\Omega)$ be a solution to the homogeneous equation $g + Kg - i\eta Sg = 0$, then u as defined by (2.1) solves the homogeneous case of the problem (D) . Hence from the uniqueness $u = 0\ in\ R^3_b \setminus \Omega$.

Now by the jump condition we have

$$-u_- = g, \ \ -\frac{\partial u_-}{\partial \nu_x} = i\eta g, \ \ on\ \partial\Omega$$

The first Green's formula implies that

$$i\eta \int_{\partial\Omega} | g |^2 d\sigma = \int_{\Omega} (| \nabla u |^2 - k^2 n^2(z) | u |^2) d\mathbf{X}.$$

Since $k^2 n^2(z)$ is real, it follows that $g = 0\ \ on\ \partial\Omega$.

3 A Dense Set in $L^2(\partial\Omega)$

Let Ω be a bounded connected object contained in R_b^3. Moreover, we assume that the boundary $\partial\Omega$ is C^2, and the outward normal is designated by ν. The relative compliment of Ω is given by $\Omega_e := R_b^3 \setminus \Omega$. For convenience we may also refer to Ω as Ω^- and Ω^e as Ω^+. $H_n^{(1)}(r)$ are Hankel functions of order n and the first kind, $J_n(r)$ are Bessel functions of order n, the $\phi_n(z)$ are the modal solutions of the depth-variable equation (1.5), and the a_n are the corresponding eigenvalues. It is not difficult to prove

Theorem 3.1: *Let λ be a complex whose imaginary part is positive . Then the family of function given by*

$$(\frac{\partial}{\partial\nu} + \lambda)[\phi_n(z)J_m(ka_nr)cos(m\theta)] \tag{3.1}$$

$$(\frac{\partial}{\partial\nu} + \lambda)[\phi_n(z)J_m(ka_nr)sin(m\theta)] \tag{3.2}$$

$$n, m = 0, 1, ..., \infty,$$

are complete in $L^2(\partial\Omega)$.

Proof: The proof varies only in several places from that given earlier by Gilbert-Xu [9] for the constant index case. First we notice that if for a given $g \in L^2(\partial\Omega)$ we have

$$\int_{\partial\Omega} g(r,z,\theta)(\frac{\partial}{\partial\nu} + \lambda)[\phi_n(z)J_m(ka_nr)cos(m\theta)]d\sigma = 0, \tag{3.3}$$

$$\int_{\partial\Omega} g(r,z,\theta)(\frac{\partial}{\partial\nu} + \lambda)[\phi_n(z)J_m(ka_nr)sin(m\theta)]d\sigma = 0, \tag{3.4}$$

for $m, n = 0, 1, ..., \infty$, then g is equivalent to the zero function in $L^2(\partial\Omega)$. If $Z_0 := \{(\mathbf{x}, z) \in R_b^3 : | \mathbf{x} | = r_0\}$ is a cylinder containing the object Ω, then for all points $(\mathbf{x}, z) \in Z_0$ the Green's function may be expanded as

$$G(z,\zeta,|\,\mathbf{x}-\xi\,|) = \frac{i}{4}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\epsilon_m\phi_n(z)\phi_n(\zeta)}{\|\,\phi_n\,\|^2}H_m^{(1)}(ka_nr)J_m(ka_nr')$$

$$[cos(m\theta)cos(m\theta') + sin(m\theta)sin(m\theta')]. \qquad (3.5)$$

where we have used the cylindrical coordinates (r,θ,z) for $(\mathbf{x},z)$ and (r',θ',ζ) for (ξ',ζ'). From (3.3) and (3.4) the function defined by

$$u(\mathbf{x},z) := \int_{\partial\Omega}(\frac{\partial}{\partial\nu_\xi}+\lambda)G(z,\zeta,|\,\mathbf{x}-\xi\,|)g(r',\zeta',\theta')d\sigma \qquad (3.6)$$

is identically zero in the unbounded component of $\mathbf{R}_b^3\setminus Z_0$. Hormander [10] showed that the uniqueness of Cauchy's problem holds for elliptic partial differential equations under fairly general conditions. For our equation (1.1) these conditions convert into $n^2(z)$ being bounded and the surface on which the Cauchy data is given may be written the form $z = f(r,\theta)$, or $r = f(z,\theta)$ with f contained in the class C^1. From the unique solvability of Cauchy's problem follows the weak unique continuation property, namely if $u(\mathbf{x},z)$ vanishes in a complete neighborhood then it vanishes identically in its domain of definition. We may conclude therefore, as in the case [9], that $u(\mathbf{x},z)$ vanishes identically in $R_b^3\setminus\overline{\Omega}$.

Using those fact concerning single and double layer potentials we recalled in (2.6) and (2.8), we may conclude that the potential defined by (3.6) satisfies the following two jump condition across the surface $\partial\Omega$:

$$u_+ - u_- = 2g, \;\; on\ \partial\Omega, \qquad (3.7)$$

and

$$(\frac{\partial u}{\partial\nu})_+ - (\frac{\partial u}{\partial\nu})_- = -2\lambda g,\ on\ \partial\Omega. \qquad (3.8)$$

From this it follows that

$$(\frac{\partial u}{\partial \nu})_- + \lambda u_- = 0,\ on\ \partial\Omega, \tag{3.9}$$

that $u(\mathbf{x}, z)$ vanishes identically in Ω^+, and that $g = 1/2(u^+ - u^-) = 0\ on\ \partial\Omega$.

Now we let S, K and K' be operators defined by (2.11) $\sim$ (2.13), and introduce certain spaces that are useful:

$$H(k,\Omega_e) := \{u : u \in C^2(\Omega_e)\cap C^1(\overline{\Omega_e}),\ where\ u\ satisfies\ (1.1),\ (1.2)\ and\ (1.6)\}$$

$$A(k,R_b^3) := \{u :\ u(\mathbf{x},z) = \int_0^{2\pi}\int_0^h g(\zeta,\theta')\sum_{n\in\Lambda}\phi_n(\zeta)\phi_n(z)e^{ik a_n \mathbf{x}\cdot\mathbf{y}}d\zeta d\theta',$$
$$where\ \ (\mathbf{x},z)\in R_b^3,\ \mathbf{y} = (cos\theta', sin\theta'),\ g\in L^2(D_1),\ \Lambda\in\mathcal{N}\}.$$

where $\mathcal{N}$ is any finite set of the natural numbers containing {0,1,2,...,N } and $D_1 := [0,h]\times[0,2\pi]$. Moreover, we set

$$T_D(k,\Omega_e) = \{u; u = u^i + u^s, u^i \in A(k,R_b^3), u^s \in H(k,\Omega_e), u = 0,\ on\,\partial\Omega\},$$

$$\Phi_D(k,\Omega_e) := \frac{\partial T_D(k,\Omega_e)}{\partial\nu}\ |_{\partial\Omega} = \{\frac{\partial u}{\partial\nu}\ |_{\partial\Omega}:\ u\in T_D(k,\Omega_e)\}.$$

Following the procedure in Gilbert-Xu [9] it is now easy to show

Theorem 3.2: *The set $\Phi_D(k,\Omega_e)$ is dense in $L^2(\partial\Omega)$.*

Proof: The theorem will be established if we can show that if $g\in L^2(\partial\Omega)$ and that

$$\int_{\partial\Omega}\overline{g}\frac{\partial u}{\partial\nu}ds = 0,\ for\ all\ u\in T_D(k,\Omega_e), \tag{3.10}$$

then g=0 in $L^2(\partial\Omega)$. From the fact that $u\in T_D(k,\Omega_e)$ and using the properties of the operators **S**, **K** and **K'** the following integral equation

$$(\mathbf{I}+\mathbf{K}'+i\mathbf{S})\frac{\partial u(\mathbf{x},z)}{\partial\nu} = 2\left(\frac{\partial u^i(\mathbf{x},z)}{\partial\nu} + iu^i(\mathbf{x},z)\right),\ (\mathbf{x},z)\in\partial\Omega, \tag{3.11}$$

holds on $\partial\Omega$. It can be shown that $(\mathbf{I}+\mathbf{K}+i\mathbf{S})$ is invertible; hence, condition (3.10) may be rewritten as

$$< (\mathbf{I}+\mathbf{K}+i\mathbf{S})^{-1}\overline{g}, \frac{\partial u^i}{\partial \nu} + iu^i >= 0. \tag{3.12}$$

Since the $u^i \in A(k, \mathbf{R}_b^3)$, we may use the Jacobi-Anger expansion combined with Theorem 3.1 to get

$$(\mathbf{I}+\mathbf{K}+i\mathbf{S})^{-1}\overline{g} = 0;$$

from which our result follows.

4 Far-field Patterns for the Helmholtz Equation in a Stratified Finite Depth Ocean

It is known [1] that the eigenvalue problem (1.5) has an infinite number of real eigenvalues and that only a finite number of these are positive. Hence, the eigenvalues α_n^2 eventually become negative, as in the constant index case, it follows because of the large argument asymptotics for the Hankel functions of the first kind that only a finite number of modes can propagate.

Let N be the largest integer such that $a_N^2 > 0$,

$$V^N := L^2[0, 2\pi] \times span\{\phi_0, \phi_1, ..., \phi_N\}$$

We define again as was done in [9]:

(1) $P: A^N \subset A(k, \mathbf{R}_b^3) \rightarrow V^N$ by g:=Pu where

$$u(\mathbf{x}, z) = \int_{D_1} g(\theta, \zeta) \sum_{n\in\Lambda} \phi_n(z)\phi_n(\zeta) e^{ika_n \mathbf{x}\cdot\mathbf{y}} d\sigma \tag{4.1}$$

where $g \in V^N$, $\mathbf{y} = (cos\theta', sin\theta')$, $(\mathbf{x}, z) \in R_b^3$. $A^N = \{u \in A(k, \mathbf{R}_b^3) : g \in V^N\}$.

(2) $F : A(k, R_b^3) \to V^N$ by $F(\theta, z, k) := Fu^i$ where $F(\theta, z, k)$ is the propagating far-field pattern of u^s for $u = u^i + u^s \in T_D(k, \Omega_e)$.

(3) $E_D(k, \Omega) = \{u : u \in C^2(\Omega) \cap C(\overline{\Omega})$, u is a solution of Helmholtz equation in Ω and u=0 on $\partial\Omega\}$

By using the results proved in last section. we can prove in the same way as the proof of Theorem 4.1 in [9] that:

Theorem 4.1:

$$V^N = [P(E_D(k, \Omega) \cap A^N)] \oplus \overline{F(A(k, R_b^3))}$$

Here instead of repeating the proof in the constant case, we give another version of the dense property for the farfield pattern using a method similar to that in Angell-Colton-Kress [2].

Let $\omega_1 = \{\mathbf{x} = (x_1, x_2) \in R^2 \mid |\mathbf{x}| = 1\}$. Let the incoming wave u^i be

$$u^i(\mathbf{x}, z; m, \alpha) = \phi_m(z) e^{ika_m \alpha \cdot \mathbf{x}}, \ m = 0, 1, ..., N, \ \alpha \in \omega_1.$$

If the corresponding scattered wave is $u^s(\mathbf{x}, z; m, \alpha)$, then

$$u^s(\mathbf{x}, z; m, \alpha) = \int_{\partial\Omega} (u^s \frac{\partial G}{\partial \nu} - G \frac{\partial u^s}{\partial \nu}) d\sigma$$

$$= \frac{i}{4\|\phi_n\|^2} e^{-i\pi/4} \sum_{n=0}^{N} (\frac{2}{\pi k a_n r})^{\frac{1}{2}} e^{ika_n r} \phi_n(z) \int_{\partial\Omega} \{u^s(\xi, \zeta; m, \alpha) \frac{\partial}{\partial \nu_\xi} (e^{-ika_n \mathbf{x}\cdot\xi} \phi_n(\zeta))$$
$$- \frac{\partial u^s(\xi, \zeta; m, \alpha)}{\partial \nu_\xi} (e^{-ika_n \mathbf{x}\cdot\xi} \phi_n(\zeta))\} d\sigma_\xi + O(\frac{1}{r^{\frac{3}{2}}}), \qquad (4.2)$$

and the corresponding far-field pattern

$$F(\mathbf{x}, z; m, \alpha) = \sum_{n=0}^{N} \phi_n(z) \int_{\partial\Omega} \{u^s(\xi, \zeta; m, \alpha) \frac{\partial}{\partial \nu_\xi} (e^{-ika_n \mathbf{x}\cdot\xi} \phi_n(\zeta))$$
$$- \frac{\partial u^s(\xi, \zeta; m, \alpha)}{\partial \nu_\xi} (e^{-ika_n \mathbf{x}\cdot\xi} \phi_n(\zeta))\} d\sigma_\xi, \qquad (4.3)$$

$$(\mathbf{x}, z) \in \omega_1 \times [0, h] =: D_1.$$

We define $\hat{F}(\mathbf{x}, z; \alpha, \zeta)$ as:

$$\hat{F}(\mathbf{x}, z; \alpha, \zeta) = \sum_{m=0}^{N} \phi_m(z) F(\mathbf{x}, \zeta; m, \alpha), \quad (\mathbf{x}, z), \ (\alpha, \zeta) \in D_1. \tag{4.4}$$

Theorem 4.2: *For any* $(\mathbf{x}, z)$, $(\alpha, \zeta) \in D_1$, *we have*

$$\hat{F}(\mathbf{x}, z; \alpha, \zeta) = \hat{F}(-\alpha, \zeta; -\mathbf{x}, z). \tag{4.5}$$

Proof: Let $F(\mathbf{x}, z; m, \alpha) = \sum_{n=0}^{N} \phi_n(z) F_n(\mathbf{x}, m, \alpha)$, then

$$\begin{aligned}
F_n(\mathbf{x}, m, \alpha) &= \int_{\partial\Omega} \{u^s(\xi, \zeta; m, \alpha) \frac{\partial}{\partial \nu_\xi}(e^{-ik\alpha_n \mathbf{x}\cdot\xi} \phi_n(\zeta)) \\
&\quad - \frac{\partial u^s(\xi, \zeta; m, \alpha)}{\partial \nu_\xi}(e^{-ik\alpha_n \mathbf{x}\cdot\xi} \phi_n(\zeta))\} d\sigma_\xi \\
&= \int_{\partial\Omega} \{u^s(\xi, \zeta; m, \alpha) \frac{\partial}{\partial \nu_\xi} u^i(\xi, \zeta; n, -\mathbf{x}) \\
&\quad - \frac{\partial u^s(\xi, \zeta; m, \alpha)}{\partial \nu_\xi} u^i(\xi, \zeta; n, -\mathbf{x})\} d\sigma_\xi \\
&= -\int_{\partial\Omega} \{u^i(\xi, \zeta; m, \alpha) \frac{\partial}{\partial \nu_\xi} u^i(\xi, \zeta; n, -\mathbf{x}) \\
&\quad - \frac{\partial u^s(\xi, \zeta; m, \alpha)}{\partial \nu_\xi} u^s(\xi, \zeta; n, -\mathbf{x})\} d\sigma_\xi \\
&= \int_{\partial\Omega} \{u^s(\xi, \zeta; m, \alpha) \frac{\partial}{\partial \nu_\xi} u^s(\xi, \zeta; n, -\mathbf{x}) \\
&\quad - \frac{\partial u^i(\xi, \zeta; m, \alpha)}{\partial \nu_\xi} u^i(\xi, \zeta; n, -\mathbf{x})\} d\sigma_\xi \\
&= -\int_{\partial\Omega} \{u^i(\xi, \zeta; m, \alpha) \frac{\partial}{\partial \nu_\xi} u^s(\xi, \zeta; n, -\mathbf{x}) \\
&\quad - \frac{\partial u^i(\xi, \zeta; m, \alpha)}{\partial \nu_\xi} u^s(\xi, \zeta; n, -\mathbf{x})\} d\sigma_\xi \\
&= F_m(-\alpha, n, -\mathbf{x}).
\end{aligned}$$

Hence,

$$\hat{F}(-\alpha,\zeta;-\mathbf{x},z)=\sum_{m=0}^{N}\phi_m(\zeta)F(-\alpha,z,-\mathbf{x})$$

$$=\sum_{m=0}^{N}\phi_m(\zeta)\sum_{n=0}^{N}\phi_n(z)F_n(-\alpha,m,-\mathbf{x})$$

$$=\sum_{m=0}^{N}\sum_{n=0}^{N}\phi_m(\zeta)\phi_n(z)F_n(\mathbf{x},n,\alpha)$$

$$=\sum_{m=0}^{N}\phi_n(z)F(\mathbf{x},\zeta,n,\alpha)$$

$$=\hat{F}(\mathbf{x},z;\alpha,\zeta).$$

Now let $\{(\alpha_n,\zeta)\}_{n=1}^{\infty}$, $\zeta\in[0,h]$ be a set of vectors on D_1 such that $\{\alpha_n\}$ has an cluster point on ω_1. Define the sets

$$\mathcal{F}=\{\hat{F}(\mathbf{x},z;\alpha_n,\zeta)\ :\ n=1,2,...,\ \zeta\in[0,h]\};$$

$$v(\xi,\zeta)=\int_{D_1}g(\mathbf{x},z)\sum_{n=0}^{N}\phi_n(z)\phi_n(\zeta)e^{ik\alpha_n\mathbf{x}\cdot\xi}d\sigma_X. \tag{4.6}$$

Theorem 4.3: *If v defined by (4.6) is not an eigenfunction of the Dirichlet problem in Ω, then the set $\mathcal{F}$ is dense in V^N.*

Proof: We need only to show that if $h(\mathbf{x},z)\in V^N$ such that

$$\int_{D_1}\hat{F}(\mathbf{x},z;\alpha_n,\zeta)\overline{h(\mathbf{x},z)}d\sigma_X=0, \tag{4.7}$$

for $n=1,2,...,\zeta\in[0,h]$, then $h(\mathbf{x},z)\equiv 0$ for $(\mathbf{x},z)\in D_1$.

As pointed out in last section, the uniqueness of Cauchy's problem holds for our problem, therefore, for any $(\alpha,\zeta)\in D_1$ (4.7) holds. Now by Theorem 4.2, we can see that

$$\int_{D_1}\hat{F}(-\alpha,\zeta;-\mathbf{x},z)\overline{h(\mathbf{x},z)}d\sigma_X=0$$

for any $(\alpha,\zeta) \in \partial\Omega$. By changing variables, it follows

$$\int_{D_1} \hat{F}(\mathbf{x}, z; \alpha, \zeta)\overline{h(-\alpha,\zeta)}d\sigma_\alpha = 0$$

Denote that $g(\alpha,\zeta) = \overline{h(-\alpha,\zeta)}$, and define the generalized Herglotz wave functions

$$v(\mathbf{x}, z) = \int_{D_1} g(\alpha,\zeta) \sum_{n=0}^{N} \phi_n(z)\phi_n(\zeta)e^{ik a_n \mathbf{x}\cdot\alpha} d\sigma_\alpha$$

with Herglotz kernel g (Gilbert-Xu [8],[9]). And define the solution $U = U^s + v$ to the problem (D) by

$$U^s(\mathbf{x}, z) = \int_{D_1} g(\alpha,\zeta) \left[\sum_{m=0}^{N} u^s(\mathbf{x},\zeta; m,\alpha)\phi_m(z)\right] d\sigma_\alpha$$

then from the representation of $u^s(\mathbf{x}, z; m, \alpha)$and the asymptotic behavior of Hankel functions, we know

$$U^s(\mathbf{x}, z) = \sum_{m=0}^{N} \phi_m(z) \int_{D_1} g(\alpha,\zeta)u^s(\mathbf{x},\zeta; m,\alpha)d\sigma_\alpha$$

has the farfield pattern

$$\sum_{m=0}^{N} \phi_m(z) \int_{D_1} g(\alpha,\zeta)F(\mathbf{x},\zeta; m,\alpha)d\sigma_\alpha$$

$$= \int_{D_1} g(\alpha,\zeta)\hat{F}(\mathbf{x}, z; \alpha,\zeta)d\sigma_\alpha$$

which vanishes. Since $U^s \in V^N$, it follows that $U^s(\mathbf{x}, z) = 0 \; in \; \Omega_e$ (cf[9]).

Moreover,

$$U = U^s + v = \int_{D_1} g(\alpha,\zeta) \sum_{m=0}^{N} \phi_m(z)[u^i(\mathbf{x},\zeta; m,\alpha) + u^s(\mathbf{x},\zeta; m,\alpha)]d\sigma_\alpha$$

$$= 0, \quad on \;\; \partial\Omega,$$

so $v = -U^s = 0 \; on \; \partial\Omega$.

Since v is not a Dirichlet eigenvalue of Ω, we can conclude that $v = 0 \; in \; \Omega$ and hence $g = 0$. This implies $h = 0$.

References

[1] Ahluwalia, D. and Keller, J.; "Exact and asymptotic representations of the sound field in a stratified ocean," *Wave Propagation and Underwater Acoustics*, Leture Notes in Physics 70, Springer, Berlin (1977)

[2] Angell, T.S., Colton, D. and Kress, R.; "Far field patterns and inverse scattering problems for imperfectly conducting obstacles. (preprint) (1988)

[3] Brekhovskiskh, L. ; ***Wave in Layered Media***, 2^{nd}ed., Academic Press, New York, (1980)

[4] Colton, D. and Kress, R.; *Integral Equation Methods in Scattering Theory*, John Wiley, New York, (1983)

[5] Colton, D. and Monk, P.; "A novel method of solving the inverse scattering problem for time-harmonic acoustic waves in the resonance region," *SIAM J. Appl. Math.* 45 (1985), 1039-1053.

[6] Duston, M., Gilbert,R.P., Verma, G.H. and Wood, D.H.; " Direct generation of normal modes by transmution theory ", ***Computational Acoustics: Algorithms and Applications, Vol.2.*** (1987), 389-402

[7] Gilbert, R.P. and Wood, D.H.; "A transmutational approach to underwater sound propagation", ***Wave Motion 8*** (1986) 383-397.

[8] Gilbert, R. P. and Xu, Y.; "Starting fields and far fields in ocean acoustics" ,*Wave Motion*, to appear, (1987)

[9] Gilbert, R. G. and Xu, Yongzhi; "Dense sets and the projection theorem for acoustic harmonic waves in homogeneous finite depth ocean", *Wave Motion*. (1988)

[10] Miranda, C.; *Partial Differential Equations of Elliptic Type*, Springer-Verlag , (1970)

[11] Xu, Yongzhi; "The propagating solution and far field patterns for acoustic harmonic waves in a finite depth ocean," preprint, (1988)