WASHINGTON SEA GRANT PROGRAM

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THREE PUBLICATIONS
RELATED TO ECHO INTEGRATION
FROM THE WASHINGTON SEA GRANT
MARINE ACOUSTICS PROGRAM

By Paul H. Moose and J. E. Ehrenberg

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WSG 71-2
A SIMPLIFIED ANALYSIS OF THE STATISTICAL CHARACTERISTICS OF THE FISH ECHO INTEGRATOR
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THE VARIANCE OF FISH POPULATION ESTIMATES USING AN ECHO INTEGRATOR
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CONTENTS

1 *A Simplified Analysis of the Statistical Characteristics of the Fish Echo Integrator*

27 *The Variance of Fish Population Estimates Using an Echo Integrator*

45 *Derivation and Numerical Evaluation of a General Variance Expression for Fish Population Estimates Using an Echo Integrator*

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A WASHINGTON SEA GRANT PUBLICATION

WSG 71-2

A SIMPLIFIED ANALYSIS OF THE STATISTICAL CHARACTERISTICS OF THE FISH ECHO INTEGRATOR

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ABSTRACT

The fish echo integrator has been developed at the University of Washington to assess the abundance of fish at various depth strata of selected Northwest regional waters. The statistical characteristics of the echo integrator are described herein, with respect to mode of operation.

ACKNOWLEDGMENT

This publication is the result of research accomplished during the summer of 1970 at the University of Washington. The work was supported by the National Science Foundation Grant GH-40 to the Washington Sea Grant Program, now a part of the National Sea Grant Program, which agency is maintained by the National Oceanic and Atmospheric Administration of the U. S. Department of Commerce.

A Simplified Analysis of the Statistical
Characteristics of the Fish Echo Integrator

I Summary

The fish echo integrator has been developed at the University under the Sea Grant Program. This device is presently being used by several groups to assess, quantitatively, the abundance of fish in selected depth strata in various regions of the Northwest. The integrator, as presently configured, outputs a random variable which corresponds to the integral of the output of a detected echo sounder signal between time-in-ping intervals corresponding to the selected depth strata. The integrated signal is summed for a number of transmissions as the vessel proceeds on the desired course. At the end of the selected number of pings, a new sum is begun. The integrator output is normalized by the number of pings and the thickness of the strata so that the final output is volts per meter per ping.

The purpose of this note is to analyze a reasonably simple model of this system and determine certain statistical characteristics about the "goodness" of the integrator as an estimator of fish abundance. In particular, we shall, focus our attention on computation of the mean and variance of the integrator signal. We shall show, within the framework and assumptions of our model that:

A.) The Integrator gives an unbiased estimate of the quantity $\mu_\beta^{(2)} \rho$ where $\mu_\beta^{(2)}$ is the average fish target strength and ρ is the number of fish per unit volume. As expected, we must know $\mu_\beta^{(2)}$ if we are to calculate ρ. Errors introduced into the estimate by errors in estimating $\mu_\beta^{(2)}$ are not considered in this note.

B.) The integrator does not provide a consistent estimate of ρ. That is, as the number of pings, P, tends to infinity, the variance does not tend to zero. The dependence of the variance of the density ρ, on the pulse length and beam pattern, on the length of the transec (the length, beam pattern and P determine the degree of overlap) and on the distribution function of the fish reflectivity coefficients for finite P is also determined. The most general result obtained for the normalized variance[1], of the integrator,

$$\eta_\rho^2 = \frac{1}{P}\left[\frac{c\delta/2}{D} + (\gamma_\beta - 1)\ /\rho DA_g\right] + \frac{1}{\rho DA_T},$$

includes the effects of both multiple echos and overlapping insonification volumes. It is interesting to note the importance of the statistics of the target reflectivity (γ_β) in this result as well as to note that as $P \to \infty$ the variance reduces to $1/\rho DA_T$. This is the variance of the density estimate if one could count each of the fish in the total insonified volume, DA_T, and consequently this would be the performance of an optimum estimation procedure.

The next section of this report presents the model of the system to be used in the calculations. In the development of the model it becomes apparent that various moments of certain sequences of random variables are important. Section III develops the required moments. In Section IV the mean and variance of the integrator output are calculated for single pings and for multiple, non-overlapping pings for arbitrary TVG and arbitrary density of fish versus depth. These results are specialized for constant fish density and 20log R TVG. With these specializations, the case of overlapping pings is considered and results are obtained in a general form (see Eq. (32)) suitable for numerical calculation and, with some approximation, in closed form (see Eq. (36)). In Section V we remark briefly on the approximations and their potential effect on

[1]The normalized variance is defined as the ratio of the variance to the square of the mean. The equation here only applied for 20log R TVG gain correction.

the results, on some problems of optimization and on future experimental and theoretical problems of importance.

System Model

The echo-sounder transmits a short pulse of acoustical energy at an ultrasonic carrier frequency. The energy is concentrated, due to the transducer aperture, into a narrow beam directed vertically downward. Energy reflected by the fish returns to the transducer which again, due to its aperture, exhibits maximum sensitivity in the same narrow beam. In reality, we should deal with spherical waves and transducer transmission and reception patterns which are continuous functions of angle and with ship pitch and roll. For our model, however, (see Figure I),we shall assume an ideal elliptical beam pattern of arthwartships width $2\Theta_2$ and fore/aft width $2\Theta_1$. Furthermore, we suppose that the transmit/receive bandwidths are sufficiently wide to pass the pulse essentially undistorted. Accordingly, the complex modulation envelope of the receiver output at time, t_k after transmission consists of the superposition of all echos from fishes between ranges (depths) $(t-\delta)/2c$ and $t/2c$ where δ and c are the pulse length and velocity of propagation, respectively.

Targets of equal size will have mean square levels at the transducer output in inverse proportion to the fourth power (neglecting absorption losses) of the time after transmission. However, the insonified volume increases in proportion to the square of the time after transmission so that the total mean square signal level is, on the average, decreasing with time squared. Thus, if the receiver power gain is increased as the square of time (20log t) after transmission, the average signal level into the integrator will be constant throughout the range gate period. These remarks are formalized as follows. Let X_k be the complex modulation envelope of the transducer output at time t_k after transmission.

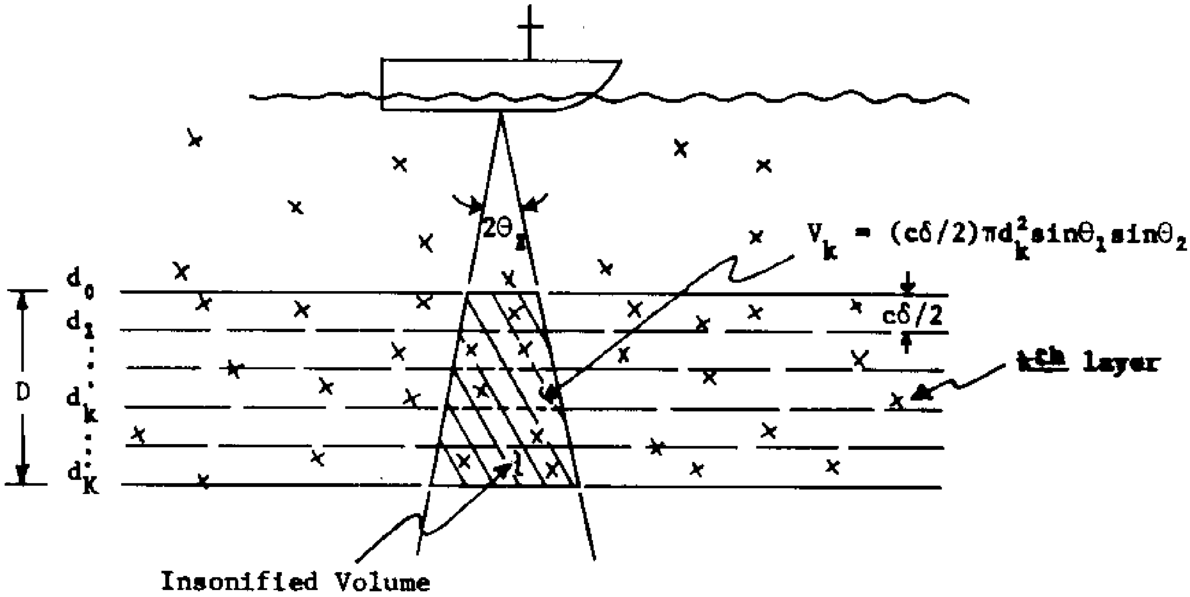


Figure 1

Approximate Geometry

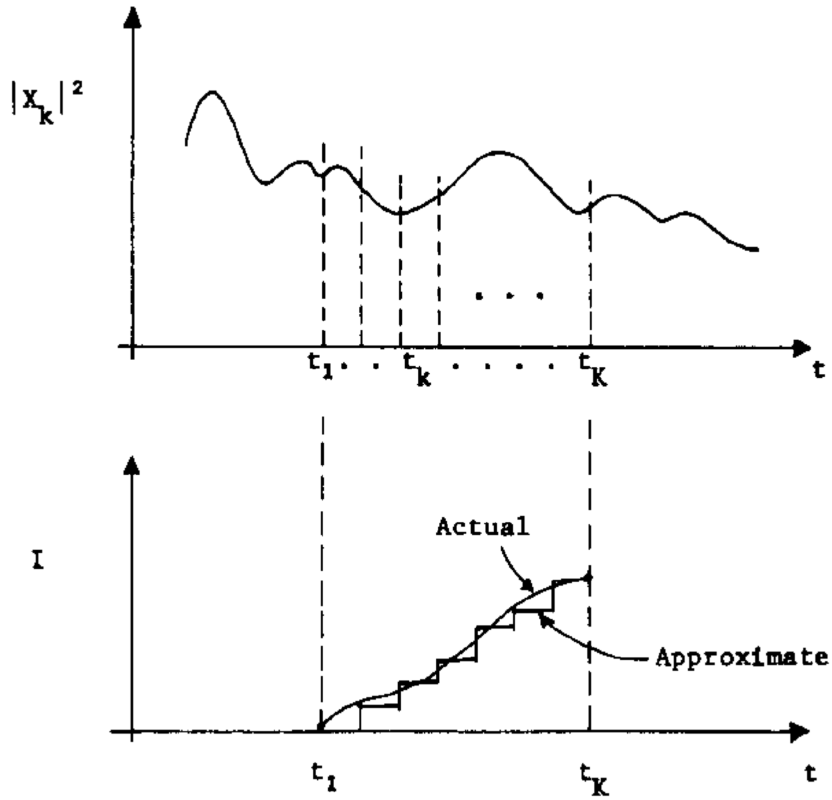


Figure 2

Integrator Input and Output Signals

Then

$$X_k = \alpha_k \sum_{n=1}^{N_k} \beta_n e^{j\phi_n} \tag{1}$$

where α_k ($\alpha_k \sim \frac{1}{t_k{}^2}$) represents the receiver output for a unit target at range corresponding to t_k, β_n is the reflection coefficient of the n^{th} fish and N_k is the number of fish insonified, that is with overlapping echos, at time t_k. (See Figure 1). We can approximate the integrator input signal for a single ping by the stairstep signal of Figure 2. Thus

$$I = \delta \sum_{k=1}^{K} g_k \left[X_k X_k^*\right] \tag{2}$$

where: $t_k - t_{k-1} = \delta$ is the pulse length. In this way we account for all the fish, i.e., $\sum_{k=1}^{K} N_k$ = total fish in the insonified depth strata yet different fish contribute to each of the samples of signal return in (1). g_k is the power gain of the receiver amplifiers at time t_k.

The sampled approximation to the integrator output may be considered accurate under the following circumstances. The sample sequence $\left\{\left[X_k X_k^*\right]\right\}$ completely describes a square law detected signal with frequencies less than $f_o < \frac{1}{2\delta}$. However, the integrator is, in effect, a low pass filter with a cutoff frequency $f_c \sim \frac{1}{K\delta}$ where $K\delta$ is the total length of the range gate. Thus, for range gates of several pulse lengths or more, the approximation should be accurate within a few percent. The sampled spectra for a single target[2], the integrator transfer function and region of contributing error are illustrated in Figure 3.

[2]This is the worst case, since it contains the most high frequencies due to the sharp leading and trailing edges of the pulse.

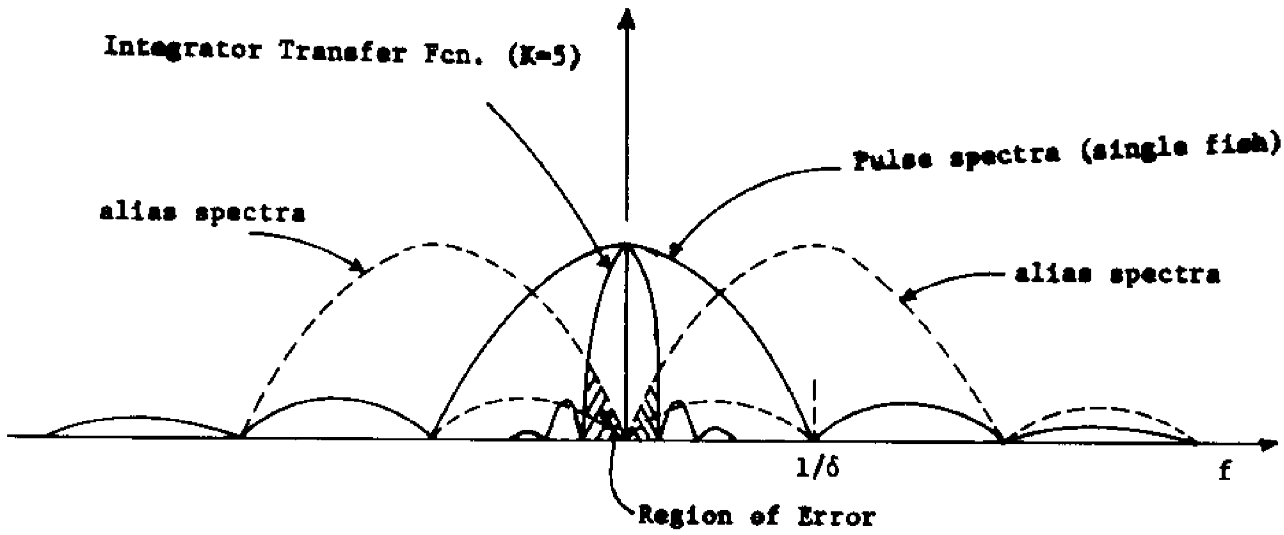


Figure 3

Spectra of Sampled Envelope (Single Fish) and Integrator Transfer Function

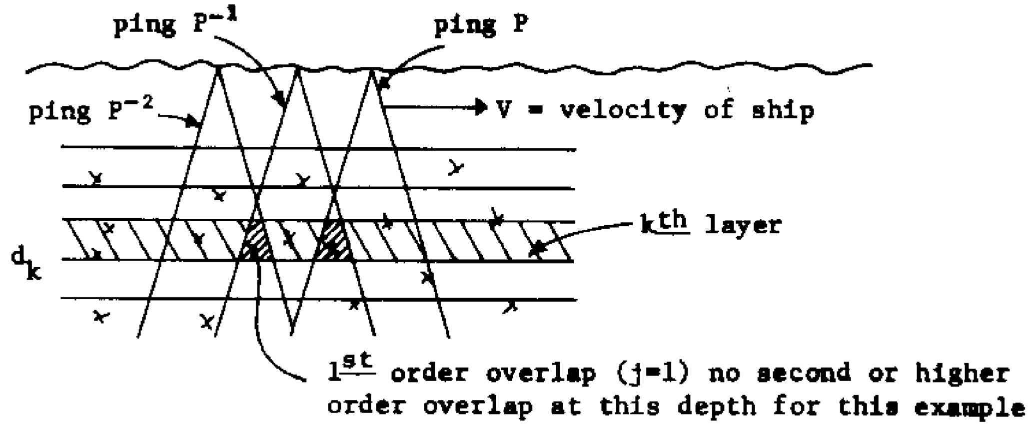


Figure 4

Illustrating the Region of Overlap

Selection of the samples one pulse length apart assures that each of the K terms of (2) are statistically independent random variables if we assume that the N_k are statistically independent, e.g., if the fish are Poisson distributed throughout the volume. Consequently the mean and variance of (2) are the sums of the means and variances of the individual terms of (2). Similarly, if P transmissions from non-overlapping regions are superimposed as the ship moves forward along its track, the mean and variance of the sum will be the sums of the means and variances from the individual soundings. If, however, the transmission repetition rate and ships velocity are selected so that there is q_{jk} fractional overlap of the k^{th} layer on the p^{th} and $p-j^{th}$ transmissions (See Figure 4), the means will still add but the variance will be increased due to the ping-to-ping correlation. Accordingly, completion of the analysis requires that we study the mean and variance of the square of various overlapping and non-overlapping Poisson sums of random variables. This study is the purpose of the next section.

III Moments of Certain Poisson Sums

The sum,

$$X = \sum_{n=1}^{N} y_n \tag{3}$$

is known as a Compound Poisson Process[3] and serves as an appropriate model not only for our problem but many other physical problems as well, including a classical model of Brownian motion which leads to the well known Weiner Process. Consequently, its properties have been studied extensively. It is assumed that the y_n are independent, identically distributed random variables. In this problem, the y_n correspond to $\alpha\beta_n e^{j\phi_n}$ and are independent and identically

[3]Emanuel Parzen, 1962, Stochastic Processes, Holden-Day, page 128.

distributed if we assume the ϕ_n are independent U : $[-\pi,\pi]$ random variables and the β_n are independent and identically distributed. Both are reasonable first-order assumptions for echos from fish.

In order to calculate the integrators' performance, we seek the mean and variance of XX*. Note that these are given by the conditional moments according to

$$\mu = \overline{XX^*} = \sum_{N=0} E[XX^*|N]\, P(N) \tag{4}$$

and

$$\sigma^2 = E\{[XX^* - \mu]^2\} = \sum_{N=0}^{\infty} E[[XX^*]^2|N]\; P(N) - \mu^2 \tag{5}$$

The conditional moments are

$$E[XX^*|N] = \sum_{n=1}^{N} E[y_n y_n^*] = N\mu_y^{(2)} \tag{6}$$

and

$$\begin{aligned} E[(XX^*)^2|N] &= \sum_{n=1}^{N} E[(y_n y_n^*)^2] + 2 \sum_{n=1}^{N} \sum_{\substack{m=1 \\ m\neq n}}^{N} E(y_n y_n^*)\; E(y_m y_m^*) \\ &= N\, \mu_y^{(4)} + 2\, N(N-1)\; (\mu_y^{(2)})^2 \end{aligned} \tag{7}$$

where $\mu_y^{(k)}$ is the k^{th} moment of the identically distributed random variables $\{y_n\}$. The terms in (7) are obtained by accounting for all non-zero terms that remain in the expected value of the quadruple sum obtained in raising (3) to the 4^{th} power.

The number of events, N, in (3) is assumed to be a Poisson random variable with Poisson parameter λ, i.e.,

$$P(N) = \frac{\lambda^N e^{-\lambda}}{N!}. \tag{8}$$

The first and second moments of (8) are

$$\left.\begin{aligned} E(N) &= \lambda \\ E\ (N^2) &= \lambda^2 + \lambda \end{aligned}\right\} . \tag{9}$$

Combining (6) & (9) with (4) and (7) & (9) with (5) we obtain

$$\left.\begin{aligned} \mu &= \lambda\ \mu_y^{(2)} \\ \sigma^2 &= \lambda\ \mu_y^{(4)} + \lambda^2(\mu_y^{(2)})^2 \end{aligned}\right\} \tag{10}$$

To deal with the problem of overlap, we must calculate the mean and variance

$$\left.\begin{aligned} \mu &= E\left[X_1^* X_1 + X_2^* X_2\right] \\ \sigma^2 &= E\ (X_1^* X_1 + X_2^* X_2 - \mu)^2 \end{aligned}\right\} \tag{11}$$

where

$$\left.\begin{aligned} X_1 &= \sum_{n=1}^{N_1} y_{1n} \\ X_2 &= \sum_{n=2}^{N_2} y_{2n} \end{aligned}\right\} \tag{12}$$

and N_1 & N_2 come from overlapping regions. (See Figure 5)

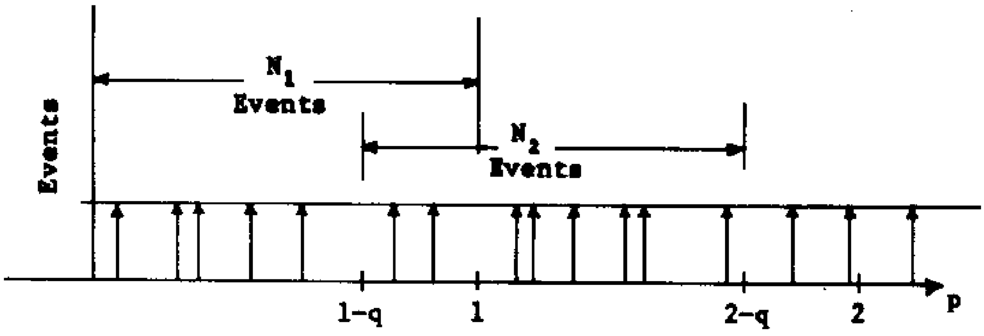


Figure 5

Overlapping Poisson Intervals

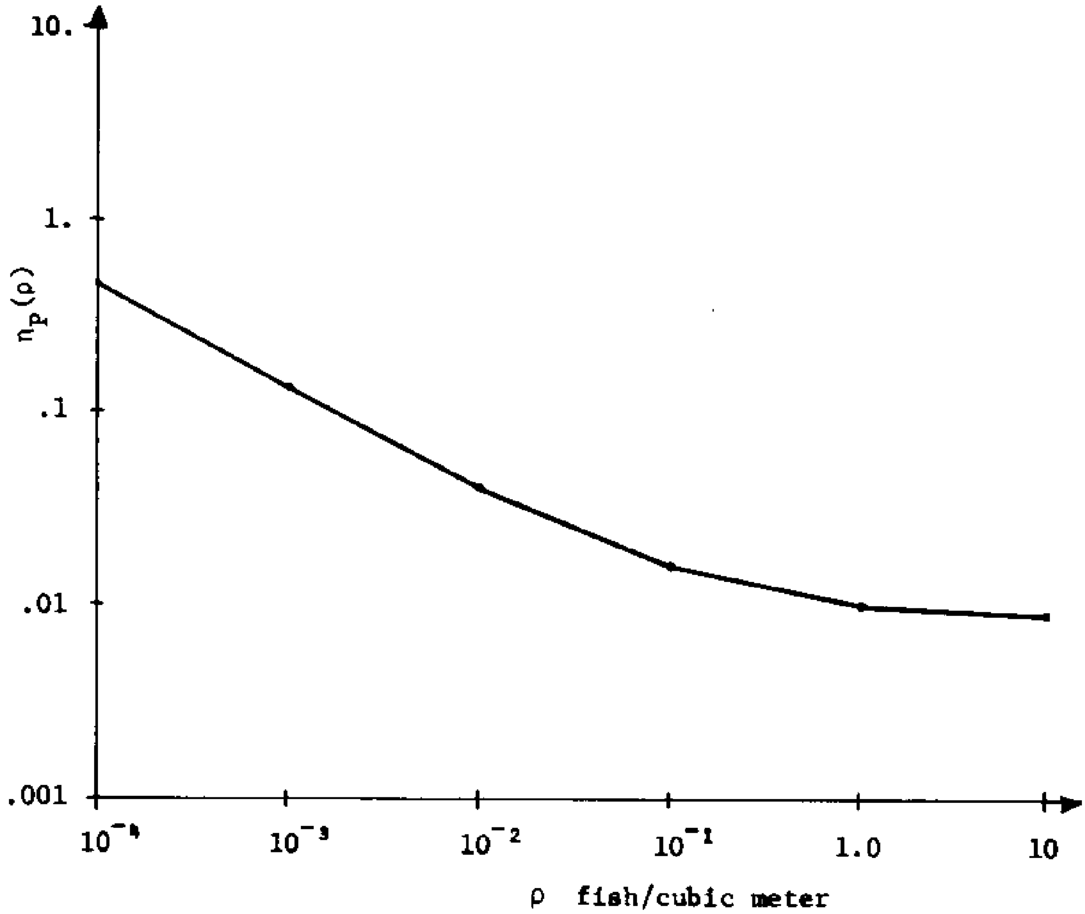


Figure 6

Relative Standard Deviation of the Estimate of ρ versus ρ for a Typical Example

The mean is just the sum of the individual means, i.e.,

$$\mu = E\left[X_1 X_1^*\right] + E\left[X_2 X_2^*\right] = 2\lambda\mu_y^{(2)} \tag{13}$$

The variances also add plus a term due to the correlation of the two samples such that

$$\sigma^2 = 2\left[\lambda\mu_y^{(4)} + \lambda^2(\mu_y^{(2)})^2\right] + 2E\left[X_1 X_1^* \; X_2 X_2^*\right] - 2\left[\lambda\mu_y^{(2)}\right]^2 \tag{14}$$

The co-variance term may be calculated as follows. Let N_1' & N_2' be the number of fish in the non-overlapping portions of the first and second intervals and Q be the number of fish in the overlapping portion. Now (12) may be rewritten as

$$\begin{aligned} X_1 &= \sum_{n=1}^{N_1'} y_{1n} + \sum_{n=N_1'+1}^{Q+N_1'} y_{1n} = X_1' + X_{1q} \\ X_2 &= \sum_{n=1}^{N_2'} y_{2n} + \sum_{n=N_2'+1}^{Q+N_2'} y_{2n} = X_2' + X_{2q} \end{aligned} \tag{15}$$

and the joint moment of (14) can be rewritten as

$$\begin{aligned} E\left[X_1 X_1^* X_2 X_2^*\right] &= E\left[X_1' X_1'^* X_2' X_2'^*\right] + E\left[X_1' X_1'^* X_{2q} X_{2q}^*\right] \\ &\quad + E\left[X_{1q} X_{1q}^* X_2' X_2'^*\right] + E\left[X_{1q} X_{1q}^* X_{2q} X_{2q}^*\right] \end{aligned} \tag{16}$$

Since the three intervals are independent, (16) reduces to

$$E\left[X_1X_1^*X_2X_2^*\right] = \left[\lambda(1-q)\ \mu_y^{(2)}\right]^2 + 2q(1-q)(\lambda\mu_y^{(2)})^2$$
$$+ E\left[X_{1q}^*X_{1q}X_{2q}X_{2q}^*\right] \quad (17)$$

since $$E(N_1) = E(N_2) = \lambda(1-q),\ E(Q) = \lambda q$$

are the average number of events expected in the three intervals of concern. The last term is evaluated (if we take the ping-to-ping phases and amplitudes to be uncorrelated) as,

$$E\left[X_{1q}X_{1q}^*X_{2q}X_{2q}^*\right] = \sum_{Q=0}^{\infty} E\left[X_{1q}X_{1q}^*X_{2q}X_{2q}^*|Q\right]P\left[Q\right] \quad (18)$$

where the conditional $4^{\underline{th}}$ moment is found to be

$$E\left[X_{1q}X_{1q}^*X_{2q}X_{2q}^*|Q\right] = Q^2(\mu_y^{(2)})^2 \quad (19)$$

such that

$$E\left[X_{1q}X_{1q}^*X_{2q}X_{2q}^*\right] = (\lambda q + (\lambda q)^2)\ (\mu_y^{(2)})^2 \quad (20)$$

Combining (20) with (17) we have

$$E(X_1X_1^*X_2X_2^*) = (\lambda^2 + \lambda q)\ (\mu_y^{(2)})^2 \quad (21)$$

so that finally, the variance of the sum of the squares of two compound Poisson processes with 100q % overlap is

$$\sigma^2 = 2\left[\lambda\mu_y^{(4)} + (\lambda q + \lambda^2)\ (\mu_y^{(2)})^2\right] \tag{22}$$

In the next section we will apply the results derived above to calculate the mean and variance of the echo integrator.

IV Mean and Variance of the Integrator Signal

The calculations presented in the previous section gave the mean and variance of a single layer for one or two overlapping pings. We can extend these results to the integrator by indexing the parameters with the layer no. and summing the means and variances from the k independent layers. The Poisson parameter λ_k is the mean no. of scatterers in the $k^{\underline{th}}$ layer which is also given by $\rho_k\ V_k$ where ρ_k is the Poisson density at the $k^{\underline{th}}$ layer and V_k is the equivalent insonified volume of the $k^{\underline{th}}$ layer. Incorporating this nomenclature along with (10) and (2) we obtain

$$\left.\begin{aligned} I &= \mu_\beta^{(2)}\delta \sum_{k=1}^{K} \rho_k\ V_k\ g_k\alpha_k^2 \\ \mathrm{Var}\ (I) &= (\mu_\beta^{(2)}\delta)^2\left[\sum_k (\rho_k\ V_k\ g_k\alpha_k^2)^2\ (1 + \gamma_\beta/\rho_k\ V_k)\right] \end{aligned}\right\} \tag{23}$$

where $\mu_\beta^{(2)}$, the second moment of the β_n (reflectivities) is just the average target stength of the fish and $\gamma_\beta = \mu_\beta^{(4)}/(\mu_\beta^{(2)})^2$. γ_β has been calculated for a number of common distributions by Ol'shevskii[4] and was found to range from 1 (β=constant) to 6 (β exponentially distributed). A Rayliegh distribution yielded a γ of 2 and a uniform distribution a γ of 1.8.

If the fish density is constant with depth over the integrated stratum, (24) reduces to

[4]V.V. Ol'shevskii, 1967, Characteristics of Sea Reverberation, p. 57. Consultants Bureau, N.Y., translated from Russian.

$$\left.\begin{aligned} I &= \mu_\beta^{(2)} \rho \left(\delta \sum_{k=1}^{K} V_k g_k \alpha_k^2\right) \\ \mathrm{Var}\ (I) &= \left(\mu_\beta^{(2)} \rho\right)^2 \left[\delta^2 \sum_{k=1} \left(V_k g_k \alpha^2\right) \ (1+\gamma_\beta/\rho V_k)\right] \end{aligned}\right\} \qquad (25)$$

Eq. (25) illustrates one of the principal features of the device, i.e., that the integrator (with square-law detection) produces an unbiased estimate of ρ, the density of fish in the layer. That is, the integrator signal per ping is proportional to the density ρ. The scale factor depends on the fish target strength, $\mu_\beta^{(2)}$, the beam pattern and pulse length, (V_k, δ; see Fig. 1), the power gain of the receiver, g_k, and the source level and transmission loss in the medium, α_k. Given these system constants (easily measured except for $\mu_\beta^{(2)}$) we can estimate ρ from I and the unbiasedness of the estimate does not depend on the choice of TVG (g_k vs. k).

The quality of the estimate is best studied by considering the relative scatter of the data about the mean. Accordingly, we consider the normalized variance (for constant ρ)

$$\eta^2 = \frac{\mathrm{Var.}\ (I)}{I^2} = \left[\sum_{k=1}^{K} \left(V_k g_k \alpha_k^2\right) \ \left(1 + \gamma_\beta/\rho V_k\right)\right] \Big/ \left[\sum_{k=1}^{K} V_k g_k \alpha_k^2\right]^2 \qquad (26)$$

and see that it does depend on the constants of the system (g_k, V_k, α_k^2) as well as the statistics of the target reflectivity as embodied in γ_β and the true value of the density of fish ρ. In fact we notice immediately that

$$\begin{aligned} &\lim_{\rho \to 0} \eta^2 = \infty \\ &\lim_{\rho \to \infty} \eta^2 = \sum_{k=1}^{K} \left(V_k g_k \alpha_k^2\right)^2 \Big/ \left[\sum_{k=1}^{K} V_k g_k \alpha_k^2\right]^2 \end{aligned} \qquad (27)$$

and that η^2 increases monotonically with γ_β. For 20log R TVG, $g_k \sim V_k^{-1}\alpha_k^{-2}$ and (26) simplifies to

$$\eta^2 = \frac{1}{K} + (\gamma_\beta/\rho)(\langle A^{-1}\rangle/D)$$

$$\text{where:} \quad \langle A^{-1}\rangle = \frac{1}{K}\sum_{k=1}^{K}\frac{1}{A_k}, \quad (A_k \approx \pi d_k^2 \sin\theta_1 \sin\theta_2) \tag{28}$$

is the "average inverse area" of the beam in the insonified region and D is the thickness of the stratum (see Fig. 1).

An even more explicit representation is obtained if there are enough pulse lengths, $c\delta/2$, in the depth stratum D so that the sum, $\langle A^{-1}\rangle$, in (28) may be replaced by an integral in which case

$$\eta^2 \approx \frac{(c\delta/2)}{D} + \frac{\gamma_\beta}{\rho D A_g} \tag{29}$$

where $A_g = d_g^2\, \pi \sin\theta_1 \sin\theta_2$ and $d_g = \left[d_o d_K\right]^{1/2}$ is the geometric mean depth of the stratum. The first term is always one or less and can be made small by using thick strata or narrow pulses or both. The second term is roughly inversely proportional to the average total number of fish insonified in the range gate (E(No. of fish) $\approx \rho\, D A_g$. We also see how the second term depends critically on the distribution function of the fish target strengths and accordingly on γ_β. If not many fish are insonified each ping, the second term could be much bigger than the first and indeed may be greater than 1 particularly for large γ_β. This situation, certainly recognized experimentally very early, led to averaging or integrating the return from a number of pings along a given transec where the fish are schooled. As mentioned previously, the means from sequential pings add directly as do the variances if they are non-overlapping. In the event of P non-overlapping pings,

$$\overline{I}_P = P\overline{I}$$

$$\mathrm{Var}\ (I_P) = P\ \mathrm{Var}\ (I) \tag{30}$$

$$\eta_P^2 = \eta^2/P$$

where $\overline{I}$, Var (I) and η^2 are as previously defined for single pings. The estimate of ρ obtained from I_P is consistent, i.e.,

$$\lim_{P \to \infty} \eta_P^2 = 0 \tag{31}$$

for an increasing number of non-overlapping pings. In order to determine the effect of overlapping pings, we can make use of the analysis of Section III for overlapping Poisson intervals. If the pings overlap, the normalized variance is increased due to the ping-to-ping correlation such that[5]

$$\eta_P^2 = \frac{1}{P}\ (\eta^2 + \frac{2}{\rho D} \sum_{j=1}^{P}\ (1-j/P)\langle A_j^{-1}\rangle) \tag{32}$$

where $\langle A_j^{-1}\rangle$ is the "average inverse area" of $j^{\underline{th}}$ order overlap and is given exactly by

$$\langle A_j^{-1}\rangle = \frac{1}{K} \sum_{k=1}^{K} q_{jk}/A_k \tag{33}$$

[5]See Appendix A for the derivation of (32)

where q_{jk} is the fractional area overlap of ping P with ping p-j at the $k^{\underline{th}}$ layer and A_k is the insonified area of the $k^{\underline{th}}$ layer.

It is interesting to consider the case where the boat does not move but continues to get returns from the same set of fish ping after ping. In this event (see eq. 29),

$$q_{jk} = 1 \; ; \; j=1,2,\text{---}P, \; k=1,2,\text{---}K$$

$$\langle A_j^{-1} \rangle = \frac{1}{K} \sum_{k=1}^{K} 1/A_k \approx A_g^{-1} \tag{34}$$

$$\eta_P^2 = \frac{1}{P}\left(\frac{c\delta/2}{D} + (\gamma_\beta - 1) \; / \; \rho D A_g\right) + 1/\rho D A_g$$

Eq. (34) illustrates that the minimum attainable normalized variance of the estimate of the number of fish in a given volume, if the boat does not move, is inversely proportional to the number of fish insonified and this limit is reached according to (34) as $P \to \mathcal{D}$. This leads us to the conclusion that the echo integrator is not a consistent estimator for ρ with increasing numbers of pings P or with increasingly short pulses δ for finite thickness depth strata, $(D < \mathcal{D})$ and finite length transecs. Our intuition would lead us to believe that the minimum normalized variance for the moving ship case would be inversely proportional to the total number of fish insonified. These suspisions are confirmed by noting, with the help of Appendix B that

$$\langle A_j^{-1} \rangle \approx \frac{1}{A_g} \left[1 - 2j \frac{A_v}{A_g}\right]; \quad j < \frac{A_g}{2A_v} \tag{35}$$

$$= \quad 0 \quad \text{Otherwise}$$

where $\quad A_v = 2\left(\frac{d_o + d_K}{2}\right)(\sin\theta_2)vT$

With some manipulation, it can be shown that

$$\eta_P^2 \approx \frac{1}{P}\left[\frac{c\delta/2}{D} + (\gamma_\beta - 1)/\rho D A_g\right] + \frac{1}{\rho D A_T} \tag{36}$$

where higher order terms in 1/P have been dropped and $A_T = PA_v$ = total area covered in the transec at the center of the depth stratum. Eq. (36) will be generally valid if we choose for A_T the lesser of PA_v and PA_g. That is, if no overlap exists the ship is moving fast enough that $A_T < A_g$ and in this case (36) reduces to (30), the case of non-overlapping pings. If the ship does not move at all, $A_T \approx A_g$ and (36) reduces to (34). Or for P = 1, $A_T \approx A_g$ and (36) reduces to (29), the single ping case. Thus the approximations used in obtaining (36) are neglible for all the previously calculated special cases. Accordingly, we accept it as our central result.

Example:

Suppose we set our range gate for 10 to 40 meters. Let our transducer pattern have an included angle of 10° and let it be circular. Let the vessel traverse a 500 meter transec at 5 knots (∿2.5 meter/sec) and transmit 2 pulses/sec., i.e., one every 1.25 meters of forward motion.

$$d_g = \sqrt{10 \times 40} = 20 \text{ meters}$$

$$A_g = \pi\, d_g^2\, (\sin 5°)^2 \approx 10 \text{ m}^2$$

$$PA_g = 4000 \text{ m}^2 \geq PA_v = 2150 \text{ m}^2$$

$$\therefore A_T = PA_v = 2150 \text{ m}^2$$

Let the amplitude distribution be Rayliegh in which case $\gamma_\beta = 2$. Let the pulses be one meter in length (∿1.2 m.s). With these parameters

$$\eta_P^2\ (\rho) = \frac{1}{400}\left[\frac{1}{30} + (2-1)/\rho.300\right] + \frac{1}{\rho(64,500)}$$

$$\eta_P^2\ (\rho) = \frac{1}{12,000} + \frac{1}{\rho}\ \frac{1}{144,000} + \frac{1}{64,500}$$

$$\eta_P^2\ (\rho) \approx (.8 + \frac{.2}{\rho}) \times 10^{-4}$$

The relative standard deviation, $\eta_p(\rho)$ is plotted versus ρ for this example in Figure 6. For $\rho > .002$, the relative accuracy of our estimate is ± 10% of the measured value. For less dense populations, the estimate deteriorates rapidly. Regardless of the population density, the estimate will have at least a ± 1% error.

V Concluding Remarks

There are a number of approximations used in the foregoing analysis which render it potentially inaccurate. Among the most important are:

1.) Sampled representation of integrator signal

2.) Ideal beam patterns, plane waves

3.) Poisson distribution of fish

4.) Independent amplitude as well phase fluctuations of the echos from the same fish on subsequent pings (see p. 12)

Items (1) and (3) are not felt to be unrealistic assumptions.[6] Item (2) could be serious if strong signals enter on side lobes of the beam patterns or if the beam patterns are very broad and the density is not constant with depth. Item (4) is justified if the ping-to-ping variation of echo amplitude (due to relative aspect, movement in the beam pattern, etc.) are large compared to the variations

[6] J. Ehrenberg has not made the sampled data approximation in his work and obtains similar results.

introduced by variation of the size of the fish being assessed. Ehrenberg (1970) has worked the problem with perfect amplitude but no phase correlation from ping-to-ping for echos from the same fish. This modifies our result (in particular Eq. (20)) and increases the variance above that given by (36). However, his result is probably pessimistic for large P. The ultimate answer to this question depends on accurately characterizing the statistical properties of β, the reflectivity of the fish. Methods for experimental determination of these properties in situ are presently being discussed.

Another matter being examined by Ehrenberg is the optimum choice of TVG (g_k vs k). Clearly (see Eq. (26)), the optimum TVG depends on γ_β and ρ as well as V_k and α_k. The other matter of interest is determining the optimum "unstructured" estimator for ρ (i.e. relief from square and integrate) as well as including the effects of noise, both stationary ambient noise and reverberation noise.

Glossary of Terms

t = time in ping

c = propagation velocity

δ = pulse length

X_k = sample of complex modulation envelope at time t_k at transducer output

α_k = receiver output for a unit target at depth d_k

β_n = reflection coefficient of $n^{\underline{th}}$ fish

ϕ_n = phase of echo from $n^{\underline{th}}$ fish

d_k = depth corresponding to time in ping t_k

g_k = power gain of receiver at time t_k

N_k = number of fish insonified (on a single ping) in the $k^{\underline{th}}$ layer $(d_k - d_{k-1})$

I = Integrator output per ping

I_P = Integrator output after P pings

q_{jk} = fractional overlap of $k^{\underline{th}}$ layer on the $p^{\underline{th}}$ and $p - j^{\underline{th}}$ pings

θ_1 = half-angle of pattern fore and aft

θ_2 = half-angle of pattern athwartships

$\overline{I}$ = Expected value of I

$E(\cdot)$ = Expected value of $(\cdot)$

$\mathrm{Var}(\cdot)$ = Variance of $(\cdot)$

$\mu_x^{(n)}$ = $n^{\underline{th}}$ moment of x

λ = Poisson parameter

ρ = Poisson parameter; fish/unit volume

V_k = volume of $k^{\underline{th}}$ layer

γ_β = $\mu_\beta^{(4)} / (\mu_\beta^{(2)})^2$

D = Thickness of depth stratum

η^2 = normalized variance of integrator output for a single ping

η_P^2 = normalized variance of integrator output for P pings

d_g = $\sqrt{d_o d_K}$ the geometric mean depth of the stratum

d_m = $\frac{d_o + d_K}{2}$, the mean depth of the stratum

A_g = $\pi d_k^2 \sin\theta_1 \sin\theta_2$; area of beam at d_g

A_v = $(2\, d_m \sin\theta_2)(v\, T)$; area covered at mean depth in one ping

A_T = total insonified area; the lessor of PA_g & PA_v

v = velocity of ship

T = time between pings

Appendix A

Derivation of Eq. 32

Consider the random variable

$$Z_p = (\zeta_1 + \zeta_2 + \cdots \zeta_p)$$

where the ζ_p are identically distributed but non-independent random variables with means $\bar{\zeta}$ and variances Var (ζ). Accordingly

$$\left.\begin{aligned} \overline{Z}_p &= P\,\overline{\zeta} \\ \mathrm{Var}(Z_p) &= E(Z_p)^2 - P^2\overline{\zeta}^2 \end{aligned}\right\}$$

The variance may be written

$$\mathrm{Var}(Z_p) = \sum_{i=1}^{P} (\overline{\zeta_i^{\,2} - \overline{\zeta}^2}) + 2 \sum_{i=2}^{P} \sum_{j=1}^{i-1} (\overline{\zeta_i \; \zeta_{i-j} - \overline{\zeta}^2})$$

$$\mathrm{Var}(Z_p) = P \; \mathrm{Var}\;(\zeta) + 2 \sum_{j=1}^{P} (P-j) \; \mathrm{Co\;Var}\; (\zeta_i \zeta_{i-j})$$

In Section III (see Eq (21)) we found that the co-variance of ζ_i and ζ_{i-j} $(\zeta_i = X_i X_i)$ is

$$\mathrm{Co\;Var}\;(\zeta_i \;\zeta_{i-j}) = \frac{q_j}{\lambda^j} (\overline{\zeta})^2$$

where q_j is the fractional overlap of the $i^{\underline{th}}$ and i-$j^{\underline{th}}$ intervals. Consequently

$$\mathrm{Var}(Z_P) = P\ \mathrm{Var}\ (\zeta) + 2(\bar{\zeta})^2 \sum_{j=1}^{P} (P-j)\ q_j/\lambda$$

The degree of overlap q_j, λ, $\bar{\zeta}$ and Var (ζ) depend on the layer number k but the layers are independent. Consequently, the total variance of $\bar{I}_P$ is the sum of the K different terms from (A-5) and similarly with the means of (A-2). Accordingly, as in (28), we obtain

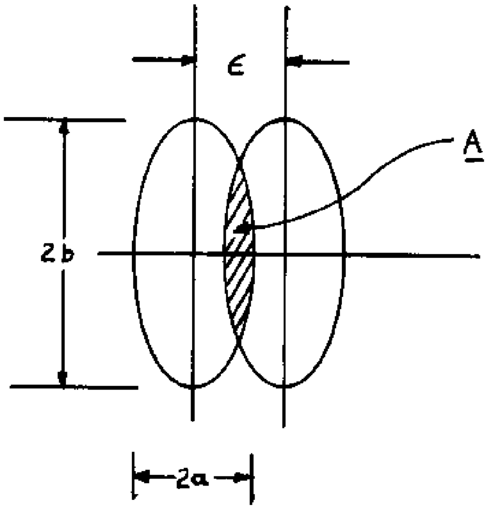
$$\eta_P^2 = \eta^2/P + \frac{2}{\rho DK} \sum_{j=1}^{P} (1-j/P) \left(\sum_{k=1}^{K} q_{jk}/A_k \right)$$

where the second term only simplifies for 20log R TVG.

Appendix B

Overlap Calculations

Consider the intersecting area of two identical displaced ellipses as shown below.



By direct integration,

$$A = 2ab\ \sin^{-1}(1-(\frac{\varepsilon}{\alpha})^2)^{1/2} - 2\ \varepsilon b(1-(\frac{\varepsilon}{\alpha})^2)^{1/2}$$

so that the fractional overlap, q, is

$$q = \frac{A}{A_{ellipse}} = \frac{2}{\pi}\left[\sin^{-1}\left[1-(\frac{\varepsilon}{\alpha})^2\right]^{1/2} - \frac{\varepsilon}{\alpha}\ (1-(\frac{\varepsilon}{\alpha})^2)^{1/2}\right]^*$$

For $\varepsilon/\alpha < 1$, (B-2) can be approximated as

$$q \approx (1 - \frac{4}{\pi}\frac{\varepsilon}{\alpha})$$

If the vessel is traveling with velocity v and pinging at a rate 1/T, then

*Interesting to note that q is independent of b, the major axis of the ellipse (athwartships beam width).

$$q_{jk} \approx \frac{1-j\ 4vT}{\pi dk\ \sin\theta_1} \quad , \quad j < \frac{\pi d_k\ \sin\theta_1}{4vT}$$

And

$$\langle A_j^{-1} \rangle = \frac{1}{K} \sum_{k=1}^{K} q_{jk} \Big/ A_k \approx \frac{1}{A_g} \left[1-2j\ \frac{A_v}{A_g}\right] , \quad j < \frac{A_g}{2A_v}$$

where $A_v = (2d_m\ \sin\theta_2) \times vT\ ,\quad d_m = \frac{d_o + d_k}{2}$

is the average area covered between transmissions by the beam athwartships at the center of the depth stratum.

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THE VARIANCE OF FISH POPULATION ESTIMATES USING AN ECHO INTEGRATOR

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ABSTRACT

In the following discussion a mathematical model is developed for the distribution of fish in a marine environment and for the echo-integrator system used to estimate the fish population. This model is then analyzed to show the mean and variance of the estimated parameter under a number of conditions.

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THE VARIANCE OF FISH POPULATION ESTIMATES USING AN ECHO INTEGRATOR

Variance in a Single Return

The analysis of variance in echo integration is started with the solution of a simplified problem: the integration of a single acoustic return from time t_0 to t. The problem is simplified further by assuming that the statistical distribution of fish in the volume is not a function of depth. The transducer is assumed to have a circular beam pattern with a uniform intensity across the beam. After this initial problem is solved, the analysis is extended to include two additional factors: multiple sounding with beam overlap and nonuniform distribution of fish in the volume being insonified. The nonuniform distribution problem is of importance in estimating the abundance of fish in schools. The simplified model for the beam pattern is used throughout the analysis. More realistic beam patterns are difficult to handle analytically. This assumption does not change appreciably the statistical conclusions of the analysis.

The echo integrator is assumed to have a time-varying-gain control of 40 log R which cancels the spreading loss. The effect of TVG on estimation variance is being investigated.

The acoustic pressure as a function of time at the transducer can be written as

$$p(t) = \sum_{m=1}^{N(t)} W(t, \tau_m, P_m, \theta_m) \tag{1}$$

where

$$W(t, \tau_m, P_m, \theta_m) = P_m \cos(\omega t + \theta_m)[u(t-\tau_m) - u(t-\tau_m-T_p)]$$

P_m is a random variable representing the strength of the signal returned from the m^{th} fish. It is a function of the fish's size, aspect angle, etc. θ_m is the phase of the signal return from the m^{th} fish. It can be assumed that θ_m is uniformly distributed from 0 to 2π and is independent of P_m. τ_m is the time at which the return from the m^{th} fish first appears at the transducer. T_p is the pulse length of the acoustic pulse. u(t) is the unit step function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

N(t) is the number of fish in the volume of water producing returns from time t_0 to time t_4 .

N(t) is a random variable which depends on the distribution of fish in the volume of water being insonified. It will be assumed that the fish are Poisson distributed in space. The properties of such a distribution are

1) $$\lim_{\Delta V \to 0} P[1 \text{ fish in } \Delta V] = \lambda \Delta V$$

where λ is the Poisson density factory.

2) The number of fish in a given volume is independent of the number of fish in any other nonoverlapping volume.

3)

$$P[N \text{ fish in } V] = \frac{(\lambda V)^N}{N!} e^{-\lambda V} \tag{2}$$

The quantity at the transducer is the number of fish returns in a specified time interval rather than the number of fish contained in a given volume. When the problem is reformulated in terms of the number of signals received in the time interval t_o to t , the distribution for N (t) becomes

$$P[N(t) = k] = \frac{\left[\int_{t_0}^{t} \gamma(t)\, dt\right]^k}{k!} e^{-\int_{t_0}^{t} \gamma(t)\, dt}$$

where

$$\gamma(t) = \lambda \frac{d}{dt} V(t) \tag{3}$$

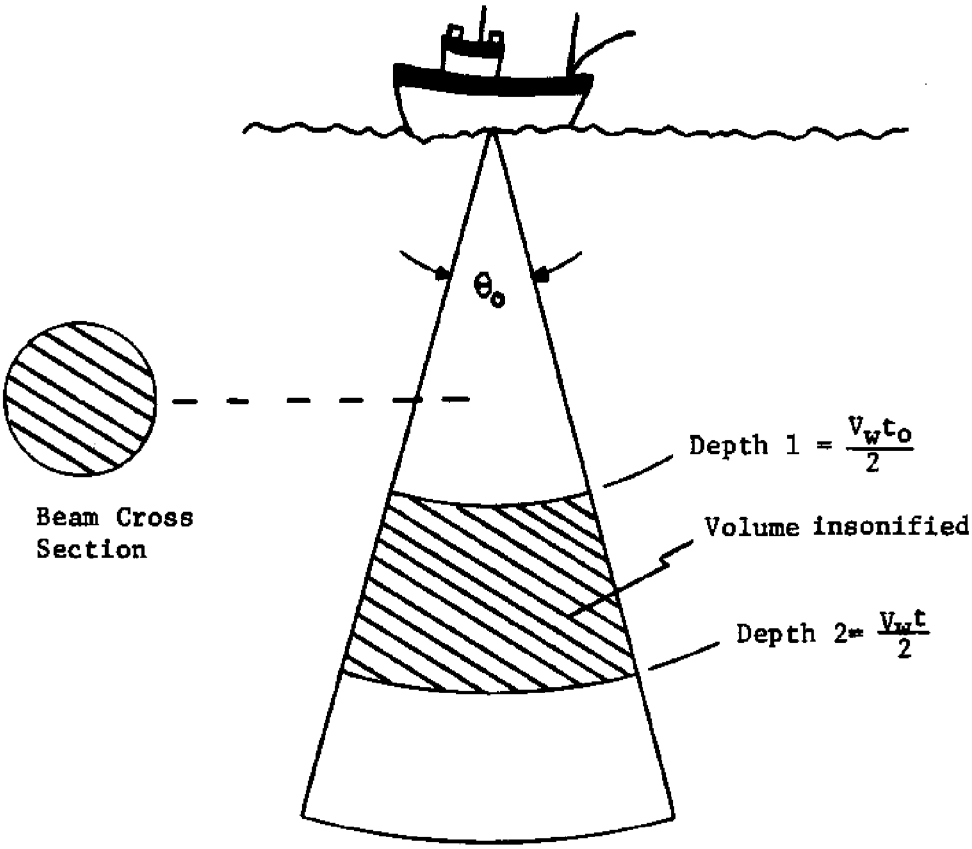
and V(t) is the volume producing returns from time t_o to t . If it is assumed that the volume ensonified by the transducer is a cone with circular cross section (Figure 1), the expression for V(t) becomes,

$$V(t) = \int_{V_w t_0/2}^{V_w t/2} dr \int_0^{\theta_0/2} d\theta \int_0^{\pi} r^2 \sin\theta\, d\phi$$
$$= \frac{\pi}{12} V_w^3 [t^3 - t_0^3][1 - \cos(\tfrac{\theta_0}{2})] \tag{4}$$

where V_w = velocity of sound in water

and

$$\gamma(t) = \frac{\lambda \pi V_w^3}{4} [1 - \cos(\tfrac{\theta_0}{2})] t^2$$



θ_o
Depth 1 = $\frac{V_w t_o}{2}$
Beam Cross Section
Volume insonified
Depth 2 = $\frac{V_w t}{2}$

FIGURE I

Processes of the form of P(t), defined in equation 1, with N(t) defined in equation 3 are referred to as filtered nonhomogeneous Poisson processes.

If it is assumed that the transducer has ideal linear response characteristics, then the voltage at the output of the transducer, X(t), is directly proportional to p(t),

$$x(t) = \sum_{m=1}^{N(t)} W(t, \tau_m, A_m, \Theta_m) \tag{6}$$

where A_m is proportion to P_m

The output of the integrator at time t is given by

$$I(t) = \int_{t_0}^{t} x^2(\alpha)\, d\alpha \tag{7}$$

The purpose of this analysis is to determine two of the statistical properties of $I(t)$; $E[I(t)]$ and $VAR[I(t)]$. By interchanging the order of integration and expectation, it follows that

$$E[I(t)] = \int_{t_0}^{t} \overline{X^2(\alpha)}\, d\alpha \tag{8}$$

and

$$VAR[I(t)] = \int_{t_0}^{t}\int_{t_0}^{t} \overline{X^2(\alpha)X^2(\beta)} - \overline{X^2(\alpha)}\,\overline{X^2(\beta)}\, d\alpha\, d\beta \tag{9}$$

The required moments can be evaluated using the joint characteristic function of X (α), X (β). The joint characteristic function of any two random variables p & q is defined as

$$\Phi_{p,q}(u_1, u_2) = \overline{e^{i(u_1 p + u_2 q)}} \tag{10}$$

Expanding the exponential in a power series,

$$\Phi_{p,q}(u_1,u_2) = 1 + iu_1\overline{p} + iu_2\overline{q} - \frac{1}{2}iu_1^2\overline{p^2} - \frac{1}{2}iu_2^2\overline{q^2} + \cdots \tag{11}$$

For the case where all odd order moments are zero (which is the case for X (α) , X (β)) , the series becomes

$$\Phi_{p,q}(u_1,u_2) = 1 - \frac{1}{2}u_1^2\overline{p^2} - \frac{1}{2}u_2^2\overline{q^2} - u_1u_2\overline{pq} + \frac{1}{4}u_1^2u_2^2\overline{p^2q^2} + \frac{1}{24}u_1^4\overline{p^4} + \frac{1}{24}u_2^4\overline{q^4} + \cdots \tag{12}$$

The series expansion of $\ln\Phi_{pq}(u_1, u_2)$ can be obtained from the above series by using the series expansion for ln (1 + x)

$$\ln(1+X) = X - \frac{X^2}{2} + \frac{1}{3}X^3 - \cdots \tag{13}$$

Therefore,

$$\ln\Phi_{pq}(u_1,u_2) = -\frac{1}{2}u_1^2\overline{p^2} - \frac{1}{2}u_2^2\overline{q^2} - u_1u_2\overline{pq} + u_1^2u_2^2\left[\frac{1}{4}\overline{p^2q^2} - \frac{1}{4}\overline{p^2}\,\overline{q^2} - \frac{1}{2}(\overline{pq})^2\right] + \cdots \tag{14}$$

It therefore follows that

$$\overline{p^2} = -\frac{\partial^2}{\partial u_1^2}\ln\Phi_{pq}(u_1,u_2)\Big|_{u_1=u_2=0} \tag{15}$$

and that

$$\overline{p^2q^2} - \overline{p^2}\,\overline{q^2} = \frac{\partial^4}{\partial u_1^2\partial u_2^2}\ln\Phi_{pq}(u_1,u_2)\Big|_{u_1=u_2=0} + 2\left[\frac{\partial^2}{\partial u_1\partial u_2}\Phi_{pq}(u_1,u_2)\Big|_{u_1=u_2=0}\right]^2 \tag{16}$$

The joint characteristic function of $X(\alpha)$, $X(\beta)$ which is derrived in Appendix A is

$$\Phi_{X(\alpha),X(\beta)}(u_1,u_2) = \exp\left\{ \int_{t_0}^{MAX(\alpha,\beta)} \gamma(\tau)\, \phi(\tau,u_1,u_2)\, d\tau \right\} \tag{17}$$

Where

$$\phi(\tau,u_1,u_2) = \overline{\left[\exp\left\{ iu_1 W(\alpha,\tau,A,\theta) + iu_2 W(\beta,\tau,A,\theta) \right\} - 1 \right]}$$

From equations 15, 16 and 17 it follows that

$$\overline{X^2(\alpha)} = -\int_{t_0}^{\alpha} \gamma(\tau) \left[\frac{\partial^2}{\partial u_1^2} \phi(\tau,u_1,u_2)\Big|_{u_1=u_2=0} \right] d\tau \tag{18}$$

and that

$$\begin{aligned} X^2(\alpha)X^2(\beta) - X^2(\alpha)X^2(\beta) = \\ \int_{t_0}^{MAX(\alpha,\beta)} \gamma(\tau) \left[\frac{\partial^4}{\partial u_1^2 \partial u_2^2} \phi(\tau,u_1,u_2)\Big|_{u_1=u_2=0} \right] d\tau \\ +2\left\{ \int_{t_0}^{MAX(\alpha,\beta)} \gamma(\tau) \left[\frac{\partial^2}{\partial u_1 \partial u_2} \phi(\tau,u_1,u_2)\Big|_{u_1=u_2=0} \right] d\tau \right\}^2 \end{aligned} \tag{19}$$

The exponential part of $\phi(\tau,u_1,u_2)$ can be expanded in a series to give

$$\begin{aligned} \phi(\tau,u_1,u_2) = iu_1 \overline{W(\alpha,\tau,A,\theta)} + iu_2 \overline{W(\beta,\tau,A,\theta)} \\ -\frac{1}{2}u_1^2 \overline{W^2(\alpha,\tau,A,\theta)} - \frac{1}{2}u_2^2 \overline{W^2(\beta,\tau,A,\theta)} \end{aligned} \tag{20}$$

and

$$\overline{X^2(\alpha)} = \int_{t_0}^{\alpha} \gamma(\tau)\, \overline{W^2(\alpha,\tau,A,\theta)}\, d\tau \tag{21}$$

$$\overline{X^2(\alpha)X^2(\beta)} - \overline{X^2(\alpha)}\;\overline{X^2(\beta)} = \int_{t_0}^{MAX(\alpha,\beta)} \gamma(\tau)\, \overline{W^2(\alpha,\tau,A,\theta)W^2(\beta,\tau,A,\theta)}\, d\tau$$

$$+ 2\left[\int_{t_0}^{MAX(\alpha,\beta)} \gamma(\tau)\, \overline{W(\alpha,\tau,A,\theta)W(\beta,\tau,A,\theta)}\, d\tau\right]^2 \tag{22}$$

The moments of $W(\alpha,\tau,A,\theta)$ and $W(\beta,\tau,A,\theta)$ can be evaluated in a straightforward manner to yield

$$\overline{W^2(\alpha,\tau,A,\theta)W^2(\beta,\tau,A,\theta)} = \overline{A^4}\left[\frac{1}{4} + \frac{1}{8}\cos\omega(\beta-\alpha)\right]\left[u(\alpha-\tau) - u(\alpha-\tau-T_p)\right]^2 \left[u(\beta-\tau) - u(\beta-\tau-T_p)\right]^2 \tag{23}$$

$$\overline{W^2(\alpha,\tau,A,\theta)} = \frac{\overline{A^2}}{2}\left[u(\alpha-\tau) - u(\alpha-\tau-T_p)\right]^2 \tag{24}$$

$$\overline{W(\alpha,\tau,A,\theta)W(\beta,\tau,A,\theta)} = \frac{\overline{A^2}}{2}\cos\omega(\alpha-\beta)\left[u(\alpha-\tau) - u(\alpha-\tau-T_p)\right]\left[u(\beta-\tau) - u(\beta-\tau-T_p)\right] \tag{25}$$

$\overline{X^2(\alpha)}$ and $\overline{X^2(\alpha)X^2(\beta)} - \overline{X^2(\alpha)}\;\overline{X^2(\beta)}$ can be obtained from substituting (23), (24) and (25) into equations (21)and (22).

$$\overline{X^2(\alpha)X^2(\beta)} - \overline{X^2(\alpha)}\;\overline{X^2(\beta)} = \overline{A^4}\left[\frac{1}{4} + \frac{1}{8}\cos\omega(\beta-\alpha)\right]\Gamma(\alpha,\beta,T_p) + \frac{\overline{A^2}}{2}\left[\cos\omega(\alpha-\beta)\quad \Gamma(\alpha,\beta,T_p)\right]^2 \tag{26}$$

and

$$X^2(\alpha) = \frac{\overline{A^2}}{2}\,\Gamma(\alpha,\alpha,T_p) \tag{27}$$

where

$$\Gamma(\alpha,\beta,T_p) = \frac{\lambda\pi V_w^3}{4}\left[1-\cos\left(\frac{\theta_0}{2}\right)\right]\int_{\beta-T_p}^{\alpha} t^2\,dt \quad \text{FOR} \begin{cases}\beta>\alpha\\ t_0<\beta-T_p\\ \beta<\alpha+T_p\end{cases}$$

$$= \frac{\lambda\pi V_w^3}{4}\left[1-\cos\left(\frac{\theta_0}{2}\right)\right]\int_{t_0}^{\alpha} t^2\,dt \quad \text{FOR} \begin{cases}\beta>\alpha\\ t_0>\beta-T_p\\ \beta<\alpha+T_p\end{cases}$$

$$= \frac{\lambda\pi V_w^3}{4}\left[1-\cos\left(\frac{\theta_0}{2}\right)\right]\int_{\alpha-T_p}^{\beta} t^2\,dt \quad \text{FOR} \begin{cases}\alpha>\beta\\ t_0<\alpha-T_p\\ \alpha<\beta+T_p\end{cases}$$

$$= \frac{\lambda\pi V_w^3}{4}\left[1-\cos\left(\frac{\theta_0}{2}\right)\right]\int_{t_0}^{\beta} t^2\,dt \quad \text{FOR} \begin{cases}\alpha>\beta\\ t_0>\alpha-T_p\\ \alpha<\beta+T_p\end{cases}$$

A simplifying assumption can be made for the case where the integration interval, $t - t_o$, is much greater than the pulse length, T_p. This assumption is generally satisfied for the University of Washington echo integrator of which the pulse length, Tp, is .5 ms corresponding to a depth in water of .375 m. For this case, it is not necessary to include the integrals (28B and 28D) corresponding to the region of integration near the start of the time interval. When the starting time of the integration t_o is much greater than T_p, the integrand in equation 28 is essentially constant over the period of integration and

$$\int_{\beta-T_p}^{\alpha} t^2\,dt \approx [\beta-T_p]^2[\alpha-\beta+T_p] \tag{28}$$

When this approximation is used to simplify the expressions for $\overline{X^2(\alpha)X^2(\beta)} - \overline{X^2(\alpha)}\,\overline{X^2(\beta)}$ and $\overline{X^2(\alpha)}$ the equations for VAR[I(t)] and E[I(t)] become

$$VAR[I(t)] = \int_{t_0}^{t} d\beta \int_{0}^{T_p} \overline{A^4} K[\beta - T_p]^2 \left[\frac{1}{4} + \frac{1}{8}\cos 2\omega\tau\right][T_p - \tau]\, d\tau$$
$$+ \int_{t_0}^{t} d\beta \int_{0}^{T_p} (\overline{A^2})^2 K^2 [\beta - T_p]^4 \left[\frac{1}{4} + \frac{1}{4}\cos 2\omega\tau\right][T_p - \tau]^2\, d\tau$$
$$+ \int_{t_0}^{t} d\beta \int_{-T_p}^{0} \overline{A^4} K \beta^2 \left[\frac{1}{4} + \frac{1}{8}\cos 2\omega\tau\right][\tau + T_p]\, d\tau$$
$$+ \int_{t_0}^{t} d\beta \int_{-T_p}^{0} (\overline{A^2})^2 K^2 \beta^4 \left[\frac{1}{4} + \frac{1}{4}\cos 2\omega\tau\right][\tau + T_p]^2\, d\tau \qquad (29)$$

where $E[I(t)] = \frac{\overline{A^2} K}{2} \int_{t_0}^{t} T_p [\alpha - T_p]^2\, d\alpha$

$$\tau = \beta - \alpha$$

$$K = \frac{\lambda \pi V_w^3}{4} \left[1 - \cos\left(\frac{\theta_e}{2}\right)\right] \qquad (30)$$

The integrals may be evaluated using the previous assumption that

$$t - t_o >> T_p$$
$$t_o >> T_p$$

along with the fact that

$$\int_{0}^{T_p} \tau^i \cos 2\omega\tau \approx 0 \qquad i = 0, 1, 2$$

for $T_p >> \frac{2\pi}{\omega}$

The resulting expressions are

$$VAR[I(t)] = \frac{\overline{A^4} K T_p^2}{12} [t^3 - t_o^3] + \frac{(\overline{A^2})^2 K^2 T_p^3}{30} [t^5 - t_o^5] \qquad (31)$$

$$E[I(t)] = \frac{\overline{A^2} K T_p}{6} [t^3 - t_o^3] \qquad (32)$$

A meaningful quantity to consider in determining the performance of the integrator is the normalized variance

$$\frac{VAR[I(t)]}{\{E[I(t)]\}^2} = \frac{3\overline{A^4}}{(\overline{A^2})^2 K [t^3 - t_o^3]} + \frac{6\, T_p [t^5 - t_o^5]}{5\, [t^3 - t_o^3]^2} \qquad (33)$$

The results can be further simplified by recalling that the volume ensonified, V(t), is

$$V(t) = \frac{\pi}{12} V_W^3 \left[t^3 - t_o^3\right]\left[1 - \cos\left(\frac{\theta}{2}\right)\right]$$
$$= \frac{K}{3\lambda}\left(t^3 - t_o^3\right)$$

Solving for K

$$K = \frac{V(t)\, 3\lambda}{(t^3 - t_o^3)} \tag{34}$$

Using this expression for K, the expected value of the integral and the normalized variance are

$$E[I(t)] = \frac{\overline{A^2}}{2}\lambda V(t) T_p \tag{35}$$

and

$$\frac{VAR[I(t)]}{\{E[I(t)]\}^2} = \frac{\overline{A^4}}{(\overline{A^2})^2 \lambda V(t)} + \frac{6}{5}\frac{T_p\,[t^5 - t_o^5]}{[t^3 - t_o^3]^2} \tag{36}$$

$\lambda V(t)$ is the expected number of fish in V(t) . The first term in the normalized variance decreases as the inverse of the number of fish. The second term decreases as the inverse of the integration interval, $t - t_0$.

Nonuniform Fish Distribution

In the original formulation of the problem, the density of fish per unit volume, λ, was assumed constant. In some situations, this will not be the case. A reasonable assumption for the fish density may be that it is a function of depth, R. For this case, the distribution of the number of signals returns in a time interval is still given by equation 3 where $\gamma(t)$ is now given by

$$\gamma(t) = \frac{d\lambda(R)}{dR}\;\frac{dR}{dt}\;\frac{dV(t)}{dt} \tag{37}$$

The rest of the analysis for finding VAR [I(t)] and E [I(t)] is the same as before.

In some cases it may be reasonable to assume that the volume insonified is divided into layers of constant density. The output of the integrator can then be written as a sum of the integrals for each layer.

$$I(t) = I_1 + I_2 + \cdots$$

where

$$I_1 = \int_{t_0}^{t_1} x^2(t)\,dt$$

$$I_2 = \int_{t_1}^{t_2} x^2(t)\,dt$$

$$\vdots$$

If the integration interval for each layer is much greater than the pulse length, T_p , the integrals are approximately independent and

$$E[I(t)] = \sum_n E[I_n] \tag{38}$$

$$VAR[I(t)] = \sum_n VAR[I_n] \tag{39}$$

The mean value of the integral, E[I(t)], is again proportionate to the number of fish in the volume of water ensonified. The total variance is the sum of the variances for the individual layers.

<u>Multiple Sounding</u>

In the previous analysis, the mean and variance of the integration of a return from a single acoustic pulse were considered. In this section, the analysis will be extended to multiple sounding. It was shown (equation 35) that the integration of a single return gives an unbiased estimate of the Poisson density factor, λ ,

$$\lambda = C\,E[I(t)]$$

where

$$C = \frac{1}{V(t)\,T_p\,\overline{A^2}/2} \tag{40}$$

For multiple sounding, the sample mean, $\hat{\lambda}_T$, of the individual return estimates, $\hat{\lambda}_i$, also gives an unbiased estimate of the density.

$$E[\hat{\lambda}_T] = \frac{1}{N} \sum_{i=1}^{N} E[\hat{\lambda}_i] = \lambda \tag{41}$$

where N is the number of individual soundings. If a different volume of water is sampled with each sounding, the individual return estimates are independent and the variance of $\hat{\lambda}_T$ is

$$\mathrm{VAR}\,\hat{\lambda}_T = \frac{1}{N^2} E\left[\left(\sum_{i=1}^{N} \hat{\lambda}_i\right)^2\right] - \lambda^2 = \frac{1}{N}\,\mathrm{VAR}\,\hat{\lambda} \tag{42}$$

The total variance is inversely proportional to the number of independent soundings. If there is beam overlap in the sampling scheme, the individual return estimates are no longer independent and equation 42 does not hold.

Insight into the beam overlap problem can be gained by determining the variance in the density estimate when the same volume of water is sampled many times. It is assumed that the same fish are present in the volume insonified during each integration. The phase of the acoustic return for any given fish during one integration is assumed independent of the phase of the return of the same fish during any other integration. The intensity of the return from a given fish is assumed to be the same for each integration. With these assumptions, the sample mean of the estimates is

$$\hat{\lambda} = \frac{1}{NC}\left[I_1 + I_2 + \cdots I_N\right]$$

where

$$I_i = \int_{t_0}^{t} x_i^2(t)\,dt$$

and

$$x_i(t) = \sum_{m=1}^{N(t)} A_m \cos(\omega t + \theta_{mi})\left[u(t-\tau_{mi}) - u(t-\tau_{mi}-T_p)\right] \tag{43}$$

The variance of $\hat{\lambda}$ is

$$\mathrm{VAR}\,\hat{\lambda} = \frac{1}{C^2 N^2}\Big\{E\left[(I_1 + I_2 + \cdots I_N)^2\right] - \left[E(I_1) + E(I_2) + \cdots E(I_N)\right]^2\Big\}$$

$$= \frac{1}{C^2 N^2}\ \mathrm{VAR}\, I + \frac{N-1}{N}\left[E(I_j I_k) - E(I_j)E(I_k)\right]_{j \neq k} \qquad (44)$$

The convariance between separate integrals, I_j, I_k can be evaluated by the same technique used to obtain the autocovariance of I. When the same assumptions that were previously used are made, the expression for normalized variance is

$$\frac{\mathrm{VAR}\,\hat{\lambda}}{[E(\hat{\lambda})]^2} = \frac{\overline{A^4}}{(\overline{A^2})^2\,\lambda V(t)} + \frac{1}{N}\left\{\frac{6\,T_p\,[t^5 - t_0^5]}{5\,[t^3 - t_0^3]^2}\right\} \qquad (45)$$

This expression can be compared with equation (36) for the single integration normalized variance. It is seen that multiple integration does not change the first term but causes the second term to go down as 1/N . This is reasonable since the first term accounts for the intensity of the fish returns which was assumed to remain the same during each integration. The second term accounts for the random phases of the returns which were assumed to be statistically independent during different integrations. Beam overlap should have the same effect on variance.

APPENDIX A:

In this appendix, the joint characteristic function for a filtered non-homogeneous Poisson process will be derived. With X(t) defined in equation 6

$$\Phi_{\alpha,\beta}(u_1,u_2) = E\left[\exp\left\{i\sum_{m=1}^{N(\beta)} g_m(\tau)\right\}\right]$$

Where $g_m(\tau) = u_1 W(\alpha,\tau_m,A_m,\theta_m) + u_2 W(\beta,\tau_m,A_m,\theta_m)$ (A1)

And $\beta \geq \alpha$

Equation A1 can be rewritten as

$$\Phi_{\alpha,\beta}(u_1,u_2) = \sum_{n=0}^{\infty} E\left[\exp\left\{i\sum_{m=1}^{N(\beta)} g_m(\tau)\right\} / N(\beta)=n\right] P[N(\beta)=n] \tag{A2}$$

The conditional expectation in A2 can be written as

$$E\left[\exp\left\{i\sum_{m=1}^{N(\beta)} g_m(\tau)\right\} / N(\beta)=n\right] = \int_{t_0}^{\beta}\int_{s_1}^{\beta}\cdots\int_{s_{n-1}}^{\beta} \left\{E\left[\exp\left\{i\sum_{m=1}^{N(\beta)} g_m(\tau)\right\} / N(\beta)=n, \tau_1=s_1,\cdots\tau_n=s_n\right]\right\}$$

$$p_{\tau_1,\tau_2\ldots\tau_n}(s_1,s_2,\cdots s_n)\, ds_1\, ds_2\cdots ds_n \tag{A3}$$

Where $\tau_1,\tau_2\cdots\tau_n$ are the arrival times of the nonhomogeneous Poisson process. The joint density function of the arrival time is*

$$p_{\tau_1,\tau_2\cdots\tau_n}(s_1,s_2\cdots s_n) = \frac{\gamma(s_1)\gamma(s_2)\cdots\gamma(s_n)\; n!}{\left[\int_{t_0}^{\beta}\gamma(t)\,dt\right]^n} \tag{A4}$$

When this density function is used in (A3), the condition expectation becomes

$$E\left[\exp\left\{i\sum_{m=1}^{N(\beta)} g_m\right\} / N(\beta)=n\right] = \frac{n!}{\left[\int_0^{\beta}\gamma(\tau)\,d\tau\right]^n}\int_{t_0}^{\beta}\int_{s_1}^{\beta}\cdots\int_{s_{n-1}}^{\beta}\prod_{m=1}^{n}\left[E\left\{\exp(i g_m(\tau))\right\}\gamma(s_m)\right] ds_1\, ds_2\cdots ds_n$$

$$= \frac{1}{\left[\int_0^{\beta}\gamma(\tau)\,d\tau\right]^n}\left[\int_{t_0}^{\beta}\gamma(\tau)\,E\left\{\exp\left[i g(\tau)\right]\right\}d\tau\right]^n \tag{A5}$$

---

* The joint density dunction for the arrival times of a homogeneous Poisson process is derrived in Stochastic Processes by Parzen. The nonhomogeneous case is left as a problem.

Substituting this conditional expectation and the density function for n into equation (A2), it follows that

$$\Phi_{X(\alpha),X(\beta)}(u_1,u_2) = \sum_{n=0}^{\infty} e^{-\int_{t_0}^{\beta}\gamma(\tau)d\tau} \frac{\left[\int_{t_0}^{\beta}\gamma(\tau)d\tau\right]^n}{n!}$$

$$\left\{\frac{1}{\int_{t_0}^{\beta}\gamma(\tau)d\tau}\int_{t_0}^{\beta}\gamma(\tau)\,E[\exp[ig(\tau)]]\,d\tau\right\}^n \quad \text{(A6)}$$

$$= e^{-\int_{t_0}^{\beta}\gamma(\tau)d\tau}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left\{\int_{t_0}^{\beta}\gamma(\tau)\,E[\exp\{ig(\tau)\}]\,d\tau\right\}^n\right] \quad \text{(A7)}$$

The summation in (A7) is just the expansion for

$$\exp\left\{\int_{t_0}^{\beta}\gamma(\tau)\,E\,[\exp\{i\,g(\tau)\}]\,d\tau\right\}$$

It therefore follows that

$$\Phi_{X(\alpha),X(\beta)}(u_1,u_2) = \exp\left\{\int_{t_0}^{\beta}\gamma(\tau)\,[E\exp\{ig(\tau)\} - 1]\,d\tau\right\} \quad \text{(A8)}$$

If $\alpha > \beta$ then

$$\Phi_{X(\alpha),X(\beta)}(u_1,u_2) = \exp\left\{\int_{t_0}^{\alpha}\gamma(\tau)\,[E\exp\{ig(\tau)\} - 1]\,d\tau\right\} \quad \text{(A9)}$$

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DERIVATION AND NUMERICAL EVALUATION OF A GENERAL VARIANCE EXPRESSION FOR FISH POPULATION ESTIMATES USING AN ECHO INTEGRATOR

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ABSTRACT

This analysis is an extension of one previously promulgated under the title, "The Variance of Fish Population Estimates Using an Echo Integrator". The principal result of the prior paper is an expression for the normalized variance of the integrated acoustic returns from fish insonified by a single transducer pulse. Some special multiple pulse problems were also considered. The prior analysis assumed a time varying gain control (TVG) of 40 log R. The results presented herein are valid for the general multiple pulse problem with a TVG of the form G Log R where G is a real variable.

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DERIVATION AND NUMERICAL EVALUATION OF A GENRAL VARIANCE EXPRESSION FOR FISH POPULATION ESTIMATES USING AN ECHO INTEGRATOR

VARIANCE OF A SINGLE PULSE IN TERMS OF A GENERAL T.V.G.

The assumptions made for this analysis are the same as those given in reference 1 except for some generalizations which should be clear from a comparison of the present and previous results. The transducer produces a pressure wave of the form

$$p_t(t) = \begin{cases} K \cos \omega t & 0 \leq t \leq T_p \\ 0 & t < 0 \text{ \& } t > T_p \end{cases} \quad (1)$$

The received pressure wave at time t is

$$p(t) = \sum_{m=1}^{N(t)} \frac{P_m}{t^2} \cos(\omega t + \theta_m)\left[u(t-\tau_m) - u(t-\tau_m-T_p)\right] \quad (2)$$

where P_m is a random variable representing the strength of the signal returned from the m^{th} fish. θ_m is the phase of the signal returned from the m^{th} fish. τ_m is the time at which the m^{th} return first appears at the transducer. N(t) is a Poisson distributed random variable representing the number of fish producing returns at the transducer through time t. The $1/t^2$ term accounts for the spreading loss of propagation. The voltage produced by the transducer, x(t), is assumed to be proportional to p (t).

$$x(t) = \sum_{m=1}^{N(t)} \frac{A_m}{t^2} \cos(\omega t + \theta_m)[u(t-\tau_m) - u(t-\tau_m-T_p)] \tag{3}$$

All commercial echo sounders have time varying gain controls (T.V.G.) of the form G log R. The case G=0 corresponds to no T.V.G. at all. G=40 is the case previously considered and it has the effect of canceling out the spreading loss of propagation. For a TVG of G log R the output of the echo integrator gated over the time interval t_o to t_F is

$$I(t) = \int_{t_o}^{t_F} t^g x^2(t)\, dt \tag{4}$$

where $g = \frac{G}{10}$

The expected value and variance of the integrator output are

$$E[I(t)] = \int_{t_o}^{t_F} t^g E[x^2(t)]\, dt \tag{5}$$

$$\mathrm{VAR}[I(t)] = \int_{t_o}^{t_F} \int_{t_o}^{t_F} t^g s^g \{E[x^2(t)x^2(s)] - E[x^2(t)]E[x^2(s)]\}\, dt\, ds \tag{6}$$

The necessary moments of $x^2(t)$ are evaluated from the joint characteristic function of x(t), x(s) (see reference 1 for details).

$$\overline{X^2(t)} = t^{-4}\frac{\overline{A^2}}{2}\Gamma(t,t) \tag{7}$$

$$E[X^2(t)X^2(s)] - E[X^2(t)]E[X^2(s)] = t^{-4}s^{-4}\Big\{\overline{A^4}\Big[\frac{1}{4} + \frac{1}{8}\cos 2\omega(t-s)\Big]\Gamma(t,s) + (\overline{A^2})^2\Big[\frac{1}{4} + \frac{1}{4}\cos 2\omega(t-s)\Big]\Gamma^2(t,s)\Big\} \tag{8}$$

where

$$\begin{aligned}\Gamma(t,s) &= \lambda K\int_{s-T_p}^{t}\tau^2 d\tau \approx \lambda K t^2[t-s+T_p] \qquad s < t+T_p,\ s > t \\ &= \lambda K\int_{t-T_p}^{s}\tau^2 d\tau \approx \lambda K s^2[s-t+T_p] \qquad t < s+T_p,\ t > s\end{aligned} \tag{9}$$

where $K = \frac{\pi V_w^3}{8}\sin\left(\frac{\theta_A}{z}\right)\sin\left(\frac{\theta_B}{z}\right)$ *

V_w - velocity of sound in water

θ_A = transducer beam angle in the fore-aft direction

θ_B = transducer beam angle athwartships

λ = fish density per unit volume

Substituting (7) and (8) into (5) and (6), the following expressions for E[I(t)] and VAR[I(t)] are obtained

$$E[I(t)] = \lambda\frac{\overline{A^2}}{2}\frac{K T_p [t_F^{g-1} - t_0^{g-1}]}{[g-1]} \tag{10}$$

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* In the original analyses, the transducer was assumed to have a circular beam pattern. The general case of an elliptical beam has been considered here.

$$\begin{aligned} VAR[I(t)] &= 2\int_{t_o}^{t_F}\int_{S-T_p}^{S} t^{g-4} S^{g-4} \Big\{ \frac{\overline{A^4}}{4} \lambda K t^2 [t - S + T_p] \\ &\quad \frac{(\overline{A^2})^2}{4} \lambda^2 K^2 t^4 [t - S + T_p]^2 \Big\} dt\, ds \\ &= \frac{\overline{A^4}\lambda K}{2} \int_{t_o}^{t_F} s^{g-4} \int_{S-T_p}^{S} t^{g-2} [t - s + T_p]\, dt\, ds \\ &\quad + \frac{(\overline{A^2})^2 \lambda^2 K^2}{2} \int_{t_o}^{t_F} s^{g-4} \int_{S-T_p}^{S} t^{g} [t - S + T_p]^2\, dt\, ds \end{aligned} \tag{11}$$

When the integrals are evaluated, the expression for VAR[I(t)] can be written as

$$\begin{aligned} VAR[I(t)] &= \Big[\frac{\overline{A^4}\lambda K T_p^2}{4(2g-5)} \left(t_F^{2g-5} - t_o^{2g-5} \right) \\ &\quad + \frac{(\overline{A^2})^2 \lambda^2 K^2 T_p^3}{6(2g-3)} \left(t_F^{2g-3} - t_o^{2g-3} \right) \Big] \Big[1 + O\left(\frac{T_p}{t}\right) \Big] \end{aligned} \tag{12}$$

where $O(\frac{T_p}{t})$ are terms of order $\frac{T_p}{t}$. Since $T_p \ll t$ for most operating conditions, these higher order terms can be neglected.

The quantity of interest is the estimated fish density, $\hat{\lambda}$, rather than the integrator output, I(t). From (10) it follows directly that an unbiased estimate of λ is

$$\hat{\lambda} = \frac{I(t)}{\frac{\overline{A^2}}{2} K T_p \frac{[t_F^{g-1} - t_o^{g-1}]}{g-1}} \tag{13}$$

The normalized variance of the estimate, $\hat{\lambda}$, is

$$\begin{aligned} \frac{VAR[\hat{\lambda}]}{\{E[\hat{\lambda}]\}^2} &= \frac{\overline{A^4}\,(g-1)^2}{(\overline{A^2})^2 (2g-5) \lambda K t_F^3} \frac{[1 - (t_o/t_F)^{2g-5}]}{[1 - (t_o/t_F)^{g-1}]^2} \\ &\quad + \frac{2(g-1)^2}{3(2g-3)} \frac{T_p}{t_F} \frac{[1 - (t_o/t_F)^{2g-3}]}{[1 - (t_o/t_F)^{g-1}]^2} \end{aligned} \tag{14}$$

MULTIPLE SOUNDING ANALYSIS

Define I_T as the output of the echo-integrator after the integration of the returns from N_p pulses. Then

$$I_T = I_1 + I_2 \cdots + I_{N_p} \tag{15}$$

where I_j is the integral of the returns from the j^{th} pulse. If it is assumed that the fish density factory, λ, is the same for all N pulses and that each pulse is range gated to the same depth interval, it follow that

$$E[I_T] = N_p E_I \tag{16}$$

where E_I is given by (10). It should be noted that (16) does not depend on the I_j's being independent or uncorrelated.

$VAR[I_T]$, however, does depend upon the correlation between the individual terms. When a different volume of water is sampled with each pulse, the I_j's are independent and

$$VAR[I_T] = N_p VAR[I] \tag{17}$$

where VAR[I] is given by (12).

The corresponding normalized variance for the fish density estimate, $\hat{\lambda}$, is

$$\frac{VAR[\hat{\lambda}]}{\{E[\hat{\lambda}]\}^2} = \frac{1}{N_p} \frac{VAR[I]}{E_I^2} \tag{18}$$

The expression for $VAR[I_T]$ is more difficult to obtain when there is overlap of the volumes of water insonified by adjacent acoustic pulses. For this analysis, it is convenient to consider the depth strata insonified to be composed of several layers. The initial and final depth of each layer is determined so as to satisfy the following two conditions: first, the average number of times each layer is insonified is an integer, and second, the depth of each layer is large compared to the pulse resolution depth, $V_w T_p/2$. This last condition insures that the integrator outputs for adjacent layers are approximately independent. Since the layers are approximately independent, it follows that the total variance is the sum of the variances for each layer.

$$VAR(I_T) = \sum_i N_i \cdot VAR(I_i) \tag{19}$$

where $VAR(I_i)$ is the variance of the integrated returns from the insonified volume in the i^{th} layer. N_i is the number of distinct insonified volumes in the i^{th} layer,

$$N_i = \frac{N_p}{n_i} \tag{20}$$

where N_p is the total number of acoustic pulses and n_i is the average number of multiple countings in the i^{th} layer. The derivation of the expression for $VAR(I_i)$ is given below.

If $x_{ij}(t)$ is defined as the signal returned from a volume in the i^{th} layer insonified by the j^{th} acoustic pulse and if I_{ij} is the corresponding integrator output, then

$$I_{ij} = \int_{t_{i-1}}^{t_i} t^g x_{ij}^2(t)\, dt \tag{21}$$

The total integrated output for this layer is

$$I_i = \sum_{j=1}^{n_i} I_{ij} \tag{22}$$

and the variance of I_i is

$$\begin{aligned} VAR(I_i) &= E[I_i^2] - \{E[I_i]\}^2 \\ &= \sum_{j=1}^{n_i} E[I_{ij}^2] + \sum_{\substack{j=1 \\ j\neq \ell}}^{n_i} \sum_{\ell=1}^{n_i} E[I_{ij} I_{i\ell}] \\ &\quad - \sum_{j=1}^{n_i} \{E[I_{ij}]\}^2 - \sum_{\substack{j=1 \\ j\neq \ell}}^{n_i} \sum_{\ell=1}^{n_i} E[I_{ij}]\, E[I_{i\ell}] \end{aligned} \tag{23}$$

When (21) is sustituted into (23), the expression for $\mathrm{VAR}(I_i)$ becomes

$$\mathrm{VAR}(I_i) = \sum_{j=1}^{n_i} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} t^q s^q \Big\{ E[X_{ij}^2(t)\, X_{ij}^2(s)] - \big(E[X_{ij}(t)]\big)^2 \big(E[X_{ij}(s)]\big)^2 \Big\}\, dt\, ds$$
$$+ \sum_{\substack{j=1 \\ j \neq \ell}}^{n_i} \sum_{\ell=1}^{n_i} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} t^q s^q \Big\{ E[X_{ij}^2(t)\, X_{i\ell}^2(s)] - \big(E[X_{ij}(t)]\big)^2 \big(E[X_{i\ell}(s)]\big)^2 \Big\}\, dt\, ds \tag{24}$$

The first sum in (24) is simply n_i times the integrator output variance for a single acoustic pulse, equation (12). The second order moments of $x_{ij}(t)$ and $x_{i\ell}(s)$ in the second term of (24) can be evaluated from their joint characteristic function. (The technique used is considered in detail in reference 1.)

$$E[X_{ij}^2(t)\, X_{i\ell}^2(s)] - \big(E[X_{ij}(t)]\big)^2 \big(E[X_{i\ell}(s)]\big)^2$$
$$= t^{-4} s^{-4} \Big\{ \overline{A_{ij}^2\, A_{i\ell}^2}\; E[\cos^2(\omega t + \theta_{ij}) \cos^2(\omega t + \theta_{i\ell})]$$
$$\Gamma(t,s) + \overline{A_{ij}^2}\; \overline{A_{i\ell}^2}\; \big(E[\cos(\omega t + \theta_{ij})$$
$$\cos(\omega t + \theta_{i\ell})]\big)^2\, \Gamma^2(t,s) \Big\} \tag{25}$$

where $\Gamma(t,s)$ is defined in (9). $A_{i\ell}$ and A_{ij} are the random amplitude factors on the ℓ^{th} and j^{th} integration respectively. At this point, it is convenient to make some additional refinements of the model. For pulse repetition rates of one or two per second it can be expected that the same sample of fish will remain in an insonified volume for several consecutive pulses. The position of the fish relative to the transducer beam and relative to one another will in general change from pulse to pulse. The changes in relative position will be assumed to cause the phases of the return signals for different pulses to be uncorrelated. That is

$$E[\theta_{ij}\,\theta_{i\ell}] = E[\theta_{ij}]E[\theta_{i\ell}]$$

and

$$E[\cos(\omega t+\theta_{ij})\cos(\omega t+\theta_{i\ell})] = 0 \qquad (26)$$

Using the assumption that $\frac{2\pi}{\omega} \ll T_p$ and equation (24), (25) and (26), the expression for VAR (I_i) becomes

$$\mathrm{VAR}(I_i) = \left[n_i\overline{A^4} + \sum_{\substack{j=1 \\ j\neq\ell}}^{n_i}\sum_{\ell=1}^{n_i} \overline{A_{ij}^2\,A_{i\ell}^2}\right]\frac{\lambda K T_p^2\left(t_i^{2g-5} - t_{i-1}^{2g-5}\right)}{4(2g-5)} + \frac{n_i(\overline{A^2})^2\lambda^2 K^2 T_p^3\left(t_i^{2g-3} - t_{i-1}^{2g-3}\right)}{6(2g-3)} \qquad (27)$$

$\overline{A_{ij}^2\,A_{i\ell}^2}$ can be expressed in terms of $\overline{A^4}$ by means of an intensity correlation coefficient, $\rho_{j\ell}(A_i^2)$, which is defined as

$$\rho_{j\ell}(A_i^2) = \frac{\overline{A_{ij}^2\,A_{i\ell}^2} - \overline{A_{ij}^2}\;\overline{A_{i\ell}^2}}{\overline{A_i^4} - (\overline{A_i^2})^2} \qquad (28)$$

The mathematical form of $\rho_{j\ell}(A_i^2)$ is considered in Appendix A. Using this definition for $\rho_{j\ell}\,(A_i^2)$, the expression for VAR (I_i) becomes

$$VAR(I_i) = \left[(n_i + C_{n_i})\overline{A^4} + (n_i^2 - n_i - C_{n_i})(\overline{A^2})^2 \right] \frac{\lambda K T_p^2 (t_i^{2g-5} - t_{i-1}^{2g-5})}{4(2g-5)}$$

$$+ \frac{n_i (\overline{A^2})^2 \lambda^2 K^2 T_p^3}{6(2g-3)} (t_i^{2g-3} - t_{i-1}^{2g-3}) \tag{29}$$

where

$$C_{n_i} = \sum_{\substack{j=1 \\ j \neq \ell}}^{n_i} \sum_{\ell=1}^{n_i} \rho_{\ell j}(A_i^2)$$

The normalized variance of the estimated fish density after the integration of N pulses can be determined from equations (16), (19), (20) and (29).

$$\frac{VAR[\hat{\lambda}]}{\{E[\hat{\lambda}]\}^2} = \frac{1}{N_p}\left[\frac{\overline{A^4}}{(\overline{A^2})^2} - 1\right] \frac{(g-1)^2}{K\lambda(2g-5)} \frac{[t_f^{2g-5} - t_0^{2g-5}]}{[t_f^{g-1} - t_0^{g-1}]^2}$$

$$+ \frac{1}{N_p} \frac{2T_p(g-1)^2}{3(2g-3)} \frac{[t_f^{2g-3} - t_0^{2g-3}]}{[t_f^{g-1} - t_0^{g-1}]^2} \tag{30}$$

$$+ \frac{1}{N_p} \frac{(g-1)^2}{\lambda K(2g-5)} \sum_i \left[\frac{C_{n_i}}{n_i} \frac{\overline{A^4}}{(\overline{A^2})^2} + n_i - \frac{C_{n_i}}{n_i}\right] \frac{[t_i^{2g-5} - t_{i-1}^{2g-5}]}{[t_f^{g-1} - t_0^{g-1}]^2}$$

Equation (30) is a general expression for the normalized variance of the density estimate. The numerical evaluation of equation (30) for any particular set of system parameters is considered in appendix B.

The purpose of this paper is to give the mathematical details of the derivation of the general variance expression for fish population estimates using echo-integration. Some conclusions which maybe drawn from the variance expressions obtained in this paper are given in reference 2.

REFERENCES

1. Ehrenberg, J.E. 1971. "The Variance of Fish Population Estimates Using an Echo Integrator." WSG 71-3.

2. Moose, P.H. and Ehrenberg, J.E. "An Expression for the Variance of the Abundance Estimates Obtained with a Fish Echo Integrator,"submitted for publication to the Journal of the Fisheries Research Board of Canada

APPENDIX A

In this appendix, the form of the intensity correlation coefficient, $\rho_{j\ell}(A^2)$, is investigated. The exact form of $\rho_{j\ell}(A^2)$ is unknown and can only be determined experimentally. Some of the factors affecting the correlation of signals on adjacent pulses are understood. They may be used to formulate a reasonable expression for $\rho_{j\ell}(A^2)$. The target strength of the individual fish change from pulse to pulse because of the change in their angle of illumination. The distribution of fish in a particular volume has a mean target strength which can differ from the target strength averaged over the total population being surveyed. An expression for A_j^2 which accounts for the two facts noted above is

$$A_j^2 = \alpha_j(\overline{A^2} + \beta) \tag{A1}$$

where α_j is a random variable accounting for the dependence of A_j^2 on an angle of illumination and β is a random variable accounting for the variation in mean target strength. β is a function of the distribution of fish which was assumed to remain the same during multiple countings. β is therefore not a function of j. Since $\overline{A_j^2} = \overline{A^2}$ it follows that $\overline{\alpha_j} = 1$ and $\overline{\beta} = 0$. It is reasonable to assume that α and β are uncorrelated since the physical effects from which they represent are unrelated.

The correlation coefficient defined by equation (28) is

$$\rho_{j\ell}(A^2) = \frac{\overline{A_j^2 A_\ell^2} - \overline{A_j^2}\,\overline{A_\ell^2}}{\overline{A^4} - (\overline{A^2})^2} \tag{A2}$$

Using the expression for A_j in (A1) and the assumed properties of α_j and β, it follows that

$$\overline{A_\ell^2 A_j^2} = \overline{\alpha_\ell \alpha_j}\,[(\overline{A^2})^2 + \overline{\beta^2}] \tag{A3}$$

$$\overline{A^4} = \overline{\alpha^2}\,[(\overline{A^2})^2 + \overline{\beta^2}] \tag{A4}$$

$$\overline{A_j^2} = \overline{A_\ell^2} = \overline{A^2} \tag{A5}$$

and that

$$\rho_{j\ell}(A^2) = \frac{\overline{\alpha_j \alpha_\ell}\,[(\overline{A^2})^2 + \overline{\beta^2}] - (\overline{A^2})^2}{\overline{\alpha^2}\,[(\overline{A^2})^2 + \overline{\beta^2}] - (\overline{A^2})^2} \tag{A6}$$

Equation (A6) can be written as the sum of two terms, one which is dependent on j and ℓ and another which is not.

$$\rho_{j\ell}(A^2) = \frac{\mathrm{cov}(\alpha_j \alpha_\ell)\,[(\overline{A^2})^2 + \overline{\beta^2}]}{\overline{\alpha^2}\,[(\overline{A^2})^2 + \overline{\beta^2}] - (\overline{A^2})^2} + \frac{\overline{\beta^2}}{\overline{\alpha^2}\,[(\overline{A^2})^2 + \overline{\beta^2}] - (\overline{A^2})^2} \tag{A7}$$

$COV(\alpha_j\alpha_\ell)$ has a maximum of $\overline{\alpha^2}$ when $j = \ell$ and decreases as $|j-\ell|$ increases. When α_j and α_ℓ are uncorrelated, the first term in (A7) goes to zero. An assumed typical plot of $\rho_{j\ell}(A^2)$ is shown in figure A1.

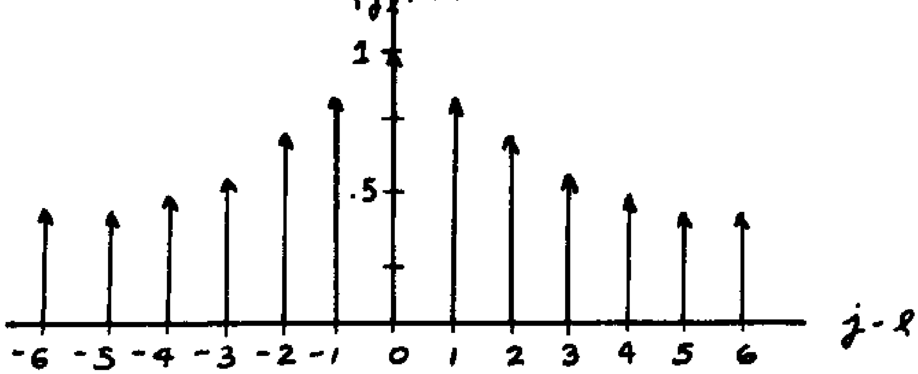


Figure A1

APPENDIX B

An expression for the normalized variance of the estimated fish density is given in eq. (30). The first step in the evaluation of (30) is to divide the depth stratum into layers with an integer amount of overlap and a depth that is large compared to the pulse resolution depth. The mathematical relationship between the system operating parameters and the volume overlap is determined below.

The average volume overlap in a depth interval d_1 to d_2 is

$$V(d_1, d_2) = \int_{d_1}^{d_2} A_{OL}(R)\, dR \tag{B1}$$

where $A_{OL}(R)$ is the overlapping area at a depth R. $A_{OL}(R)$ is given by

$$A_{OL}(R) = \sum_{\substack{\ell \\ \ell \neq k}}^{N_p} \sum_{k}^{N_p} A_{\ell k}(R) \tag{B2}$$

where $A_{\ell k}(R)$ is the amount of overlap between the eliptical beams of ℓ^{th} and k^{th} transducer pulses at a depth R and N_p is the total number to transmitted pulses. The geometry necessary to determine $A_{\ell k}(R)$ in terms of the system parameters is shown in figure B1.

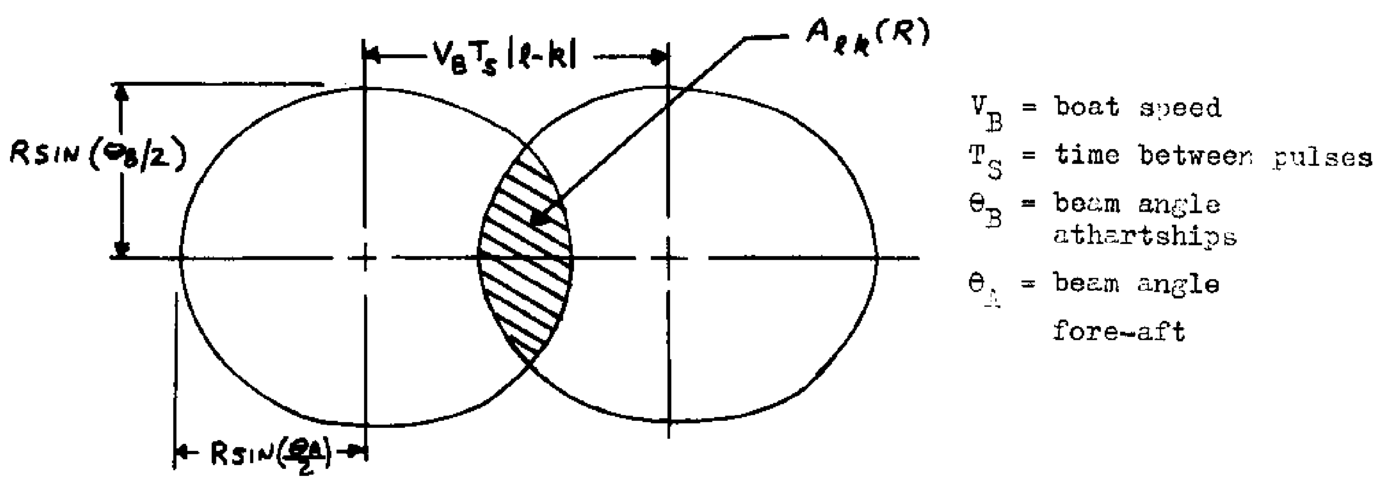


Figure B1

The resulting expression for $A_{\ell k}(R)$ is

$$A_{\ell k}(R) = 2R^2 \sin\left(\frac{\theta_B}{2}\right) \sin\left(\frac{\theta_A}{2}\right) \sin^{-1}\left[1 - \left(\frac{|\ell-k| V_B T_s}{2R\sin\left(\frac{\theta_A}{2}\right)}\right)^2\right]^{1/2}$$

$$- R\sin\left(\frac{\theta_B}{2}\right)|\ell-k|\, V_B T_s \left[1 - \left(\frac{|\ell-k| V_B T_s}{2R\sin\left(\frac{\theta_A}{2}\right)}\right)^2\right]^{1/2} \qquad \text{(B3)}$$

$$\text{FOR} \quad \frac{|\ell-k| V_B T_s}{2R\sin\left(\frac{\theta_A}{2}\right)} \leq 1$$

$$\simeq 0 \qquad\qquad \text{FOR} \quad \frac{|\ell-k| V_B T_s}{2R\sin\left(\frac{\theta_A}{2}\right)} > 1$$

For any set of operating conditions it is possible to obtain the proper division of the depth stratum into layer by incrementing the limits of integration in (B1) until the desired conditions are satisfied. The evaluation of (30) is straightforward once the depth stratum has been divided into layers.

A Fortran IV program has been written to evaluate (30) and plot the normalized variance as a function of fish density. A listing of the program and a typical output is given at the back of this appendix.

```
      PROGRAM FISHV(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C     THIS PROGRAM DETERMINES THE NORMALIZED VARIANCE OF THE ECHO
C     INTEGRATION SYSTEM FOR A SPECIFIED SET OF INPUT CONDITIONS
C
      DIMENSION F(50),COUNT(50),PVAR(50),PLAMDA(50)
      DIMENSION NSCALE(5),IMAGE(867)
      COMMON NP,RI,RF,THETAA,THETAB,DELD,PIE,X,RM,DP
      PIE = 3.1415926535
      NTD = 50
      READ(5,90) NDATA
C     NDATA = NUMBER OF SETS OF INPUT DATA
C     UP TO FIVE (5) SETS OF DATA CAN BE EVALUATED
      CALL OMIT(3)
       NSCALE(1) = 1
      NSCALE(2) = 0
      NSCALE(3) = 0
      NSCALE(4)= 0
      NSCALE(5) = 0
C     THE NEXT TWO INSTRUCTIONS DETERMINE THE PLOT GRID
      CALL PLOT1(NSCALE,5,10,7,14)
      CALL PLOT2(IMAGE,2.,-5.,0.,-5.)
      DO 40 IDATA = 1,NDATA
      READ(5,100) RI,RF,TVG,RM,PDUR
      READ(5,100)ANGFA,ANGAS,VBOAT,RATES,ALAMDA
      READ(5,110) NP,NPOINT,STEP
C     RI AND RF = INITIAL AND FINAL DEPTH IN METERS
C     TVG = TIME VARING GAIN CONTROL (DB)
C     RM = MOMENT RATIO - E(V**4)/(E(V**2))**2
C     PDUR = SONAR PULSE DURATION IN SEC.
C     ANGFA AND ANGAS = BEAM ANGLES, FORE-AFT AND ATHWARTSHIPS ( IN DEGREES)
C     VBOAT = BOAT VELOCITY IN KNOTS
C     ALAMDA = FIRST DENSITY TO BE EVALUATED
C     NPOINT = NUMBER OF DENSITY POINTS TO BE CALCULATED
C     STEP = MULTIPLING FACTOR FOR INCREMENTING ALAMDA (ALAMDA = ALAMDA*STEP)
C     NP = TOTAL NUMBER OF ACOUSTIC PULSES
      DELD = (VBOAT*.51444)/RATES
      DP = PDUR*750.
      THETAA = ANGFA*PIE/180.
      THETAB = ANGAS*PIE/180.
      CALL LEVELS(F,COUNT,NTD,L)
C     LEVELS DEVIDES DEPTH INTERVAL INTO SUBINTERVALS - EACH SUBINTERVAL HAS
C     AN INTEGER NUMBER OF MULTIPLE COUNTINGS
      X = (TVG - 40.)/10.
      WRITE(6,120)
      WRITE(6,250)
      WRITE(6,130)
      WRITE(6,140)RI,RF
      WRITE(6,250)
      WRITE(6,150)
      WRITE(6,160) TVG,PDUR,RM
      WRITE(6,250)
      WRITE(6,170)
      WRITE(6,175)
      WRITE(6,160) ANGFA,ANGAS,VBOAT
      WRITE(6,250)
```

```
      WRITE(6,180)
      WRITE(6,190) RATES,NP
      WRITE(6,250)
      WRITE(6,200)
      WRITE(6,210)
      DO 30 I = 1,L
      IF(I.EQ.1) GO TO 20
      DS = F(I-1)
      GO TO 30
   20 DS = RI
   30 WRITE(6,220) DS,F(I),COUNT(I)
      WRITE(6,230)
      CALL INTVAR(F,COUNT,L,VART1,VART2,NTD)
C     INTVAR ELALUATES THE EXPRESSION FOR NORMALIZED VARIANCE
      DO 10 I=1,NPOINT
      A1 = 1./((ALAMDA*RF**3    *PIE)*SIN(THETAA/2.)*SIN(THETAB/2.))
      VAR = A1*VART1 + VART2
      WRITE(6,240) ALAMDA,VAR
      ALAMDA= ALAMDA*STEP
      PVAR(I) = ALOG10(VAR)
   10 PLAMDA(I) = ALOG10(ALAMDA)
C
C     THE NEXT SET OF INSTRUCTIONS PLOT THE CURVE
      DATA IHOL1,IHOL2,IHOL3,IHOL4,IHOL5,IHOL6/6R.00001,6R.0001 ,6R.001
     1 ,6R.01    ,6R.1     ,6R1.     /
      IMAGE( 18) = IHOL1
      IMAGE(188) = IHOL2
      IMAGE(358) = IHOL3
      IMAGE(528) = IHOL4
      IMAGE(698) = IHOL5
      IMAGE(834) = IHOL6
      GO TO (300,301,302,303,304),IDATA
  300 CALL PLOT3(1H1,PLAMDA,PVAR,50)
      GO TO 40
  301 CALL PLOT3(1H2,PLAMDA,PVAR,50)
      GO TO 40
  302 CALL PLOT3(1H3,PLAMDA,PVAR,50)
      GO TO 40
  303 CALL PLOT3(1H4,PLAMDA,PVAR,50)
      GO TO 40
  304 CALL PLOT3(1H5,PLAMDA,PVAR,50)
   40 CONTINUE
      PRINT 2
      CALL PLOT4(20,20HNORMALIZED VARIANCE          )
      WRITE (6,4)
      WRITE (6,5)
    2 FORMAT(1H1)
    4 FORMAT(*             .00001          .0001          .001          .01
     1          .1             1.          10           1000*)
    5 FORMAT(*-            FISH DENSITY PER CUBIC METER *)
C
   90 FORMAT(I10)
  110 FORMAT (2I10,1E12.4)
  100 FORMAT (5E12.4)
  120 FORMAT(*1 ECHO INTEGRATOR VARIANCE AS A FUNCTION OF FISH DENSITY*)
  130 FORMAT(*- INITIAL DEPTH (METERS)    FINAL DEPTH (METERS)*)
```

```
140 FORMAT(1H0,1E18.5,1E25.5)
150 FORMAT(*-         TVG(DB)        PULSE LENGTH(SEC)    MOMENT RATIO*)
160 FORMAT(1H0,1E16.5,2E20.5)
170 FORMAT(*-                BEAM ANGLE-DEGREES*)
175 FORMAT(*        FORE-AFT                 ATHWARTSHIPS      BOAT SPEED -
   1KNOTS*)
180 FORMAT(*- SOUNDING RATE PER SEC    NUMBER OF SOUNDINGS*)
190 FORMAT(1H0,1E18.5,I22)
200 FORMAT(*- THE DEPTH INTERVAL HAS BEEN SPLIT INTO LAYERS WITH AN IN
   1TERGER NUMBER */*  OF COUNTINGS, THE DIVISION IS LISTED BELOW*)
210 FORMAT(*-    INITIAL DEPTH   FINAL DEPTH      NUMBER OF */
   1      *      (METERS)        (METERS)        COUNTINGS* )
220 FORMAT (1H0,3E15.4)
230 FORMAT(*1 FISH DENSITY       NORMALIZED */
   1      *  PER METER CU.       VARIANCE *)
240 FORMAT(1H0,2E14.5)
250 FORMAT(*-*)
    END
```

```
      SUBROUTINE INTVAR(F,COUNT,L,VART1,VART2,NTD)
      DIMENSION F(NTD),COUNT(NTD)
      COMMON NP,RI,RF,THETAA,THETAB,DELD,PIE,X,RM,DP
      SUM3 = 0.
      SUM4 = 0.
      DO 70 M = 1,L
      I   = INT(COUNT(M ) + .5)
      AI  = I
      IF(M.EQ.1) GO TO 50
      DS  = F(M-1)
      GO TO 60
   50 DS = RI
   60 SUM3 = SUM3 + ((F(M)/RF)**(2.*X+3.))*(1.-(DS/F(M))**(2.*X+3.))
     1 *(C(I)/AI)
      SUM4 = SUM4 + ((F(M)/RF)**(2.*X+3.))*(1.-(DS/F(M))**(2.*X+3.))
     1 *(AI-1.-C(I)/AI)
   70 CONTINUE
      TR = RI/RF
      ANP = NP
      VAR1=(X+3.)**2*(1.-TR**(2.*X+3.))/((2.*X+3.)*(1.-TR**(X+3.))**2)
      VAR2=(X+3.)**2*(1.-TR**(2.*X+5.))/((2.*X+5.)*(1.-TR**(X+3.))**2)
      VAR3 = (((X+3.)**2)/((2.*X+3.)*(1.-TR**(X+3.))**2))*SUM3
      VAR4 = (((X+3.)**2)/((2.*X+3.)*(1.-TR**(X+3.))**2))*SUM4
      VART1 = (RM*VAR1+RM*VAR3 + VAR4)/ANP
      VART2 = (VAR2*2.*DP)/(3.*RF*ANP)
      RETURN
      END
```

```
      SUBROUTINE LEVELS (F,COUNT,NTD,L)
C
C  F(L) IS THE END DEPTH OF THE L SUBINTERVAL
C  COUNT(L) IS THE AVERAGE NUMBER OF COUNTINGS FROM DEPTH L-1 TO L
C
      DIMENSION F(NTD),COUNT(NTD)
      COMMON NP,RI,RF,THETAA,THETAB,DELD,PIE,X,RM,DP
      ANG = THETAA /2.
      ANP = NP
      ANTD = NTD
      DELR = (RF-RI)/ANTD
      DO 8 L = 1,NTD
      AL = L
      SUM1 = 0.
      SUM2 = 0.
      AA = DELD/(2.*SIN(ANG)*(RI +AL*DELR))
      NEND = INT(1./AA)
      SUMEND = 0.
      NENDP = NEND + 1.
      IF(NEND.LT.1) GO TO 140
      DO 120 IEND = 1,NEND
      DO 110 JEND = 1,NENDP
      IF(JEND.EQ.IEND) GO TO 110
      DIFEND = IABS(IEND-JEND)
      IF(AA*DIFEND.GT.1.) GO TO 110
      SUMEND = SUMEND + PIE/2.-ASIN(AA*DIFEND)-AA*DIFEND*((1.-(AA*
     1  DIFEND)**2)**.5)
  110 CONTINUE
  120 CONTINUE
  140 CONTINUE
      AI = 1.
    3 IF(AI*AA .GT. 1.) GO TO 5
      SUM1 = SUM1 + PIE/2. - ASIN(AI*AA)-AI*AA*((1.-(AA*AI)**2)**.5)
      AI = AI + 1.
      GO TO 3
    5 F(L) = (2./(PIE*ANP))*((ANP-2.*NEND)*SUM1 + SUMEND)
  150 FORMAT (1H0,2E15.6)
    8 CONTINUE
       NCOUNT = 1 + INT(F(1))
      ACOUNT = NCOUNT
      K = 1
      AK = 1.
      L = 1
   10 SUMN = 0.
      SUMD = 0.
   20 SUMN = SUMN + ((RI + AK*DELR)**2)*F(K)
      SUMD = SUMD + ((RI + AK*DELR)**2)
      RATIO = SUMN/SUMD
      K = K +1
      IF (INT(RATIO) .GT.NP-1) GO TO 50
      AK = AK + 1.
      IF(RATIO .GE.ACOUNT) GO TO 30
      IF( RI +(AK-1.)*DELR .GT. RF) GO TO 40
      GO TO 20
   30 COUNT(L) = RATIO +1.
      F(L) = RI + (AK-1.)*DELR
```

```
      NCOUNT = INT(RATIO ) +1
      ACOUNT = NCOUNT
      IF (L.EQ.1) GO TO 33
      DS = F(L-1)
      GO TO 36
   33 DS = RI
   36 CONTINUE
      IF((F(L) -DS     ).LT.(10.*DP)) GO TO 10
      L = L + 1
      GO TO 10
   40 COUNT(L)  = RATIO  +1.
      F(L) = RF
      GO TO 60
   50 F(L) = RF
      COUNT(L) = ANP
   60 L = L +1
      DO 70 II = L,NTD
      F(II) = 0.
   70 COUNT(II) = 0.
      L = L - 1
      RETURN
      END
      FUNCTION C(I)
      C = I*(I - 1)
      RETURN
      END
```

ECHO INTEGRATOR VARIANCE AS A FUNCTION OF FISH DENSITY

INITIAL DEPTH (METERS)	FINAL DEPTH (METERS)
1.00000E+01	4.00000E+01

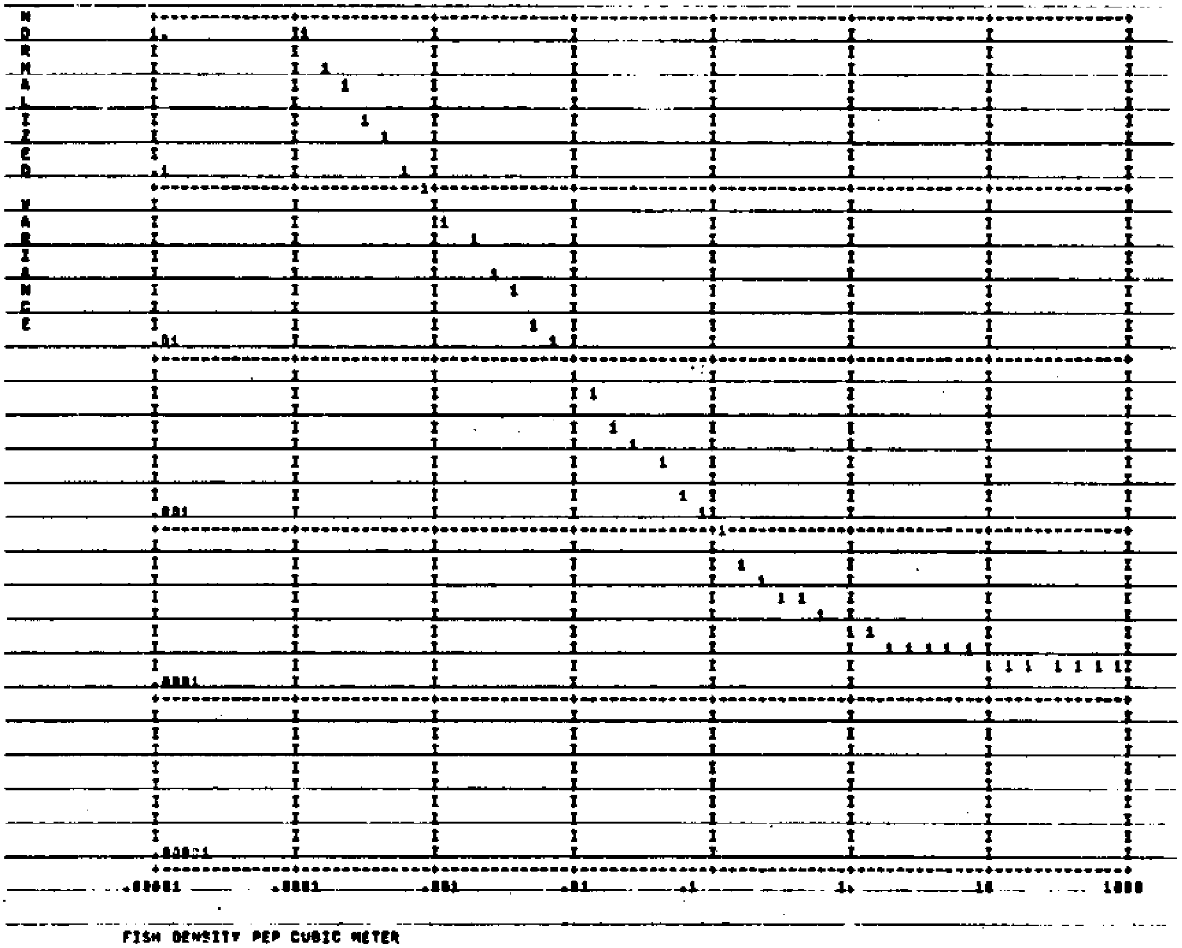
TVG(DB)	PULSE LENGTH(SEC)	MOMENT RATIO
2.00000E+01	1.00000E-03	2.00000E+00

BEAM ANGLE-DEGREES FORE-AFT	ATHWARTSHIPS	BOAT SPEED - KNOTS
2.30000E+01	2.30000E+01	4.00000E+00

SOUNDING RATE PER SEC	NUMBER OF SOUNDINGS
2.00000E+00	100

THE DEPTH INTERVAL HAS BEEN SPLIT INTO LAYERS WITH AN INTERGER NUMBER OF COUNTINGS, THE DIVISION IS LISTED BELOW

INITIAL DEPTH (METERS)	FINAL DEPTH (METERS)	NUMBER OF COUNTINGS
1.0000E+01	2.2600E+01	3.0397E+00
2.2600E+01	3.7600E+01	5.0092E+00
3.7600E+01	4.0000E+01	5.3707E+00


NORMALIZED VARIANCE
.1
.01
.001
.0001
.00001
.00001
.0001
.001
.01
.1
1.
10
1000
FISH DENSITY PER CUBIC METER