Stock Assessment in Aggregating Fisheries

Marc Mangel
Departments of Mathematics and
Agricultural Economics
University of California
Davis, CA 95616

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Abstract

In many different fisheries, catch occurs only when the fish are found in dense aggregates. For these cases, catch per unit effort is most likely a poor estimator for population. An alternative procedure for population estimation uses recent developments in search theory as applied to fisheries. Three points are treated in this paper. The first addresses the question of what one can say about stock level in a region where there is effort but no catch. Second, a simulation that compares catch per unit effort and search estimates for population is discussed and results are presented. Third, the results of an empirical study on Pacific Ocean Perch in Rennell Sound, British Columbia, Canada, are presented.

1. Introduction

Many methods currently in use for the estimation of fish stock biomass employ the assumption that catch (C) is proportional to population level (N) according to

$$C = qEN \quad (1)$$

Here E is fishing effort and q is a catchability coefficient. Clearly C/E, catch per unit effort (CPUE), is proportional to N. In order to estimate N absolutely, the value of q must be known. This is a difficult problem (e.g. Clark (1976), Ricker (1975), Seber (1982)).

The CPUE model can be called the "spilled jello" model. Namely, imagine a bowl of jello spread uniformly over the dining room table. In this case each point of the table is the same and (ignoring depletion) a spoon running along the table accumulates jello according to the time (effort) spent on the table.

On the other hand, many kinds of fish are caught when they form aggregations. Tuna and groundfish are examples. In this case, a better model is the "spilled fruit jello" model, in which the pieces of fruit play the role of schools of fish. One would guess that CPUE, as given by equation (1) is not a good estimate for these kinds of populations. The main questions, then, is what should replace equation 1?

Recent work by Allen (personal communication) Mangel (1982a,b), Mangel and Clark (1983), and Mangel and Beder (1983) shows that search theory can be applied to fisheries and that stock estimates can be obtained using search data. These methods require 1) that the search for fish is a major source of uncertainty in the overall fishing operation, and 2) that the ease of finding fish decreases as the stock level decreases. If these conditions are met, then search theory can provide new ways of estimating population levels. In this paper, some of these concepts are illustrated in three ways. The next section addresses the following question from a theoretical viewpoint: Consider a region in which some effort was expended but no fish were found. What can be said about the stock level in this region?

In section 3, the ideas of search theory are illustrated in a simulation for the exploitation of a stock that is fished only when it is aggregated. Search estimates for stock level are compared with CPUE estimates. In section 4, the ideas of search theory are illustrated in an empirical study of Pacific ocean perch (Sebastes alutes) in Rennell Sound, off Graham Island, British Columbia, Canada.

In all sections, the basic assumption is that the fish are captured when they form aggregations. Consequently, many of the questions refer to estimating the number and average size of the aggregations.

2. Estimating Stock Level When No Fish Are Caught

Imagine a region in which some effort, measured by search time, is expended but no fish are caught. What can be said about the stock level in the region? One cannot rule out the presence of fish; instead the quantity that must be found is the probability distribution for the number of aggregations in the region. This is a posterior probability distribution, since it is found after the effort is expended. Define $g(n;N_m,t)$ by

$g(n;N_m,t)$ = Probability that the region contains n aggregations, given that the maximum possible number of aggregations is N_m and a search time of t hours yielded no catch. (2)

The value of N_m can be determined, for example, from the habitable volume of the region V and the typical volume of an aggregation v_a according to

$$N_m = \frac{V}{v_a} \quad . \tag{3}$$

In addition to V and N_m the following parameters are needed:

s = average speed of a vessel searching for aggregations of fish (m/hr)

w = sweep width of the net (m^2)

k = number of vessels searching . (4)

Finally, the prior (i.e. before any search) distribution for the number of aggregations is needed. Define $g_0(n)$ by

$g_0(n)$ = Probability that the region contains n aggregations, determined before fishing occurs . (5)

In the literature of Bayesian decision theory, $g_0(n)$ is called a prior probability distribution. In appendix 1, it is shown that

$$g(n;N_m,t) = \frac{g_0(n)e^{-nk\delta t}}{\sum_{n=0}^{N_m} g_0(n)e^{-nk\delta t}} \quad . \tag{6}$$

In this equation, δ is a parameter given by the formula

$$\delta = \frac{w\ s}{V}\ . \tag{7}$$

In Appendix 1, it is shown that the quantity $p = 1 - e^{-k\delta t}$ is the probability of finding at least one aggregation with search effort t.

One choice for $g_0(n)$ is the uniform prior,

$$g_0(n) = \frac{1}{N_m} \text{ for all } n \tag{8}$$

which assumes that all values of n are equally likely. With this choice for $g_0(n)$, equation (6) becomes

$$g(n;N_m,t) = \frac{(1-e^{-k\delta t})e^{-kn\delta t}}{1 - e^{-(N_m+1)k\delta t}}\ . \tag{9}$$

Equation (9) is derived in Appendix 1, where an explicit formula for the average value of n is given.

Figures 1 and 2 show results of calculations using $1-e^{-k\delta t} = .1$, $.01$ and $N_m = 20, 50$. Figure 3 shows a proposed prior distribution $g_0(n)$ for the case in which the quality of the region may be bad, average, or good.

Software that runs on an Apple II+ microcomputer for the employment of these methods is available from the author.

Figure 1. The posterior distribution $g(n',N_m,t)$ when $1 - e^{-k\delta t} = .1, .01$ and $N_m = 20$.

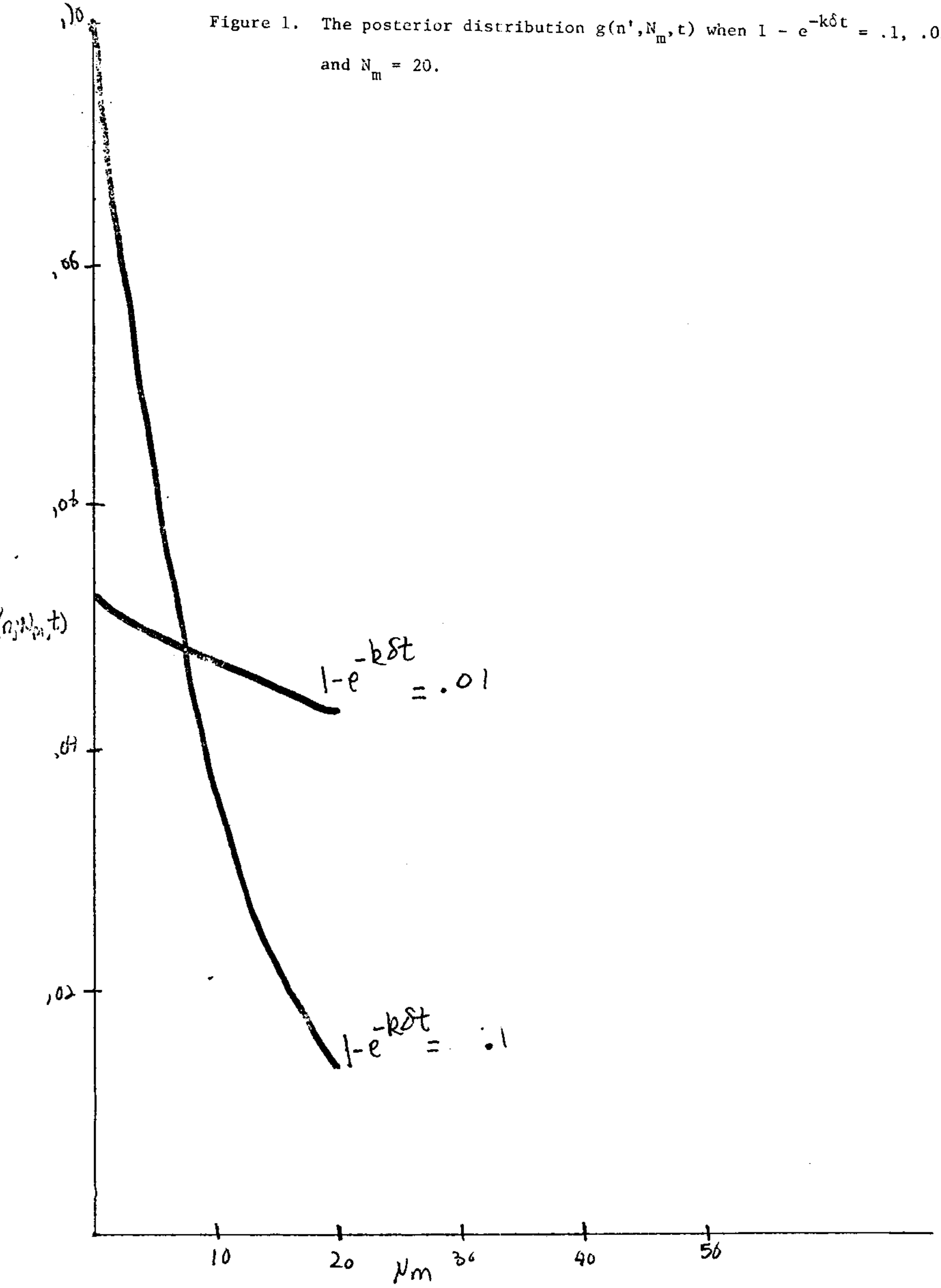
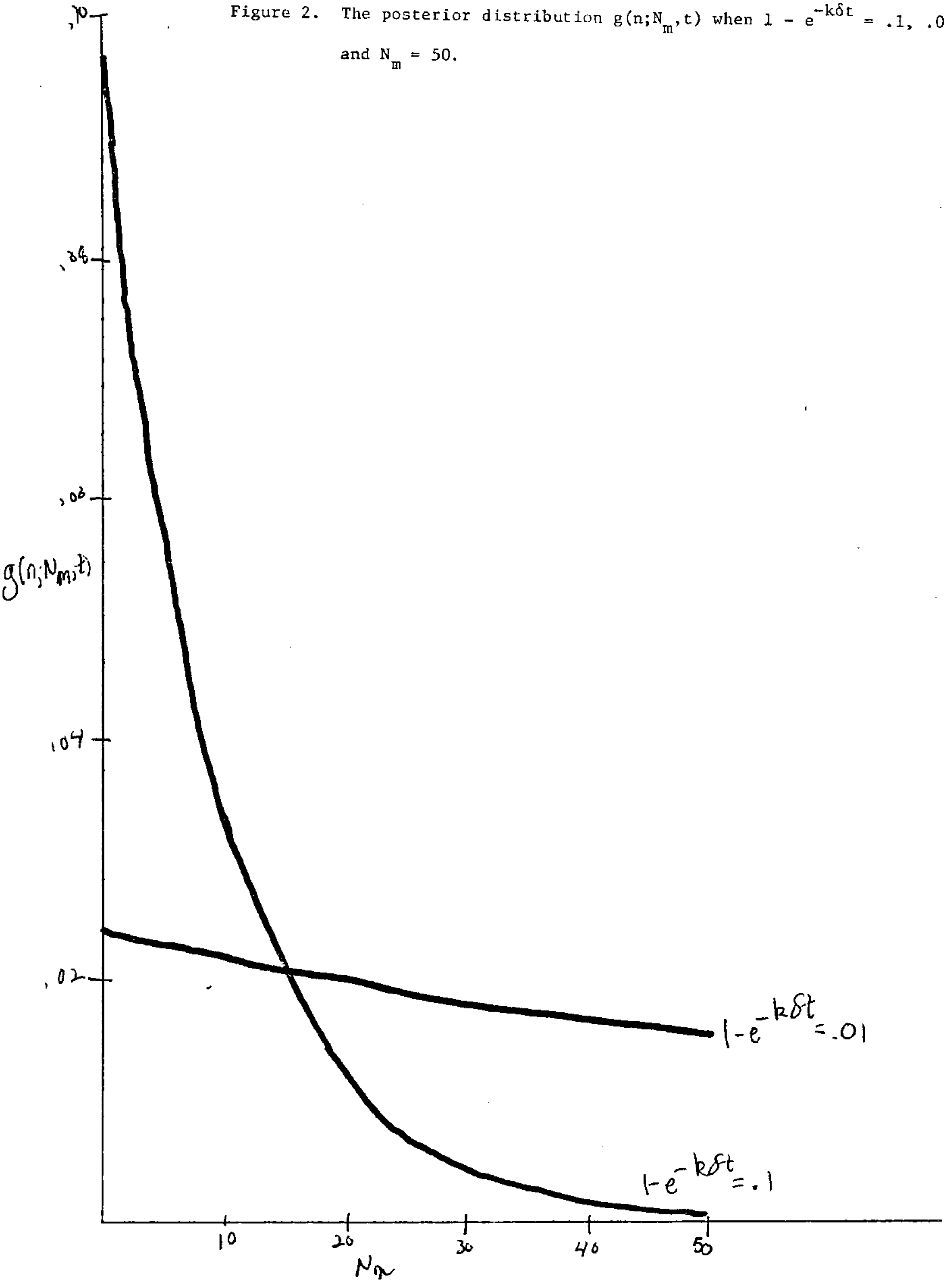


Figure 2. The posterior distribution $g(n;N_m,t)$ when $1 - e^{-k\delta t} = .1, .01$ and $N_m = 50$.



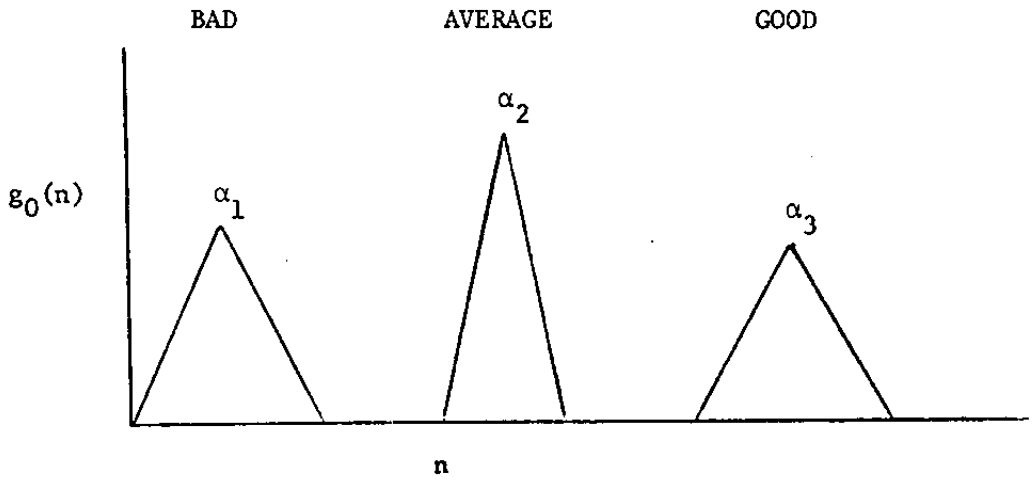


Figure 3. Proposed prior distribution that can be used when there is a probability α_1 that the region is bad, α_2 that the region is average, and α_3 that the region is good.

3. Simulation Study of Stock Assessment Methods

At the outset of the empirical study on Pacific Ocean Perch (POP), Sebates alutus, the author learned that off California in 1982-83 there was a controversy about the size of the POP stock. Fishermen thought that the stock was plentiful, while managers did not (P. Leipsig, personal communication). This observation will take on new importance once the simulation results are described.

The simulation described in this section models the processes of fishing for an aggregated species and estimating its biomass. The following rules are employed:

1. Each run of the simulation lasts 7 days of fishing.
2. Each day of fishing consists of 12 hours during which the fish may be caught.
3. At the start of each day of fishing, the current biomass is randomly broken into N_p aggregations, where N_p is a Poisson random variable with parameter λ. The search for aggregations is characterized by

$$\text{Probability}\{\text{search time} \leq t\} = 1 - e^{-(N_p - N_f)\delta t}$$

   Here N_f is the current number of aggregations already found.
4. When an aggregation of biomass B_p is found the fishing time is determined by the number of sets that can be made before the aggregation is exhausted. At the end of the day of fishing, the stock biomass is reduced by the catch.
5. At the end of the week of fishing, the original biomass in the region is estimated from total catch, the catchability coefficient Q, and the parameter δ described in the last section. These estimates are compared with the true biomass at the start of the week. The

catchability coefficient Q was determined as follows. Six "survey" runs of the simulation were made in which the initial biomass, (B_0), catch (H) and set time (TH) were measured. Q was then estimated using ordinary least squares from the formula (see Appendix 1)

$$H = B_0(1-e^{-QTH}) .$$

Figure 4 shows a flow chart of the simulation. The variables of the flow chart are the following ones:

D = day of the week $(1 \leq D < 7)$

TA = time needed to drop and pull the net

NV = net volume

δ = search parameter

Q = catchability coefficient

$\hat{B}_0$ = initial biomass of the stock

B_0 = current biomass of the stock

L_0 = parameter used to find λ

B_C = parameter used to find λ

NF = number of aggregations found in the current day

H = catch in the current day

$\overline{TS}$ = total search time

$\overline{TH}$ = total set time

TM = 12 hours, maximum search and fishing time in one day

NP = number of aggregations

$P(\lambda)$ = Poisson distribution with parameter λ

BP = biomass of an aggregation

TY = time to find an aggregation

$\mathcal{E}(p)$ = Exponential distribution with parameter p

TR = defined quantity (time remaining in fishing day)

TQ = time to harvest an aggregation of biomass B_p

H = harvest

HT = total harvest (in the week)

E1 = estimate of initial biomass using CPUE

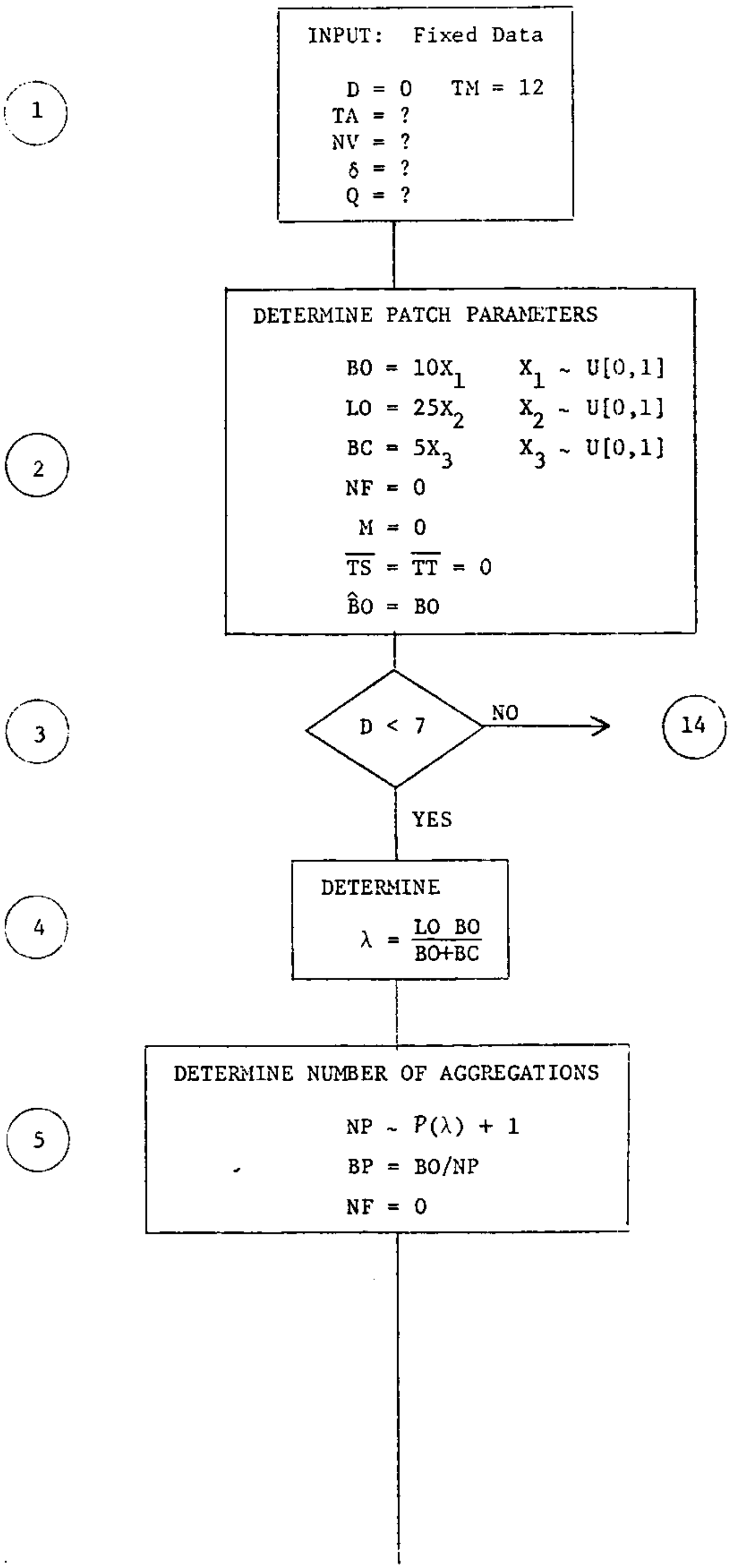
E2 = estimate of initial biomass using search theory.

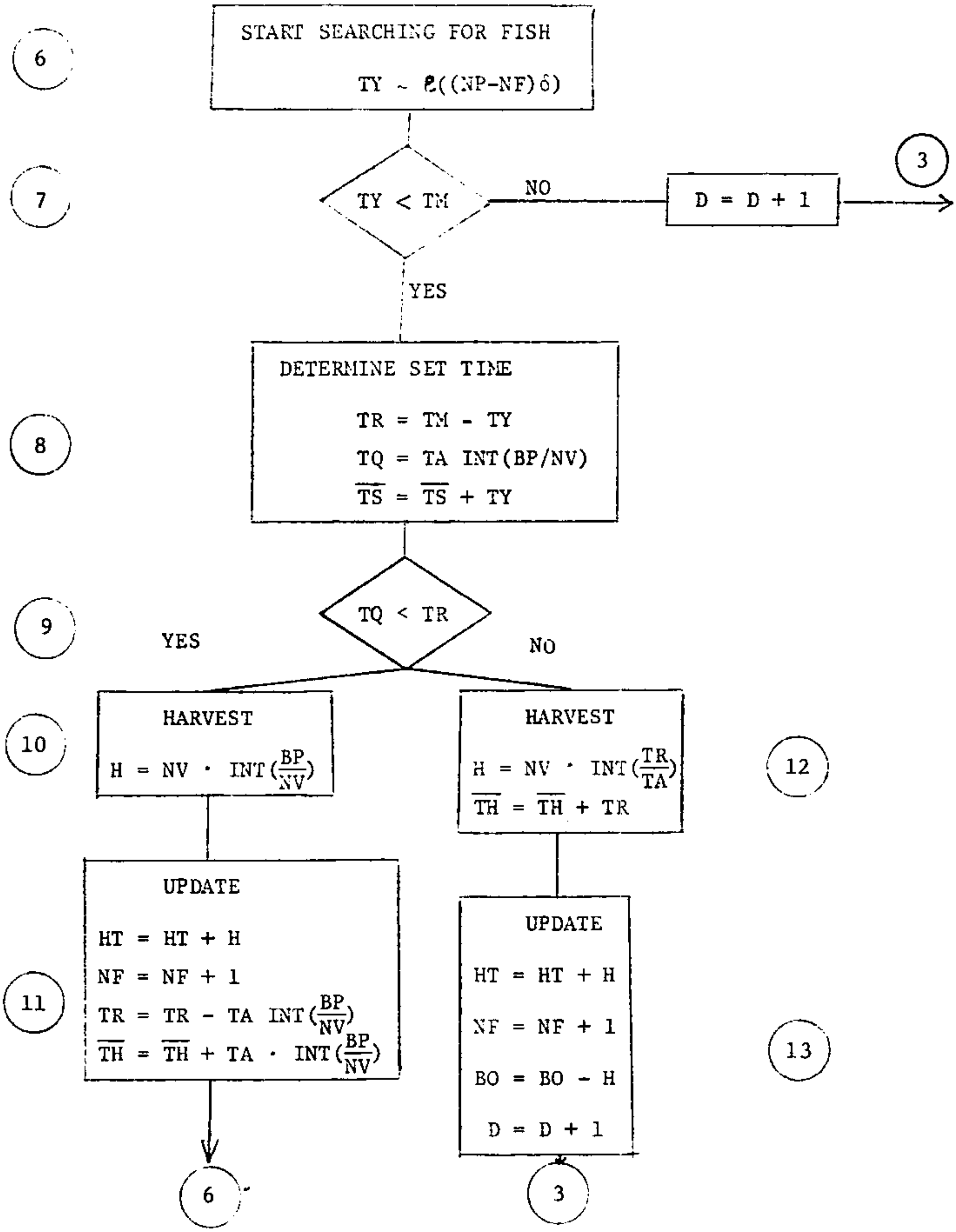
$X_i \sim U[0,1] = X_i$ is a uniformly distributed random number, $0 \leq x_i \leq 1$

INT(X) = integer part of X

ε_i = Relative error in initial biomass estimate

Figure 4. Flow chart of the simulation




6
START SEARCHING FOR FISH
TY ~ ℰ((NP-NF)δ)
7
TY < TM
NO
D = D + 1
3
YES
DETERMINE SET TIME
8
TR = TM - TY
TQ = TA INT(BP/NV)
$\overline{TS} = \overline{TS} + TY$
9
TQ < TR
YES
NO
HARVEST
10
$H = NV \cdot INT(\frac{BP}{NV})$
HARVEST
$H = NV \cdot INT(\frac{TR}{TA})$
$\overline{TH} = \overline{TH} + TR$
12
UPDATE
HT = HT + H
NF = NF + 1
11
$TR = TR - TA\ INT(\frac{BP}{NV})$
$\overline{TH} = \overline{TH} + TA \cdot INT(\frac{BP}{NV})$
UPDATE
HT = HT + H
NF = NF + 1
13
BO = BO - H
D = D + 1
6
3

14

ESTIMATION

$$E1 = HT/(1-EXP(-Q\overline{TH})$$

$$E2 = HT/(1-EXP(-Q\overline{TS}))$$

$$\varepsilon_i = \frac{|E_i - \hat{B}O|}{\hat{B}O}$$

↓

REPEAT

For each set of parameters, 5000 simulations were run. All parameters except TA, the set times were fixed. The three values chosen for TA were 1 hour, 2 hours, and 4 hours. Figures 5, 6, and 7 show the frequency of the relative errors ε_i, as a percentage of B_0. Study of the figures leads to the following conclusion: the search estimate is more likely to be way off, but given that it is not way off, it is more accurate than the CPUE estimate. This conclusion is highlighted using the data from the figures in Table (3.1).

The cause of the higher frequency of large errors with the search estimates is unknown and is currently under investigation.

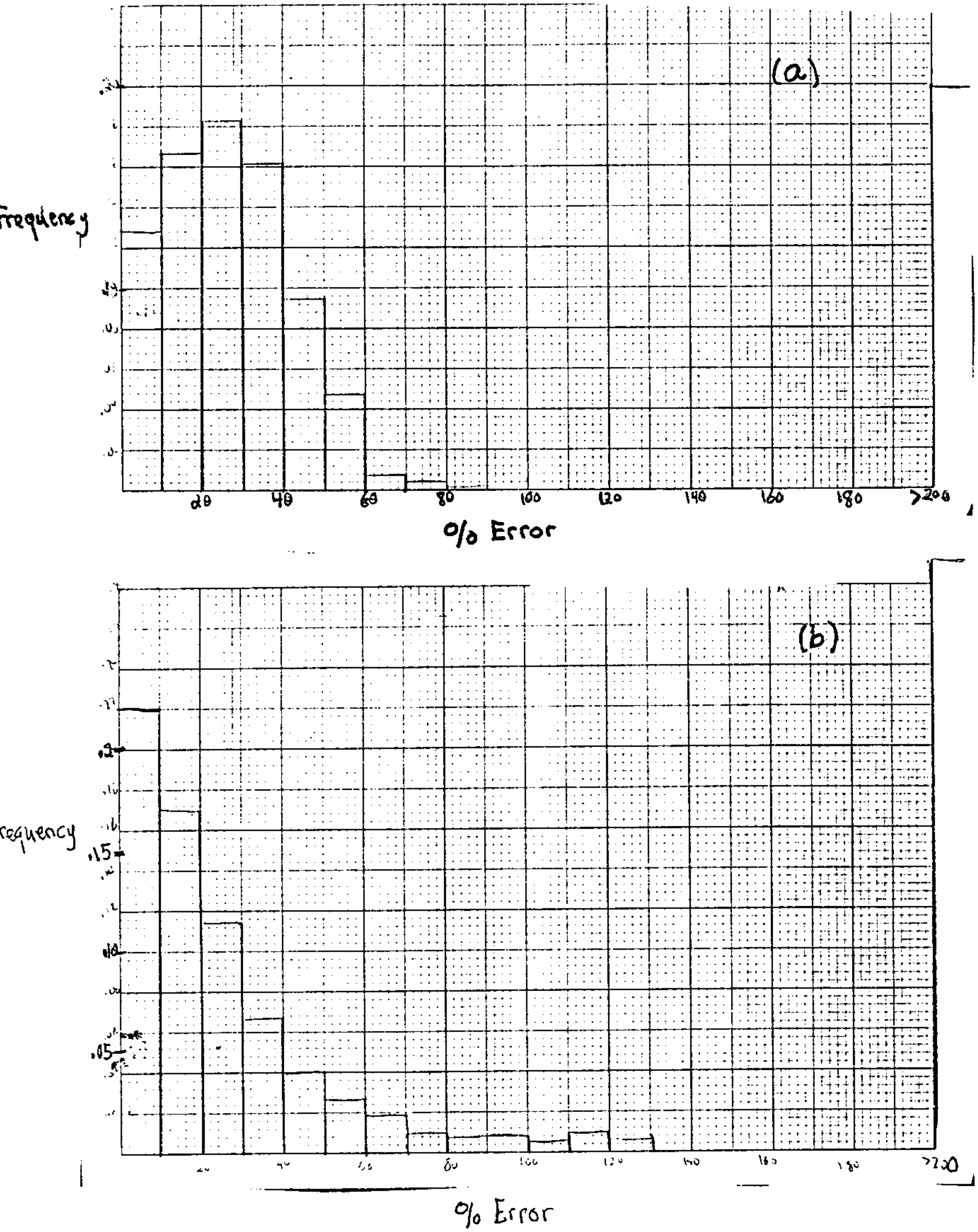


Figure 5. Comparison of the errors in CPUE (panel a) and search (panel b) estimates for biomass, TA = 1 hour.

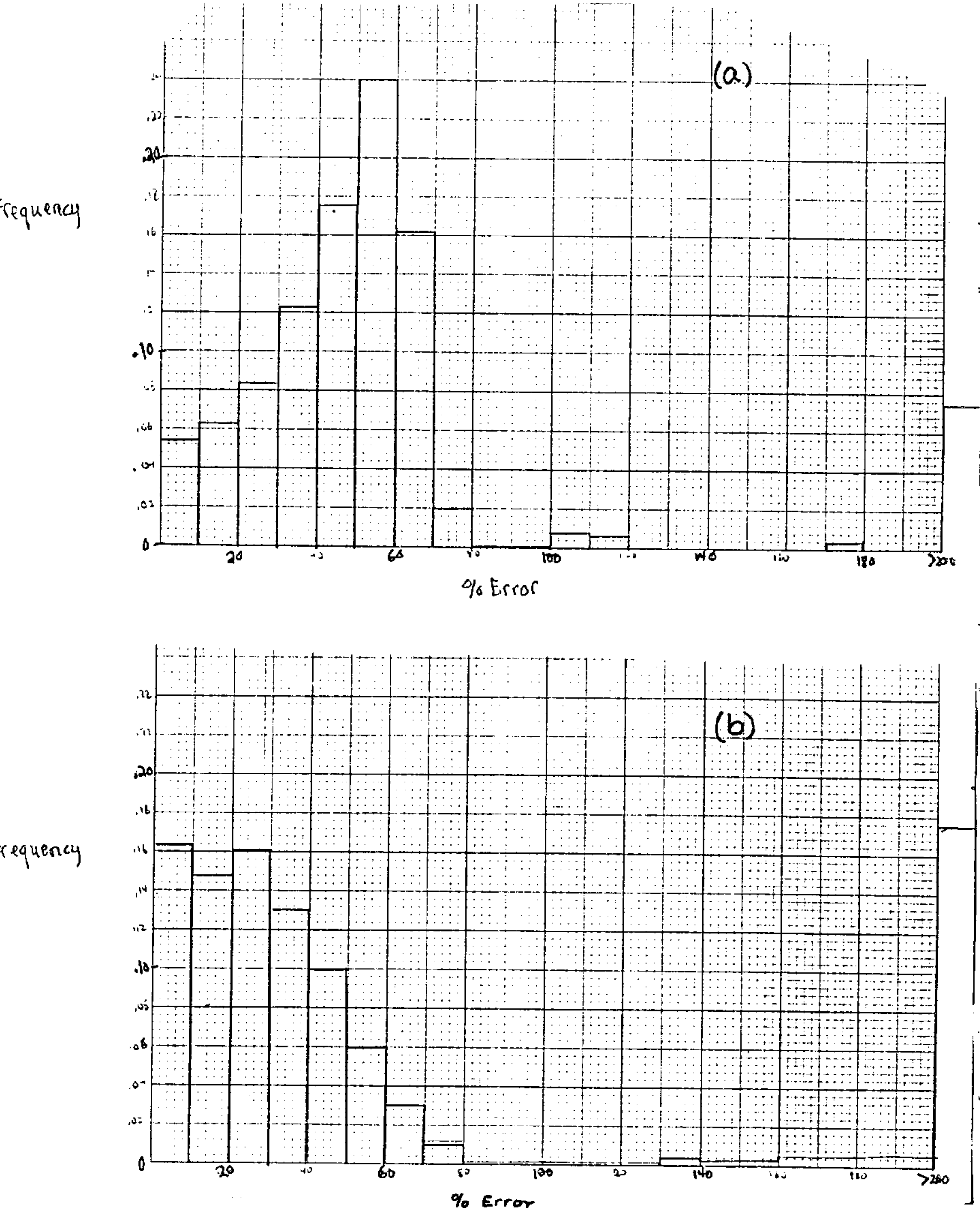


Figure 6. Comparison of the errors in CPUE (panel a) and search (panel b) estimates for biomass, TA = 2 hours

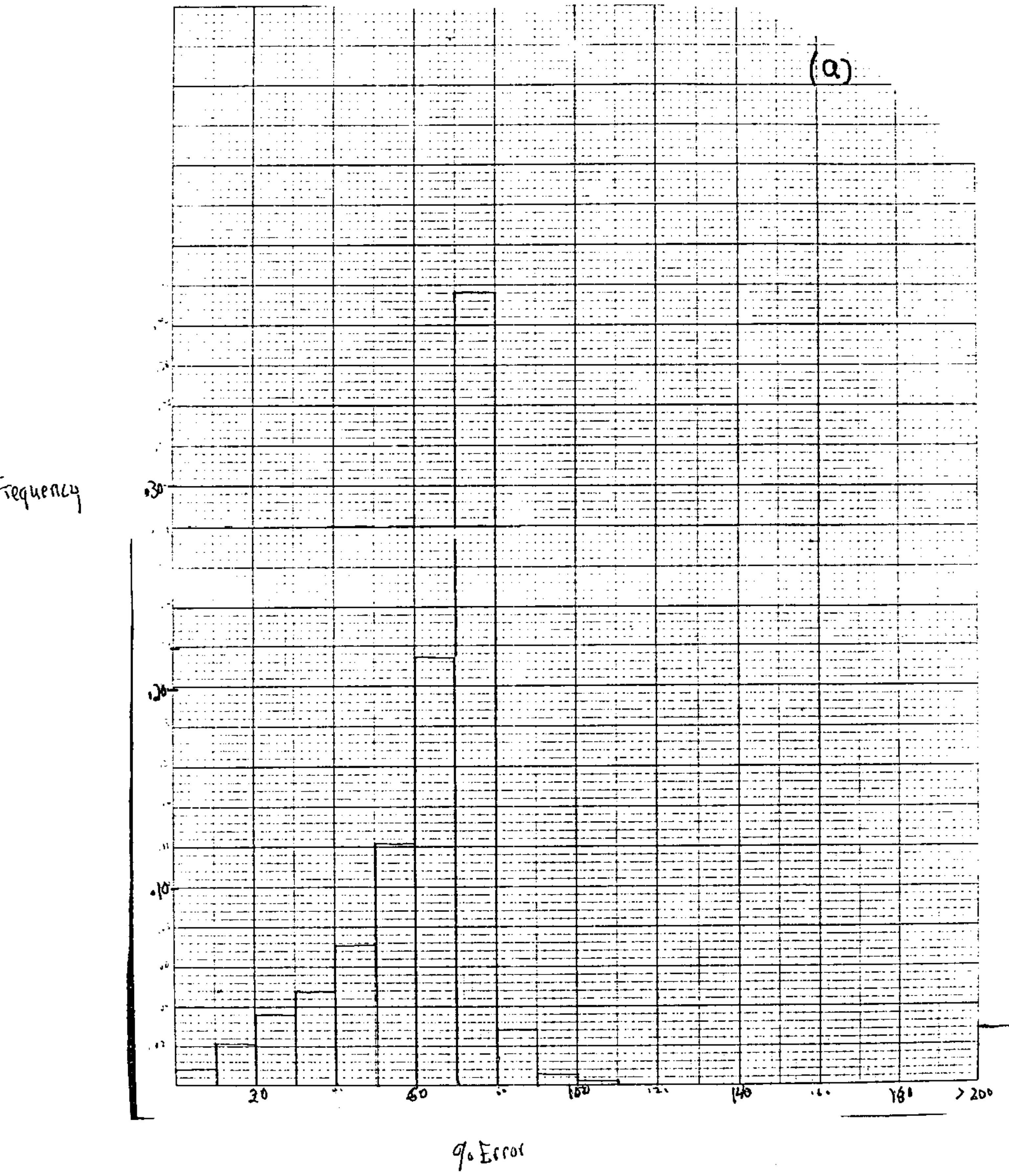


Figure 7. Comparison of the errors in CPUE (panel a) and search (panel b) estimates for biomass, TA = 4 hours.

(Figure 7 cont.)

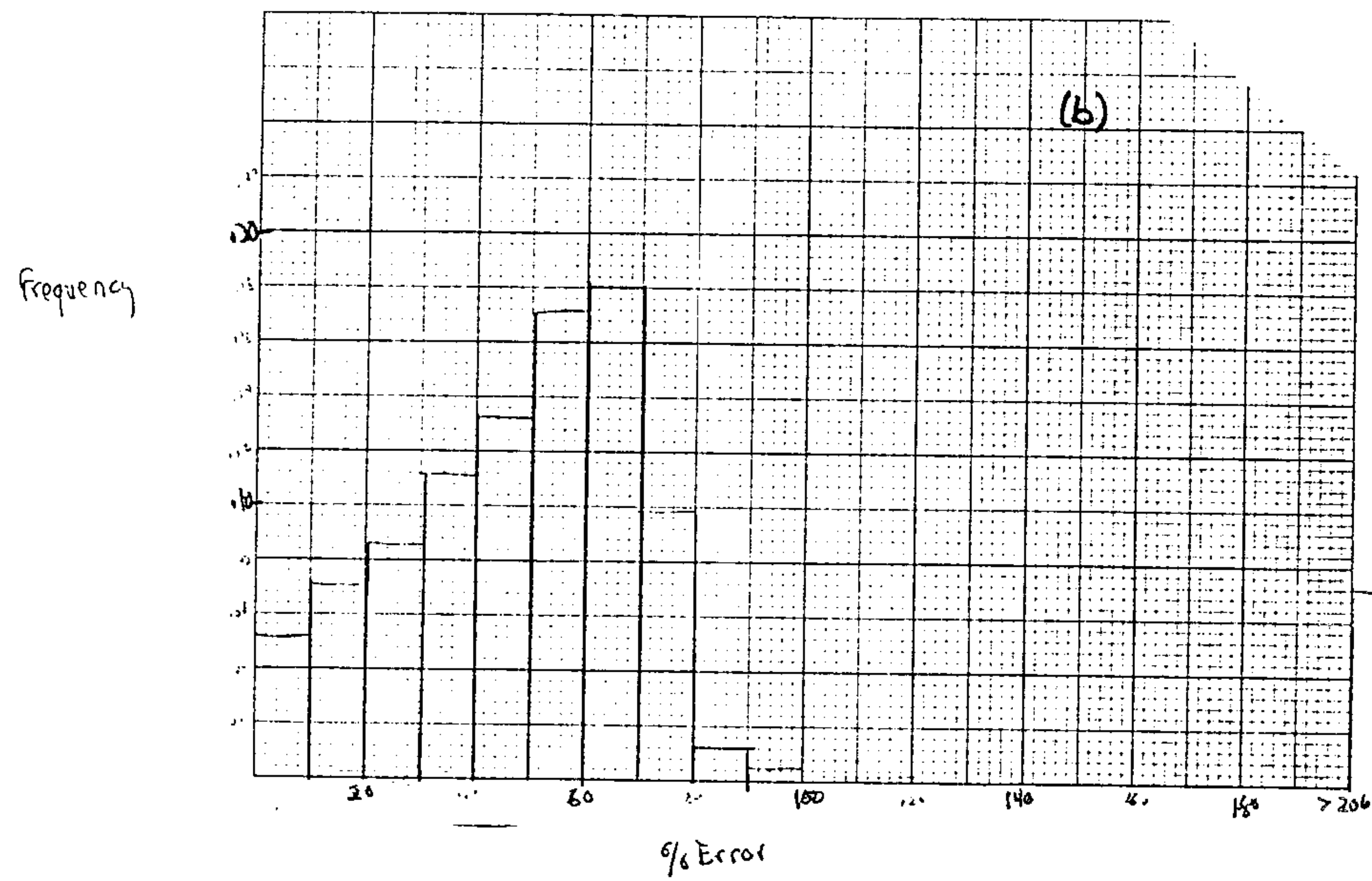


Table 3.1

Comparison of Search and CPUE Estimates for the Biomass of the Stock

TA	Fraction of trials with error					
	< 10%		< 50%		> 200%	
	CPUE	Search	CPUE	Search	CPUE	Search
1 hour	.13	.24	.72	.62	.20	.29
2 hours	.05	.16	.50	.70	.07	.17
4 hours	.01	.05	.19	.47	.03	.07

The results presented here indicate that if the fishermen think that it is an average year, then the estimate obtained using search data from them may be reasonably accurate. If the fishermen believe that the stock is very abundant, then there is a bigger chance that the estimate is considerably off and it may be advisable to run an independent biomass survey. The controversy regarding perch in California mentioned at the start of this section may be an example of the phenomenon discovered in these simulations.

4. Empirical Study of Pacific Ocean Perch in Rennell Sound, British Columbia, Canada

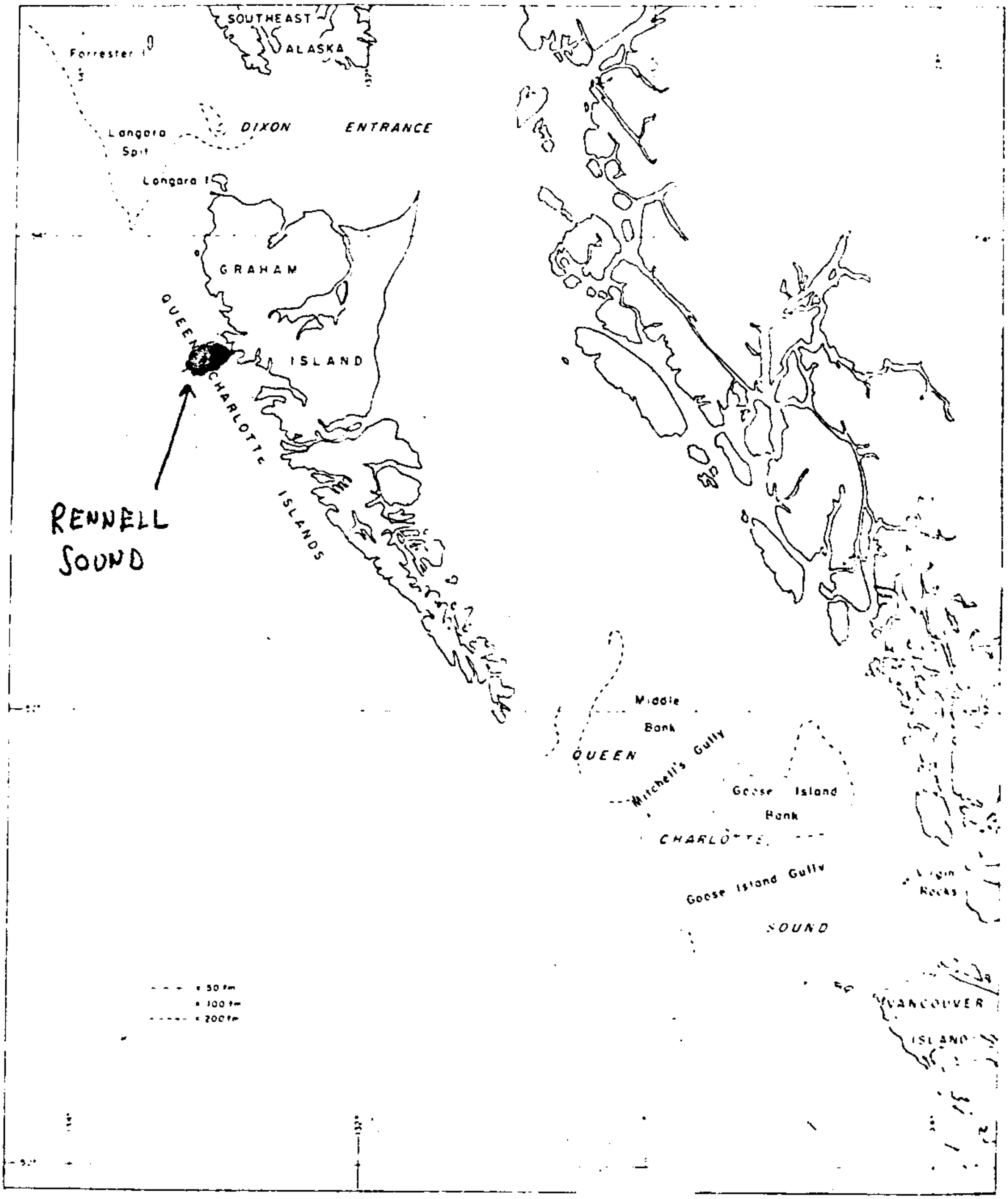
Rennell Sound is located on the west side of Graham Island in the Queen Charlotte Islands (Figure 8). The data on POP catch on Rennell sound are almost ideal for an empirical study of the usefulness of the search methods. First, it is essentially a closed population, apparently the isolated remains of a once much more predominant stock. There is little recruitment, so that the fishery is, in this case, exploiting an exhaustible resource. Second, the data set is sufficiently large to be broken into a subset for estimation of parameters and one for testing the methods. Third, the entire fishable volume of the Sound is known (Leaman and Nagtegaal, 1982).

POP were first exploited in Rennell Sound in 1976. They are caught only when they form aggregations during the daylight. The log book data kept at the Pacific Biological Station, Nanaimo, British Columbia were used as a source of catch and effort information with the following rules:

1. Only sets that contained more than 85% POP were included in the initial analysis. This restriction was relaxed and will be discussed later.

2. Each trip report was assumed to represent the encounter with one aggregation of POP.

Figure 8. Map showing the location of Rennell Sound.



The parameter δ was calculated from the formula

$$\delta = \frac{sw}{V_F} \quad . \tag{11}$$

Here s is the speed of a ship while trawling, w is the sweep width of the net, and V_F is the fishable volume of the sound. V_F was computed from the data given by Leaman and Nagtegaal (1982). The value $V_F = 12.4 \times 10^9$ m^3 was used. The value of s chosen was s = 6.5 n.mi/hr = 1.2×10^4 m/hr.

3. The number of hours available for search was computed according to

$$\text{Search time} = \text{Days fished} - \text{set time} - \text{dark hours} \, . \tag{10}$$

Dark hours were computed using sunrise/sunset tables from an almanac.

By using equation (10) to compute search times, one implicitly assumes that fishing and searching are the only activites. If skippers logged the actual search time, then equation (10) would not be needed and one could find search time through dockside interviews. The sweep width of the net was taken to be the area of the net opening. By doing this, one assumes that fish within the opening are captured with certainty. In this case, the net opening was taken to be rectangular with dimensions 6 m by 18 m (area = 108 m^2). Consequently, δ is given by

$$\delta = \frac{(1.2\times10^4 \text{ m/hr})(1.08\times10^2 \text{ m}^2)}{1.24\times10^{10} \text{ m}^3} \tag{12}$$

$$= 1.05 \times 10^{-4} \text{ hr}^{-1} \, .$$

In the calculations, δ was chosen to be 1×10^{-4} hr^{-1}. Data from 1976 and the first half of 1977 were used to obtain estimates of the initial biomass B_0

and of the catch per set (C_s). The thirteen data points are given in appendix 2. Each data point gives an estimate of B_0, according to

$$B_0 = \text{Catch}/(1-\exp(-\delta \cdot \text{Search Time})). \tag{14}$$

The estimates for B_0 were combined to given the average and variance

$$\begin{aligned} \hat{B}_0 &= 32809 \text{ tons} \\ \sigma_{B_0} &= 24853 \text{ tons} \end{aligned} \tag{15}$$

so that the coefficient of variation is .76. The average catch per set, based on these data, was computed from the nominal value of one hour per set and was found to be

$$C_s = 2.4 \text{ tons/set}. \tag{16}$$

These results are used in a predictive way as follows. The estimate for the biomass remaining at the start of July 1977 is $\hat{B}_R = 32468$ tons. The main predictive question is the following: If the fishing effort in a future year is given, can the catch be predicted? Here effort is measured by time on the ground. An algorithm for estimating the catch, given effort, is described in Appendix 3. Table 3.1 shows the effort and predicted and observed catches from July 1977 to 1982. There is no entry for 1979 because there were no sets with more than 85% POP in 1979. Except for 1978, the predicted catch is reasonably close to the observed catch. Note that for 1978, the predicted catch is much too high, given the level of effort. In 1978, many vessels fished a short time in Rennell Sound, caught a little POP (or none) and never fished there again. Consequently, these vessels can be viewed as experimenting or trying out the ground and did not obtain a catch commensurate with their effort.

The model used to obtain Table 3.1 does not include "learning to fish the ground". Models in which learning by the skipper is incorporated are presently being developed.

Table 3.2 shows the results of analogous calculations, except that the biomass estimate at the start of each year was modified by substracting the biomass of POP taken in sets with less than 85% POP and the actual harvest (rather than predicted harvest). This procedure yields virtually the same values as those shown in Table 3.1. In summary, the population estimate and catch predictions derived using search theory appear to be reasonably accurate.

Evidence to further support the interpretation of the difference between the 1978 prediction and observation is given in Table 3.3, which shows the number of vessels in the data set for each year.

Table 3.1

Predicted and Observed POP Catches in Rennell Sound

Year	Biomass at the Start*	Effort (daylight hrs)	Catch		Error**	
			Predicted	Observed	Absolute	%
6-12/77	32468 t	280 hrs	385 t	430 t	45 t	10%
1978	32083	240	328	119	209	-176
1980	31755	47	64	51	13	- 25
1981	31691	48	66	43	23	- 53
1982	31625	48	65	79	14	18

*Biomass at the start of year y + 1 = Biomass at the start of year y

- Predicted catch in year y

**Absolute error = |Predicted catch - Observed catch|

$$\% \text{ error} = \left[\frac{\text{Predicted catch - observed catch}}{\text{observed catch}}\right] \times 100$$

Table 3.2

Predicted and Observed POP Catches in Rennell Sound, including depletion from sets that were less than 85% POP

Year	Biomass at the Start	Catch Predicted	Catch Observed	Error Absolute	Error %	POP taken in sets<85% POP*
6-12/77	32468 t	385 t	430 t	45 t	10%	242 t
1978	31796	328	119	209	-176	160
1980	31517	64	51	13	- 25	126
1981	31342	65	43	22	- 51	135
1982	31164	65	79	14	18	87

*105 t were taken in 1979

Table 3.3

Number of Vessels Taking More than 85% POP

Sets

Year	Number of Vessels	Number of New Vessels	Fractional Effort and Catch by New Vessels	
			Effort	Catch
1976 and Jan-June, 1977	7	-	-	-
July-Dec 1977	5	0	-	-
1978	8	3	12%	3.9%
1980	3	2	79%	92%
1981	2	0	-	-
1982	3	1	30%	46%

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Appendix 1 Derivation of the Results from Section 2.

In order to derive equation (6), observe that

$$\text{Probability}\{\text{number of aggregations is n, given that none were found with effort t}\}$$

$$= \frac{\text{Prob}\{\text{number is n and none were found}\}}{\text{Prob}\{\text{none were found}\}} \, . \tag{A1}$$

According to the theory of Mangel and Clark (1983) and Mangel and Beder (1983), the number of aggregations encountered with effort t is a binomial random variable with parameters n and $1 - e^{-k\delta t}$. Thus,

$$\begin{aligned} &\text{Prob}\{\text{finding no aggregations with effort t when n are present}\} \\ &= (1-(1-e^{-k\delta t}))^n \\ &= e^{-nk\delta t} \, . \end{aligned} \tag{A2}$$

Using equation (A2) in (A1), gives

$$g(n;N_m,t) = \frac{g_0(n)e^{-nk\delta t}}{\sum_{n=0}^{N_m} g_0(n)e^{-nk\delta t}} \tag{A3}$$

which is equation (6).

For the special case in which $g_0(n) = \frac{1}{N_m}$, in order to evaluate the denominator one must consider the sum

$$\sum_{n=0}^{N_m} e^{-nk\delta t} \, . \tag{A4}$$

In order to evaluate this sum, use the identity

$$\sum_{n=0}^{N_m} (1-p)^n = \frac{1-(1-p)^{N_m+1}}{p} \tag{A5}$$

with $p = 1 - e^{-k\delta t}$. Then

$$\sum_{n=0}^{N_m} e^{-nk\delta t} = \frac{1-e^{-(N_m+1)k\delta t}}{1-e^{-k\delta t}} \; . \tag{A6}$$

Using (A6) in (A3) given equation 9.

Mangel and Beder (1983) consider an inference problem for the situation in which n aggregations are found with a search effort t. They show that an estimate for the initial number of aggregations n_0 is

$$n_0 = \mathrm{Int}[n/1-e^{-k\delta t}] . \tag{A7}$$

Here Int(x) deontes the integer part of x. If the total catch from the n aggregations is H, so that the average catch is H/n, then an estimate for the initial biomass of the stock is

$$\begin{aligned} B_0 = (\tfrac{H}{n})\cdot n \; &= H \;\mathrm{Int}\left[\frac{n}{1-e^{-k\delta t}}\right] \\ &\approx \frac{H}{1-e^{-k\delta t}} \; . \end{aligned} \tag{A8}$$

By way of contrast, observe that the standard CPUE analysis proceeds by assuming that H(t), the harvest after t hours of fishing, and B_0 are related by

$$\frac{dH}{dt} = kq(B_0-H) \qquad H(0) = 0 . \tag{A9}$$

Here q is the catchability coeeficient.

The solution of this equation is

$$H(t) = B_0(1-e^{-kqt}). \tag{A10}$$

Consequently, if the harvest with fishing effort t is $\tilde{H}$, then an estimate of B_0 is

$$B_0 = \tilde{H}/1-e^{-kqt} . \tag{A11}$$

Appendix 2 Data Used in the Empirical Study

Table A2.1 gives the data on POP catch from 1976 and the first half of 1977.

Table A2.1

Search Time and Catch Data from

1976 and Jan-July, 1977

Search Time (hrs)	Catch (tons)
13.9	49
4.0	37.8
4.5	26.3
5.6	24.6
1.3	8.2
11.7	23.4
31.5	47.1
22.6	36.9
11.1	30.5
16.5	13.2
5.0	3.0
30.3	92.5
23.1	24.2

Total Catch = 341 tons

Table A2.2 gives the data on POP catch from July, 1977 to 1982

Table A2.2

Catch and Effort Data on POP

Year	Effort (Days fished)	Catch (t)	Year	Effort	Catch
July-Dec 1977	1.5	12.3	1978(cont)	2.5	11.1
	2.5	34.3		1.0	1.8
	3.8	63.6		2.0	8.4
	.5	.7		2.0	6.9
	2	59.1		.4	1.9
	3	54.1		.2	2.8
	1	26.5			
	.5	.5	1980	.3	.1
	2.5	60.1		.5	3.5
	4.0	97.1		.8	27.9
	2	21.6		.3	2.5
				2	16.5
1978	.5	1.2			
	.3	2.3	1981	1	17.1
	.5	4.3		1	13.0
	2	15.8		2	12.5
	1	.6			
	2	6.7	1982	1	13.4
	.5	3.5		.8	22.3
	.5	10.2		1	13.4
	1.0	11.4		1.2	36.3
	.3	.2			
	1.5	29.9			

Appendix 3 Algorithm Used to Predict Catch

This appendix contains a description of the algorithm used to predict POP catch in section 4. To start one is given $\hat{B}$ = the biomass estimate at the start of the period of interest, C_s = the catch per one hour set, and E = fishing effort. The first step is to convert E to daylight hours. If E is measured in days, the daylight hours, T_E are estimated by

$$T_E = 12 \cdot E. \tag{A12}$$

Assume that t_s of the total hours are spent searching (t_s is, of course, unknown). Then the expected number of sets is

$$\frac{\hat{B}(1-e^{-\delta t_s})}{C_s}. \tag{A13}$$

Since the set time in the fishery is nominally 1 hour, the time spent fishing is

$$t_F = \frac{\hat{B}(1-e^{-\delta t_s})}{C_s}. \tag{A14}$$

Assuming that the fishing and searching are the only daylight activites means that $T_E = t_F + t_s$ so that

$$T_E - t_s - \frac{\hat{B}(1-e^{-\delta t_s})}{C_s} = 0. \tag{A15}$$

Solving equation (A15) for t_s* allows one to estimate the harvest from the equation

$$\hat{H} = \hat{B}(1-e^{-\delta t_s}). \tag{A16}$$

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*A computer program to do this is available from the author.