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The Numerical Computation of Free Surface Flows

in a Wetland Environment

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Mathematical Notation

β_1, β_2	=	momentum correction factors to account for vertical averaging
$\Delta x, \Delta y$	=	distance between nodes in the x- and y-directions, [L]
Δt	=	time step, [T]
$\delta x, \delta y, \delta z$	=	dimensions of an infinitesimal element in space, [L]
$\hat{H}$	=	the height of the upstream flow above the crest of the weir, [L]
ρ	=	fluid (water) density, $\frac{[M]}{[L]^3}$
ρ_a	=	density of air, $\frac{[M]}{[L]^3}$
τ^b	=	bottom shear stress, $\frac{[M]}{[L][T]^2}$
τ_x^s, τ_y^s	=	shear stress due to wind in the x- and y-directions, $\frac{[M]}{[L][T]^2}$
τ_{yx}	=	shear stress in the x-direction acting on the y-plane, $\frac{[M]}{[L][T]^2}$
θ	=	wind stress coeficient (≈ 0.0026)
φ	=	the angle between the wind direction and the y-axis
ξ	=	water level elevation from the reference plane, [L]
ω	=	angular frequency of forcing
a	=	amplitude of forcing, [L]
C	=	Chezy coeficient, [L]
C_d	=	discharge coeficient for a broad crested weir
f	=	Coriolis parameter, $\frac{1}{[T]}$
g	=	acceleration of gravity, $\frac{[L]}{[T]^2}$
H	=	$\xi + h$, [L]
$\hat{H}$	=	H upstream from the weir, [L]
h	=	bottom elevation from the reference plane, [L]
n, j, k	=	indices for t, x and y, respectively
J	=	Bessel function of the first kind with order i
L	=	length of the system, [L]
M_i	=	mass entering system, [M]
M_o	=	mass leaving system, [M]
M_p	=	percent mass conserved, $\frac{[M]}{[M]}$
M_t	=	mass in the system at time t, [M]
M_{t_0}	=	initial mass in the system, [M]
$\mathbf{n}$	=	normal vector to the system boundary
n	=	porosity, $\frac{[L]^3}{[L]^3}$
Q	=	flow over the weir, $\frac{[L]^3}{[T]}$
S_0	=	bottom slope, $\frac{[L]}{[L]}$

t = time, [T]

U, V = vertically averaged velocities in the x- and y-directions , $\frac{[L]}{[T]}$

u, v, w = the velocities in the x, y, and z Cartesian coordinate directions, $\frac{[L]}{[T]}$

w = wind speed, $\frac{[L]}{[T]}$

x, y, z = Caresian coordinate directions

Y = height of the crest of the weir from the channel bottom, [L]

Y_i = Bessel function of the second kind with order i

Introduction

Transport due to the movement of water is one of the most important processes governing the fate of constituents in a wetland environment. As a result, a firm understanding of the hydrodynamic processes in a wetland environment is necessary in order to develop effective management strategies for wetland wastewater treatment.

In this paper, the equations of motion describing two-dimensional unsteady free-surface flow (shallow water equations or vertical averaged primitive equations) in a wetland environment are derived from basic principles. A numerical model based on the time centered implicit scheme presented by Leendertse [1967, et seq.] is developed. The motivation for developing the model lies in the need for a well written, structured code that can be easily incorporated as part of an optimization model to determine management strategies for wetland wastewater treatment.

Literature Review

The study of unsteady flow in open channels dates back to the 1600's. Newton [1687] analyzed celerity waves reaching the conclusion that the propagation of celerity waves is proportional to the square root of the wavelength. Lagrange [1788] was the first to develop the theory that the celerity waves along canals can be expressed as $C = (gH)^{\frac{1}{2}}$. In addition, Cauchy, A.L., Green, G., Laplace, P. S., and several others have studied flow in open channels. However, not until recently, with the development of the digital computer, has large scale simulation of flow in open channels been possible.

Simulation of open channel flow has been addressed in models of overall wetland ecosystems (Loucks [1981], Parker [1974]). These models address in great detail, nutrient cycling and biomass, however, typically fail to adequately address wetland hydrodynamics. Kadlec [1986] asserts that, of the basic ecosystem simulators reviewed by Mitsh [1983], "None of these approaches are sufficient to describe overland flow of a thin water sheet impeded by wetland vegetation".

In recognition of this, Kadlec [1986], presented a model for overland flow through vegetated areas. Although not explicitly stated, a kinematic routing approach was taken. Frictional and gravitational effects were assumed to dominate the momentum balance of the system. The resulting momentum equation was analogous to Manning's equation. In conjunction with an equation of continuity, a complete system of equations describing overland flow was formulated. A striking feature of the model was the introduction of a porosity term in the equation of continuity to account for the mass of water replaced by vegetation. A one dimensional algorithm was developed and validated. The two dimensional analog for the momentum equation was not presented and as a result, the model was inappropriate for the description many systems which can only be described adequately by a two dimensional model. In addition, for many systems, the assumption of negligible wind shear may not be valid.

The approach taken by Kadlec [1986] was not new to the modelling of free-surface flow. A great deal of successful research in the modelling free-surface flow has been performed in the past [Leendertse 1967, et seq.; Abbott, 1967; Lynch and Gray, 1979]. In these studies, the equations of motion were left in their more general form (i.e. the shallow water equations). The shallow water equations are sufficient to describe a much broader range of phenomenon (e.g. forcing due to wind shear) than addressed by Kadlec [1986]. Solutions to the shallow water equations will be addressed here.

The shallow water equations (vertically averaged primitive equations) are a set of nonlinear partial differential equations which are notoriously difficult to solve. Specifically, two equations for momentum and an equation of continuity describe the state of the system (i.e. velocities and water surface elevations) as a function of the main driving forces; wind, Coriolis, non-point source or sinks (e.g. precipitation, infiltration and evapotranspiration), and surface flow to and from the system. The equations are often simplified so a solution may be more easily obtained.

Sheng *et al.*[1978] compared the results from a rigid lid model with a full three-dimensional model. For a rigid lid approximation, $\frac{\partial \xi}{\partial t} = 0$, where ξ is the water surface elevation. The state variables thus become pressures and velocities. In a controlled wetland environment, $\frac{\partial \xi}{\partial t} = 0$ is most probably a good approximation. However, because the ultimate goal is optimal control of wetland wastewater treatment processes, the state of the system, in this case ξ, must be reflected in the equations of motions. As a result, the rigid lid approximation does not adequately describe the state of the system for the purposes of this study.

Several finite element models of two-dimensional free-surface flow have been developed [Grotkop, 1973;Norton et al., 1973;Connor and Wang, 1974;Connor and Brebbia, 1976;Kawahara, 1977;Koutitas, 1978;Herrling,1982;Walters and Cheng, 1979]. Gray [1980, 1982] critiqued several finite element models of free-surface flow. He examined the phenomenon of artificial and numerical damping. The use of excessive numerical or artificial damping to obtain stability and accuracy in the finite element simulation of free-surface flows is quite frequent. As a consequence, the parametric values of a validated model which relies on excessive artificial or numerical damping would most likely not reflect the parameters of the real system for which the model was validated. The use of finite elements in the simulation of free-surface flows is thus tenuous. Nonetheless, because of the popularity of the finite elements in the field of water resources, a brief overview of several models and their difficulties is presented.

One of the earliest finite element models of the shallow water equations was developed by Grotkop [1973]. He incorporated six-node, triangular-prism, space-time finite elements. Nonlinear terms were linearized using a lagging method and as a result, the solution for each time level involved solving a set of linear equations. The implicit weighting of the finite elements produced excessive numerical damping [Gray, 1982].

Norton et al. [1973], and Walters and Cheng [1979] developed models using isoparametric elements in space where the velocities and surface elevations were calculated using quadratic and linear basis functions, respectively. Solutions involved solving a set of nonlinear equations for each time step using a Newton-Raphson algorithm. The solution algorithm was not computationally efficient. Other algorithms have been shown to be an order of magnitude faster. In addition, Gray [1982] stated that "There is no requirement of higher order interpolation on velocity than surface elevation...".

Connor and Wang [1974] developed a model using linear triangular elements. The model is known by the acronym CAFE. The solution to the finite element approximations incorporated a lagging method similar to that of the Grotkop model. Despite its popularity, the model relies on excessive numerical damping to reduce node to node oscillations [Gray, 1982].

Kawahara [1977] developed a finite element model which involved a split-level explicit time stepping scheme. Gray [1982] found that the numerical damping in this model resulted from truncation error in the time derivative approximations. The solution was shown to converge to a different set of partial differential equations as Δt approached 0 and number of elements approached infinity.

Herling [1978] presented a model using "hybrid" finite elements. The use of "hybrid" elements reduced the number of equations in the elemental matrices, with linear triangular elements, from nine equations to three creating a much more efficient algorithm.

More recently [Lynch and Gray, 1979;Kinnmark and Gray, 1982; Kinnmark and Gray, 1984; Kinnmark, 1986], several numerical models of the modified wave equation have been presented. In these models, the governing equations were first transformed to a single second-order, in time, partial differential equation (modified wave equation. This equation was then coupled with the primitive momentum equation and solved using finite elements. Gray [1982] found that this approach did not introduce the numerical damping discussed earlier. In addition, Gray found that solutions agreed with analytical solutions and that excessive node-to-node oscillations were not a problem.

The collocation finite element method is said to offer advantages in the solution of some classes of nonlinear problems. Halabi and Shen [1981] applied the collocation method to the primitive equations. The model incorporated bi-cubic rectangular elements in an orthogonal curvilinear coordinate system. The model was said to be more efficient than finite difference methods, and to compare well with analytical solutions.

Jamart and Winter [1978] solve the vertically-integrated, time-dependent primitive equations using Fourier decomposition. The state variables were Fourier decomposed where they appeared linearly, and the nonlinear terms for advection and friction were evaluated iteratively. A variational principle together with finite elements was then used to solve the decomposed set of elliptic partial differential equations. The model was applied to the Hood Canal, Washington.

Finite element solutions offer advantages, namely, the ability to easily simulate complex geometry, and the ease in which boundary conditions are incorporated. However, despite these advantages and the claims of success amongst researchers, work demonstrating that the physics of two-dimensional vertically averaged flow can be simulated using existing finite element models is limited [Gray, 1982]. As a consequence, finite elements have gained little popularity in the solution of the shallow water equations as compared with finite difference methods. This is also due in part to the stability, accuracy, and simplicity of several finite difference algorithms for the solution of the shallow water equations.

The class of finite difference models most widely used for predictive purposes in the modelling of two-dimensional free-surface flows are the time centered implicit schemes (leapfrog in time, staggered grid) [Abbot, 1981]. Leendertse [1967, et seq.] presents the most widely used of these models. Its popularity is due in part to several extensive publications, produced by Leendertse, designed to permit an independent observer to reproduce the results. The solution method is almost identical to another introduced by Abbot [1967]. However, Abbott addressed a simplified quasi-linear version of primitive equations. In these time centered implicit schemes, numerical operators are used to decompose the equations of motion into what can be thought of as two one-dimensional problems. The one-dimensional problems result in a tridiagonal system for each row in the finite difference grid. Because the resulting systems are tridiagonal, the algorithm is very efficient. In addition, equations for each row may be solved for independently, reducing computer storage requirements. The method is also unconditionally stable allowing for large time steps.

Other implicit finite difference solutions which decompose the system into a set of nonlinear algebraic equations offer no advantages over the Leendertse solution. In fact, these algorithms are very inefficient and require more computer storage. In contrast, explicit finite difference schemes are very efficient with computer storage. However, to maintain stability, the Courant condition cannot be violated. For the purposes of this study, explicit schemes result in unreasonable restrictions on how the system is discretized.

The goal of this paper is to present a numerical model of the shallow water equations that can be incorporated as part of an optimal management model for wetland wastewater treatment. The governing equations are derived from first principles and the time centered implicit scheme as presented by Leendertse [1967, et seq.] is used as a solution method. A preliminary verification of the model, to insure that it conserves mass can reproduce known analytical solutions, is presented.

Model Formulation

The shallow water equations are derived from the basic principles of conservation of mass (continuity) and momentum. All assumptions are explicitly stated.

Conservation of Mass

The principle of conservation of mass requires that the total mass flowing out of an element in space in a given time must be equal to the reduction of mass within the element in that time.

Let ρ = fluid density

n = porosity of the element

u, v, w = the velocities in the x, y, and z Cartesian coordinate directions

$\delta x, \delta y, \delta z$ = the dimensions of an infinitesimal element in space

The rate of change of mass in the x-direction is then [McDowell and O'Connor, 1977; Leendertse, 1970]

$$\rho u n \delta y \delta z - \left[\rho u + \frac{\partial(\rho u)}{\partial x}\delta x\right] n \delta y \delta z = -\frac{\partial(\rho u)}{\partial x} n \delta x \delta y \delta z$$

The sum of this term and similar terms for the y- and z -directions can be equated to the time rate of change of mass within the element to yield

$$-\frac{\partial \rho}{\partial t}\delta x \delta y \delta z = \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\delta x \delta y \delta z$$

which can be written as

$$-\frac{\partial \rho}{\partial t} = \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$

If the fluid is assumed to be incompressible and have a constant density, then

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial y} = \frac{\partial \rho}{\partial z} = 0$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (1)$$

The assumption of incompressibility is very good for water in open channels. However, the assumption of a constant density fluid is often violated. For the purposes of this study, density gradients are assumed to be negligible and equation 1 valid.

Conservation of Momentum

The forces acting on an element in any given direction can be equated to the rate of change of momentum in that direction. Specifically, there are components of shear stress and pressure forces acting on the faces of the element. In addition, there are body forces due to gravity and Coriolis. Let τ_{yx} be the component of shear stress in the x-direction, acting on a plane perpendicular to the y-axis. Then the resultant force acting in the x-direction is the sum of the pressure, shear stress and body forces

$$\left[-\frac{\partial p}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}-\frac{fv}{n}\right] n\delta x\delta y\delta z$$

where p is the pressure and f is the Coriolis parameter. This can be equated to the mass in the element multiplied by its acceleration in the x-direction

$$\rho\frac{du}{dt}\delta x\delta y\delta z=\left[-\frac{\partial p}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}-\frac{fv}{n}\right] n\delta x\delta y\delta z \tag{2}$$

Similar expressions can be written for the y- and z -directions. Note that $\frac{du}{dt}$ is defined as

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt} \tag{3}$$

The equations presented thus far apply at a point in space and time. It will be assumed that the local average of components due to turbulent fluctuations is zero [McDowell and O'Connor, 1977]. Rather than adopt a new notation, u, v, and w will be considered to reflect local mean conditions.

The shear stress τ_{yx} is typically small except near steep banks and is omitted from equation 2. If the vertical components of force due to changes in momentum and gradients of vertical shear stress are negligible compared with the force due to gravity then equations 2 and 3 reduce to

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv+n\frac{\partial p}{\rho\partial x}-n\frac{\partial \tau_{zx}}{\rho\partial z}=0 \tag{4}$$

and for the y- and z-directions

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu+n\frac{\partial p}{\rho\partial y}-n\frac{\partial \tau_{zx}}{\rho\partial z}=0 \tag{5}$$

$$\frac{\partial p}{\partial z}+\rho g=0 \tag{6}$$

Averaging Over the Depth

If equations 1,4,5 and 6 are averaged over the depth of flow, the result is the shallow water, or primitive equations. These equations describe flows which are two -dimensional in plan.

Let ξ = the water level elevation from the reference plane

h = bottom elevation from the reference plane

H = $h+\xi$

Then equation 6 integrated over the vertical is

$$p_s - p + \int_{-h}^{\xi} \rho g dz = 0 \tag{7}$$

where p_s refers to the pressure at the water surface. If it is assumed that the density is constant (i.e. $\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial z} = 0$) then as a consequence of equation 7

$$\frac{\partial p}{\partial x} = g\rho \frac{\partial \xi}{\partial x}$$

and equation 4 becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + ng\frac{\partial \xi}{\partial x} - n\frac{\partial \tau_{zx}}{\rho \partial z} = 0$$

Averaging over the vertical

$$\frac{1}{H}\int_{-h}^{\xi}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + ng\frac{\partial \xi}{\partial x} - n\frac{\partial \tau_{zx}}{\rho \partial z}\right)dz = 0$$

Evaluating the terms in the integral individually

$$\frac{n}{\rho H}\int_{-h}^{\xi}\frac{\partial \tau_{zx}}{\partial z}dz = n\left[\frac{1}{\rho H}\tau_x^s - \frac{1}{\rho H}\tau_x^b\right]$$

where τ^b and τ^s denote the bottom and surface shear stress defined for the x- and y-directions as

$$\tau_x^b = g\frac{U(U^2+V^2)^{\frac{1}{2}}}{n^2C^2H}$$

$$\tau_x^s = \theta \rho_a w^2 \sin\varphi$$

$$\tau_y^b = g\frac{V(U^2+V^2)^{\frac{1}{2}}}{n^2C^2H}$$

$$\tau_y^s = \theta \rho_a w^2 \cos\varphi$$

where C is the Chezy coefficient, θ is the wind stress coeficient ≈ 0.0026 [Leendertse, 1967 et seq.], ρ_a is the atmospheric density, w is the wind speed, φ is the angle between the wind direction and the y-axis, and U and V are now the depth averaged x-direction and y-direction velocities. Note that the porosity, n, is already implicit in these velocities requiring that τ_x^b be divided by n^2 to scale the friction to the velocities in the pore space.

The surface gradient, Coriolis, acceleration and advective terms averaged over the vertical become

$$\frac{n}{H}\int_{-h}^{\xi} g\frac{\partial \xi}{\partial x}dz = ng\frac{\partial \xi}{\partial x}$$

$$\frac{1}{H}\int_{-h}^{\xi}(fv)dz = fV$$

$$\frac{1}{H}\int_{-h}^{\xi}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)dz = \frac{\partial U}{\partial t} + \beta_1 U\frac{\partial U}{\partial x} + \beta_2 V\frac{\partial U}{\partial y}$$

$$U = \frac{1}{H}\int_{-h}^{\xi} udz$$

$$V = \frac{1}{H}\int_{-h}^{\xi} vdz$$

where β_1 and β_2 are factors incorporated to account for the lack of information regarding the vertical velocity profile. In practice, the nonlinear terms are often either omitted, or β_1 and β_2 are chosen as unity [Leendertse, 1970;McDowell and O'Connor, 1977;Henderson, 1966].

Similarly the equation for continuity averaged over the vertical is

$$\frac{1}{H}\int_{-h}^{\xi}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dz = 0$$

Given that $(w)_\xi = n\frac{\partial \xi}{\partial t}$ and $(w)_{-h} = 0$, the result is

$$\frac{n\partial \xi}{\partial t} + \frac{\partial(HU)}{\partial x} + \frac{\partial(HV)}{\partial y} = 0$$

The resulting equations of motion applied to a point in plan for a homogeneous incompressible liquid become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + ng\frac{\partial \xi}{\partial x} + g\frac{u(u^2+v^2)^{\frac{1}{2}}}{nC^2H} - \frac{n}{\rho H}\tau_x^s = 0 \tag{8}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - fu + ng\frac{\partial \xi}{\partial y} + g\frac{v(u^2+v^2)^{\frac{1}{2}}}{nC^2H} - \frac{n}{\rho H}\tau_y^s = 0 \tag{9}$$

$$n\frac{\partial \xi}{\partial t} + \frac{\partial (Hu)}{\partial x} + \frac{\partial (Hv)}{\partial y} = 0 \tag{10}$$

where U and V have been replaced with u and v for notational convenience.

Numerical Transformation

The time centered implicit scheme presented by Leendertse [1967, et seq.] is used to transform equations 8, 9, and 10. The numerical scheme involves solving for u and ξ at time step $n+\frac{1}{2}$, and v and ξ at time step $n+1$. At each half time step, a tridiagonal system of linear equations for each row in the finite difference grid is solved. An alternative notation to that presented by Leendertse [1967, et seq.] is used to more clearly define how a solution to the transformed equations is obtained.

The numerical approximation for equation 10 at time step $n+\frac{1}{2}$ is [Leendertse, 1970]

$$\begin{aligned} 0 =& \frac{2n_{j,k}}{\Delta t}\left(\xi_{j,k}^{n+\frac{1}{2}} - \xi_{j,k}^{n}\right) \\ &+ \frac{1}{2\Delta x}\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j+\frac{1}{2},k-\frac{1}{2}} + \xi_{j+1,k} + \xi_{j,k}\right)^{n} u_{j+\frac{1}{2},k}^{n+\frac{1}{2}} \\ &- \frac{1}{2\Delta x}\left(h_{j-\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k-\frac{1}{2}} + \xi_{j-1,k} + \xi_{j,k}\right)^{n} u_{j-\frac{1}{2},k}^{n+\frac{1}{2}} \\ &+ \frac{1}{2\Delta y}\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k+\frac{1}{2}} + \xi_{j,k+1} + \xi_{j,k}\right)^{n} v_{j,k+\frac{1}{2}}^{n} \\ &- \frac{1}{2\Delta y}\left(h_{j+\frac{1}{2},k-\frac{1}{2}} + h_{j-\frac{1}{2},k-\frac{1}{2}} + \xi_{j,k-1} + \xi_{j,k}\right)^{n} v_{j,k-\frac{1}{2}}^{n} \end{aligned} \tag{11}$$

where j, k, and n index the discretized system for x, y and t, respectively. Similarly, the numerical approximation for equation 8 at time step $n+\frac{1}{2}$ is [Leendertse, 1970]

$$\begin{aligned} 0 =& \frac{1}{\Delta t}\left(u_{j+\frac{1}{2},k}^{n+\frac{1}{2}} - u_{j+\frac{1}{2},k}^{n-\frac{1}{2}}\right) - \left(\frac{f}{4} - \frac{u_{j+\frac{1}{2},k+1}^{n-\frac{1}{2}} - u_{j+\frac{1}{2},k-1}^{n-\frac{1}{2}}}{8\Delta y}\right)\bar{v} \\ &+ \frac{1}{2\Delta x}\left(u_{j+\frac{3}{2},k}^{n-\frac{1}{2}} - u_{j-\frac{1}{2},k}^{n-\frac{1}{2}}\right) u_{j+\frac{1}{2},k}^{n+\frac{1}{2}} + \frac{n_{j,k}g}{2\Delta x}\left(\xi_{j+1,k}^{n+\frac{1}{2}} - \xi_{j,k}^{n+\frac{1}{2}} + \xi_{j+1,k}^{n-\frac{1}{2}} - \xi_{j,k}^{n-\frac{1}{2}}\right) \\ &+ \frac{4g\left(u_{j+\frac{1}{2},k}^{n+\frac{1}{2}} + u_{j+\frac{1}{2},k}^{n-\frac{1}{2}}\right)\left\{\left(u_{j+\frac{1}{2},k}^{n-\frac{1}{2}}\right)^{2} + \frac{1}{16}\bar{v}^{2}\right\}^{\frac{1}{2}}}{n_{j,k}\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j+\frac{1}{2},k-\frac{1}{2}} + \xi_{j+1,k} + \xi_{j,k}\right)^{n}\left(C_{j+1,k} + C_{j,k}\right)^{2,n}} \\ &- \frac{2n_{j,k}\tau_{x}^{s}}{\rho\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j+\frac{1}{2},k-\frac{1}{2}} + \xi_{j+1,k} + \xi_{j,k}\right)^{n}} \end{aligned} \tag{12}$$

$$\bar{v} = \left(v_{j+1,k+\frac{1}{2}} + v_{j,k+\frac{1}{2}} + v_{j+1,k-\frac{1}{2}} + v_{j,k-\frac{1}{2}}\right)^{n}$$

Note that in equations 11 and 12, the water surface elevation, ξ, only appears at integer numbers of j and k (e.g. $\xi_{j,k+1}$), while u appears at integer numbers of k and half steps of j, and v appears at integer numbers of j and half steps of k.

The numerical approximation for equation 10 at time step $n+1$ is [Leendertse, 1970]

$$
\begin{aligned}
0 = &\frac{2n_{j,k}}{\Delta t}\left(\xi_{j,k}^{n+1} - \xi_{j,k}^{n+\frac{1}{2}}\right) \\
&+ \frac{1}{2\Delta x}\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j+\frac{1}{2},k-\frac{1}{2}} + \xi_{j+1,k} + \xi_{j,k}\right)^{n+\frac{1}{2}} u_{j+\frac{1}{2},k}^{n+\frac{1}{2}} \\
&- \frac{1}{2\Delta x}\left(h_{j-\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k-\frac{1}{2}} + \xi_{j-1,k} + \xi_{j,k}\right)^{n+\frac{1}{2}} u_{j-\frac{1}{2},k}^{n+\frac{1}{2}} \\
&+ \frac{1}{2\Delta y}\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k+\frac{1}{2}} + \xi_{j,k+1} + \xi_{j,k}\right)^{n+\frac{1}{2}} v_{j,k+\frac{1}{2}}^{n+1} \\
&- \frac{1}{2\Delta y}\left(h_{j+\frac{1}{2},k-\frac{1}{2}} + h_{j-\frac{1}{2},k-\frac{1}{2}} + \xi_{j,k-1} + \xi_{j,k}\right)^{n+\frac{1}{2}} v_{j,k-\frac{1}{2}}^{n+1}
\end{aligned}
\tag{13}
$$

The numerical approximation for equation 9 at time step $n+1$ is [Leendertse, 1970]

$$
\begin{aligned}
0 = &\frac{1}{\Delta t}\left(v_{j,k+\frac{1}{2}}^{n+1} - v_{j,k+\frac{1}{2}}^{n}\right) - \left(\frac{f}{4} - \frac{v_{j+1,k+\frac{1}{2}}^{n} - v_{j-1,k+\frac{1}{2}}^{n}}{8\Delta x}\right)\bar{u} \\
&+ \frac{1}{2\Delta y}\left(v_{j,k+\frac{3}{2}}^{n} - v_{j,k-\frac{1}{2}}^{n}\right) v_{j,k+\frac{1}{2}}^{n+1} + \frac{n_{j,k}g}{2\Delta y}\left(\xi_{j,k+1}^{n+1} - \xi_{j,k}^{n+1} + \xi_{j,k+1}^{n} - \xi_{j,k}^{n}\right) \\
&+ \frac{4g\left(v_{j,k+\frac{1}{2}}^{n+1} + v_{j,k+\frac{1}{2}}^{n}\right)\left\{\left(v_{j,k+\frac{1}{2}}^{n}\right)^2 + \frac{1}{16}\bar{u}^2\right\}^{\frac{1}{2}}}{n_{j,k}\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k+\frac{1}{2}} + \xi_{j,k+1} + \xi_{j,k}\right)^{n+\frac{1}{2}}\left(C_{j,k+1} + C_{j,k}\right)^{2,n+\frac{1}{2}}} \\
&- \frac{2n_{j,k}\tau_y^s}{\rho\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k+\frac{1}{2}} + \xi_{j,k+1} + \xi_{j,k}\right)^{n+\frac{1}{2}}}
\end{aligned}
\tag{14}
$$

$$
\bar{u} = \left(u_{j+\frac{1}{2},k+1} + u_{j-\frac{1}{2},k+1} + u_{j+\frac{1}{2},k} + u_{j-\frac{1}{2},k}\right)^{n+\frac{1}{2}}
$$

Equations 11 and 12 are solved simultaneously for ξ and u at time step $n+\frac{1}{2}$ given information at time steps n and $n-\frac{1}{2}$. With the solution for $n+\frac{1}{2}$, equations 13 and 14 are used to solve for ξ and v at time step $n+1$. The algorithm can then be repeated solving for ξ and u in equations 11 and 12.

To obtain a solution, equation 11 is simplified as follows

$$
\alpha_j u_{j-\frac{1}{2},k}^{n+\frac{1}{2}} + \beta\xi_{j,k}^{n+\frac{1}{2}} + \chi_j u_{j+\frac{1}{2},k}^{n+\frac{1}{2}} = A_j, \ \forall j
$$

where

$$
\alpha_j = -\frac{\Delta t}{4\Delta x}\left(h_{j-\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k-\frac{1}{2}} + \xi_{j-1,k} + \xi_{j,k}\right)^{n}
$$

$$
\beta = n_{j,k}
$$

$$\chi_j = \frac{\Delta t}{4\Delta x}\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j+\frac{1}{2},k-\frac{1}{2}} + \xi_{j+1,k} + \xi_{j,k}\right)^n$$

$$A_j = n_{j,k}\xi^n_{j,k} - \frac{\Delta t}{4\Delta y}\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k+\frac{1}{2}} + \xi_{j,k+1} + \xi_{j,k}\right)^n v^n_{j,k+\frac{1}{2}}$$
$$+ \frac{\Delta t}{4\Delta y}\left(h_{j+\frac{1}{2},k-\frac{1}{2}} + h_{j-\frac{1}{2},k-\frac{1}{2}} + \xi_{j,k} + \xi_{j,k-1}\right)^n v^n_{j,k-\frac{1}{2}}$$

Equation 12 is simplified to yield

$$\tilde{\alpha}\xi^{n+\frac{1}{2}}_{j,k} + \tilde{\beta}_j u^{n+\frac{1}{2}}_{j+\frac{1}{2},k} + \tilde{\chi}\xi^{n+\frac{1}{2}}_{j+1,k} = B_j,\ \forall j$$

where

$$-\tilde{\alpha} = \tilde{\chi} = n_{j,k}\frac{g\Delta t}{2\Delta x}$$

$$\tilde{\beta}_j = 1 + \frac{\Delta t}{2\Delta x}\left(u^{n-\frac{1}{2}}_{j+\frac{3}{2},k} - u^{n-\frac{1}{2}}_{j-\frac{1}{2},k}\right)$$
$$+ \frac{4g\Delta t\left\{\left(u^{n-\frac{1}{2}}_{j+\frac{1}{2},k}\right)^2 + \frac{1}{16}\bar{v}^2\right\}^{\frac{1}{2}}}{n_{j,k}\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j+\frac{1}{2},k-\frac{1}{2}} + \xi_{j+1,k} + \xi_{j,k}\right)^n\left(C_{j+1,k} + C_{j,k}\right)^{2,n}}$$

$$B_j = u^{n-\frac{1}{2}}_{j+\frac{1}{2},k} + \Delta t\left(\frac{f}{4} - \frac{u^{n-\frac{1}{2}}_{j+\frac{1}{2},k+1} - u^{n-\frac{1}{2}}_{j+\frac{1}{2},k-1}}{8\Delta y}\right)\bar{v}$$
$$- \frac{4g\Delta t u^{n-\frac{1}{2}}_{j+\frac{1}{2},k}\left\{\left(u^{n-\frac{1}{2}}_{j+\frac{1}{2},k}\right)^2 + \frac{1}{16}\bar{v}^2\right\}^{\frac{1}{2}}}{n_{j,k}\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j+\frac{1}{2},k-\frac{1}{2}} + \xi_{j+1,k} + \xi_{j,k}\right)^n\left(C_{j+1,k} + C_{j,k}\right)^{2,n}}$$
$$+ \frac{2n_{j,k}\Delta t\tau^s_x}{\rho\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j+\frac{1}{2},k-\frac{1}{2}} + \xi_{j+1,k} + \xi_{j,k}\right)^n} - n_{j,k}\frac{g\Delta t}{2\Delta x}\left(\xi^{n-\frac{1}{2}}_{j+1,k} - \xi^{n-\frac{1}{2}}_{j,k}\right)$$

Using this notation, the solution to 11 and 12 forms a tridiagonal system of linear equations for every row k. For a given row k, the coefficients α, β, χ, $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\chi}$ are evaluated $\forall j$. Given a tidal boundary at $j = 0$ (i.e. ξ_0 is known) and a closed boundary at $j = m$ (i.e. a known velocity $u = 0$ at $j = m + \frac{1}{2}$) then the matrix system that results is

$$
\begin{bmatrix}
\tilde{\beta}_0 & \tilde{\chi} & 0 & 0 & 0 & \dots & 0 \\
\alpha_1 & \beta & \chi_1 & 0 & 0 & \dots & 0 \\
0 & \tilde{\alpha} & \beta_1 & \tilde{\chi} & 0 & \dots & 0 \\
0 & 0 & \alpha_2 & \beta & \chi_2 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & \dots & \dots & \alpha_m & \beta
\end{bmatrix}
\begin{bmatrix}
u_{\frac{1}{2}} \\ \xi_1 \\ u_{\frac{3}{2}} \\ \xi_2 \\ \vdots \\ \vdots \\ \xi_m
\end{bmatrix}^{n+\frac{1}{2}}
=
\begin{bmatrix}
B_0 \\ A_1 \\ B_1 \\ A_2 \\ \vdots \\ \vdots \\ A_m
\end{bmatrix}^{n}
+
\begin{bmatrix}
-\tilde{\alpha}_0\xi_0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ -\chi_m u_{m+\frac{1}{2}}
\end{bmatrix}^{n+\frac{1}{2}}
, \forall k
$$

Equations 13 is simplified to

$$\gamma_k v_{j,k-\frac{1}{2}}^{n+1} + \delta \xi_{j,k}^{n+1} + \epsilon_k v_{j,k+\frac{1}{2}}^{n+1} = D_k, \ \forall k$$

where

$$\gamma_k = -\frac{\Delta t}{4\Delta y}\left(h_{j+\frac{1}{2},k-\frac{1}{2}} + h_{j-\frac{1}{2},k-\frac{1}{2}} + \xi_{j,k-1} + \xi_{j,k}\right)^{n+\frac{1}{2}}$$

$$\delta = n_{j,k}$$

$$\epsilon_k = \frac{\Delta t}{4\Delta y}\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k+\frac{1}{2}} + \xi_{j,k+1} + \xi_{j,k}\right)^{n+\frac{1}{2}}$$

$$
\begin{aligned}
D_k = n_{j,k}\xi_{j,k}^{n+\frac{1}{2}} &- \frac{\Delta t}{4\Delta x}\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j+\frac{1}{2},k-\frac{1}{2}} + \xi_{j+1,k} + \xi_{j,k}\right)^{n+\frac{1}{2}} u_{j+\frac{1}{2},k}^{n+\frac{1}{2}} \\
&+ \frac{\Delta t}{4\Delta x}\left(h_{j-\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k-\frac{1}{2}} + \xi_{j-1,k} + \xi_{j,k}\right)^{n+\frac{1}{2}} u_{j-\frac{1}{2},k}^{n+\frac{1}{2}}
\end{aligned}
$$

and equations 14 is simplified to

$$\tilde{\gamma}\xi_{j,k}^{n+1} + \tilde{\delta}_k v_{j,k+\frac{1}{2}}^{n+1} + \tilde{\epsilon}\xi_{j,k+\frac{1}{2}}^{n+1} = E_k, \ \forall k$$

where

$$-\tilde{\gamma} = \tilde{\epsilon} = n_{j,k}\frac{g\Delta t}{2\Delta y}$$

$$
\begin{aligned}
\tilde{\delta}_k = 1 &+ \frac{\Delta t}{2\Delta y}\left(v_{j,k+\frac{3}{2}}^{n} - v_{j,k-\frac{1}{2}}^{n}\right) \\
&+ \frac{4g\Delta t\left\{\left(v_{j,k+\frac{1}{2}}^{n}\right)^2 + \frac{1}{16}\tilde{u}^2\right\}^{\frac{1}{2}}}{n_{j,k}\left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k+\frac{1}{2}} + \xi_{j,k+1} + \xi_{j,k}\right)^{n+\frac{1}{2}}\left(C_{j,k+1} + C_{j,k}\right)^{2,n+\frac{1}{2}}}
\end{aligned}
$$

$$E_k = v^n_{j,k+\frac{1}{2}} + \Delta t \left(\frac{f}{4} - \frac{v^n_{j+1,k+\frac{1}{2}} - v^n_{j-1,k+\frac{1}{2}}}{8\Delta x} \right) \bar{u}$$

$$- \frac{4g\Delta t v^n_{j,k+\frac{1}{2}} \left\{ \left(v^n_{j,k+\frac{1}{2}} \right)^2 + \frac{1}{16}\bar{u}^2 \right\}^{\frac{1}{2}}}{n_{j,k} \left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k+\frac{1}{2}} + \xi_{j,k+1} + \xi_{j,k} \right)^{n+\frac{1}{3}} \left(C_{j,k+1} + C_{j,k} \right)^{2,n+\frac{1}{2}}}$$

$$+ \frac{2n_{j,k}\Delta t \tau^*_y}{\rho \left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k+\frac{1}{2}} + \xi_{j,k+1} + \xi_{j,k} \right)^{n+\frac{1}{2}}} - n_{j,k} \frac{g\Delta t}{2\Delta y} \left(\xi^n_{j,k+1} - \xi^n_{j,k} \right)$$

For equations 13 and 14, given closed boundaries at $k = 0$ and $k = l$ (boundary conditions will be discussed in detail in the next section) the matrix system that results is

$$\begin{bmatrix} \delta & \epsilon_0 & 0 & 0 & 0 & \dots & 0 \\ \tilde{\gamma} & \tilde{\delta}_0 & \tilde{\epsilon} & 0 & 0 & \dots & 0 \\ 0 & \gamma_1 & \delta & \epsilon_1 & 0 & \dots & 0 \\ 0 & 0 & \tilde{\gamma} & \tilde{\delta}_1 & \tilde{\epsilon} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & \gamma_l & \delta \end{bmatrix} \begin{bmatrix} \xi_0 \\ v_{\frac{1}{2}} \\ \xi_1 \\ v_{\frac{3}{2}} \\ \vdots \\ \vdots \\ \xi_l \end{bmatrix}^{n+1} = \begin{bmatrix} D_0 \\ E_0 \\ D_1 \\ E_1 \\ \vdots \\ \vdots \\ D_l \end{bmatrix}^{n+\frac{1}{2}} + \begin{bmatrix} -\gamma v_{-\frac{1}{2}} \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ -\epsilon_l v_{l+\frac{1}{2}} \end{bmatrix}^{n+1} , \forall j$$

Because the boundaries at $k = 0$ and $k = l$ are closed, the velocities in the right hand side vector are 0.

Boundary Conditions

Boundary value problems require detailed information on the state of the system along the boundaries. Often, detailed information on the state of the system is not known. Thus the specification of boundary conditions becomes a difficult process. In addition, the finite difference equations as presented are not adequate to describe certain boundary conditions. Modifications must be made to account for different boundary conditions and each modification may vary slightly depending on the system geometry.

Known Velocity

For a boundary where $\mathbf{v} \bullet \mathbf{n}$ is known, where $\mathbf{n}$ is the unit vector normal to the boundary and $\mathbf{v} = \mathbf{v}(u, v)$, the equation for continuity at time step $n + \frac{1}{2}$ is evaluated as

$$\beta\xi_{0,k} + \chi_0 u_{\frac{1}{2},k} = A_0 - \alpha_0 u_{-\frac{1}{2},k}$$

$$\alpha u_{\frac{1}{2},k} + \beta_0 \xi_{m,k} = A_m - \chi_m u_{m+\frac{1}{2},k}$$

where $u_{-\frac{1}{2},k}$ and $u_{m+\frac{1}{2},k}$ are known velocities and are equal to zero if the boundary is closed.

Weir Boundary

When the boundary may be represented as a weir, u on the boundary becomes a function of ξ at the previous time step . For a broad crested weir the flow over the weir is given as (French, 1985)

$$Q = C_d \frac{2}{3}(2g)^{\frac{1}{2}} \hat{H}^{\frac{3}{2}} b$$

$$C_d = \frac{0.65}{\left(1 + \frac{\hat{H}}{Y}\right)^{\frac{1}{2}}}$$

where $\hat{H}$ is the height of the upstream flow above the crest of the weir, Y is the height of the crest of the weir from the channel bottom, Q is the flow over the weir, b is the width of the weir, and g is the acceleration of gravity. The boundary value is then calculated by evaluating $\hat{H} = \xi - Y$ where Y is given, calculating Q, and determining the appropriate velocity from

$$u = \frac{Q}{\Delta y H}$$

or

$$v = \frac{Q}{\Delta x H}$$

The division by $\Delta x H$ or $\Delta y H$ distributes the flow over the computational grid at the boundary converting the flow rate to a velocity.

Unless information is known about velocities parallel to the boundary, they are usually taken to be zero (e.g., closed, tidal and weir boundary conditions). As a consequence, $A_j = \xi_{j,k}$ when $v_{j,k+\frac{1}{2}} = 0$ and $v_{j,k-\frac{1}{2}} = 0$ on the boundary.

Known Flow

A boundary where a known flow is given can be addressed in a similar manner as the weir boundary. The flow rate is converted to a velocity by dividing through by $\Delta x H$ or $\Delta y H$. It is important to note that specified velocities are vector quantities possessing both a magnitude and direction. The sign for the boundary condition must be chosen correctly to give the needed inflow or outflow.

Hydrodynamic Model Verification

The verification of a hydrodynamic model is a difficult process. Very few analytical solutions are available and those that are available are restricted to simplified geometries and boundary conditions. For the purposes of this study, verification of the hydrodynamic model consisted of reproducing known analytical solutions, and demonstrating conservation mass and momentum.

Problem 1: Sinusoidal Forcing in One-Dimension

A one-dimensional problem was formulated with the following boundary conditions

$$\xi(L,t) = a\cos(\omega t)$$

$$u(0,t) = 0$$

where a is the amplitude of forcing, ω is the angular frequency of forcing, and L is the length of the system.

The analytical solution for the problem specified is given as (Walters and Cheng, 1980)

$$\xi = a\frac{\cos(kx)}{\cos(kL)}\cos(\omega t)$$

$$u = ac\frac{\sin(kx)}{h\cos(kL)}\sin(\omega t)$$

where $k = \frac{\omega}{(gh)^{.5}}$, and $c = (gh)^{.5}$. The solution is only valid in the absence of friction.

Results

A rectangular basin was specified with a width of 2 km, L = 4 km, h = 10 m, a = 0.1 m, and a period of 1 hour for the forcing at the open boundary. The numerical model was solved with $C = 10^6$, Δt = 300sec, and Δx = 500m. Figures 1A and 1B show good agreement between time varying analytical and numerical solutions for elevations and velocities at 2 different spatial locations in the system. The poor fit at early times was due to the selection of inappropriate initial conditions.

Problem 2: Sinusoidal Forcing with a Specified a Bottom Slope

Problem 2 addressed the effects of a non-zero bottom slope. For the boundary conditions specified in problem 1, the analytical solution with zero friction and bottom slope S_0 is given as (Walters and Cheng, 1980)

$$\xi = [AJ_0(2(kx)^{.5}) + BY_0(2(kx)^{.5})]cos(\omega t)$$

$$u = -\left(\frac{g}{S_0 x}\right)[AJ_1(2(kx)^{.5}) + BY_1(2(kx)^{.5})]sin(\omega t)$$

where $h = S_0 x$, $A = \frac{aY_1(2(kx_0)^{.5})}{D}$, $B = -\frac{aJ_1(2(kx_0)^{.5})}{D}$, $D = Y_1(2(kx_0)^{.5})J_0(2(kx_1)^{.5}) - Y_0(2(kx_1)^{.5})J_1(2(kx_0)^{.5})$, $k = \frac{\omega^2}{S_0 g}$, x_1 is the location of the open boundary, $x_0 = x_1 - L$, $S_0 = \frac{dh}{dx}$, and J and Y are Bessel functions of the first and second kind, respectively, where the subscripts denote their order.

Results

A rectangular basin was specified with a width of 2 km, L = 4 km, h = 10 m at the deep end, a = 0.01 m, S_0 = 0.001, and a period of 1 hour for the forcing at the open boundary. The numerical model was solved with $C = 10^6$, Δt = 300sec, and Δx = 500m. As with problem 1, Figures 2A and 2B show good agreement between time varying analytical and numerical solutions for elevations and velocity at 2 different spatial locations in the system.

Problem 3: Conservation of Mass

The good agreement between numerical and analytical solutions in problems 1 and 2 verify that the model conserves mass and momentum under simplified conditions. More complex geometries than those of problems 1 and 2 require that the computer perform additional computations which need to be verified. A majority of these calculations were checked by hand. In addition, conservation of mass for a long term simulation was analyzed using the following relation for the percent mass conserved in a system

$$M_p = 100\frac{(M_t - \sum_t M_i + \sum_t M_o)}{M_{t_0}}$$

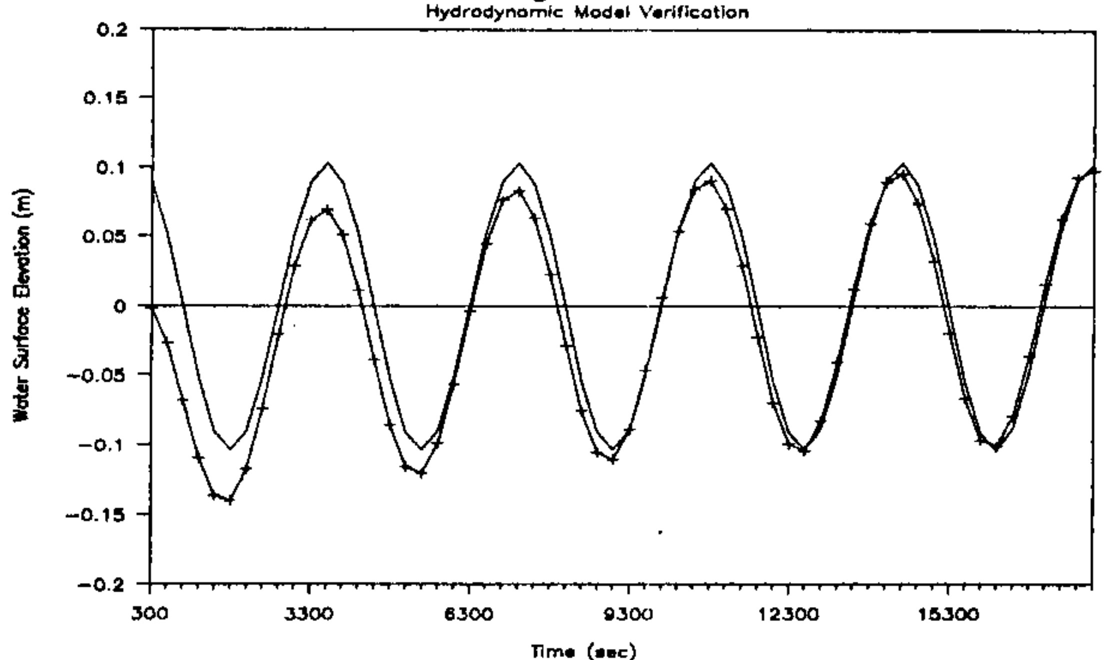
where M_p is the percent mass conserved, M_t is the mass in the system at time t, M_o is the mass leaving the system, M_i is the mass entering the system, and M_{t_0} is the initial mass in the system.

Results

A computational grid of the Allen Marsh of the Arcata Wastewater Treatment Facilities was used for simulation (not shown). The influent boundary condition was specified as 100,000 $\frac{cm^3}{s}$ and the effluent boundary was specified as a weir with $Y = 1.556\ m$. The bottom elevations were chosen as a constant $h = 0$ and the initial water surface elevations

Figure 1A

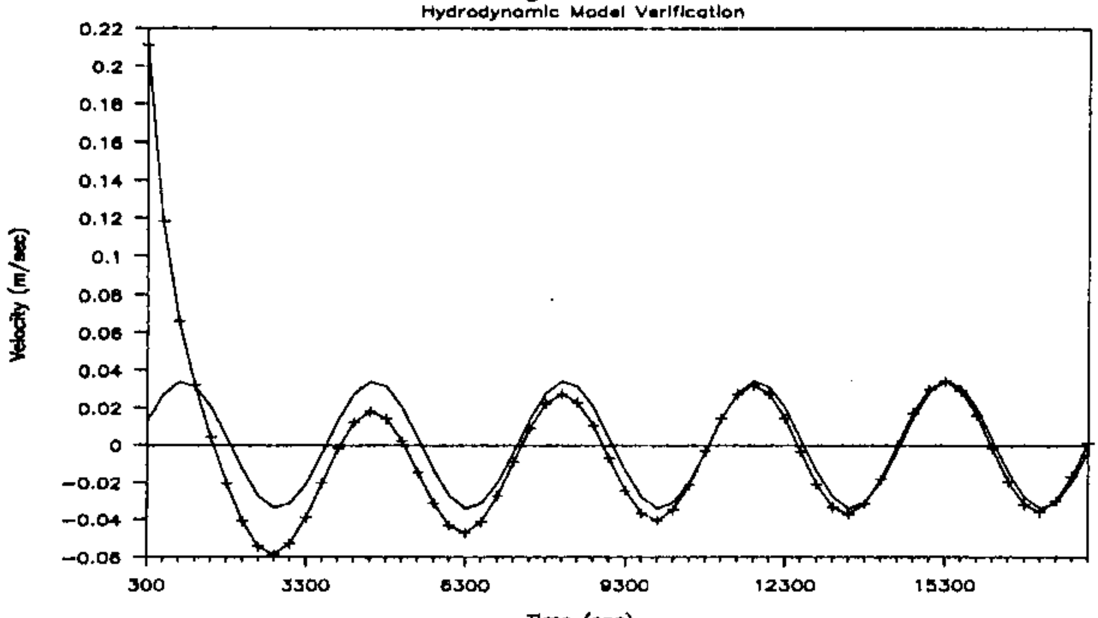
Hydrodynamic Model Verification



Analytical Solution + Numerical Solution

Figure 1B

Hydrodynamic Model Verification



Analytical Solution + Numerical Solution

Figure 2A

Hydrodynamic Model Verification

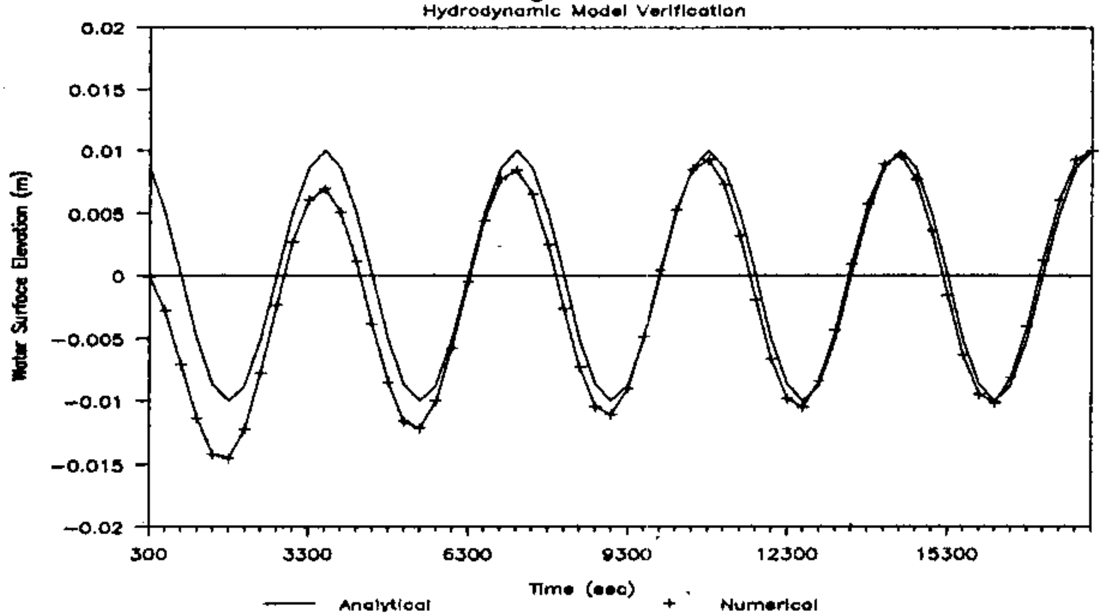
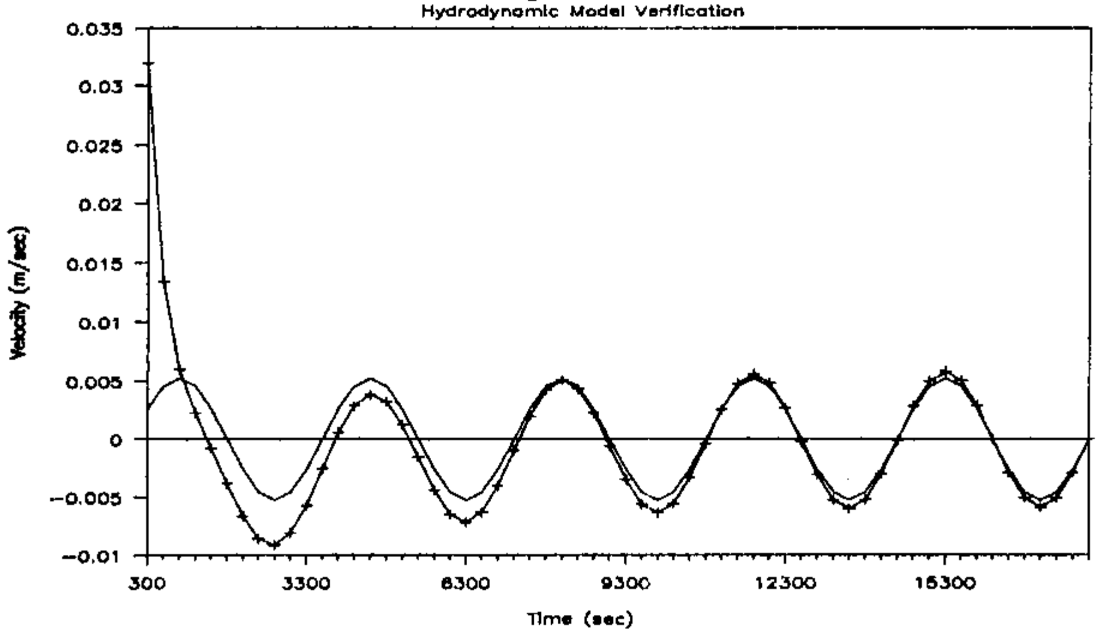


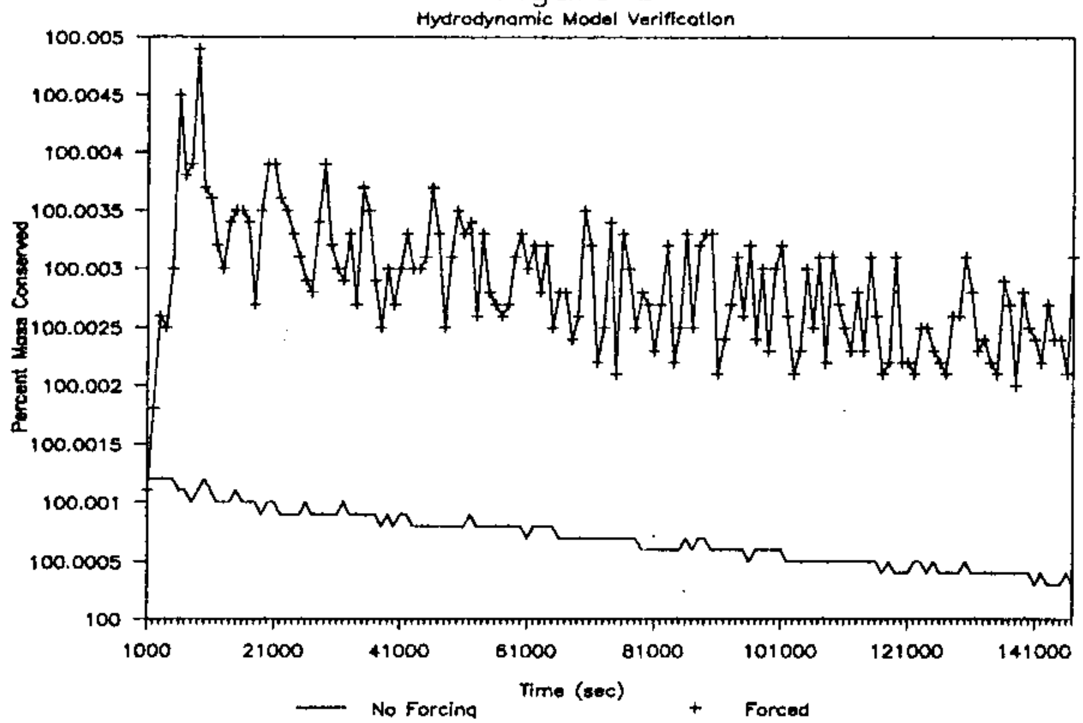
Figure 2B

Hydrodynamic Model Verification



was specified as 16.67 m. The values correspond to estimated values for the Allen Marsh obtained from field studies and the Arcata Dept. of Public Works. A Δt of 250 s was used. Plots of the percent mass conserved as a function of time are given in Figure 3. The Figure gives the percent mass conserved with no forcing (i.e., no flux boundary conditions). The loss of mass is due solely to computer precision. The Figure also gives percent mass conserved for the problem with the forcing that has been described. The results indicate that the model conserves mass within an acceptable tolerance.

Figure 3


Hydrodynamic Model Verification
100.005
100.0045
100.004
100.0035
100.003
100.0025
100.002
100.0015
100.001
100.0005
100
Percent Mass Conserved
1000
21000
41000
61000
81000
101000
121000
141000
Time (sec)
No Forcing
Forced

Conclusions and Recommendations

The equations of motion describing two-dimensional unsteady free-surface flow in a wetland environment were derived from basic principles. A numerical model based on the time centered implicit scheme presented by Leendertse [1967, *et seq.*] was developed. A computer model was formulated to solve the numerically transformed equations. The model was verified, under limited conditions, by comparison of model solutions to known analytical solutions. In addition, the model was shown to conserved mass for a more general case.

Recommendations

The following recommendations for further research are presented:

1. Develop and link, with the model presented, a computer model for mass transport.
2. Investigate assumptions of completely mixed flow for a number of different geometries and boundary conditions.
3. If applicable, calibrate the model for the Arcata System.
4. Using the model, investigate treatment processes in the Arcata System.

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