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FAR-FIELD MATCHING FOR TIDAL CALCULATIONS IN NEARSHORE REGIONS

Carl Pearson

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PREFACE

Wave motion in estuaries and harbors has been a problem of practical importance for many years. Commercial ship operators and private boat owners, designers of marinas, as well as certain governmental agencies have been concerned with the response of harbors and estuaries to various types of wave forms.

In response to this interest, ocean engineers have traditionally used small-scale hydraulic models for predicting wave motion in harbors and other near-shore environments. More recently, however, high speed digital computer models have been developed for the same purpose. In fact, during the past few years, several Sea Grant institutions, including the University of Washington, have addressed the problem of developing computer programs for the prediction of estuarine and harbor wave motion. As work in this area progressed, it became apparent that an adequate treatment of wave motion in semi-enclosed bodies of water required not only the application of sophisticated mathematical techniques, but also a reconsideration of certain fundamental aspects of the problem formulation.

This report describes one such aspect: namely the proper matching of far-field (open sea) solutions and near-shore environment solutions. Admittedly, the discussion presented here is more technical than in most Sea Grant reports. However, the subject matter is relevant to a problem which concerns a number of Sea Grant researchers and, for that reason, we felt that it would be appropriate to present this material to the scientific community in the form of a Sea Grant report.

FAR-FIELD MATCHING FOR TIDAL CALCULATIONS IN NEAR-SHORE REGIONS

1. INTRODUCTION AND ABSTRACT

Previous work [1] dealing with tidal current calculations in bays and estuaries has shown that the solution is sensitive to the tidal boundary condition imposed at the mouth of the bay. This sensitivity arises from the fact that the instantaneous tidal height is affected by tidal wave reflection from the local shoreline configuration, so that from the mathematical point of view the boundary condition includes part of the problem solution. In principle, the difficulty can be resolved by replacing the boundary condition at the mouth of the bay by a matching requirement between far-field and near-shore solutions; for tidal problems, such a procedure is complicated by the importance of the Coriolis term as well as by bathymetric and topographic complexities.

In this report, we describe a number of matching methods which can be used. These methods are based on integral equations, finite elements, variational methods, transform methods, series solutions, or on a combination of such methods. Since non-tidal harbor response problems are also of interest, we include some discussion of methods which are limited in their ability to incorporate Coriolis effects. It appears that most of the approaches to be discussed are computationally feasible; implementation for a particular estuary region will be reported on separately. Here we restrict ourselves to derivations and to such numerical experimentation as will help elucidate specific properties of certain methods.

Except where stated, we will use shallow water theory, and a time factor $e^{-i\omega t}$, throughout. We anticipate that non-linear effects will be important in the near-shore region, and will be handled by an iterative technique of the kind discussed in [1]. The far-field region will be assumed sufficiently deep that a linear calculation is adequate. The effect of the earth's curvature is neglected, but could be included at the cost of some complexity in formulation.

2. REFLECTION OF PLANE WAVE

In the sequel, we will make use of the solution to the following problem. Let a plane wave of angular frequency ω be propagating shoreward in a region of constant depth h_o; the shoreline coincides with the x-axis as shown in Fig. (1).

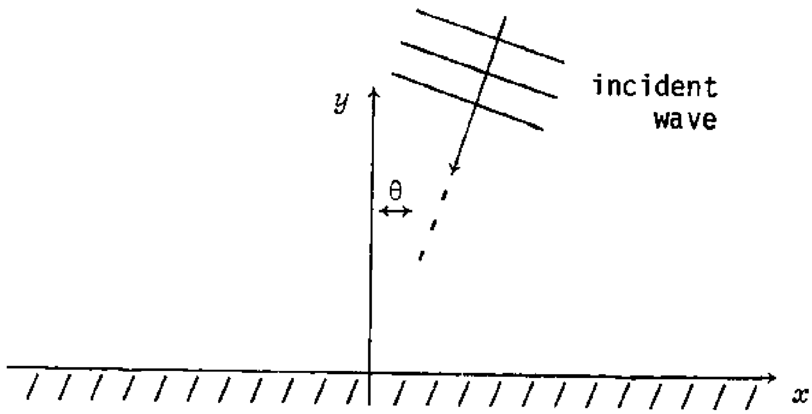

Figure (1)

Near the shore, the depth becomes $h = h(y)$, a function of y only. If u and v are the velocities in the x- and y-directions, ζ the water height, and f the Coriolis parameter, linearized shallow water theory requires [2]

$$\begin{aligned} u_t - fv &= -g\zeta_x \\ v_t + fu &= -g\zeta_y \\ (hu)_x + (hv)_y &= -\zeta_t \end{aligned} \tag{1}$$

where a subscript indicates a partial derivative. Here t is time and g the acceleration of gravity. For harmonic motion, we write u, v, ζ as the real parts of $Ue^{-i\omega t}$, $Ve^{-i\omega t}$, $Ze^{-i\omega t}$ respectively, where U, V, Z are complex functions of x and y. Equation (1) becomes

$$\begin{aligned} i\omega U + fV &= gZ_x \\ -i\omega V + fU &= -gZ_y \\ (hU)_x + (hV)_y &= i\omega Z \end{aligned} \tag{2}$$

from which

$$U = \frac{g}{\omega^2 - f^2}(-i\omega Z_x + fZ_y)$$

$$V = \frac{g}{\omega^2 - f^2}(-i\omega Z_y - fZ_x) \qquad (3)$$

$$[d(Z_x + \frac{if}{\omega} Z_y)]_x + [d(Z_y - \frac{if}{\omega} Z_x)]_y + k_o^2 Z = 0$$

where $d = h/h_o$, $k_o^2 = (\omega^2 - f^2)/(gh_o)$. If V is to vanish at $y = 0$, we require

$$i\omega Z_y + fZ_x = 0 \quad \text{at} \quad y = 0 \quad . \qquad (4)$$

An incident plane wave of unit amplitude can be written as (cf. Fig. (1))

$$Z_I = e^{-ik_o(x \sin\theta_o + y \cos\theta_o)} \qquad (5)$$

and it is now desired to obtain a solution of the third of Eqs. (3) subject to conditions (4) and (5).

Since $d = d(y)$, separation of variables is feasible. We write

$$Z = e^{-i(k_o \sin\theta_o)x} \psi(y) \qquad (6)$$

(cf. Snell's law for ray propagation) and substitute into Eqs. (3) and(4) to obtain

$$(d\psi')' + [k_o^2(1 - d \sin^2\theta_o) - d' \frac{fk_o}{\omega} \sin\theta_o]\psi = 0 \qquad (7)$$

with

$$\psi' - (\frac{fk_o}{\omega} \sin\theta_0)\psi = 0 \quad \text{at} \quad y = 0 \qquad (8)$$

where a prime means d/dy. (If $d(0) = 0$, Eq. (8) must be replaced by a finiteness condition at $y = 0$.) For certain choices of $d(y)$, Eq. (7) can be solved analytically (e.g., in terms of hypergeometric functions if d is linear); however, we will let $d(y)$ be arbitrary and content ourselves with a numerical solution. Take $\psi(0) = 1$; then the resulting numerical solution can be obtained via standard methods, and since $d \sim 1$ for sufficiently large y, it will be asymptotic to a linear combination of

$$e^{-(ik_o \cos\theta_o)y}, \quad e^{(ik_o \cos\theta_o)y} \tag{9}$$

as $y \to \infty$. Moreover (with $\psi(0) = 1$), ψ will be real, so that the coefficients of this linear combination must be complex conjugates of one another. Thus we merely need divide our $\psi(0) = 1$ solution by the coefficient of the first term in (9) to give the solution corresponding to Eq. (5); the second term, then, represents an exiting wave of unit amplitude and so of energy equal to that of Z_I.

An alternative numerical approach--more cumbersome, but also more picturesque--would be to approximate the depth profile by a sequence of constant depth steps, as in Fig. (2).

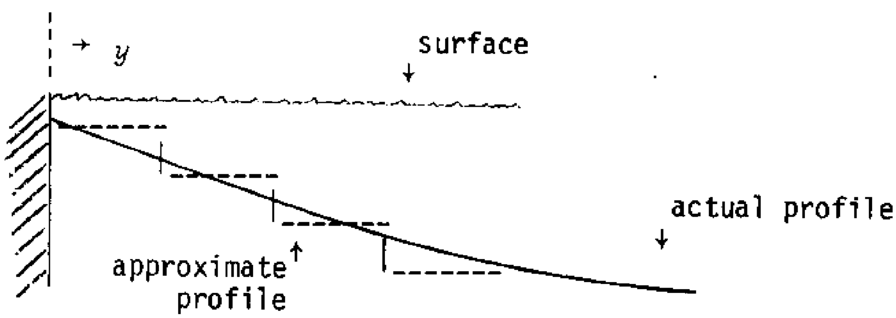

Figure (2)

Let the depth be h_j for $y_j < y < y_{j+1}$. Define $k_j^2 = (\omega^2 - f^2)/(gh_j)$, and represent Z in this y-interval by

$$Z_j(x,y) = A_j \, e^{ik_j(-x \sin_j - y \cos\theta_j)} + B_j \, e^{ik_j(-x \sin\theta_j + y \cos\theta_j)}$$

where A_j and B_j are complex constants, and where $k_j \sin\theta_j = k_o \sin\theta_o$ (Snell's law).

Continuity of Z and (hV) at the transition point y_j requires

$$Z_j(x,y) = Z_{j-1}(x,y) \qquad \text{for } y = y_j$$

and

$$h_j[i\omega \frac{\partial Z_j}{\partial y} + f \frac{\partial Z_j}{\partial x}] = h_{j-1}[i\omega \frac{\partial Z_{j-1}}{\partial y} + f \frac{\partial Z_{j-1}}{\partial x}] \quad \text{for} \quad y = y_j \quad .$$

These equations, together with conditions (4) and (5), are adequate to determine all A_j and B_j values. As a check, it must turn out that $|A_j| = |B_j|$.

3. REMARKS ON KELVIN WAVES

If $d = 1$ for all $y > 0$ in the problem of Sec. (2), we find

$$\psi = e^{-ik_o\cos\theta_o y} + \left\{\frac{\omega\cos\theta_o - if\sin\theta_o}{\omega\cos\theta_o + if\sin\theta_o}\right\} e^{ik_o\cos\theta_o y} \qquad (10)$$

which is a combination of an incident and a reflected (with phase change) plane wave. No Kelvin wave is generated. If the coastline has a bend in it, or is irregular, then in general a Kelvin wave will be produced (see for example [3] and [4]). It does not seem to have been pointed out in the literature that if the disturbance source is at a finite rather than an infinite distance, then a Kelvin wave can be generated by reflection at even a plane boundary.

Consider the half plane $y > 0$, with h constant, and let there be a source

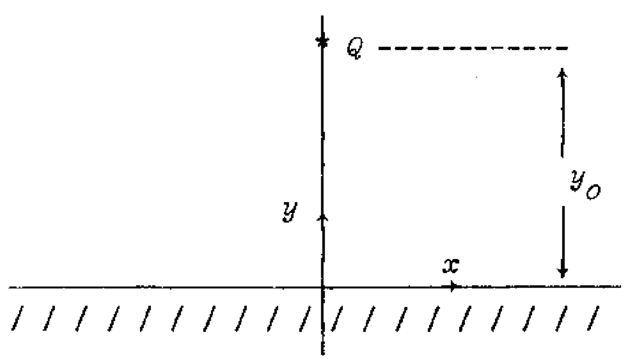

Figure (3)

at Q as shown in Fig. (3). Using the notation of Sec. (2), the governing equation is now

$$Z_{xx} + Z_{yy} + k^2 Z = R\ \delta(x)\ \delta(y - y_o) \tag{11}$$

where δ is the delta function, $k^2 = (\omega^2 - f^2)/(gh)$, and R is the source strength factor. Were there no barrier at $y = 0$, the solution to Eq. (11) would be given by

$$Z = -\frac{iR}{4} H_o^{(1)} \left[k \sqrt{x^2 + (y - y_o)^2}\right] \tag{12}$$

where the Hankel function of the first kind, $H_o^{(1)}$, is required in order that waves be outgoing at ∞ (the time factor is again $e^{-i\omega t}$). Because of the barrier, the actual solution differs from this, and must satisfy condition (4) at $y = 0$.

Denote the Fourier transform of Z by $\overline{Z}$, defined by

$$\overline{Z}(\lambda,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z(x,y)\ e^{i\lambda x} dx \quad . \tag{13}$$

Then a transform in x of Eq. (11) yields

$$\overline{Z}_{yy} - (\lambda^2 - k^2)\overline{Z} = \frac{R}{\sqrt{2\pi}}\ \delta(y - y_o) \quad . \tag{14}$$

It follows that

$$\overline{Z} = A\ e^{-\sqrt{\lambda^2 - k^2}\ y} \quad \text{for} \quad y > y_o \tag{15}$$

$$\overline{Z} = B \cosh \sqrt{\lambda^2 - k^2}\ y + C \sinh \sqrt{\lambda^2 - k^2}\ y \quad \text{for} \quad y < y_o \tag{16}$$

where the branch cut and inversion contour in the λ-plane are as shown in Fig. (4).

(One way to see that the designated inversion path is appropriate is to note that if dissipation terms are incorporated in Eqs. (2)--say $(-\varepsilon u)$ and $(-\varepsilon v)$ on the right hand sides of the first two equations, respectively,

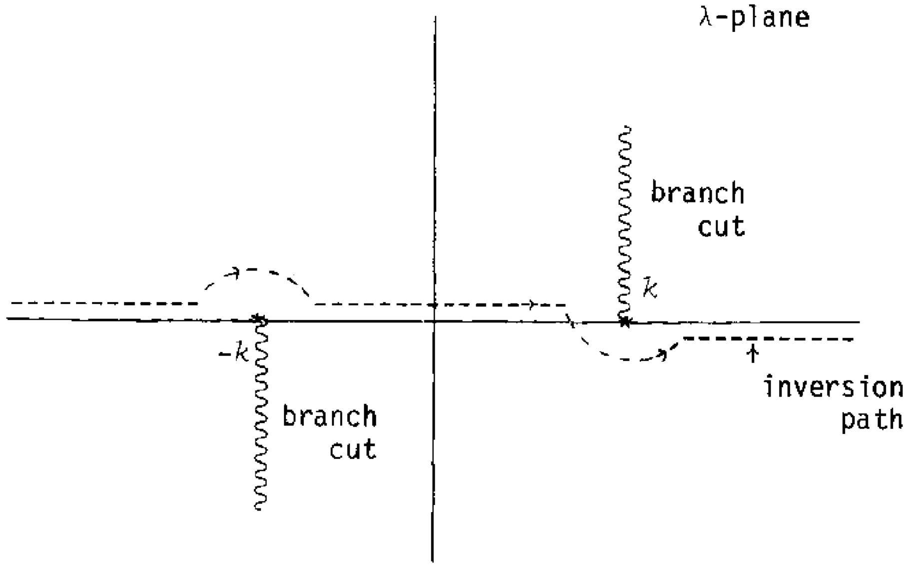

Figure (4)

where $\varepsilon > 0$ is small--then k^2 will have a small positive imaginary part, so that the inversion path will pass above the replacement for $(-k)$ and below that for $(+k)$; in the limit, as the dissipation vanishes, the indentations become as shown. A second method is to observe that, as shown in the sequel, this choice--and no other choice--satisfies the radiation condition previously alluded to). In Eqs. (15) and (16), we interpret $\sqrt{\lambda^2 - k^2}$ as positive real for λ real with $\lambda > k$; the choice of branch cuts removes any ambiguity elsewhere.

A transform of Eq. (4) leads to

$$i\omega\, \overline{Z}y - i\lambda\, f\overline{Z} = 0 \qquad \text{for} \quad y = 0$$

whence

$$C = \frac{f\lambda}{\omega\sqrt{\lambda^2 - k^2}}\, B \quad . \tag{17}$$

At $y = y_o$, the values of $\overline{Z}$ on the two sides of y_o, as given by Eqs. (15) and (16), must coincide; moreover, Eq. (14) requires the discontinuity in $\overline{Z}_y$ across y_o to equal $R/\sqrt{2\pi}$. Thus we are able to determine all of A, B, C, and it is easily found in particular that, for $y < y_o$,

$$\bar{Z} = -\frac{R}{2\sqrt{2\pi}\,\sqrt{\lambda^2 - k^2}}\left[\left\{e^{-\sqrt{\lambda^2 - k^2}(y-y_o)} + e^{-\sqrt{\lambda^2 - k^2}(y+y_o)}\right\}\right.$$

$$\left. - \frac{2f\lambda}{\omega\sqrt{\lambda^2 - k^2} + f\lambda}\, e^{-\sqrt{\lambda^2 - k^2}(y+y_o)}\right] . \qquad (18)$$

As a check on the calculation, we note that if $f = 0$, then[1] the inverse of Eq. (18) yields

$$Z = -\frac{Ri}{4}\left[H_o^{(1)}\left(k\sqrt{x^2 + (y - y_o)^2}\right) + H_o^{(1)}\left(k\sqrt{x^2 + (y + y_o)^2}\right)\right] \qquad (19)$$

which represents the effect of a source plus its image.

Even if $f \neq 0$, the first part of Eq. (18) leads again to Eq. (19) (with, of course, $k^2 = (\omega^2 - f^2)/(gh)$), so that, in part, we have a source wave plus a specularly reflected wave. Inversion of the second part of Eq. (18) does not lead to a simple function; however, we are primarily interested in the

---

(1) For $\alpha > 0$,

$$I = \int_{-\infty}^{\infty} \frac{e^{-i\lambda x - \sqrt{\lambda^2 - k^2}\,\alpha}}{\sqrt{\lambda^2 - k^2}}\, d\lambda = \pi i H_o^{(1)}\left(k\sqrt{\alpha^2 + x^2}\right) ,$$

where the branch cuts and inversion path are as in Figure (4). This identity can, of course, be deduced from the fact that the field must be that given by a source plus its image, using Eq. (12); alternatively, it is easy to show that $\alpha I_x - x I_\alpha = 0$, so that I is a function of $(x^2 + \alpha^2)$; then set one of α or x equal to zero, etc.

asymptotic behavior (large x) in any event, and this we can obtain directly. In the λ-plane, there is a pole at $\lambda = -k/\sqrt{1 - f^2/\omega^2}$, and the inversion path passes above this pole. If $x < 0$, we deform the path of integration so as to wrap around the cut through $\lambda = k$; if $x > 0$, we wrap it around the cut through $\lambda = -k$, picking up in this latter process the contribution from the pole. The branch cut integrals are easily evaluated asymptotically, and the final result is (for $f \neq 0$)

(a) for $x < 0$:

$$Z \sim R\frac{e^{-ikx + i\pi/4}}{\sqrt{2\pi k|x|}} \tag{20}$$

(b) for $x > 0$:

$$Z \sim \frac{if\omega R}{\omega^2 - f^2}\, e^{i\frac{\omega}{c}x - (y+y_o)\frac{f}{c}} + \frac{Re^{ikx + i\pi/4}}{\sqrt{\pi k|x|}} \tag{21}$$

where $c = \sqrt{gh}$. The first term in Eq. (21) represents a non-decaying Kelvin wave traveling to the right. The other terms in Eqs. (20) and (21) decay as x becomes large, as expected.

4. MATCHING

We consider now a matching technique applicable to the (fairly general) situation depicted in Figure (5).

For $y > y_1$, let $h = h_o =$ constant. In the regions defined by $\{y < y_1, x > x_2\}$ and $\{y < y_1, x < x_1\}$, let $h = h(y)$. In the remaining region R, which includes the estuary and its entrance region, we will have $h(x,y)$; there may also be islands.

We choose y_1, $|x_1|$, and $|x_2|$ sufficiently large that along the lines C_1, C_2, C_3, C_4, the value of Z will be that obtained in Sec. (2), i.e., $Z = Z_1 = e^{-i(k_o \sin\theta_o)x}\psi_1(y)$, where we take $\psi_1(y)$ as known, with $k_o^2 = (\omega^2 - f^2)/c_o^2$. The possibility of such a choice is based on the expectation that the effect of the estuary will diminish with distance from it (more or less in proportion to inverse square root of distance, in fact, if the estuary effect can be modelled by a superposition of

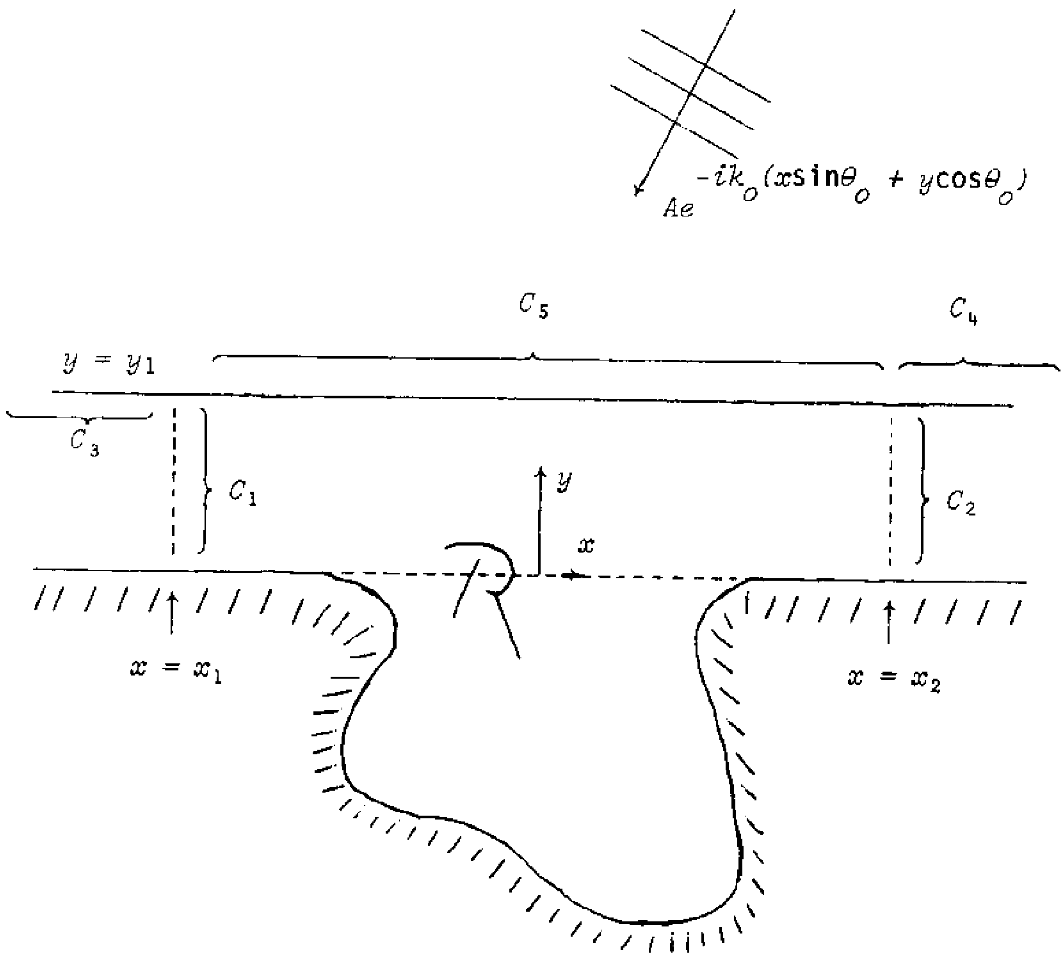

Figure (5)

secondary sources). Of course, the calculation of ψ_1 is carried out for a geometry in which there is no estuary, and in which the $h(y)$ function is considered to be valid for all x.

Along the line C_5, Z can differ significantly from Z_1. In the region S defined by $\{y > y_1\}$, write $Z = Z_1 + \phi$. Then ϕ satisfies the equation (in S)

$$\phi_{xx} + \phi_{yy} + k_o^2\phi = 0 \tag{22}$$

Denote the (as yet unknown) value of ϕ_y on y_1 by $p(x)$; for $x < x_1$ or for $x > x_2$, we have $p(x) = 0$, as a consequence of the above assumptions.

A Fourier transform in x of Eq. (22) leads to

$$\Phi(\lambda) = -\frac{P(\lambda)}{\sqrt{\lambda^2 - k_o^2}} e^{-\sqrt{\lambda^2 - k_o^2}\,(y - y_1)} \tag{23}$$

where Φ and P are the transforms of ϕ and p respectively, and where we contemplate the inversion path of Sec. (3). Using the footnoted inversion integral of Sec. (3), and the convolution theorem, Eq. (23) implies that

$$\phi(x,y) = -\frac{i}{2}\int_{x_1}^{x_2} p(\zeta) H_o^{(1)}\left(k_o\sqrt{(y - y_1)^2 + (x - \zeta)^2}\right) d\zeta \tag{24}$$

where we use the fact that $p(x)$ vanishes outside of (x_1, x_2).

The composite values of Z and Z_y on C_5 are given by $Z = Z_1 + \phi$, and $Z_y = (Z_1)_y + p$, respectively; note that if p is known on C_5, so is ϕ, via Eq. (24). These values of Z and Z_y on C_5 must correspond to those obtained from whatever solution process (e.g., finite elements) is used in R. To achieve this correspondence, several methods are possible. The simplest might be iterative--guess Z on C_5 (close to an actual tidal measurement, for the frequency of interest, for example) and determine the associated values of Z inside R; compute Z_y on C_5 for this "inside" solution and so calculate p on C_5 from $p = Z_y - (Z_1)_y$; determine ϕ from Eq. (24) and so a revised value of Z on C_5; iterate to convergence. (It may be feasible to neglect non-linearities inside R during most or all of this iteration.) Alternatively, since Eq. (24) is linear, the ϕ values over a set of mesh points on C_5 are linear combinations of the p values, so that correspondence between inside and outside solutions can be obtained by adding an extra set of linear equations to those being used to solve for Z in R by the method of ref. [1], say.

We note, incidentally, that Coriolis effects are fully included in the procedure of this section.

5. MATCHING IN A NARROW CHANNEL

It sometimes occurs (as in Hood Canal, Washington State) that one is interested in analyzing the tidal motion in a region in which the entrance to the portion of interest is along a relatively long and narrow channel (Figure (6).

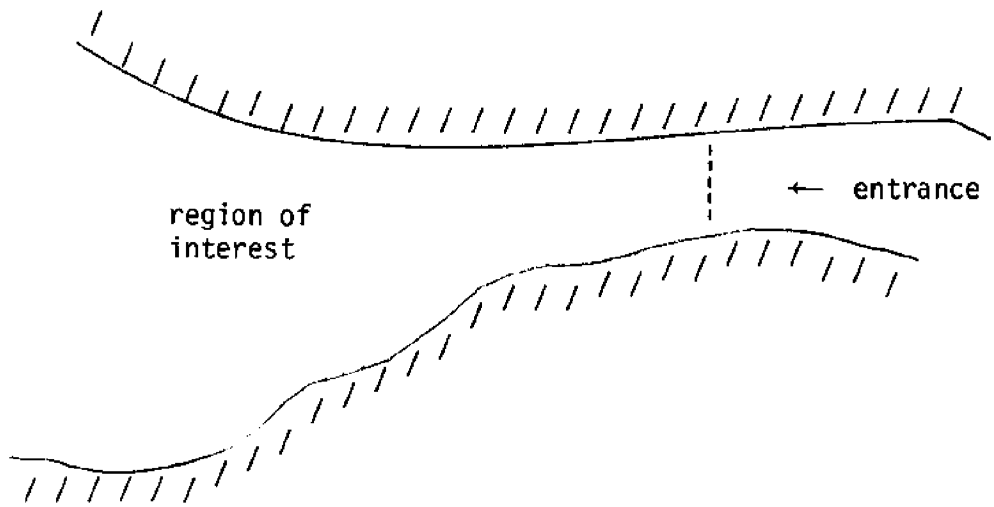

Figure (6)

Here the geometry is quite different from that of Sec. (4), and it may be unduly laborious to include the seaward portion of the channel. One approach to this kind of situation would be to say that the "directed" entrance implies a low velocity transverse to the channel axis, which predisposes towards a Kelvin wave incidence. Analytical solutions are available for Kelvin wave reflection from the ends of channels of constant width and depth, but with variable end geometries [5]. An important result is that the net phase gradient across the entrance is not sensitive to the details of the reflection geometry, but depends only on the gross features of the basin; it follows that it is possible to estimate rather simply the appropriate phase gradient at the entrance and so provide the desired matching. For details, refer to [5].

6. CYLINDER SCATTERING VIA AN INTEGRAL EQUATION

There are situations in which an integral equation may be used to solve all or part of a tidal problem, and it is useful to obtain some experience concerning numerical methods and accuracies for such problems. In doing this, it is desirable to choose prototype problems of simple form, and, preferably, ones for which analytical solutions are available.

As a first problem of this kind, consider the scattering of a plane wave by a circular island; the water depth is given the constant value h_o. Referring to Figure (7), using polar coordinates as shown,

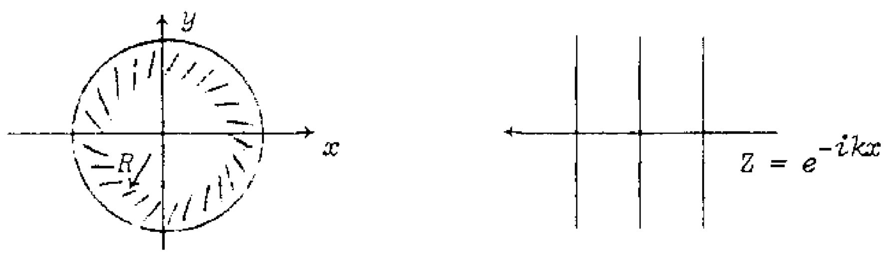

Figure (7)

and with $k^2 = (\omega^2 - f^2)/gh_o$ (we use the notation of Sec. (2)), we write $Z = Z_I + \phi(r,\theta)$ where

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r2}\phi_{\theta\theta} + k^2\phi = 0 \tag{25}$$

The boundary condition is that

$$i\omega\phi_r - \frac{f}{r}\phi_\theta = (-\omega k \cos\theta + ifk \sin\theta)\, e^{-ikr} \cos \tag{26}$$

for $r = R$. The exact solution is expressible in series form as

$$\phi = A_o H_o^{(1)}(kr) + \sum_1^\infty (A_n e^{in\theta} + B_n e^{-in\theta}) H_n^{(1)}(kr) \tag{27}$$

Insertion of the boundary conditions permits the coefficients A_j and B_j to be determined; the process is straightforward, and the final result for the scattered wave is

$$\phi - \frac{J_o'(kR)}{H_o^{(1)}{}'(kR)} H_o^{(1)}(kr) + \sum_{n=1}^\infty \left(\frac{-i\omega kR J_n'(kR) + inf\, J_n(kR)}{i\omega kR\, H_n^{(1)}{}'(kR) - inf\, H_n^{(1)}(kR)} \right. e^{in\theta}$$

$$+ \frac{-i\omega kR\, J_n'(kR) - inf\, J_n(kR)}{i\omega kR\, H_n^{(1)\prime}(kR) + inf\, H_n \quad (kR)}\, e^{-in\theta} \Bigg]\, e^{-i\frac{\pi}{2}n}\, H_n^{(1)}(kr) \tag{28}$$

Here the derivatives of the various Bessel functions can be simplified by use of the usual recursion formulas, if desired.

To solve this same problem by an integral equation, we can proceed as follows. In Fig. (8), let there be a source at Q, and consider an observation point at P (there is no difficulty in permitting the island to be non-circular, and we do so; for the present, however, we do not permit "corners").

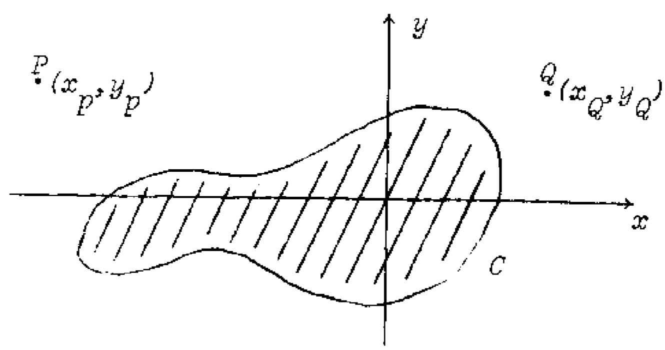

Figure (8)

Consider now the problem of finding a function ψ satisfying

$$\phi_{xx} + \phi_{yy} + k^2\psi = A\ \delta(x - x_Q)\delta(y - y_Q) \tag{29}$$

with

$$-i\omega\ \psi_n + f\ \psi_s = 0 \qquad \text{on } C \tag{30}$$

where the subscripts n and s refer to partial derivatives in the normal and tangential directions, as indicated in Fig. (8). Define a Green's function G by

$$G(x,y;x_p,y_p) = -\frac{i}{4} H_o^{(1)}(kr_G) \tag{31}$$

where $r_G^2 = (x - x_p)^2 + (y - y_p)^2$. Then

$$G_{xx} + G_{yy} + k^2 G = \delta(x - x_p)\delta(y - y_p) \quad . \tag{32}$$

If we multiply Eq. (29) by G, Eq. (32) by ψ, subtract, and integrate over the region exterior to the island, then use of the divergence theorem and the radiation condition leads readily to the result

$$\psi(P) = A\ G(Q) + \int_C (G\psi_n - \psi G_n) ds \tag{33}$$

where $\psi(P) = \psi(x_p, y_p)$ and $G(Q) = G(x_Q, y_Q; x_p, y_p)$. Using Eq. (30) gives

$$\psi(P) = A\ G(Q) - \int_C (\frac{if}{\omega} \psi_s G + \psi G_n) ds \tag{34}$$

Observe that the quantity $AG(Q)$ is equal to $AG(x_Q, y_Q; x_p, y_p)$ $= AG(x_p, y_p; x_Q, y_Q)$ and so is the value of the field ψ if the island is removed. It follows that if the field ψ is produced by a number of sources, or by a source at ∞ (i.e., a plane wave), then a more general version of Eq. (34) is

$$\psi(P) = \psi_o(P) - \int_C (\frac{if}{\omega} \psi_s G + \psi G_n) ds \tag{35}$$

where ψ_o is the value of ψ with no island.

Equation (35) will yield $\psi(P)$ if ψ is known on C. The value of ψ on C must be obtained from the solution of an integral equation, which in turn is obtained by allowing P to approach C (Fig. (9)). Let ρ denote the local radius of curvature, and consider a point P_o near C as shown. Then

$$G(P, P_o) = -\frac{i}{4}\left[J_o(kr) + i\left\{\frac{2}{\pi}(\gamma + \ln\frac{kr}{2})J_o(kr) + \frac{1}{\pi}(\frac{kr}{2})^2 + \ldots\right\}\right] \tag{36}$$

where $r = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta_1}$, and $\frac{\partial r}{\partial r_2} = \frac{r_2 - r_1 \cos\theta_1}{r}$. It follows that the only contribution from $\partial G/\partial n$, near $\theta_1 = 0$, in Eq. (34), will

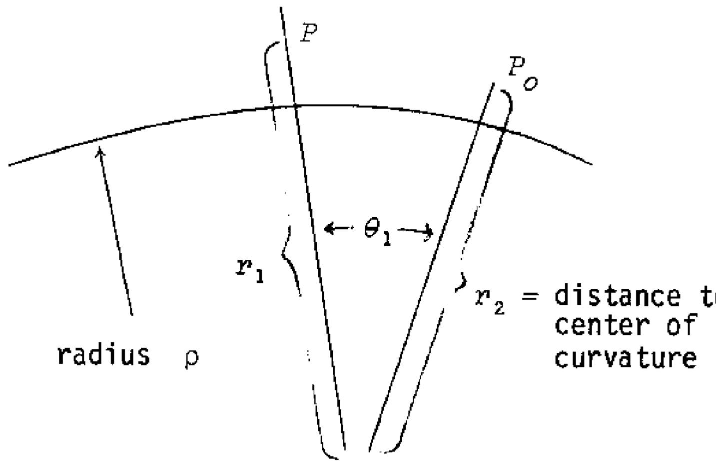

Figure (9)

arise in the form

$$-\frac{1}{2\pi}\int \frac{\rho - r_1 \cos\theta}{\rho^2 + r_1^2 - \rho r_1 \cos\theta}\,\psi ds \tag{37}$$

If $\theta_1 \neq 0$, the first factor in the integrand $\rightarrow \frac{1}{2\rho}$ as $r_1 \rightarrow \rho$; however, at $\theta = 0$, this first factor $\rightarrow \infty$ as $r_1 \rightarrow \rho$; we thus expect a δ-function behavior. Near $\theta_1 = 0$, write $\cos\theta_1 \simeq 1 - \frac{1}{2}\theta_1^2$, and set $r_1 = \rho(1 + \varepsilon)$. The first factor is then approximated by

$$\frac{1}{2\rho} - \frac{\varepsilon}{\rho(\varepsilon^2 + \theta^2)} \rightarrow \frac{1}{2\rho} - \frac{\pi}{\rho}\,\delta(\theta)$$

Consequently, the contribution from a ds interval centered on P, where P is now a point on C, becomes

$$\psi(P)\left[\frac{1}{2} - \frac{ds}{4\pi\rho}\right] \quad . \tag{38}$$

Thus the desired integral equation, in mesh-point form (c.f., Fig. (10)) is

$$\left(\frac{1}{2} + \frac{ds}{4\pi\rho_m}\right)\psi_m = \psi_{om} - \sum_{j\neq m}\left\{\frac{if}{\omega}\,G_{mj}(\psi_s)_j - \frac{ki\,ds}{4}\,\psi_m H_1^{(1)}(kr_{mj})\frac{(x_j - x_m)n_1 + (y_j - y_m)n_2}{r_{mj}}\right\}$$

(39)

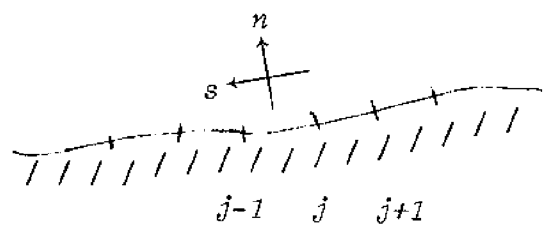

Figure (10)

where n_1 and n_2 are the x- and y-components of the outward unit normal vector, and where the remainder of the notation is clear from the context.

Several numerical experiments were carried out for the case of plane-wave scattering by a circular cylinder, with $f = 0$. As a typical result, let $\rho k = 2\pi\rho/\lambda = 1$ (where λ is wave length); this might be considered intermediate wave length scattering. With 25 mesh points around the boundary, the values of ψ on the boundary obtained by solving Eq. (39) agreed to within better than four significant figures with those obtained from Eq. (28); using these boundary values of ψ, the values of ψ outside C (out to about 30 island radii) also agreed with the exact results to about four significant figures. If the term $ds/(4\pi\rho_n)$ is omitted from Eq. (39), agreement to only two significant figures is obtained.

In general, adequate accuracy was obtained if the mesh point spacing is small compared to both the incident wave-length and the length scale of the island.

A perspective plot of wave configuration, at various values of time, was obtained by use of the Los Alamos PICTURE program and the CALCOMP plotter; Figs. (11a) through (11k) give the results (the time difference corresponded to ωt intervals of $2\pi/10$). Figure (12) depicts a magnified plot of the wave pattern, near the island, for $\omega t = 4$.

7. WEDGE SCATTERING VIA AN INTEGRAL EQUATION

In some cases, it may be useful to consider an exterior region of wedge-shape rather than of half-plane shape; consequently it is of interest to consider both exact and integral equation solutions for such regions.

In terms of polar coordinates (r,θ), let the region exterior to a wedge be defined by $\Omega < \theta < 2\pi - \Omega$, as shown in Fig. (13). The total

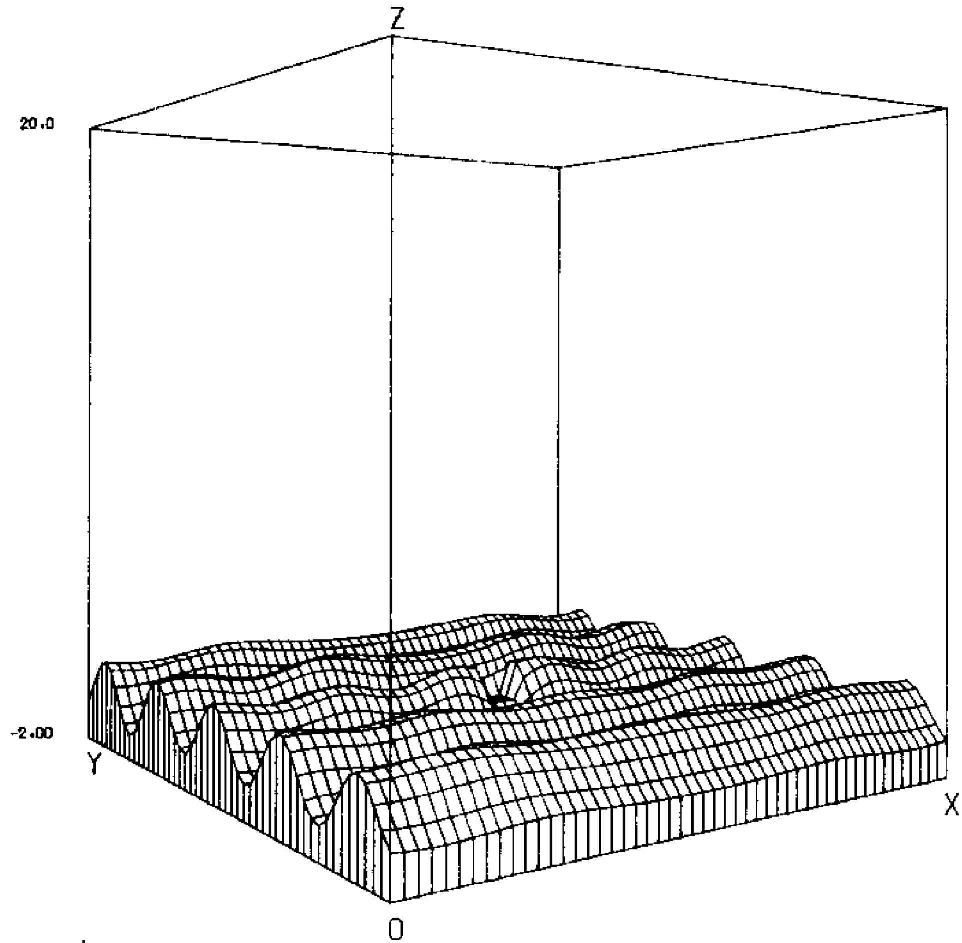

Figure (11a)

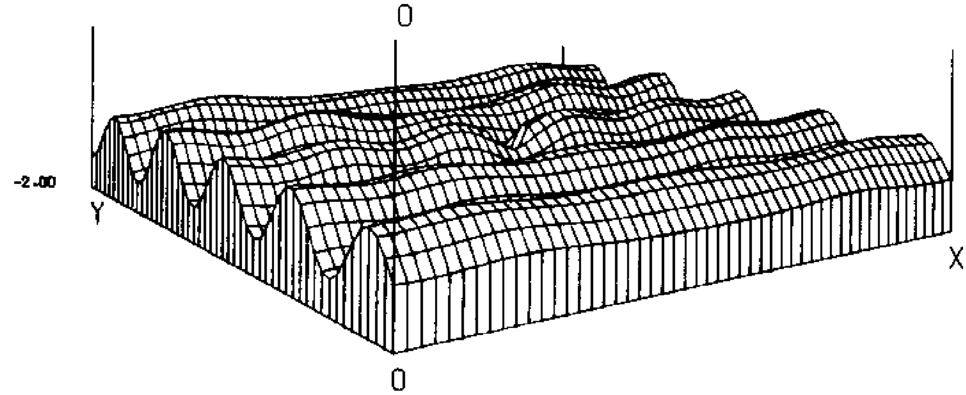

Figure (11b)

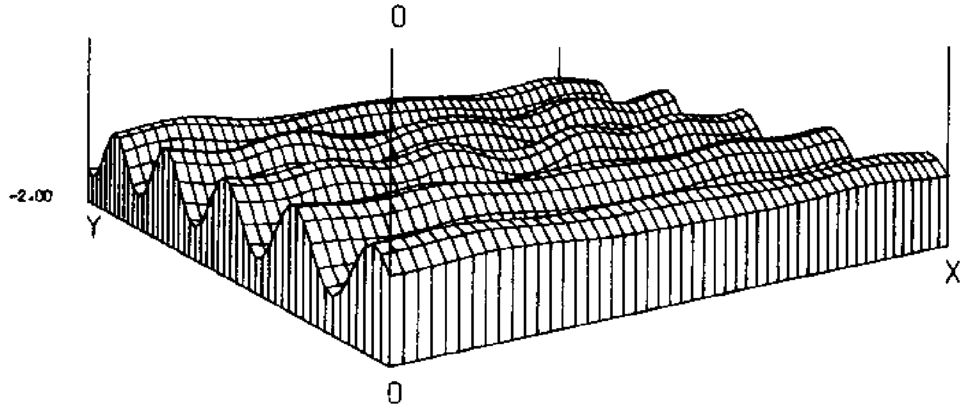

Figure (11c)

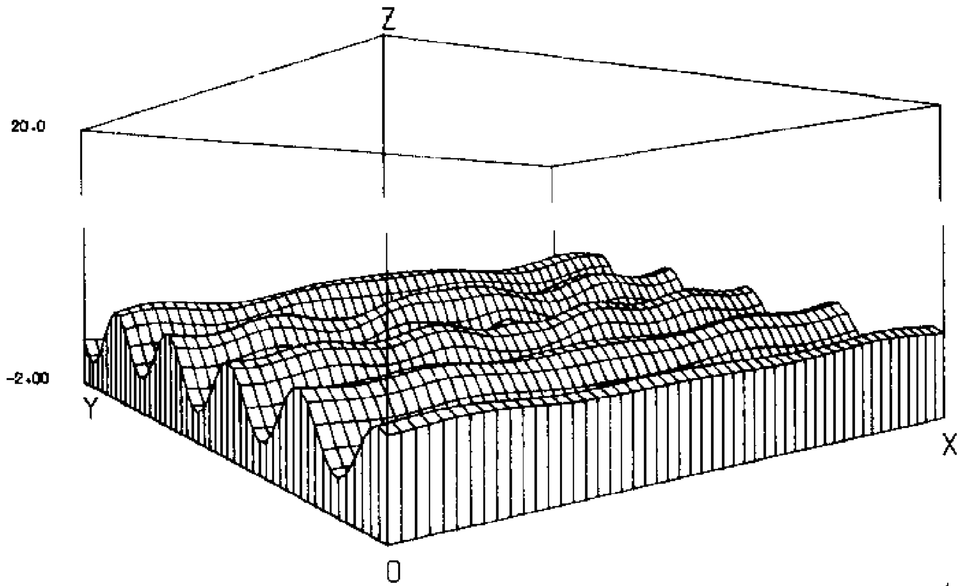

Figure (11d)

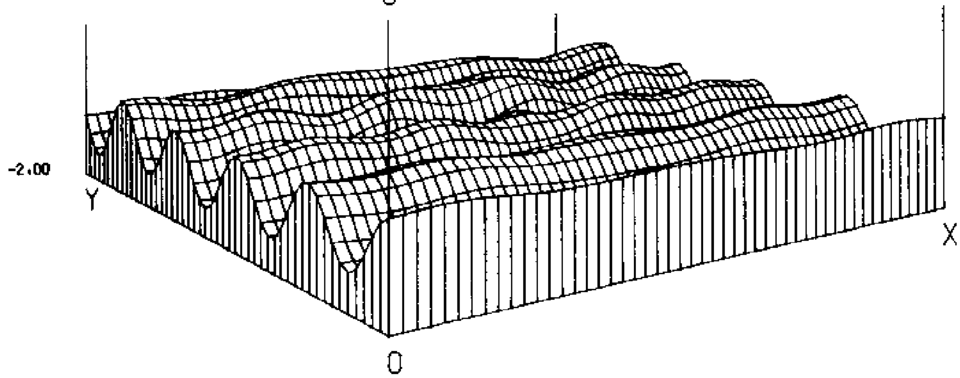

Figure (11e)

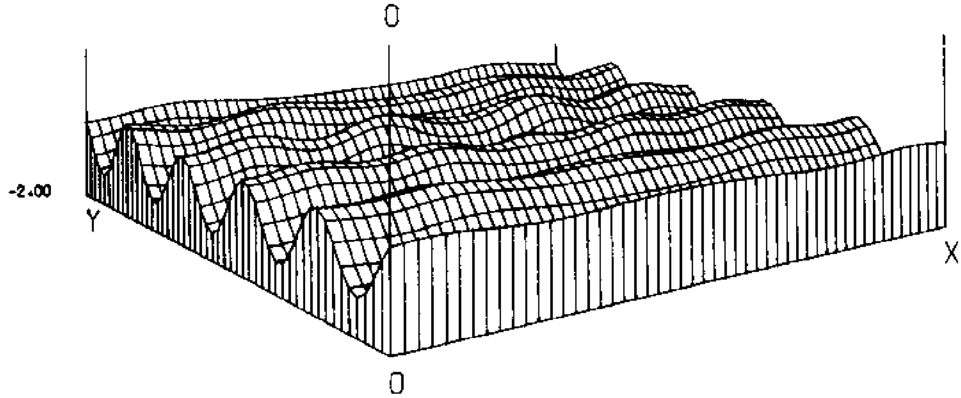

Figure (11f)

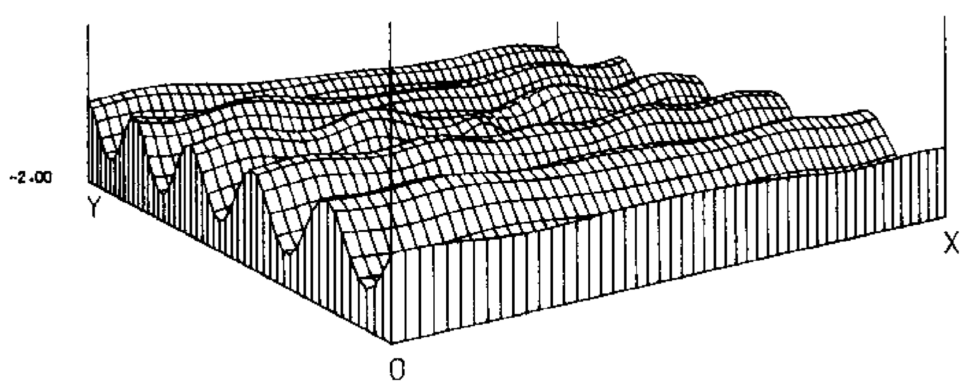

Figure (11g)

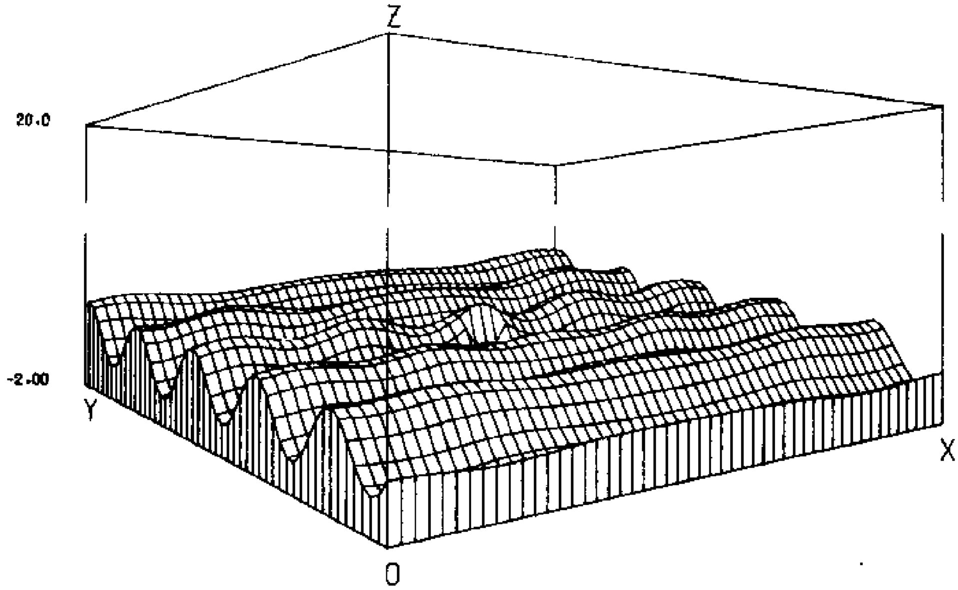

Figure (11h)

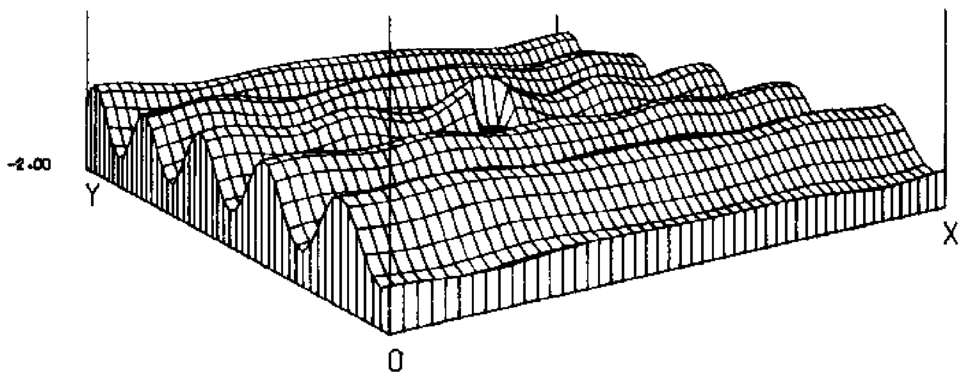

Figure (11i)

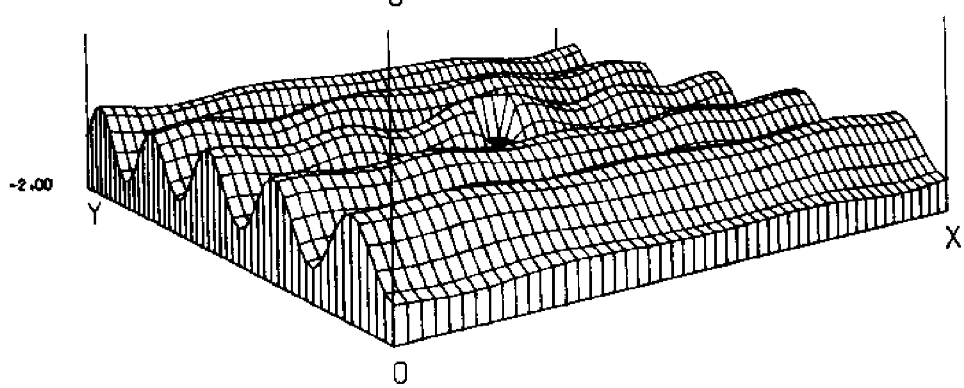

Figure (11j)

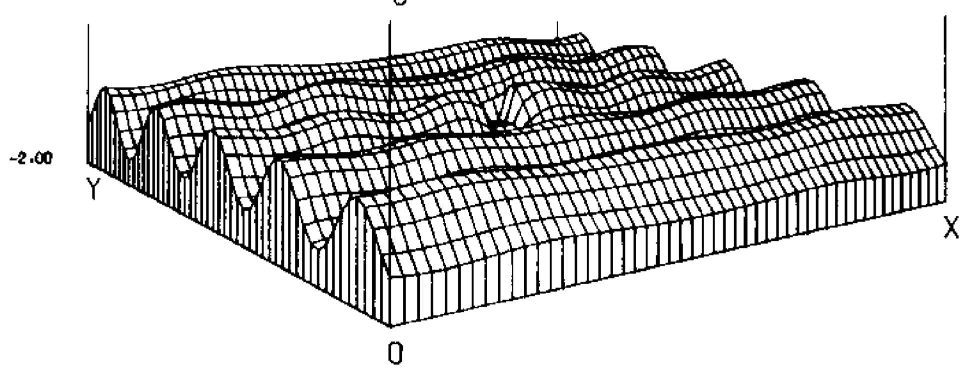

Figure (11k)

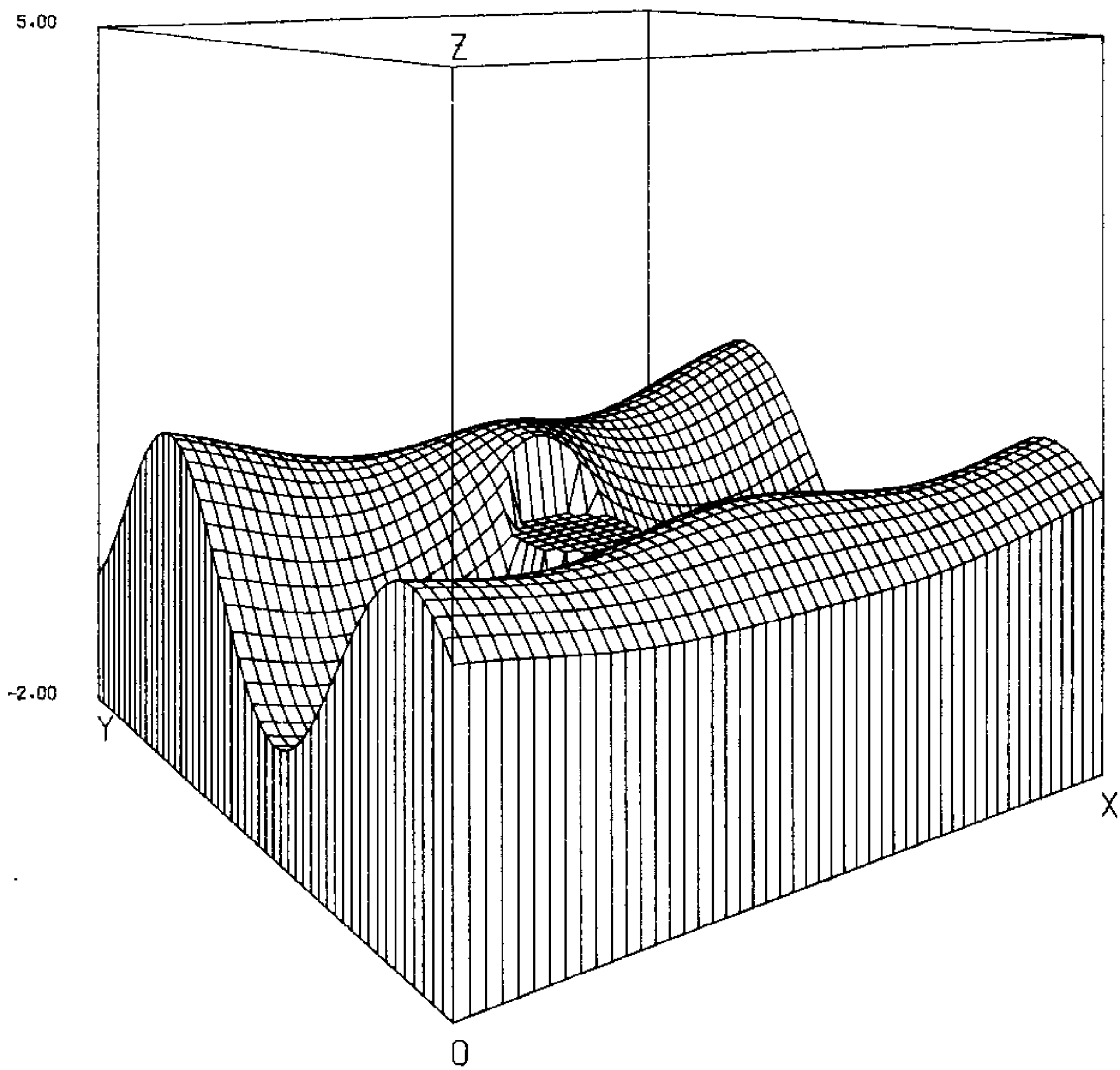

Figure (12)

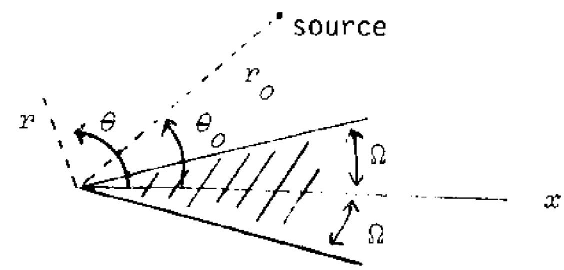

Figure (13)

exterior angle is $\alpha = 2\pi - 2\Omega$. Let there be a source of strength P at (r_o, θ_o), and let $Z(r,\theta)$ satisfy the reduced wave equation

$$Z_{rr} + \frac{1}{r} Z_r + \frac{1}{r^2} Z_{\theta\theta} + k^2 Z = \frac{P}{r_o} \delta(r - r_o)\delta(\theta - \theta_o) \tag{40}$$

with

$$Z_\theta = 0 \qquad \text{for } \theta = \Omega,\ 2\pi - \Omega \quad . \tag{41}$$

(To permit a simple series solution, Coriolis effects are not included in this example.) Write

$$Z = \frac{1}{2} \beta_o(r) + \sum_1^\infty \beta_n(r) \cos \frac{n\pi(\theta - \Omega)}{\alpha} \tag{42}$$

and it follows that

$$\beta_n'' + \frac{1}{r} \beta_n - \frac{n^2\pi^2}{\alpha^2 r^2} \beta_n + k^2\beta_n = \frac{P}{r_o} \delta(r - r_o) \cdot \frac{2}{\alpha} \cos \frac{n\pi}{\alpha} (\theta_o - \Omega)$$

whence, for $r < r_o$,

$$\beta_n = -iP \frac{\pi}{\alpha} \cos \frac{n\pi}{\alpha} (\theta_o - \Omega)\, H^{(1)}_{n\pi/\alpha}(kr_o)\, J_{n\pi/\alpha}(kr) \tag{43}$$

In combination with Eq. (42), Eq. (43) provides the field due to wedge scattering for a concentrated source at a finite distance. If r_o is permitted to approach ∞, with P set equal to $4i\sqrt{\pi k r_o/2}\, e^{i\pi/4 - ikr_o}$, the result will yield the solution for scattering due to a plane wave Z_I

where

$$Z_I = e^{-ikr\cos(\theta - \theta_o)} \tag{44}$$

A straightforward calculation gives MacDonald's solution (1902):

$$Z = \frac{2\pi}{\alpha} \sum_0^\infty \varepsilon_m e^{-i\frac{m\pi^2}{2\alpha}} \cos\frac{m\pi}{\alpha}(\theta_o - \Omega) \cos\frac{m\pi}{\alpha}(\theta - \Omega) J_{m\pi/\alpha}(kr) \tag{45}$$

where $\varepsilon_o = 1$, $\varepsilon_j = 2$ for $j \neq 0$. A number of special solutions for similar problems, with different boundary conditions, are given in Ch. 6 of [6].

For small r,

$$Z \sim \frac{2\pi}{\alpha} + \frac{4}{\Gamma(\pi/\alpha)} \left(\frac{rk}{2}\right)^{\pi/\alpha} e^{-i\frac{\pi^2}{2\alpha}} \cos\frac{\pi}{\alpha}(\theta_o - \Omega) \cos\frac{\pi}{\alpha}(\theta - \Omega) \tag{46}$$

and, in particular,

$$Z(0,\theta) = \frac{2\pi}{\alpha} \tag{47}$$

In Eq. (45), the Bessel functions may be expressed in terms of contour integrals involving exponential functions. If the orders of integration and summation are interchanged, we obtain two useful formulas--one in terms of the familiar Sommerfeld contours, and one based on Schäfli's integral:

$$Z = \frac{1}{4i\alpha}\int_{C_1+C_2} e^{ikr\cos t}\left\{\cot\frac{\pi}{2\alpha}(\pi - t - \theta + \theta_o) + \cot\frac{\pi}{2\alpha}(\pi - t - \theta - \theta_o + \Omega)\right\}dt \tag{48}$$

where C_1 and C_2 are shown in Fig. (14), and

$$Z = \frac{1}{2i\alpha}\int_{-\infty}^{(0+)} \left[2 + \frac{e^{i\frac{\pi}{\alpha}(-\frac{\pi}{2} + \theta_o + \theta - \Omega)}}{t^{\frac{\pi}{\alpha}} - e^{i\frac{\pi}{\alpha}(-\frac{\pi}{2} + \theta_o + \theta - \Omega)}} + \frac{e^{i\frac{\pi}{\alpha}(-\frac{\pi}{2} + \theta_o - \theta)}}{t^{\frac{\pi}{\alpha}} - e^{i\frac{\pi}{\alpha}(-\frac{\pi}{2} + \theta_o - \theta)}} + \frac{e^{i\frac{\pi}{\alpha}(-\frac{\pi}{2} - \theta_o + \theta)}}{t^{\frac{\pi}{\alpha}} - e^{i\frac{\pi}{\alpha}(-\frac{\pi}{2} - \theta_o + \theta)}} + \frac{e^{i\frac{\pi}{\alpha}(-\frac{\pi}{2} - \theta_o - \theta + 2\Omega)}}{t^{\frac{\pi}{\alpha}} - e^{i\frac{\pi}{\alpha}(-\frac{\pi}{2} - \theta_o - \theta + 2\Omega)}}\right] \frac{e^{\frac{1}{2}kr(t - \frac{1}{t})}}{t}\,dt \tag{49}$$

where the contour is shown in Fig. (15).

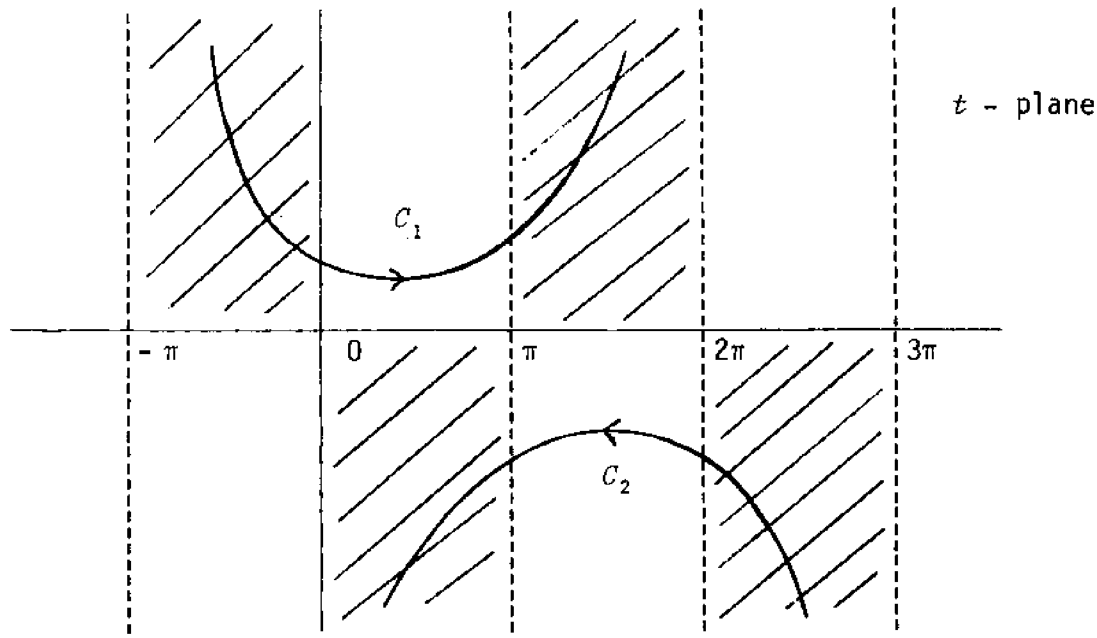

Figure (14)

Contours C_1 and C_2 for Eq. (48)

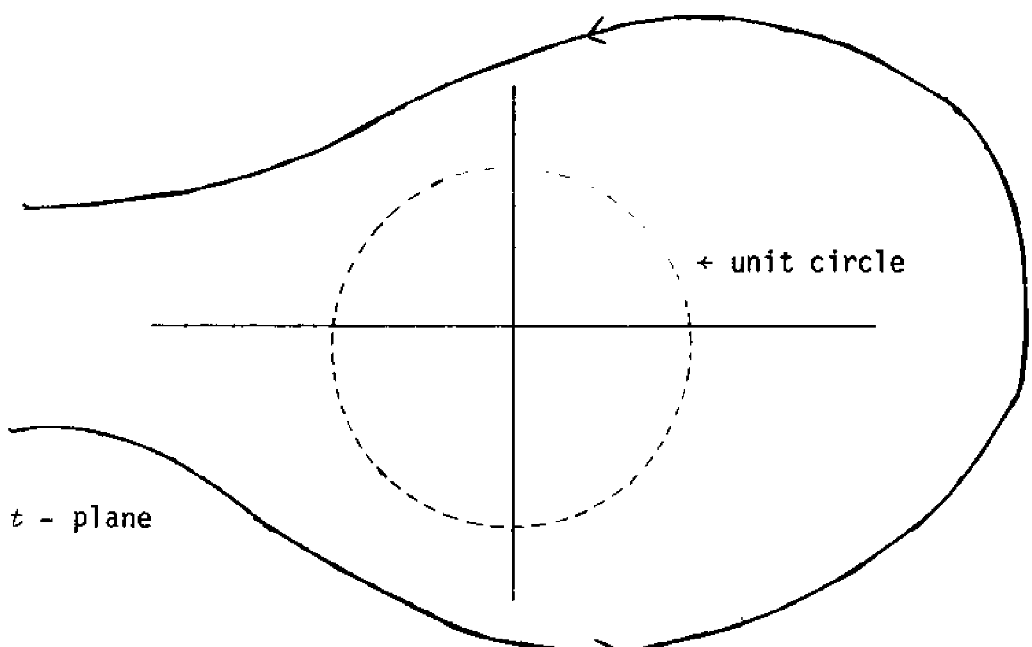

Figure (15)
Contour for Eq. (49)

The integrand in Eq. (49) has poles which lie on the unit circle shown in Fig. (15); moreover there are saddle points at $t = \pm i$. If the contour is deformed so as to pass through these saddle points, the residues at the poles provide reflected waves, and the saddle point contributions (only the one at $t = i$ is of importance, since the factor in square brackets vanishes at $t = -i$) provide diffracted waves. Calculation shows that the diffracted field is of the form of a cylindrical wave emanating from the corner; for $kr \gg$,

$$Z_{diff} \sim -\sqrt{\frac{2\pi}{kr}}\, \frac{e^{i(\frac{\pi}{4}+kr)}}{2\alpha} \sin\frac{\pi^2}{\alpha} \left\{ \frac{1}{\cos\frac{\pi}{\alpha}(\theta+\theta_o) - \cos\frac{\pi^2}{\alpha}} + \frac{1}{\cos\frac{\pi}{\alpha}(\theta+\theta_o - 2\Omega) - \cos\frac{\pi^2}{\alpha}} \right\} \qquad (50)$$

The same result can be obtained by manipulation of the Sommerfeld contours of Fig. (14) into those of Fig. (16). The inner contour is traversed twice in opposite directions; the geometric optics field is obtained from the residues arising from poles between $t = 0$ and $t = \pi$.

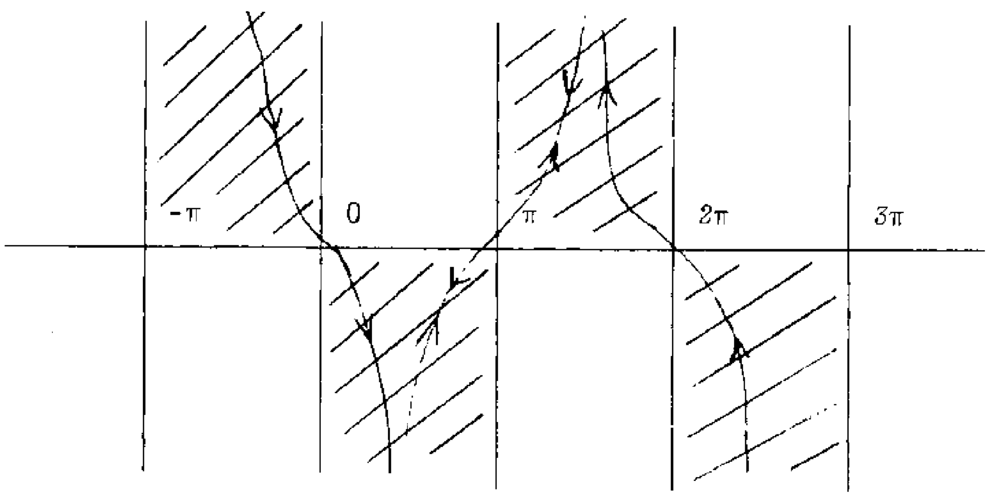

Figure (16)

For directions along the boundary of geometrical optics regions, poles and saddle points may be close to one another or may even coalesce. It is then necessary to revise the standard saddle point calculation appropriately [7]; the details are rather technical, and we will here give only one particular result. Let $\Omega = \pi/4$, $\theta_o = \pi/2$, and take $\theta = \pi$, which corresponds to the boundary of the geometric reflection region from the upper face. Then a rather tedious calculation leads to

$$Z \sim \frac{-2}{3\sqrt{3}} \cdot \frac{1}{\sqrt{2\pi kr}} e^{i(kr + \frac{\pi}{4})} + \frac{1}{2} e^{ikr} + 1 \tag{51}$$

where it is of interest to note that the last term represents a standing wave.

Turning now to an integral equation formulation, that of Sec. (6) is seen to be directly applicable, since no restriction is there placed on the physical boundary. A value of $\alpha = \frac{5}{6}\pi$ (i.e., $\Omega = \frac{7}{12}\pi$) was chosen for computer experimentation, with $\theta = \frac{11}{12}\pi$ (so that the incoming plane wave is perpendicular to one face of the wedge). The geometry is shown in Fig. (17).

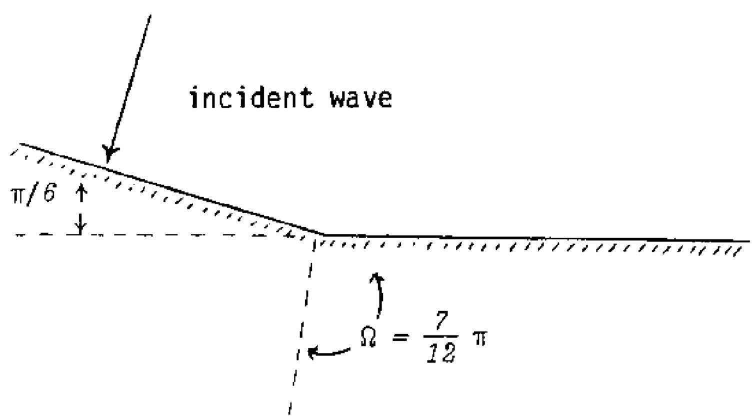

Figure (17)

To solve the integral equation numerically, a number of equally spaced mesh points were inserted along the boundary, half on either side of the apex. (Outside of these mesh points, the value of Z was taken as that which would have existed in the absence of the other face of the wedge.)

A number of linear algebraic equations resulted, which were solved by elimination, and the results were compared with the exact solution (for large r, an asymptotic representation, as previously explained, was used), for values of θ corresponding to the two faces. To obtain good agreement, it was necessary to use a large number of mesh points (note, by the way, that the problem is essentially independent of wave length), and this raised awkward problems of large-matrix storage. Consequently, elimination as a solution method was replaced by iteration (no matrix storage is then necessary), and adequate accuracy was obtained with 200 mesh points on each side, 10 per wave length interval. For $\theta = \Omega$, typical results are:

r	Z, exact	Z, integral equation
10	2.0449 - i .0449	2.0438 - i .03381
10.1	1.9120 + i .5556	1.9170 + i .5654
10.2	1.5895 + i 1.1195	1.5993 + i 1.1246
10.3	1.1198 + i 1.5896	1.1308 + i 1.5884
10.4	.5565 + i 1.9118	.5650 + i 1.9047
10.5	- .04382 + i 2.0438	- .0407 + i 2.033
10.6	- .6277 + i 1.9630	- .6311 + i 1.9524
10.7	-1.1477 + i 1.6727	-1.1565 + i 1.666
10.8	-1.5636 + i 1.2033	-1.5747 + i 1.2026
10.9	-1.8420 + i .6085	-1.8517 + i .6140
11	-1.9572 - i .0428	-1.9621 - i .0328

Consider finally the possibility of matching, for a wedge-shaped outer region, using a method analogous to that of Sec. (4). For analogous generality, we would want the depth to depend on θ, and Coriolis effects should be included. Unfortunately, the resulting problem is difficult to solve in terms of reasonably simple analytical expressions (for reference, Eqs. (3) are replaced by

$$U = \frac{g}{\omega^2 - f^2}\left(-i\omega\, Z_r + \frac{f}{r}\, Z_\theta\right)$$

$$V = \frac{g}{\omega^2 - f^2}\left(-\frac{i\omega}{r}\, Z_\theta - f\, Z_r\right) \qquad (52)$$

$$[d(Z_r + \frac{if}{\omega r} Z_\theta)]_r + \frac{d}{r}(Z_r + \frac{if}{\omega r} Z_\theta) + \frac{1}{r}[d(\frac{1}{r} Z_\theta - \frac{if}{\omega} Z_r)]_\theta + k_o^2 Z = 0$$

and Eq. (4) is replaced by

$$\frac{i\omega}{r} Z_\theta + fZ_r = 0 \tag{53}$$

at $\theta = \Omega,\ 2\pi - \Omega$). Thus for practical purposes it appears that, if a compact series representation is desired for matching purposes, it is appropriate to restrict attention to the case in which h can be taken as constant and f neglected outside the region of interest. We are then dealing with the series expansion (42), where matching with an inner solution across a constant-radius curve $r = R$ is desired; the solution can be expressed in terms of Z_r on this line and the method of Eq. (4) applied directly.

8. INTEGRAL EQUATIONS AND NON-UNIFORM DEPTH

Let the region of interest be bounded by a material boundary C and a boundary C_∞ at ∞, and let there be a source at a point Q. A typical situation is shown in Fig. (18):

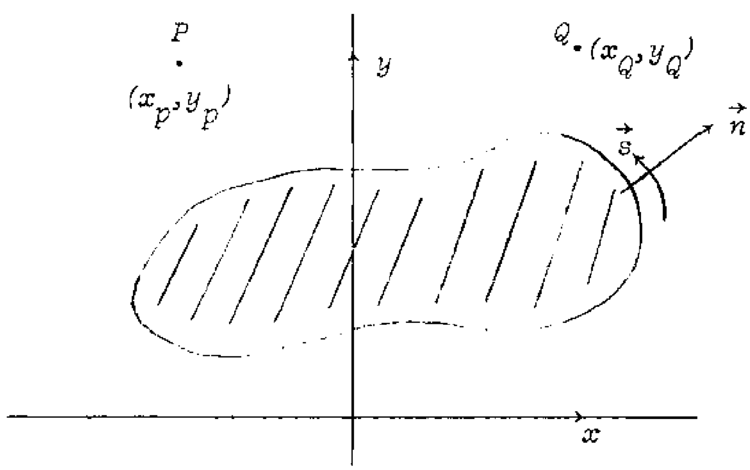

Figure (18)

The governing equation is obtained by incorporating a source term in Eq. (3):

$$[d(Z_x + \frac{if}{\omega} Z_y)]_x + [d(Z_y - \frac{if}{\omega} Z_x)]_y + k_o^2 Z = B\delta(x - x_Q)\delta(y - y_Q) \quad (54)$$

where (x_Q, y_Q) are the coordinates of the source point, and B is a (complex) strength factor. As before, k_o^2 is based on the reference depth h_o, viz; $k_o^2 = (\omega^2 - f^2)/(gh_o)$, and $d = h/h_o$. At ∞, we take $d = 1$. We also need a Green's function, and as in Eqs. (31) and 32) we require

$$G_{xx} + G_{yy} + k_o^2 G = \delta(x - x_p)\delta(y - y_p) \quad (55)$$

where (x_p, y_p) are the coordinates of the observation point P.

Multiply Eq. (54) by G and integrate over the region R contained between C and C_∞, using the divergence theorem and Eq. (55) to obtain

$$\int_{C+C_\infty} \left[d(G \frac{\partial Z}{\partial n'} - Z \frac{\partial G}{\partial n'}) + \frac{if}{\omega} GZ(d_x n_2' - d_y n_1')\right] ds$$

$$+ \int_R \left[k_o^2 ZG(1 - d) + \frac{if}{\omega} Z(G_x d_y - G_y d_x) + Z(G_x d_x + G_y d_y)\right] dA$$

$$+ d(P)\ Z(P) = B\ G(Q) \quad (56)$$

where ds is the element of arc length and dA the element of area. Here n_1', n_2' are the components of the unit outward normal vector from R, and $\partial/\partial n'$ is the derivative in that outward normal direction. The radiation condition and the fact that $d \sim 1$ together imply that the integral on C_∞ vanishes.

It is usually more convenient to choose the positive normal derivative $\partial/\partial n$ as outward from the solid boundary. If this is done, if the tangential derivative direction $\partial/\partial s$ is taken as shown in Fig. (18), and if the no-flow condition $\partial Z/\partial n = -(if/\omega)\partial Z/\partial s$ is used, Eq. (56) becomes

$$d(P)\ Z(P) + \int_{R'} \left[k_o ZG(1-d) + \frac{if}{\omega} Z(G_x d_y - G_y d_x) + Z(G_x d_x + G_y d_y)\right] dA$$

$$= BG(Q) - \int_C \left[d\left(\frac{if}{\omega} G \frac{\partial Z}{\partial s} + Z \frac{\partial G}{\partial n}\right) + \frac{if}{\omega} GZ \frac{\partial d}{\partial s}\right] ds \qquad (57)$$

We have also replaced the region R of integration by that sub-region of R in which $d \neq 1$ (remember that $d \to 1$ at ∞), since only in R' is the area integral non-zero. If Z were known on C and in R', Eq. (57) would permit its determination everywhere; to find Z on C and in R', we have to solve an integral equation, using as unknowns the values of Z on mesh points in R' and on C (on C, we need of course the limiting form of Eq. (57), obtained as in Sec. (6)). As a special case, note that d could be zero along part or all of C; if zero along all of C, no limiting form of Eq. (57) as $P \to C$ need be considered.

A special case of what is essentially Eq. (57) has been solved by Lautenbacher [8], who takes $f = 0$, sets $d = 0$ on C, and considers plane wave scattering by circular or elliptical islands. The derivation in [8] is somewhat different, in that Z is treated as the difference between the plane wave input function and the actual solution; the effect is to replace the right-hand side of Eq. (54) by the result of applying the differential operator to the plane wave. We note however that it would appear simpler to use the above formulation and permit $Q \to \infty$, with B increasing so as to yield the derived plane wave input (measured in regions where $d \sim 1$).

The integral equation based on Eq. (57) could presumably be applied directly to the case of present interest, in which C is a coast-line of indefinite extent, indented by an estuary (if there is river flow across a portion of C, an appropriate minor change must be made in Eq. (57)). Based on previous remarks, it appears that iteration rather than elimination would be the solution method of choice in this integral equation. If a portion of the region is shallow, non-linear terms could be important, and this might require a modal analysis, with iteration, of the kind used in the finite element approach of ref. [1].

Although this direct integral equation approach appears quite feasible, it is not clear that there would be advantages of either speed or accuracy over the finite element method of ref. [1]. However, the use of this method for an outer region (where linearity is adequate), coupled with finite elements for an inner region (much as in Sec. (4)), seems promising, particularly if the (distant) coast-line differs importantly from a straight line. The idea would be to use Eq. (56) rather than Eq. (57), and to obtain from the integral equation a linear relationship between values of Z on the matching boundary and values of $\partial Z/\partial n$ there; similar relationships obtained for the finite element region would then permit matching (cf Sec. (6)). (As a simplification, it might be possible to set $d = 0$ on C; sufficiently far from the estuary.)

9. VARIATIONAL APPROACH TO MATCHING

To illustrate the basic idea, consider the problem of the scattering of an incident plane wave Z_I by an island (cf Fig. (19)). We use the notation of Sec. (2), and, as before, require $d \sim 1$ sufficiently far from the island and in particular outside a large circle C across which matching is required. Let the regions between the island (contour B) and C, and outside C, be denoted by R_1 and R_2 respectively. We write $Z = Z_I + \phi_1$ in R_1, and $Z = Z_I + \phi_2$ in R_2. On B, we require $d\, \partial Z/\partial n = 0$, so $d(\partial Z_I/\partial n + \partial \phi_1/\partial n) = 0$, where $\partial/\partial n$ denotes (here and in the sequel) the outward normal derivative from R_1. In this example, we neglect Coriolis effects, and set $f = 0$.

In R_2, we approximate ϕ_2 by the first $2N+1$ terms of the series (27):

$$\phi_2 \simeq A_o H_o^{(1)}(k_o r) + \sum_1^N (A_n e^{in\theta} + B_n e^{-in\theta})\, H_n^{(1)}(k_o r) \qquad (58)$$

and we consider the coefficients $A_o, A_1, B_1, \ldots, A_N, B_N$ to be among the unknowns. Note that whatever values are chosen for those coefficients, ϕ_2 is a solution of the wave equation outside C. In R_1, ϕ_1 must satisfy the equation (cf Eq. (3)).

$$(d\phi_{1x})_x + (d\phi_{1y})_y + k_o^2\phi = -(dZ_{Ix})_x - (dZ_{Iy})_y - k_o^2 Z_I = \psi \tag{59}$$

say, where ψ is known. Construct now the variational integral

$$I = \int_{R_1} \left\{\frac{1}{2} d\left[\phi_{1x}^2 + \phi_{1y}^2\right] - \frac{1}{2} k_o^2\phi_1^2 + \psi\phi_1\right\} dA + \int_B d \frac{\partial Z_I}{\partial n} \phi_1 ds$$

Then

$$\delta I = \int_{R_1} \left[-\left\{(d\phi_{1x})_x + (d\phi_{1y})_y + k_o^2\phi_1\right\} + \psi\right] \delta\phi_1 \, dA$$

$$+ \int_B \alpha\left(\frac{\partial\phi_1}{\partial n} + \frac{\partial Z_I}{\partial n}\right) \delta\phi_1 ds + \int_C \frac{\partial\phi_1}{\partial r} \delta\phi_1 ds \tag{60}$$

where we have used the facts that $d = 1$ on C, and that $\partial/\partial n = \partial/\partial r$ on C. Across C, we want $\partial\phi_1/\partial r = \partial\phi_2/\partial r$, and $\phi_1 = \phi_2$. Define therefore

$$J = \int_C \frac{\partial\phi_2}{\partial r} \phi_1 \, ds$$

so that

$$\delta J = \int_C \left[\frac{\partial\delta\phi_2}{\partial} \phi_1 + \frac{\partial\phi_2}{\partial r} \delta\phi_1\right] ds \tag{61}$$

and consider finally the requirement

$$\delta I - \delta J + \int_C \phi_1 \frac{\partial}{\partial r} \delta\phi_2 \, ds = 0 \tag{62}$$

for all $\delta\phi_1$, and for all $\delta\phi_2$ obtainable by varying the A_j and B_j coefficients in Eq. (58). It is clear that all requirements of the problem will be satisfied, within the degree of approximation afforded by Eq. (58).

If finite elements are used inside C, then we will (for linear elements, say) obtain one equation for each nodal unknown, and also one equation for each of the A_j and B_j, so that the total number of equations equals the number of unknowns. Both non-linearities and Coriolis effects could be incorporated, at least by iteration. (For large systems, it may well be more efficient in terms of storage and speed to solve the linear equation set iteratively also.) For regions more general than that considered in this example, one needs an approximate analytical solution outside the finite element portion, and this may not be readily available in, say, wedge-shaped regions with Coriolis effects included. Consequently this approach, although effective when applicable, may be of limited usefulness.

10. BIBLIOGRAPHY

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