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NONLINEAR RESPONSE OF MARINE VEHICLES TO STOCHASTIC SIGNALS: A REVIEW

by

C. DUTHOIT

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INTRODUCTION

The design of ocean structures, similar to that of ships, is for the most part based on empirical rules and codes, and relies heavily on past experience. Although such experience may be extensive in the case of ships, the design of which is the result of centuries of mostly careful and slow evolution, it is necessarily limited for ocean structures, as exemplified by various recent failures. The need for a more rationally-based design procedure for ocean structures is now well recognized.

Ocean structures operate in an environment which is random in nature. The development and implementation of probabilistic models for predicting the loads acting on an ocean structure have been the subject of extensive research in recent years. On the other hand, it is the stochastic prediction of the response of an ocean structure to environmental loads which provides the significant information necessary for its rational design; this area has yet to be fully investigated.

Whereas linear system theory is a well developed body of knowledge, the application of which is relatively straightforward, the severe limitations of linear models are now recognized in many situations involving ocean structures.

Nonlinearities play an important role in the design of moored floating structures. In particular, the response to loads in unusual or extreme conditions, which constitutes an essential part of the design process, is essentially governed by nonlinear effects.

The aim of this report is to review the currently available methods for predicting the response of nonlinear dynamic systems to stochastic excitations, together with their relative advantages and limitations, and particular reference to ocean engineering applications

The problem may, in general, be stated as follows. Let us consider a nonlinear dynamic system described by the following input-output relationship[1]:

$$\mathcal{F}\{y(t)\} = \mathbf{x}(t) \tag{0.1}$$

where $\mathbf{x}(t)$ and $y(t)$ are the excitation and response stochastic processes, and $\mathcal{F}$ is a nonlinear autonomous (i.e. time-invariant) and generally causal functional, therefore:

$$\mathcal{F}\{y(t)\} = \mathcal{F}(y(\tau);\ \tau \le t) = \mathbf{x}(t) \tag{0.2}$$

It is required to derive some statistical properties of the response process $y(t)$ when the deterministic system $\mathcal{F}$ and stochastic excitation $\mathbf{x}(t)$ are given.

Such a general theory of the same scope as the linear theory, is not yet available, and thus progress toward a satisfactory stochastic theory of ship and platform motions has been rather slow.

[1]Upper-case Roman symbols denote scalar functions and operators, lower-case Roman symbols denote scalar variables, the same notations in bold are used for vectorial and matrix quantities, whereas calligraphic Roman symbols represent functionals.

Nonlinear stochastic modelling is a relatively new and difficult field, drawing on the latest advances in nonlinear system theory and stochastic processes. All the existing approaches are, in some way, limited in scope, by assuming particular properties of both the excitation (Gaussian, harmonic, white noise, etc.) and the system (time-invariance, memoryless, analytic, etc.); moreover, most of these techniques yield a limited description of the response (generally second-order properties or probabilistic structure).

1- EXACT METHODS

Historically, the mathematical approaches to nonlinear transformations of Gaussian processes evolved quite independently from two fundamental theories: the spectral analysis mostly in connection with signal processing in the field of communication, and the Fokker-Planck equation in the theory of Brownian motions. Both theories are reviewed below (sections 1-1 and 1-3). In addition, several fundamental theorems related to the behavior of memoryless systems are examined (section 1-2).

1-1- Spectral Analysis Theory: Linear Systems -

The theory governing the behavior of linear autonomous dynamic systems:

$$\mathcal{L}\{y(t)\} = x(t) \tag{1.1}$$

driven by stationary Gaussian signals is a well developed body of knowledge and we will not reproduce it here [Lin, 1967, Papoulis, 1984, Lin et al., 1986].

Let us simply state that the spectral analysis of linear systems originated during the first half of the century, and culminated with the much celebrated work of Rice [1944 & 1945].

The method is essentially based on the Wiener-Khintchine relations [Wiener, 1930] and the convolution theorem together with the assumption of stationarity of the response process. They yield relatively straightforward frequency domain expressions (power spectrum) of the second-order statistics of linear autonomous transformations of stationary Gaussian processes.

For future reference it is essential to emphasize that the response of such transformations may be entirely described by its mean value and second-order statistics. These second-order statistics can be evaluated in the frequency domain through the power spectrum or in the time domain through the autocorrelation function.

Statistical moments of order 1 and 2, however, do not sufficiently describe the behavior of random processes whenever deviation from normality is substantial, even if the transformation is linear. Furthermore, associated with this difficulty is the fact that nonlinear transformations, which are to be discussed below, do not preserve Gaussian character.

The application of linear spectral theory to describe the motions of ships in irregular waves appeared with the work of St. Denis and Pierson [1953]. Later, St. Denis [1973, 1974a, 1974b & 1975] and Yamanouchi [1974] discussed the limitations of linear models in various situations involving offshore platforms as well as ships.

1-2- Nonlinear Static Systems -

Although the theory of nonlinear system with memory remains our primary concern, it is instructive to examine the, quite extensive, litterature on static transformations in order to gain further insight into the difficulties associated with these memory effects.

Nonlinear static systems can be understood as nonlinear transformations without memory, therefore the response of such systems only depends upon present values of the excitation process:

$$F(y(t)) = x(t) \tag{1.2}$$

In the case of a Gaussian excitation, Price's [1958 & 1964, Baum, 1969] and Bussgang's [1952] theorems yields useful information on output second-order statistics. Whereas the case of a general excitation can be found in Abramson [1967]. Generalization of Price's theorem to the functional version of (1.2) can be found in Gorman and Zaborszky [1968]. While, the response probability distribution can be easily obtained by a simple transformation in most cases [Bendat, 1985].

A number of important contributions to the theory of memoryless transformations of stochastic processes are reproduced in Haddad [1975].

1-3- Fokker-Planck Equation: Nonlinear Dynamic Systems -

The origins of the Fokker-Planck equation are intimately linked to the theory of Brownian motions, named after an English botanist Robert Brown who observed in 1827 that small particles suspended in fluids undergo erratic movements.

The very first satisfactory statistical theory of Brownian motions appeared with Einstein in 1905 through the diffusion equation. However it was not until a decade later that the combined works of Smoluchowski, Fokker, Planck, and Ornstein among others lead to considerable generalization of Einstein's pioneering work: the Fokker-Planck equation. Whereas, further mathematical aspects of the theory were examined by

Wiener, Kolmogorov, and others [Uhlenbeck, and Ornstein, 1930, Ming Chen Wang, and Uhlenbeck, 1945]

Essentially, the method relies on the fact that the response of a discrete dynamic system subjected to a Gaussian white noise behaves as a continuous multidimensional Markov process. Then, it is possible to show that Markov processes must satisfy a consistency equation the Chapman-Kolmogorov or Smoluchowski equation:

$$p(y_3, t_3 \mid y_1, t_1) = \int_{-\infty}^{\infty} p(y_3, t_3 \mid y_2, t_2)\, p(y_2, t_2 \mid y_1, t_1)\, dy_2 \qquad t_1 < t_2 < t_3 \qquad (1.3)$$

The Smoluchowski equation, in turn, leads to the response transition probability density function $p(z, s \mid y, t)$, solution of a partial differential equation; the Fokker-Planck-Kolmogorov equation [Caughey, 1971, Ming Chen Wang and Uhlenbeck, 1945]:

$$\partial p/\partial t = -\sum_i \partial[A_i(t,y)p]/\partial y_i + \sum_i \sum_j \partial^2[B_{ij}(t,y)p]/2\partial y_i \partial y_j \qquad (1.4)$$

The steady-state solution of equation (1.4) is the probability density function $p(y)$.

The appealing aspect of an approach based on the Fokker-Planck-Kolmogorov equation is that the derived solution is an exact one. However, the assumptions underlying the existence of an analytic stationary solution to equation (1.4) are quite restrictive: in general the nonlinearities are required to be of static nature only, and the excitation is a Gaussian process the spectral density of which is that of a white noise [Caughey, 1963a & 1971].

A precise survey of the nonlinear systems which can be solved exactly by means of the Fokker-Planck equation can be found in Caughey [1971, 1982a & 1982b] and Ludwig [1975].

Concerning second-order properties of the response, virtually all the Markov processes the power spectrum of which can be evaluated exactly are the one which are Gaussian [Ming Chen Wang and Uhlenbeck, 1945]. Therefore, the case of second-order statistics of nonlinear transformations remains unsolved at least from the theory of Markov processes.

Roberts [1981] derived the amplitude distribution of the slow drift oscillations of moored vessels from the Fokker-Planck equation.

2- APPROXIMATE METHODS

Elaborating upon the two fundamental techniques described in the previous section, more general nonlinear transformations of random processes, for which no exact or closed-form solutions are known, can be discussed.

Methods which relies on the theory of Markov processes and Itô stochastic calculus are described in section 2-3, while those essentially based on spectral analysis and functional calculus are addressed in sections 2-2 (linearization), 2-4 (perturbations), and 2-5 (functional series).

Let us start this chapter on approximate methods with a rather systematic, but yet cumbersome technique which consists in simulating the equations of motion in the time domain.

2-1- Time Domain Simulation -

Time domain digital simulation of the equations of motion remains the foremost way of predicting the response of a nonlinear system to some prescribed input.

Apart from its systematic aspect, such a method exhibits many well-known drawbacks, mostly linked to the prohibitive amount of calculations necessary. ~~In particular, the case of a~~ stochastic excitation, a spectral or probabilistic description of the response becomes rather cumbersome using a time domain simulation, the nature of which is essentially deterministic. Clearly, pre- and post-processing of the time simulation of equations of motion generally include respectively simulation of the excitation power spectrum and spectral analysis of the response time series. Both of which must be handled with care and are quite demanding in computer capacity if done properly.

For these reasons, stochastic frequency domain techniques are generally preferred, whenever possible, to the more expensive time domain techniques.

Nevertheless, it has been applied to the description of ship motions. Dalzell [1971, 1973] showed, through a time-stepping procedure, that for most of the practical dynamic range, the distribution of roll maxima does not correspond to the distribution of the maxima of a random Gaussian process predicted by theory, i.e. the Cartwright and Longuet-Higgins distribution. Pérez y Pérez [1974] modeled the motions of a steered ship in waves by a linear convolution integral and frequency independent nonlinearities considered to be part of the exciting forces, these nonlinearities being associated with the rudder forces, viscous roll damping and restoring forces and moments. When compared with experiments, the roll motion prediction proves to be not as accurate as the yaw and rudder motion predictions.

Besides digital computer simulations of the equations of motion, analog measurements of electronic or electro-mechanical circuits are possible and may prove to be useful when the system in hand can be modeled correctly in that manner [Broch, 1977].

2-2- Equivalent Linearization -

Linearizing the system comes next among the available techniques. Basically, an "equivalent" linear system:

$$\mathcal{L}_{eq}\{\tilde{y}(t)\} = x(t) \tag{2.1}$$

is substituted to the original nonlinear one; equations (0.1). The price to be paid for such a drastic simplification of the nonlinear model lies in the choice of a linearization procedure, which does not follow any strict guidelines, as well as in an incomplete description of the system which ignores the specific features[2] of nonlinear systems: the system is thus globally assumed to behave as a linear one.

In the case of a deterministic excitation, Kryloff and Bogoliuboff [1947] invoque equivalent energy balance during one cycle.

Drawing upon Kryloff and Bogoliuboff work, the stochastic case is generally handled in replacing the original nonlinear equations describing the system, by equivalent linear equations which minimize the mean square error:

$$E[(\mathcal{F}\{y(t)\} - \mathcal{L}_{eq}\{y(t)\})^T(\mathcal{F}\{y(t)\} - \mathcal{L}_{eq}\{y(t)\})] \tag{2.2}$$

[2]Nonlinearities usually have two different effects. The first of these leads to a response which differs only quantitatively from the linear response (amplitude modulation), while the second one induces phenomena which are not predictable within the framework of a linear approach, such as non-Gaussian response to Gaussian processes and certain types of dynamic instabilities, sub- or superharmonic responses (frequency modulation), bifurcations...

in the mean square sense preclude accurate prediction of statistics other than second order ones? Moreover, the available litterature seems to indicate that the method of equivalent linearization tends to underestimate the response statistics [Lin, et al., 1986] although no theoretical justification appears to support this observation.

This method has been applied to ship rolling in random waves by Kaplan [1966] (viscous damping) and Vassilopoulos [1971] (viscous damping and nonlinear restoring moment). A variation of this technique involves the use of a describing function in considering the ship as a feedback system [Flower & Mackerdichian, 1978].

Following the same idea, it is also possible to substitute an "equivalent" nonlinear system to the original system when, for example, the "equivalent" nonlinear system belongs to a class of problems which can be solved (exactly or not) [Caughey, 1984]. A similar methodology has been proposed by Jaunet [1984] where a simplified nonlinear model (cascading systems: linear with memory + nonlinear without memory) of ship rolling in irregular seas is identified from experimental data .

2-3- White Noise Spectrum Excitation -

The mathematical idealization of white noise excitation[4] allows the use of techniques based on the Fokker-Planck equation as well as Itô calculus. In this section, nonlinear systems, for which no exact solutions are known, are considered and approximate response statistics evaluated through these techniques.

[4]Such an idealization may be justified provided that the correlation time of the actual excitation process is small when compared with the relaxation time of the dynamic transformation [Lin, 1967, Lin et al., 1986].

2-3-1- Fokker-Planck Equation -

The foremost and maybe most restrictive hypothesis related to the application of the Fokker-Planck equation is that the excitation power spectrum is that of a white noise. In principle, this assumption may be removed if one recalls that coloured spectrum may be whitened whenever they are factorizable[5] [Schetzen, 1980, Papoulis, 1984]. However, this possibility is more theoretical than it first appears since it leads to such an excessive complication of the associated Fokker-Planck equation that even numerical solutions are not easily obtained.

Therefore, one is left with approximate methods which may take more or less into account the shape of the excitation spectrum. One such method is a stochastic averaging procedure proposed by Stratonovitch [1963, 1967] and applied with some success to the nonlinear roll of ships by Roberts [1982a, 1982b, 1984 & 1985].

When the Fokker-Planck equation (1.4) associated with the nonlinear system in hand cannot be solved exactly, a number of approximate techniques exist. The method of equivalent nonlinearization (section 2-2) can be invoqued [Caughey, 1984 & 1986] and has been applied to the case of an oscillator with nonlinear damping by Kirk [1974]. Haddara [1974] used a perturbation method to solve the Fokker-Planck equation in the case of the roll motion of a ship.

The Fokker-Planck equation can also be solved numerically. The method of finite differences is used by Ochi [1984, 1986], in the case of the Duffing oscillator with nonlinear damping driven by a Gaussian white-noise

[5] A procedure due to Wiener [1949].

excitation which was assumed to model the surge motion of a tension leg platform in heavy seas. A Galerkin method with Hermite polynomial expansion can also be used. Wen [1975] employed such a numerical scheme in the case of nonstationary excitations, while Taudin and Rocaboy [1986] proposed a similar technique in the case of multi-degree-of-freedom marine structures subjected to general wave excitations. Taudin and Rocaboy [1986] emphasized that this numerical scheme is about as consuming in computer time as a time domain simulation, moreover numerical instabilities may arise and therefore such a technique does not seem, at least for the time being, to be very useful.

2-3-2- Cumulant Closure -

The statistical moments and cumulants of the response of nonlinear systems to white noise excitation can be computed through Itô stochastic calculus [Itô, 1951a & 1951b] from the governing equations. Successive coupled equations in the response power moments generally result. Therefore, an appropriate closure scheme must be used in order to obtain approximate closed-form solutions. One such closure scheme relies upon neglecting cumulants after some prescribed order which represent higher and higher order measures of the process deviation from normality [Crandall, 1980, Wu, and Lin, 1984].

Such a closure scheme seems to yield accurate results [Crandall, 1980, Wu, and Lin, 1984] but has not yield any application to ocean engineering problems so far.

2-4- Perturbation Techniques -

The general feature of perturbation methods is to substitute an infinite number of linear systems to a nonlinear one through an expansion in terms of a "small" parameter describing the magnitude of the nonlinearities. In this way, the nonlinear features of the system disappear. On the other hand, the system response is now expressed in terms of a, generally infinite, series [Crandall, 1963]. The difficulty of evaluation of each term generally increases geometrically as its order in the series.

Two fundamental questions then arise; namely the convergence of the series, as well as the number of terms necessary to get an accurate description of the solution. The answer to the former is generally not easy although one may get a good approximation of the solution from the knowledge of the first few terms only, even when the series diverges. Once again, the nonlinearities must remain weak in order to insure both convergence and accurate prediction of the solution with a limited number of terms.

Such an expansion procedure has been applied to the nonlinear rolling of ships in random seas, first in the case of viscous damping [Yamanouchi, 1964 & 1966] and later in the case of a static nonlinearity [Flower, 1976]. Both papers discuss the influence of the nonlinearity on the response power spectrum.

Among the perturbation techniques, multiple scale methods (spatial and/or temporal) may be applied to second order low frequency excitation problems [Triantafyllou, 1979 & 1982, Agnon & Mei, 1983]. Basically, the general idea underlying the work of Triantafyllou [1979 & 1982] is that the motions of a floating body may be splitted into two components: a small amplitude, quickly varying motion and a large amplitude, slowly varying motion. The main hypothesis being to assume that the two motions may be treated separately, the solution is then written as the sum of the solutions of the two linear problems.

2-5- Functional Series Representation Methods -

In most derivations of the equations of motion of floating bodies such as ships and offshore structures, the system is conveniently reduced to a set of second-order differential equations with frequency-dependent coefficients[6], whether linearity is assumed or not. The simplicity of such a description is only apparent; it actually represents integral equations in the time domain, the physical interpretation of which lies in the fact that a structure freely floating in waves is a space-time system. Various approximations and integrations are made to reduce it to a time system [Tick, 1959]. The price to be paid to allow such a simplification resides in the memory effect which appears as we get rid of the space dimensions and is mathematically described by the integral equations over the past history of the motion.

When the further assumption of linearity is made, the system is compactly described by its impulse response matrix in the time domain and by its harmonic response matrix in the frequency domain. The fundamental importance of these concepts in ship hydrodynamics have been stressed by Cummins [1962] and later by Bishop, Burcher & Price [1973].

For the nonlinear case, a perturbation technique would represent a natural generalization of this procedure to handle the non-linearities. It may be intuitively thought not as a regular expansion in power series, but rather

[6]This dependency is omitted or neglected without much justification among virtually all the litterature reviewed.

as a "power series with memory", namely, a Volterra functional series representation technique.

2-5-1- Volterra Functional Series -

The functional series representation of differential, integral and integro-differential equations originated with the work of Volterra by the end of last century (see e.g. Volterra [1930] and Barrett [1980b] for a comprehensive bibliography). Essentially, the input-output relation of a given analytic system is expanded in a functional power series[7] which can be formally obtained, by analogy with Taylor series, through successive functional derivatives, as defined by Volterra [1930]:

$$y(t) = \sum_{n=1}^{\infty} \mathcal{H}_n\{x(t)\} \tag{2.4}$$

where the n-th order Volterra functional is defined by:

$$\mathcal{H}_n\{x(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\tau_2,\ldots,\tau_n)x(t-\tau_1)x(t-\tau_2)\ldots x(t-\tau_n)d\tau_1 d\tau_2 \ldots d\tau_n \tag{2.5}$$

Such a power series is called a Volterra series, and the kernel $h_n(\tau_1,\tau_2,\ldots,\tau_n)$ is called the n-th order Volterra kernel. In the case of a causal system, the kernel h_n must vanish whenever any of its argument is greater than the time variable t.

It is a "power series with memory". That it is a power series can be readily seen by noting that the n-th order Volterra functional $\mathcal{H}_n\{x(t)\}$ is a

[7]Therefore, it does not provide anything more than what can otherwise be obtained through a perturbation technique (section 2-4). The real advantage of the Volterra series model lies in the formalism as well as the rather advanced body of results related to its application.

homogeneous functional of order n, it is a series with memory since $\mathcal{H}_n\{x(t)\}$ is a n-fold convolution in time. Clearly, the generalized n-th order Volterra functional $\mathcal{H}_n\{x_1(t),...,x_n(t)\}$ is n-linear, by analogy with multilinear function theory.

Wiener [1958] later applied Volterra's description of general functional relationships to nonlinear communication problems. The first systematic study of the application of the Volterra functional model to physical systems appeared with the work of Barrett [1963].

Vassilopoulos [1967] discussed the applicability of the Volterra and Wiener series to the motions of a ship in irregular seas modeled as a nonlinear autonomous system, together with particular applications to the cases of wave-induced ship resistance in random seas, as well as uncoupled nonlinear motions such as roll.

• Determination of the Volterra kernels

The problem of determining the kernels $h_n(\tau_1,\tau_2,...,\tau_n)$ or equivalently the transfer functions[8] $H_n(\omega_1,\omega_2,...,\omega_n)$ can be understood in two different ways depending on the problem in hand.

In the first case, the Volterra kernels are to be determined from knowledge of a general functional relationship of the type (0.1). This inversion problem admits a solution whenever the functional $\mathcal{F}$ is analytic, and its linear part stable (therefore invertible [Barrett, 1963]). In this case, several methods are possible.

In the direct expansion method, the system equations are manipulated until they are brought into the form of the Volterra series expansion (2.4)

[8]The transfer functions are n-fold Fourier transforms of the kernels, and the Fourier transform is one-to-one.

[Bedrosian & Rice, 1971]. In the case of nonlinear autonomous deterministic systems[9], these identities lead to algebraic equations in terms of the transfer functions of any order of the functionals $\mathcal{H}_n$ and $\mathcal{F}$ [Barrett, 1963, Parente, 1970]. Such an approach yields directly the transfer functions H_n via frequency association.

The harmonic input method relies on the specific properties of Volterra transfer functions. Clearly, they can be understood as harmonic response functions [Bedrosian & Rice, 1971].

In the second case, the Volterra series is determined from simultaneous measurements of the input and the output functions. This is the case of identification. Schetzen [1965] proposed a method of measuring the Volterra kernels of nonlinear systems. Essentially, this approach is based on the n-linear properties of n-th order Volterra functional $\mathcal{H}_n$.

- Statistical and Probabilistic Properties of the Response

Once the Volterra series is entirely determined, it is possible to evaluate statistical properties of its response $y(t)$.

Most of the existing works deal with the prediction of second-order statistics[10] of nonlinear autonomous systems driven by ergodic random Gaussian processes. Essentially, the response autocorrelation function is evaluated first:

$$R_{yy}(\tau) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E[\mathcal{H}_n\{x(t)\}\mathcal{H}_m\{x(t+\tau)\}] \tag{2.6}$$

[9]Assuming a deterministic system does not preclude the possibility of stochastic excitations.

[10]Essentially because the Volterra and Wiener functional models were first applied in connection with signal processing and mean-square estimation in the field of communication.

The response power density spectrum may then be obtained by Fourier transform of the above equation (2.6). A direct approach leads to the result derived by Rudko & Weiner [1978]. Clearly, if x(t) is of order ε, the response power density spectrum $S_{yy}(\omega)$ thus obtained direclty appears as a power series of ε, keeping in mind the homogeneous property of functional $\mathcal{H}_n$. For example, $E[\mathcal{H}_n\{x(t)\}\mathcal{H}_m\{x(t+\tau)\}]$ will be of order ε^{m+n}.

In an alternative approach, proposed by Mircea & Sinnreich [1969], the terms in (2.6) are reordered, resulting in a single series which turns out to be closely related to the Wiener-Hermite expansion (section 2-5-2).

Dalzell [1976b &1982] investigated the applicability of the Volterra series model to nonlinear ship rolling, and more in detail of the third degree Volterra functional [Dalzell, 1982], where the transfer functions are evaluated using the so-called harmonic input method, while the roll spectrum is evaluated through the Mircea-Sinnreich series.

Dalzell [1976a] and Dalzell and Kim [1979], used this technique again in the problem of added ship resistance in irregular waves and lateral drift forces and moment [Kim and Dalzell, 1981, Dalzell, 1986].

Borgman [1982] described the general procedure to take into account various type of nonlinearities (wave theory, structure motion, drag force, mooring and free surface effects).

Finally, Bouche & Olagnon [1985a &1985b] analyzed the vibrations of a fixed circular cylinder in waves where the transfer functions were evaluated by a direct expansion procedure.

Although second order statistics are essential they only yield an incomplete description of the response of nonlinear systems, principally when deviation from linearity is significant. In particular, accurate prediction of the probabilistic structure of the response necessarily involves evaluation of higher order statistics. Although there does not appear to exist any general formula, the first few output cumulants of general Volterra series can be computed from Bedrosian & Rice [1971], which in turn, may lead to an Edgeworth-series type probability distribution. Yet, quite surprisingly, there does not seem to exist any application of these ideas.

In the case of second-order autonomous stochastic systems, the probabilistic description of the response is possible through the Kac-Siegert method [Kac and Siegert, 1947], either from the probabilistic density function of the response [Neal, 1974] or from its mean upcrossing frequency [Naess, 1985]. Ultimately, extreme-value behavior of the marine structure response may be obtained.

2-5-2- Wiener-Hermite Functional Series -

Nonlinear system representation by Voterra functionals is but one technique among the functional representation techniques.

Two basic difficulties are associated with the practical application of the Volterra functional series. The first difficulty arises with the measurement (identification) of the Volterra kernels/transfer functions of a physical system, whereas the latter one with the question of convergence of the resulting series.

To circumvent these problems, Wiener constructed a new set of orthogonal functionals $\mathcal{K}_n\{x(t)\}$, with respect to a Gaussian white noise input, determined from the Volterra functionals [Wiener, 1958, Barrett, 1963, Schetzen, 1980, Rugh, 1981]. The orthogonality relations are:

$$E[\mathcal{K}_n\{x(t)\}\mathcal{K}_m\{x(t+\tau)\}] = 0 \qquad n \neq m \tag{2.7}$$

Orthogonalization of the Volterra series (2.4), through a Gram-schmidt orthogonalization procedure, leads to the Wiener series:

$$y(t) = \sum_{n=1}^{\infty} \mathcal{K}_n\{x(t)\} \tag{2.8}$$

Because the convergence of an orthogonal series is a convergence in the mean, the class of nonlinear systems that can be described by the Wiener functionals is much larger than the class that can be described by a Volterra series [Schetzen, 1980].

Yet another and simpler derivation of the Wiener functionals expansion relative to a Gaussian white noise can be achieved through expansion in some set of orthogonal polynomials. A suitable choice of orthogonal functional polynomials are the Grad Hermite polynomials $he^{(n)}$ [Barrett, 1963 & 1964]:

$$\mathcal{K}_n\{x(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} k_n(\tau_1,\tau_2,...,\tau_n) he^{(n)}(x; t-\tau_1, ..., t-\tau_n) d\tau_1 ... d\tau_n \tag{2.9}$$

where the Wiener kernels $k_n(\tau_1,\tau_2,...,\tau_n)$ may be determined through the orthogonality condition (2.7):

$$n!\, S_0^n\, k_n(\tau_1,\tau_2,...,\tau_n) = E[y(t) he^{(n)}(x; t-\tau_1, t-\tau_2, ..., t-\tau_n)] \tag{2.10}$$

The Wiener-Hermite expansion can be easily generalized to the case of a non-white Gaussian excitation process by redefining appropriate Hermite functionals [Barrett, 1980a & 1982]

Clearly, the Wiener-Hermite functional series can be understood as an orthogonal expansion with memory relative to a Gaussian input process.

It will be now shown how advantage can be taken from such an orthogonal expansion in both the measurement of the kernels and transfer functions and the derivation of output second-order statistics.

- Determination of the Wiener-Hermite kernels

The Wiener kernels $k_n(\tau_1,\tau_2,...,\tau_n)$ and transfer functions $K_n(\omega_1,\omega_2,...,\omega_n)$ can be determined from knowledge of the Volterra kernels and transfer functions [Barrett, 1980a & 1982, Rugh, 1981]

However, one of the appealing aspects of these orthogonal expansions lies in the relative simplicity to identify the corresponding kernels and transfer functions:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} n!\, k_n(\sigma_1,...,\sigma_n) R_{xx}(\sigma_1-\tau_1)...R_{xx}(\sigma_n-\tau_n) d\sigma_1...d\sigma_n = R_{y_{he}(n)(x)}(\tau_1,...,\tau_n) \tag{2.11}$$

$$n!\, K_n(\omega_1,...,\omega_n) = S_{y_{he}(n)(x)}(\omega_1,...,\omega_n)/S_{xx}(\omega_1)...S_{xx}(\omega_n) \tag{2.12}$$

The use of the Wiener expansion in analog identification of nonlinear systems, driven by stationary Gaussian white noise, appeared with the work of Lee and Schetzen [1965] (equation (2.11)). Whereas French and Butz [1973] showed how the Wiener may be measured in the frequency domain through equation (2.12). Extensive applications to physiological systems can be found in Marmarelis and Marmarelis [1978] and Marmarelis [1979].

The application of identification techniques of linear autonomous systems to ship and offshore structure motions is well established. Essentially, cross-spectrum techniques (equation (2.12)) are used in order to evaluate the linear transfer function. The superiority of the cross-spectrum over the auto-spectrum technique is not only that it yields complete response characteristics, including both amplitude and phase relations, but also that cross-spectrum is free from the effect of any orthogonal[11] noise included in the response [Yamanouchi, 1974].

In this respect, such an approach, together with an auto-spectrum technique allows a convenient measure of the adequacy of the linear model through coherence functions [Bendat, 1982, 1983 & 1985].

The cross-spectrum approach (2.12) to the identification of nonlinear systems has been mentioned for some time in the ocean engineering litterature, but without any reference to its mathematical foundation: the Wiener-Hermite expansion.

For example, as early as 1961, Tick [1961] and Hasselman [1966] showed that the nonlinear transfer functions can be obtained from high order statistical moments of the ship motions (equations (2.12)). This underlines the fact that when a nonlinear system is driven by a stationary Gaussian noise, the output is, in general non-Gaussian and therefore it cannot be anymore reasonably described by its first two moments alone.

Dalzell [1974] demonstrated by identification that a second order Volterra polynomial model is a reasonable representation of the added ship resistance produced by waves. He used identification techniques to evaluate the linear and quadratic kernels through the cross-spectrum and cross-bispectrum respectively. Assuming the Gaussian random wave model valid, Neal [1974] and Borresen [1978] treated the nonlinear response of a ship up to the second order, i.e. taking the low-frequency excitations into account.

[11] Orthogonal in the sense of equation (2.7).

A quite general and advanced review of the identification techniques of second and third order Volterra systems can be found in Bendat [1985], where particular emphasis is placed on square-law (with or without sign) and cubic systems.

A somewhat systematic approach to identify a general transformation giving the inline and transverse forces on a vertical cylinder element in random waves has been undertaken by Vugts and Bouquet [1985]. The adequacy of the models to describe the relationship between input and output is evaluated by a total coherence function [Bendat, 1985]. This work lead to a revalidation of the Morison equation, together with the associated dilemma of an appropriate selection of the empirical coefficients. Moreover, among their conclusions was emphasized the coefficient dependency on the input conditions which is in contradiction with the "black box" concept.

- Statistical and Probabilistic Properties of the Response

Keeping in mind the orthogonality of the Wiener-Hermite expansion (equation (2.8)), the response autocorrelation function can be written as the single sum:

$$R_{yy}(\tau) = \sum_{n=1}^{\infty} E[\mathcal{K}_n\{x(t)\}\mathcal{K}_n\{x(t+\tau)\}] \tag{2.13}$$

The response power density spectrum may then be obtained by Fourier transform of the above equation (2.13) [Barrett, 1980a]:

$$S_{yy}(\omega) = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} n!|K_n(\omega_1,\ldots,\omega_n)|^2 S_{xx}(\omega_1)\ldots S_{xx}(\omega_n)\delta(\omega-\omega_1-\ldots-\omega_n)d\omega_1\ldots d\omega_n \tag{2.14}$$

where δ denotes the Dirac distribution. This expression turns out to coincide with the response power density spectrum derived by Mircea and Sinnreich [1969] and Bedrosian and Rice [1971] and simply consists in a rearangement of the terms in the Volterra series. It is interesting to mention that equation (2.14) provides a decomposition of the output power spectrum into its frequency components [Barrett, 1982].

In order to illustrate this frequency resolution, let us consider the simple and idealized case of the narrow-band excitation spectrum with modal frequency ω_0 and mean square S_0: $S_{xx}(\omega) = S_0\delta(\omega-\omega_0)$, upon substitution in equation (2.14) the response power density spectrum results:

$$S_{yy}(\omega) = \sum_{n=1}^{\infty} n!|K_n(\omega_0,...,\omega_0)|^2\, S_0^n\, \delta(\omega-n\omega_0) \qquad (2.15)$$

It is clear from this example that nonlinear response occurs not only at the frequency of excitation but at superharmonic frequencies multiple of the fundamental one.

The output frequency spectrum $S_{yy}(\omega)$ of analytic nonlinear systems can be evaluated either as a power series (by Fourier transform of equation (2.6)) or as the single sum (2.14). In practice such series must be truncated, thus (2.6) yields the lowest order nonlinear terms whereas (2.14) yields the first few harmonics. Clearly, choice of either series should depend upon the problem in hand.

CONCLUSION

A general theory for nonlinear response to stochastic processes should satisfy three attributes: simplicity of implementation, accuracy of the resulting response statistics, and versatility of the method. Clearly, such a theory is not available, and one is left with the various techniques described and the associated dilemma of choosing the one which holds the most promises with regard to the problem in hand.

Furthermore, the singular lack of comprehensive full-scale as well as model measurements of situations involving marine vehicles greatly reduces the possibility of assessing the validity and adequacy of these techniques.

Therefore, extensive experiments, involving both load and response measurements, should be undertaken with the double objective of identifying the parameters of the dynamic system and characterizing the techniques the most adequate.

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