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AXISYMMETRICAL VIBRATIONS of UNDERWATER SPHERICAL SHELLS

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Axisymmetric Vibrations of Underwater Spherical Shells

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Abstract

Free vibrations of spherical shells in water have been investigated. By applying Hamilton's principle, a pair of basic coupled equations of motion is derived based on the bending theory. For harmonic motions, these equations are combined into a single sixth-order nonhomogeneous differential equation of motion in normal displacement. For a shell vibrating in water, the displacement of the shell and the hydrodynamic pressure of the water field form an interaction problem which characterizes all vibration problems of underwater structures and is solved by introducing the velocity potential of the water field. At the interface of the shell and water, it is assumed that the normal velocity of the shell is equal to that of the water field. The frequency equations for axisymmetric free vibrations are derived and the mode shapes are obtained. In each case examples are given and the results are plotted.

Introduction

The first work on the vibrations of a complete spherical shell is attributed to Lamb [1]*. He has investigated the shell vibrations in a vacuum on the basis of membrane theory and has concluded that for axi-symmetrical vibrations the amplitudes of the normal and the tangential displacements of the n^{th} mode are proportional to Legendre polynomials and associated Legendre polynomials of the n^{th} degree, respectively. On the basis of membrane theory, Baker [2, 3] has extended Lamb's study [1] by solving a particular initial value problem, and Junger [4] has discussed the case of vibrations of a spherical shell in fluid medium.

In this paper the free vibrations of a complete spherical shell in water will be investigated. A pair of basic coupled equations of motion is derived based on the bending theory by applying Hamilton's principle as has been done by Hayek [5]. After derivation of the equations, however, Hayek only considered the free vibrations of a complete spherical shell in a vacuum and a special case of harmonic forced vibration in water. For harmonic motions the basic coupled equations of this investigation are combined into a single sixth-order nonhomogeneous differential equation of motion in normal displacement.

For a shell vibrating in water, the displacement of the shell and the hydrodynamic pressure of the water field form an interacting problem. In this paper, the interacting problem is solved by introducing the velocity potential of the water field. The hydrodynamic pressure can be related to the velocity potential by using Bernoulli's

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*Numbers in brackets designate References at the end of paper.

equation for unsteady, irrotational flow of a nonviscous, incompressible fluid. With the assumption that the normal velocity of the shell is equal to that of the water field at the surface of the shell, the hydrodynamic pressure can be expressed in terms of the normal displacement of the shell.

The general frequency equation is derived, from which the frequency equations for the vibration in water based on the membrane theory and for the vibrations in a vacuum based on either bending or membrane theory can be obtained as special cases. The mode shapes of the shell are obtained in two groups called upper and lower branches. In the lower branch two special mode shapes are discussed. Examples are given and results are plotted for free vibrations both in water and in a vacuum.

1. Equations of Motion

The basic equations for axisymmetric, nontorsional vibrations of a spherical shell in water can be derived by use of Hamilton's principle. In the following the equations of motion are first derived for general motions in terms of normal and tangential displacements w and u in two coupled equations. If the shell performs harmonic motion, all the unknowns are harmonic variations in time, and the equations for general motion can be reduced to a pair of coupled equations of motion in normal functions W and U of normal and tangential displacements. Both equations for general motion and harmonic motion are written in operator form. Furthermore, for the harmonic motion the unknown U is eliminated to give a single sixth-order differential equation in W. This single equation can be applied to find the natural frequencies and the responses due to harmonic force excitations. The equations for general motion will be used for the aperiodic forced and free vibrations.

The same derivation has been used by Hayek [5] to give equations of motion in operator form for harmonic motion. Since a number of errors appear in the derivation and in the final equations of [5], the correct equations are derived as follows:

Consider an elastic spherical thin shell of thickness h, radius R, vibrating in water. The spherical coordinates (r,θ,ϕ) together with the sign convention are shown in Fig. 1. The strain energy density per unit surface of the shell derived by Novozhilov [6] is

$$\begin{aligned}\overline{V} &= \frac{Eh}{2(1-\nu^2)} [(\varepsilon_\theta+\varepsilon_\phi)^2-2(1-\nu)(\varepsilon_\theta\varepsilon_\phi-\frac{\varepsilon_{\theta\phi}^2}{4})] \\ &+ \frac{Eh^3}{24(1-\nu^2)} [(\kappa_\theta+\kappa_\phi)^2-2(1-\nu)(\kappa_\theta\kappa_\phi-\kappa_{\theta\phi}^2)]\end{aligned} \tag{1}$$

In Eq. (1), the first term represents the strain energy density of extension and shear; the second term that of bending and torsion. For axisymmetric vibrations, the strain-displacement relations [6] are

$$\begin{aligned}\varepsilon_\theta &= \frac{1}{R} [w + \frac{\partial u}{\partial\theta}] \\ \varepsilon_\phi &= \frac{1}{R} [w + u \cot \theta] \\ \varepsilon_{\theta\phi} &= 0 \\ \kappa_\theta &= \frac{1}{R^2} [- \frac{\partial^2 w}{\partial\theta^2} + \frac{\partial u}{\partial\theta}] \\ \kappa_\phi &= \frac{1}{R^2} [- \frac{\partial w}{\partial\theta} + u] \cot \theta \\ \kappa_{\theta\phi} &= 0\end{aligned} \tag{2}$$

The total strain energy of the shell is obtained by integrating the strain energy density over the entire middle surface of the shell as

$$V = \int_s \bar{V}\, ds = 2\pi R^2 \int_o^{\pi} \bar{V} \sin\theta\, d\theta \tag{3}$$

The total kinetic energy of the shell is

$$T = \frac{1}{2}\rho_s \int_v (u^2+w^2)dV = \pi\rho_s hR^2 \int_o^{\pi} (u^2+w^2)\sin\theta\, d\theta \tag{4}$$

The potential function of the hydrodynamic pressure p_a and surface forces f is

$$X = \int_s (p_a+f)w\,ds = 2\pi R^2 \int_o^{\pi} (p_a+f)w \sin\theta\, d\theta \tag{5}$$

By using Hamilton's principle,

$$\delta \int_{t_1}^{t_2} (T - V + X)dt = 0$$

together with Eqs. (1) to (5), the shell equations of motion are obtained as follows:

$$(1+\varepsilon)\left[\frac{\partial^2 u}{\partial\theta^2} + \cot\theta\,\frac{\partial u}{\partial\theta} - (\nu+\cot^2\theta)u\right] - \varepsilon\,\frac{\partial^3 w}{\partial\theta^3} - \varepsilon\cot\theta\,\frac{\partial^2 w}{\partial\theta^2}$$

$$+ (1+\nu+\varepsilon\nu+\varepsilon\cot^2\theta)\,\frac{\partial w}{\partial\theta} - \frac{1-\nu^2}{E}\,\rho_s\,R^2\,\frac{\partial^2 u}{\partial t^2} = 0 \tag{6a}$$

$$-\varepsilon\,\frac{\partial^3 u}{\partial\theta^3} - 2\varepsilon\cot\theta\,\frac{\partial^2 u}{\partial\theta^2} + [(1+\nu)(1+\varepsilon) + \varepsilon\cot^2\theta]\frac{\partial u}{\partial\theta} - [\varepsilon\cot^3\theta$$

$$+ 3\varepsilon\cot\theta - (1+\varepsilon)(1+\nu)\cot\theta]u + \varepsilon\left[\frac{\partial^4 w}{\partial\theta^4} + 2\cot\theta\,\frac{\partial^3 w}{\partial\theta^3}\right.$$

$$\left.- (1+\nu+\cot^2\theta)\frac{\partial^2 w}{\partial\theta^2} + (2\cot\theta + \cot^3\theta - \nu\cot\theta)\frac{\partial w}{\partial\theta}\right]$$

$$+ 2(1+\nu)w + \frac{1-\nu^2}{E}\,\rho_s\,R^2\,\frac{\partial^2 w}{\partial t^2} - \frac{1-\nu^2}{Eh}\,R^2(p_a+f) = 0 \tag{6b}$$

Changing the independent variable through the relation

$$x = \cos\theta, \qquad 0 \le \theta \le \pi\ ; \quad -1 \le x \le 1 \tag{7}$$

and introducing the differential operator

$$\nabla^2 = \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial}{\partial x}\right] = (1-x^2)\frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x} = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} \qquad (8)$$

we may rewrite Eqs. (6) in a compact form as

$$L_{uu}\, u + L_{uw}\, w = -\frac{1-\nu^2}{E}\rho_s R^2 \ddot{u} \qquad (9a)$$

$$L_{wu}\, u + L_{ww}\, w = -\frac{1-\nu^2}{E}\rho_s R^2 \ddot{w} + \frac{1-\nu^2}{Eh} R^2 (p_a + f) \qquad (9b)$$

where the differential operators are

$$
\begin{aligned}
L_{uu} &= -(1+\varepsilon)\left[(1-\nu) + \nabla^2 - \frac{1}{1-x^2}\right] \\
L_{uw} &= -(1-x^2)^{\frac{1}{2}}\frac{d}{dx}\left[\varepsilon(1-\nu) - (1+\nu) + \varepsilon\nabla^2\right] \\
L_{wu} &= \left[\varepsilon(1-\nu) - (1+\nu) + \varepsilon\nabla^2\right]\frac{d}{dx}(1-x^2)^{\frac{1}{2}} \\
L_{ww} &= \varepsilon\nabla^4 + \varepsilon(1-\nu)\nabla^2 + 2(1+\nu)
\end{aligned}
\qquad (10)
$$

Eqs. (9) will be used for motions which are aperiodic in time. For harmonic motions for which

$$
\begin{aligned}
u(x,t) &= U(x)\cos\omega t \\
w(x,t) &= W(x)\cos\omega t \\
p_a(x,t) &= P_a(x)\cos\omega t
\end{aligned}
\qquad (11)
$$

and

$$f(x,t) = F(x)\cos\omega t$$

Eqs. (9) may be transformed into

$$L_{uu}U + L_{uw}\, W = \Omega^2 U \qquad (12a)$$

$$L_{wu}U + L_{ww}\, W = \Omega^2 W + \frac{1-\nu^2}{Eh} R^2 (P_a + F) \qquad (12b)$$

where the frequency parameter Ω^2 is defined as

$$\Omega^2 = \frac{1-\nu^2}{E}\rho_s R^2 \omega^2 \qquad (13)$$

If we operate

$$[\varepsilon(1-\nu) - (1+\nu) + \varepsilon\nabla^2] \frac{\partial}{\partial x} (1-x^2)^{\frac{1}{2}}$$

on Eq. (12a), operate

$$[(1+\varepsilon)(1-\nu+\nabla^2) + \Omega^2]$$

on Eq. (12b), and add the results together, U is eliminated and the equation of motion in W is obtained as follows:

$$[\nabla^6 + a\nabla^4 + b\nabla^2 + c]\, W = \frac{1-\nu^2}{Eh} R^2 (d\nabla^2 + e)(P_a + F) \tag{14}$$

where

$$\begin{aligned} a &= 4 + \Omega^2 \\ b &= 5 - \nu^2 + \xi(1-\nu^2) - (\xi+\nu)\Omega^2 \\ c &= 2(1-\nu^2)(1+\xi) + [(1+3\nu)\xi-1+\nu]\Omega^2-\xi\Omega^4 \\ d &= 1 + \xi \\ e &= (1-\nu)(1+\xi) + \xi\Omega^2 \\ \xi &= \frac{1}{\varepsilon} = 12(R/h)^2 \end{aligned} \tag{15}$$

In order to solve the equation of motion (14) for a given forcing function F, it is necessary to derive an equation expressing the hydrodynamic pressure P_a in terms of the normal displacement of the shell W.

2. Hydrodynamic Pressure

The velocity potential of the water field can be related to the hydrodynamic pressure by using Bernoulli's equation for unsteady, irrotational flow of a nonviscous, incompressible fluid. The velocity potential can also be related to the normal displacement of the shell by the assumption that the normal velocity of the shell and that of water are the same at the surface of the shell. Then the hydrodynamic pressure of the water field and the normal displacement of the shell can be related together. For a complete spherical shell vibrating in an infinite water field, the velocity potential $\Phi(r,\theta,t)$ of this field satisfies Laplace's equation,

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta} = 0 \tag{16}$$

and the boundary conditions

(a) $\Phi \to 0$ as $r \to \infty$

(b) Φ finite at $\theta = 0$ and $\theta = \pi$

(c) $\left.\frac{\partial\Phi}{\partial r}\right|_{r=R} = -\dot{w}$

By the method of separation of variables, the substitution of

$\Phi = R(r)\ \Theta\ (\theta)\ T\ (t)$

into Eq. (16) yields [7]

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{n(n+1)}{r^2} R = 0 \tag{17}$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\ \frac{d\Theta}{d\theta} + n(n+1)\Theta = 0 \tag{18}$$

Eqs. (17) and (18) are in the form of Euler's equation and Legendre's equation, respectively. Their solutions are

$$R = E_{11}\ r^n + E_{12}\ r^{-(n+1)}$$

$$\Theta = E_{21}P_n(\cos\theta) + E_{22}Q_n(\cos\theta) = E_{21}P_n(x) + E_{22}Q_n(x)$$

The boundary condition (a) requires that $E_{11} = 0$. Since $Q_n(x) \to \infty$ as $x \to \pm 1$, and $P_n(x) \to \infty$ as $x \to -1$ unless n = integer [8], the boundary condition (b) yields $E_{22} = 0$ and n = integer. Thus

$$\Phi = T \sum_{n=0}^{\infty} E_n\ r^{-(n+1)} P_n(x) \tag{19}$$

The boundary condition (c) yields

$$\dot{w} = T \sum_{n=0}^{\infty} E_n (n+1) R^{-(n+2)} P_n(x) \tag{20}$$

Multiplying Eq. (20) by $P_n(x)$, integrating over x from -1 to 1, and applying the orthogonality condition of $P_n(x)$ as

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0 & \text{for } m \neq n \\ \frac{2}{2n+1} & \text{for } m = n \end{cases} \tag{21}$$

result in

$$T\, E_n = \frac{2n+1}{2n+2} R^{n+2} D_n(t) \tag{22}$$

where

$$D_n(t) = \int_{-1}^{1} \dot{w}(x,t)\, P_n(x) dx \tag{23}$$

Introducing Eq. (22) into Eq. (19), we obtain the velocity potential in (r,x,t) as

$$\Phi(r,x,t) = \sum_{n=0}^{\infty} \frac{2n+1}{2n+2} R^{n+2} D_n(t)\; r^{-(n+1)}\; P_n(x) \tag{24}$$

The relation between the velocity potential and the hydrodynamic pressure,

$$p_a = -\rho \frac{\partial \Phi}{\partial t}\Big|_{r=R} \tag{25}$$

which is obtained according to Lamb [9] becomes

$$p_a = -\rho R \sum_{n=0}^{\infty} \frac{2n+1}{2n+2} \frac{d}{dt} D_n(t)\; P_n(x) \tag{26}$$

For the harmonic motions p_a and w are given in Eq. (11). Differentiating Eq. (23) with respect to t gives

$$\frac{d}{dt} D_n(t) = \int_{-1}^{1} \ddot{w}(x,t)\; P_n(x) dx = -\omega^2 \cos w t \int_{-1}^{1} W(x)\; P_n(x) dx$$

Then the hydrodynamic pressure acting on the surface of the shell is obtained as follows:

$$P_a(x) = \omega^2 \rho R \sum_{n=0}^{\infty} \frac{2n+1}{2n+2} C_n P_n(x) \tag{27}$$

where

$$C_n = \int_{-1}^{1} W(x) P_n(x) dx \tag{28}$$

3. Frequency Equations

For the free vibration of the shell in a vacuum, the basic differential equation is homogeneous. The frequency equations are obtained by applying the boundary conditions to the solution of the basic equation. In the case of the shell vibrating in water, even for the so called free vibration, there is hydrodynamic pressure acting on the surface of the shell. Hence, the displacement of the shell and the hydrodynamic pressure of the water field form an interacting problem. Fortunately, the hydrodynamic pressure can be related to the displacement of the shell by introducing the velocity potential of the water field. This renders the interacting problem solvable.

Substituting the hydrodynamic pressure from Eq. (27) into the equation of motion (14), and setting $F = 0$ result in

$$[\nabla^6 + a\nabla^4 + b\nabla^2 + c]W(x) = \Omega^2 \frac{R\rho}{h\rho_s} \sum_{n=0}^{\infty} \frac{2n+1}{2n+2} C_n [d\cdot\nabla^2 + e] P_n(x) \tag{29}$$

In Eq. (29) the hydrodynamic pressure has been expressed in terms of Legendre polynomials. Based on the identities of Legendre polynomials as

$$\nabla^2 P_n(x) = -n(n+1) P_n(x) = -\lambda_n P_n(x)$$

$$\nabla^4 P_n(x) = \nabla^2[-\lambda_n P_n(x)] = \lambda_n^2 P_n(x)$$

$$\nabla^6 P_n(x) = \nabla^4[-\lambda_n P_n(x)] = -\lambda_n^3 P_n(x)$$

the operators at the left hand side of Eq. (29) indicate that the normal displacement $W(x)$ can be expressed in terms of Legendre polynomial expansion, that is

$$W(x) = \sum_{n=0}^{\infty} A_n P_n(x) \tag{30}$$

In a different manner Lamb [1] obtained the same conclusion by comparing his simpler equations of motion based on the membrane theory with Legendre differential equation. From Eq. (28) we have

$$C_n = \int_{-1}^{1} W(x) P_n(x) dx = \frac{2}{2n+1} A_n \tag{31}$$

Substituting Eqs. (30) and (31) into Eq. (29) yields

$$\sum_{n=0}^{\infty} A_n [-\lambda_n^3 + a\lambda_n^2 - b\lambda_n + c] P_n(x)$$

$$= \Omega^2 \frac{R\rho}{h\rho_s} \sum_{n=0}^{\infty} \frac{2n+1}{2n+2} A_n \frac{2}{2n+1} [-d\ \lambda_n + e] P_n(x) \tag{32}$$

Due to the orthogonality condition of $P_n(x)$, we may equate the coefficients of Eq. (32) to give the frequency equation as

$$\lambda_n^3 - a\lambda_n^2 + b\lambda_n - c - \Omega^2 \frac{R\rho}{h\rho_s} \frac{1}{n+1} (d\lambda_n - e) = 0 \tag{33}$$

Introducing the expressions of a, b, c, d and e from Eq. (15) into Eq. (33), and rearranging give

$$\Omega^4 \xi [1 + \frac{R\rho}{h\rho_s} \frac{1}{n+1}] - \Omega^2 [\lambda_n^2 + \lambda_n(\xi+\nu) + (1+3\nu)\xi - 1 + \nu$$

$$+ \frac{R\rho}{h\rho_s} \frac{1+\xi}{n+1} (\lambda_n - 1 + \nu)]$$

$$+ \{\lambda_n^3 - 4\lambda_n^2 + \lambda_n [5 - \nu^2 + \xi(1-\nu^2)] - 2(1-\nu^2)(1+\xi)\} = 0 \tag{34}$$

For each value of the integer n, Eq. (34) gives two distinct roots in Ω^2 which correspond to two distinct frequencies. The greater roots form the upper branch denoted by Ω_u^2; the others form the lower branch, Ω_ℓ^2. For $n = 0$, the roots are

$$\Omega_u^2 = \frac{2(1+\nu)}{1 + \frac{R\rho}{h\rho_s}} \tag{35a}$$

$$\Omega_\ell^2 = (1+\varepsilon)(-1+\nu) \tag{35b}$$

For $n = 1$, the roots are

$$\Omega_u^2 = (1+\varepsilon)(1+\nu)\left(1 + \frac{2}{1 + \frac{R\rho}{2h\rho_s}}\right) \tag{36a}$$

$$\Omega_\ell^2 = 0 \tag{36b}$$

Eq. (34) is the most general frequency equation for a complete spherical shell vibrating in water which takes into account the effects of membrane, bending and hydrodynamic pressure. Some special cases can be reduced from Eq. (34) as follows:

Case A. Vibration in water--Membrane Theory

For membrane theory $h \to 0$. Thus, multiplying Eq. (34) by $\frac{h^3}{12R^3}$ and taking the limit $h \to 0$ result in

$$\Omega^4 + \Omega^2(-\lambda_n + 1 - \nu) = 0 \tag{37a}$$

Case B. Vibration in a vacuum--Bending Theory

Substituting the mass density of water $\rho = 0$ into Eq. (34) gives

$$\Omega^4\xi - \Omega^2[\lambda_n^2 + \lambda_n(\xi+\nu) + (1 + 3\nu)\xi - 1 + \nu]$$
$$+ \{\lambda_n^3 - 4\lambda_n^2 + \lambda_n[5-\nu^2+\xi(1-\nu^2)] - 2(1-\nu^2)(1+\xi)\} = 0 \tag{37b}$$

Eq. (37b) is the same as Hayek's result [5].

Case C. Vibration in a vacuum--Membrane Theory

For membrane theory $h \to 0$, therefore

$$\xi = 12(R/h)^2 \to \infty \quad \text{or} \quad \frac{1}{\xi} \to 0$$

Multiplying Eq. (37b) by $\frac{1}{\xi}$ and taking the limit $\frac{1}{\xi} \to 0$ result in

$$\Omega^4 - \Omega^2[\lambda_n + 1 + 3\nu] + (1-\nu^2)(\lambda_n-2) = 0 \qquad (37c)$$

The roots of Eq. (37c) in Ω^2 are exactly the same as Baker's work [2].

4. Natural Frequencies

Numerical examples for natural frequencies are presented for a complete spherical steel shell vibrating both in a vacuum and in an infinite water field, for which Poisson's ratio $\nu = 0.3$ and the ratio of thickness to radius h/R=0 ~ 0.05. The density ratio of water to steel shell $\rho/\rho_s = 0.1304$. The first eleven values of the natural frequency parameters Ω^2 for each branch, computed from Eq. (34), are given in Table 1 for h/R=0.03. For comparison, the values of Ω^2 for the vibration in a vacuum are included in the last two columns of Table 1.

Note that the first root for the lower branch in Table 1 is a negative value which yields an imaginary frequency. Actually for this frequency the corresponding mode does not exist. For the lower branch the second root is zero. Therefore, the corresponding mode is merely a rigid body translation of the entire shell.

Numerical results are plotted in the solid and the dotted lines which denote vibrations in water and vacuum, respectively. In Fig. 2, the natural frequency parameter Ω^2 is plotted versus the ratios of thickness to radius h/R for various mode number n. In Figs. 3 and 4, the parameter Ω^2 is plotted versus the mode number n for various ratios of thickness to radius h/R. Fig. 3 emphasizes the lower branch, while Fig. 4 emphasizes the upper branch.

5. Mode Shapes

The normal displacement $W(x)$ has been expressed in terms of Legendre polynomials as shown in Eq. (30). After substituting $W(x)$ into equation of motion (12a), an observation is made on the basis of the form of the operators L_{uu} and L_{uw} of Eq. (10) and the following identities for Legendre polynomials and associated Legendre polynomials

$$\nabla^2 P_n(x) = -\lambda_n P_n(x)$$

$$P_n^1(x) = -(1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_n(x) \tag{38}$$

$$[\nabla^2 - \frac{1}{1-x^2}] P_n^1(x) = -\lambda_n P_n^1(x)$$

This observation indicates that the tangential displacement $U(x)$ can be expressed in terms of associated Legendre polynomials. Thus we may assume that the mode shapes $W_n(x)$ and $U_n(x)$ for the n^{th} mode of the shell are proportional to the appropriate Legendre polynomials of degree n,

$$W_n(x) = a_n P_n(x)$$

$$U_n(x) = b_n P_n^1(x) \tag{39}$$

Lamb [1], in obtaining normal displacement in terms of Legendre polynomials obtained the tangential displacement in terms of associated Legendre polynomials simultaneously.

Since $W_n(x)$ and $U_n(x)$ are mode shapes associated with the same mode number n, the amplitudes a_n and b_n must be related as follows: Substituting Eq. (39) into equation of motion (12a), and using the identities as shown in Eq. (38) result in

$$-(1+\varepsilon)b_n(1-\nu-\lambda_n)P_n^1(x) - a_n[\varepsilon(1-\nu)-(1+\nu)-\varepsilon\lambda_n][-P_n^1(x)] = \Omega^2 b_n P_n^1(x) \quad (40)$$

Equating the coefficients gives

$$b_n[(1+\varepsilon)(\lambda_n-1+\nu)-\Omega^2] = a_n[\varepsilon(\lambda_n-1+\nu)+1+\nu] \quad (41)$$

Thus, we define

$$H_n = \frac{b_n}{a_n} = \frac{\varepsilon(\lambda_n-1+\nu)+1+\nu}{(1+\varepsilon)(\lambda_n-1+\nu)-\Omega^2} \quad (42)$$

Let us normalize the normal displacement at the north pole, $x = 1$. Since $P_n(1) = 1$, Eq. (39) gives

$$a_n = 1 \quad (43)$$

with one exception for $n = 0$ of the lower branch to be discussed. Thus the mode shapes become

$$W_n(x) = P_n(x)$$
$$U_n(x) = H_n P_n^1(x) \quad (44)$$

where H_n are given in Eq. (42). Since for each value of the integer n there are two distinct frequencies, there must also be two corresponding mode shapes. Thus Eq. (42) gives two distinct values for H_n, those corresponding to the upper branch are denoted by H_n^u, the others corresponding to the lower branch are denoted by H_n^ℓ.

In Section 4, we have pointed out that there are two special frequencies. The first special case is for $n = 0$ of the lower branch, which yields the imaginary frequency. Substituting $\Omega_\ell^2 = (1+\varepsilon)(-1+\nu)$ from Eq. (35b) into Eq. (41) gives

$$a_o = 0 \quad (45)$$

Furthermore, $P_0^1(x) = 0$. Therefore, it can be concluded that this mode has no displacements in both normal and tangential directions. This proves that corresponding to the imaginary frequency the mode does not exist. In solving a particular initial value problem, Baker [2] has arrived at the same result of $a_0 = 0$. The second special case is for $n = 1$ of the lower branch. Substituting the zero value of Ω_ℓ^2 from Eq. (36b) into Eq. (42) gives

$$H_1 = 1 \tag{46}$$

Thus, Eq. (44) yields

$$\begin{aligned} W_1(x) &= P_1(x) = x = \cos\theta \\ U_1(x) &= P_1^1(x) = -(1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_1(x) = -\sin\theta \end{aligned} \tag{47}$$

At any section θ, the vertical displacement at any point is

$$W_1 \cos\theta - U_1 \sin\theta = 1 \tag{48}$$

and the horizontal displacement at any point is

$$W_1 \sin\theta + U_1 \cos\theta = 0 \tag{49}$$

These conditions prove that this mode is merely a rigid body translation of the entire shell.

The first eleven values of H_n, computed from Eq. (42), are given in Table 2 for $\nu = 0.3$, $h/R = 0.03$ and $\rho/\rho_s = 0.1304$. For comparison, the values of H_n for the vibration in vacuum are included in the last two columns of Table 2.

All the mode shapes can be determined from Eq. (44) except for $n = 0$ of the lower branch, which is a nonexisting mode. Eq. (44)

indicates that the n^{th} normal mode shape of the shell is exactly the same as Legendre polynomial of degree n. It may be mentioned that this relation holds for both the cases of shell vibration in water and in vacuum. The n^{th} tangential mode shape is proportional to associated Legendre polynomial of degree n with the proportional factor H_n. Table 2 indicates that for the lower branch the values of H_n^{ℓ} for the vibration in vacuum are close to that in water, while for the upper branch they differ greatly. The first six modes are shown in Fig. 5 to 15 for the shell vibrating in water based on the bending theory for $\nu = 0.3$, $h/R = 0.03$ and $\rho/\rho_s = 0.1304$. The solid and the dotted lines denote the displaced and the equilibrium positions, respectively.

Acknowledgement

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Table 1. Natural Frequency Parameters Ω^2 (Bending Theory) for ν=0.3, h/R=0.03

n	Vibration in Water Ω_u^2	Vibration in Water Ω_ℓ^2	Vibration in Vacuum Ω^2	Vibration in Vacuum Ω_ℓ^2
0	0.486	-0.700	2.600	-0.700
1	2.119	0.0	3.900	0.0
2	6.120	0.243	7.411	0.492
3	12.190	0.361	13.216	0.695
4	20.262	0.445	21.132	0.798
5	30.327	0.521	31.092	0.876
6	42.384	0.603	43.071	0.962
7	56.434	0.705	57.061	1.076
8	72.479	0.839	73.056	1.234
9	90.520	1.019	91.056	1.454
10	110.557	1.261	111.058	1.751

Table 2. Amplitude of Tangential Displacement H_n for Unit Normal Displacement (Bending Theory) for ν=0.3, h/R = 0.03

n	Vibration in Water H_n^u	Vibration in Water H_n^ℓ	Vibration in Vacuum H_n^u	Vibration in Vacuum H_n^ℓ
0	-1.096	-	-0.394	-
1	-1.588	1.000	-0.500	1.000
2	-1.587	0.257	-0.616	0.270
3	-1.463	0.119	-0.679	0.122
4	-1.355	0.069	-0.711	0.070
5	-1.270	0.045	-0.728	0.045
6	-1.205	0.032	-0.737	0.032
7	-1.154	0.024	-0.742	0.024
8	-1.112	0.019	-0.746	0.018
9	-1.077	0.015	-0.747	0.015
10	-1.048	0.012	-0.747	0.012

References

1. H. Lamb, "On the Vibrations of a Spherical Shell," Proc. London Math. Soc., vol. 14, 1882, pp. 50-56.

2. W. E. Baker, "Axisymmetric Modes of Vibration of Thin Spherical Shell," J. Acoust. Soc. Am., vol. 33, no. 12, 1961, pp. 1749-1758.

3. W. E. Baker, W. C. L. Hu, and T. R. Jackson, "Elastic Response of Thin Spherical Shells to Axisymmetric Blast Loading," Trans. ASME, vol. 4, Dec. 1966, pp. 800-806.

4. M. C. Junger, "Vibrations of Elastic Shells in a Fluid Medium and the Associated Radiation of Sound," J. Appl. Mech., vol. 74, 1952, pp. 439-445.

5. S. Hayek, "Vibrations of a Spherical Shell in an Acoustic Medium," J. Acoust. Soc. Am., vol. 40, no. 2, 1966, pp. 342-348.

6. V. V. Novozhilov, Thin Shell Theory, 2nd ed. P. Noordhoff Ltd., Groningen, The Netherlands, 1964, p. 47, and p. 23.

7. P. Moon and D. E. Spencer, Field Theory Handbook, Springer-Verlag, Berlin, 1961, p. 26.

8. W. Magnus, F. Oberhettinger, and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, 3rd ed., Springer-Verlag, New York, 1966, p. 198, p. 192.

9. H. Lamb, Hydrodynamics, 6th ed., Dover Publications, Inc., New York, 1945, p. 363.

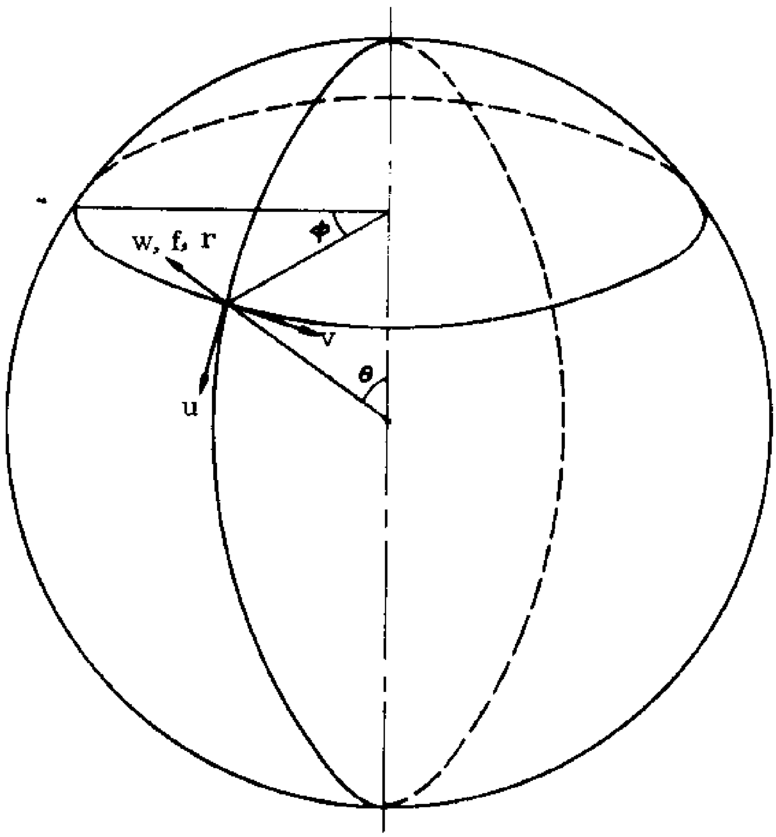

Fig. 1 Spherical coordinates

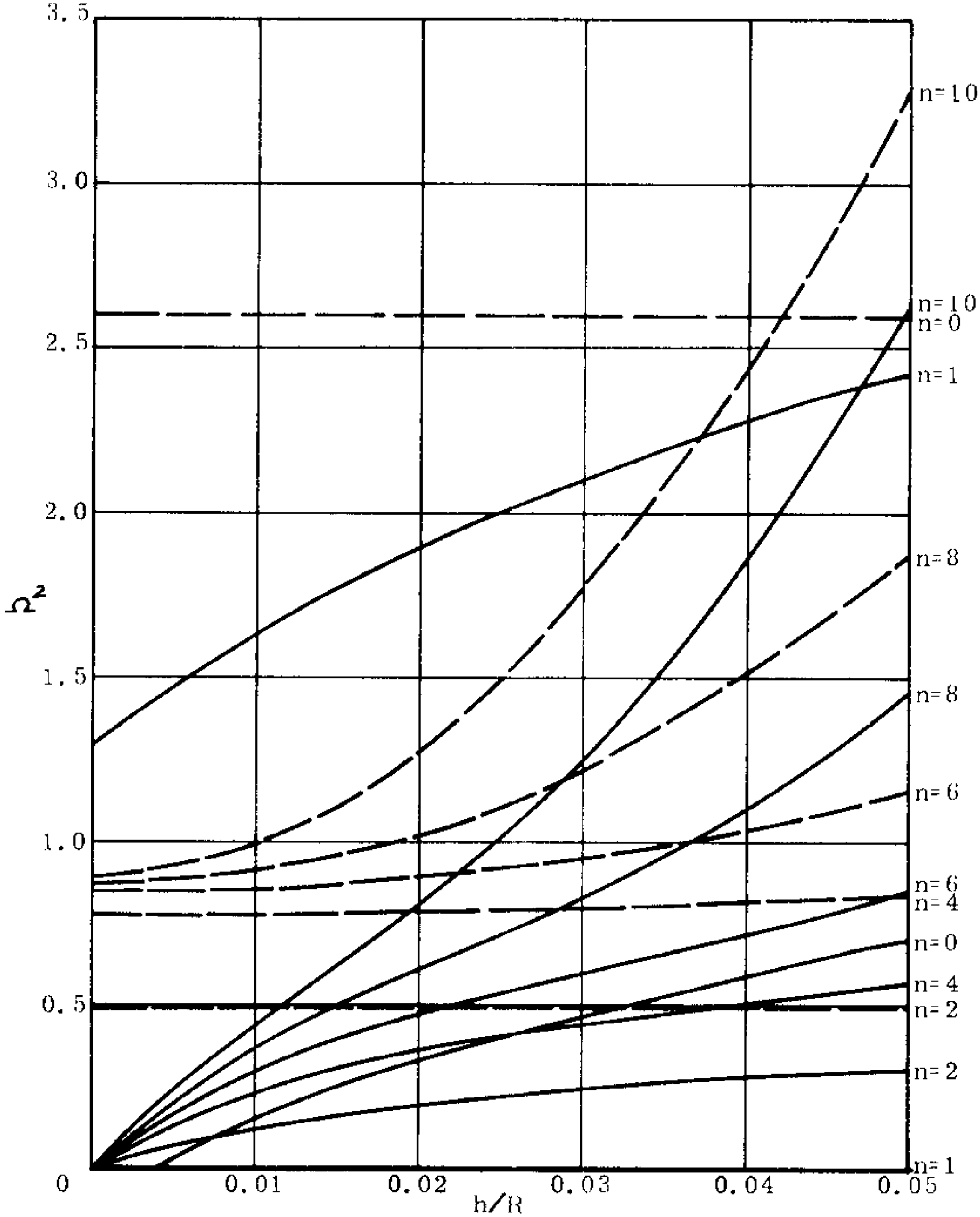

Fig. 2 Natural frequency parameter Ω^2 vs h/R for a complete spherical shell

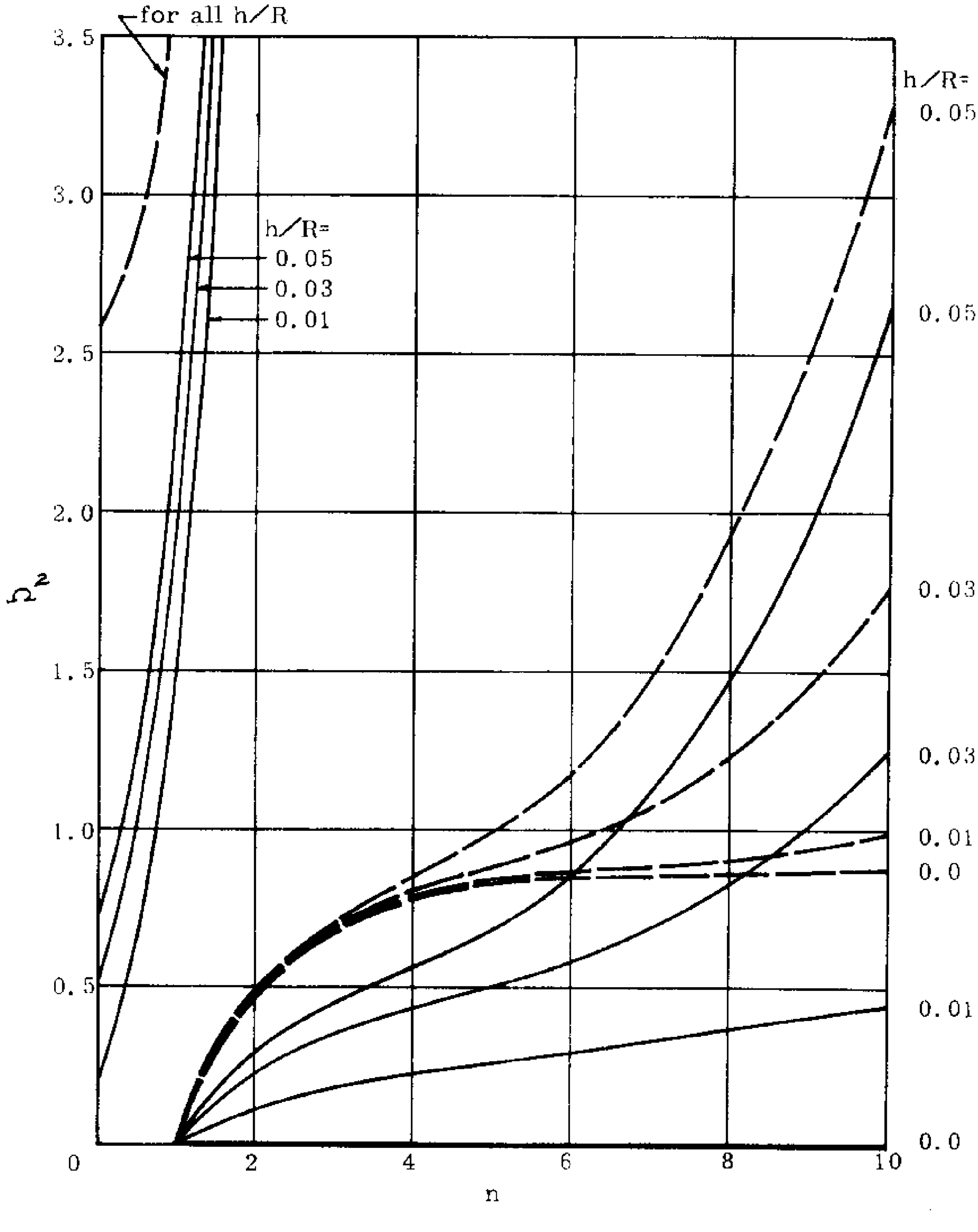

Fig. 3 Natural frequency parameter Ω^2 vs mode number n for a complete spherical shell

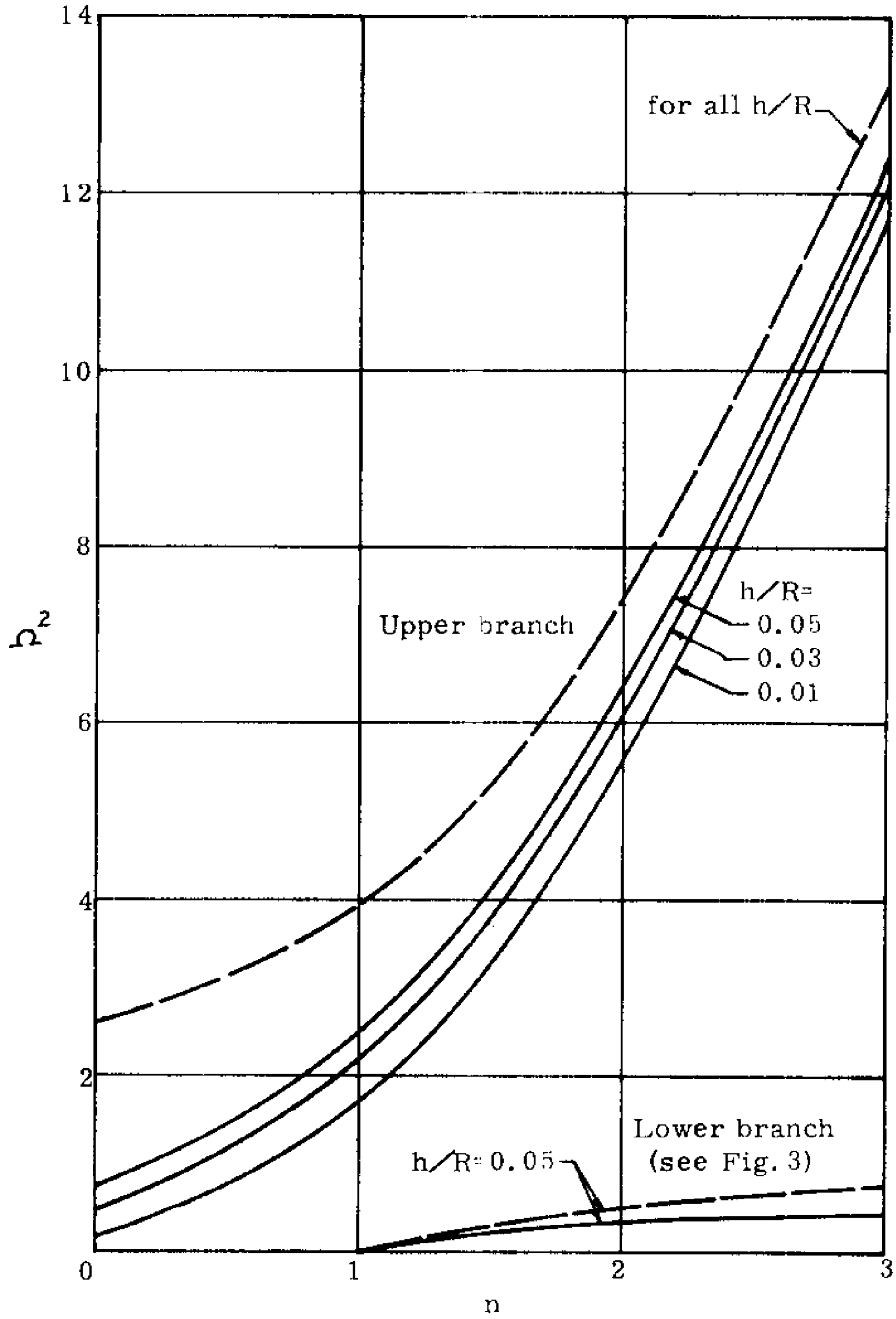

Fig. 4 Natural frequency parameter Ω^2 vs mode number n for a complete spherical shell--Upper branch

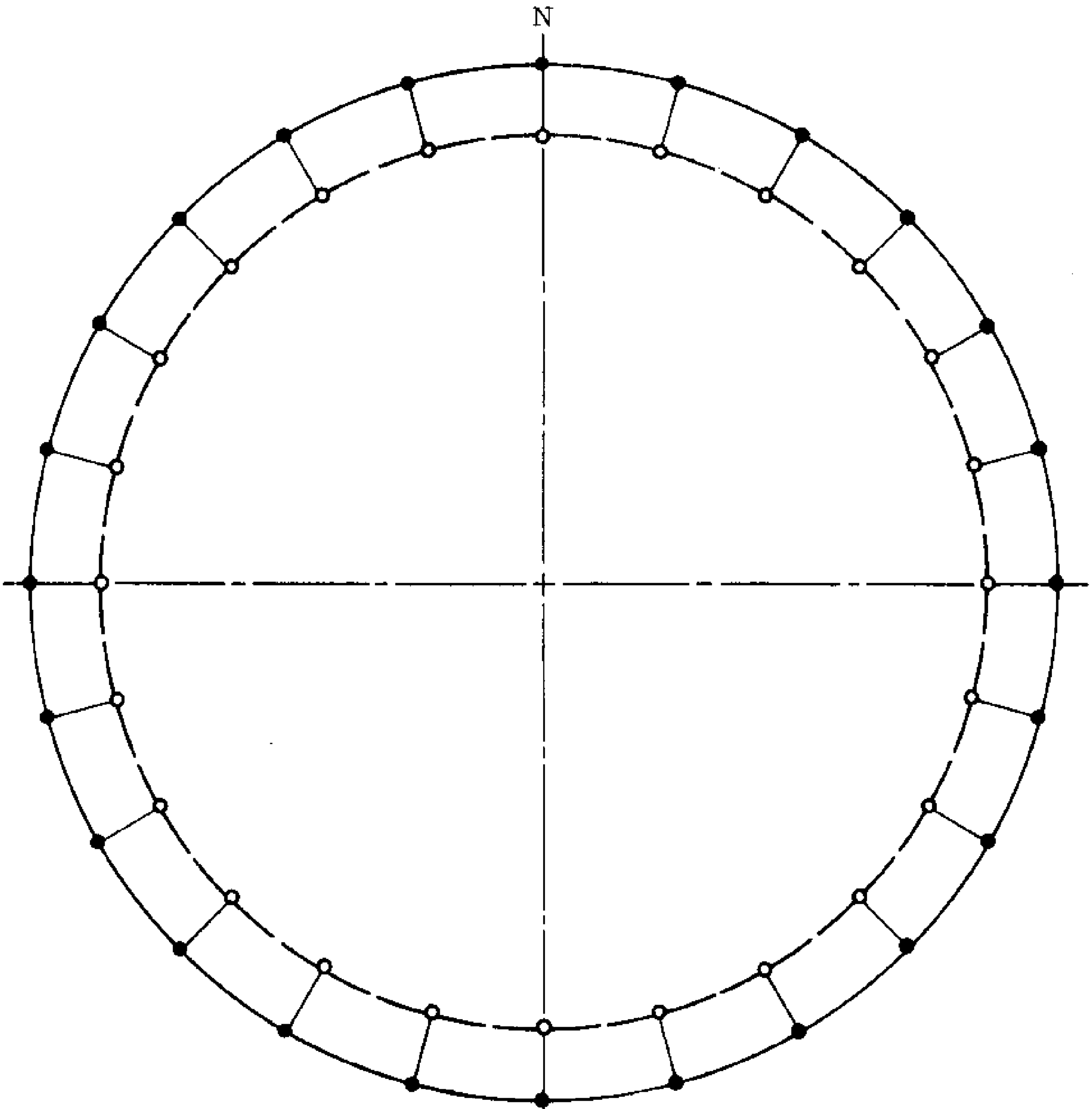

Fig. 5 The mode shape for n=0 of the upper branch

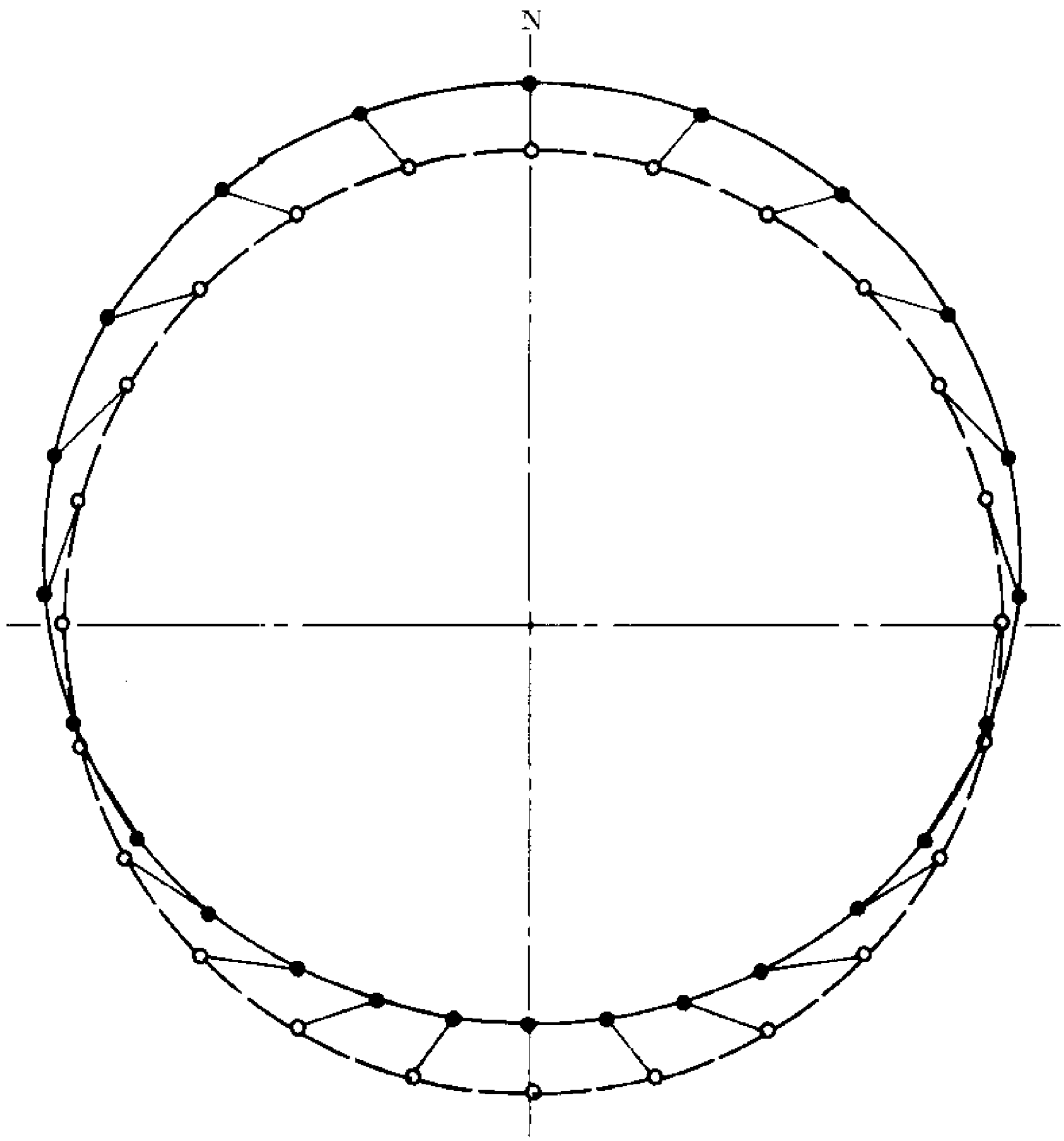

Fig. 6 The mode shape for n=1 of the upper branch

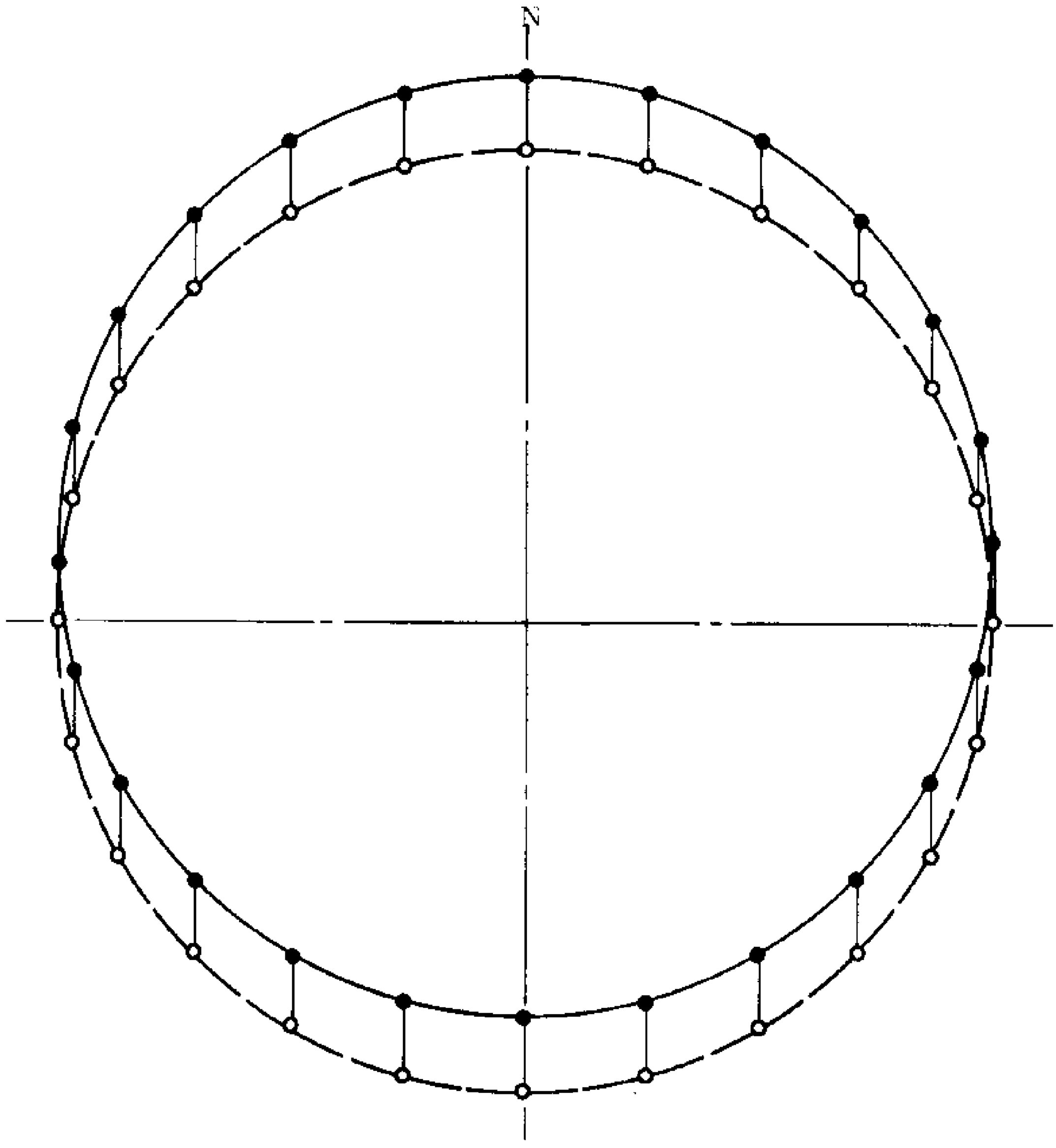

Fig. 7 The mode shape for n=1 of the lower branch

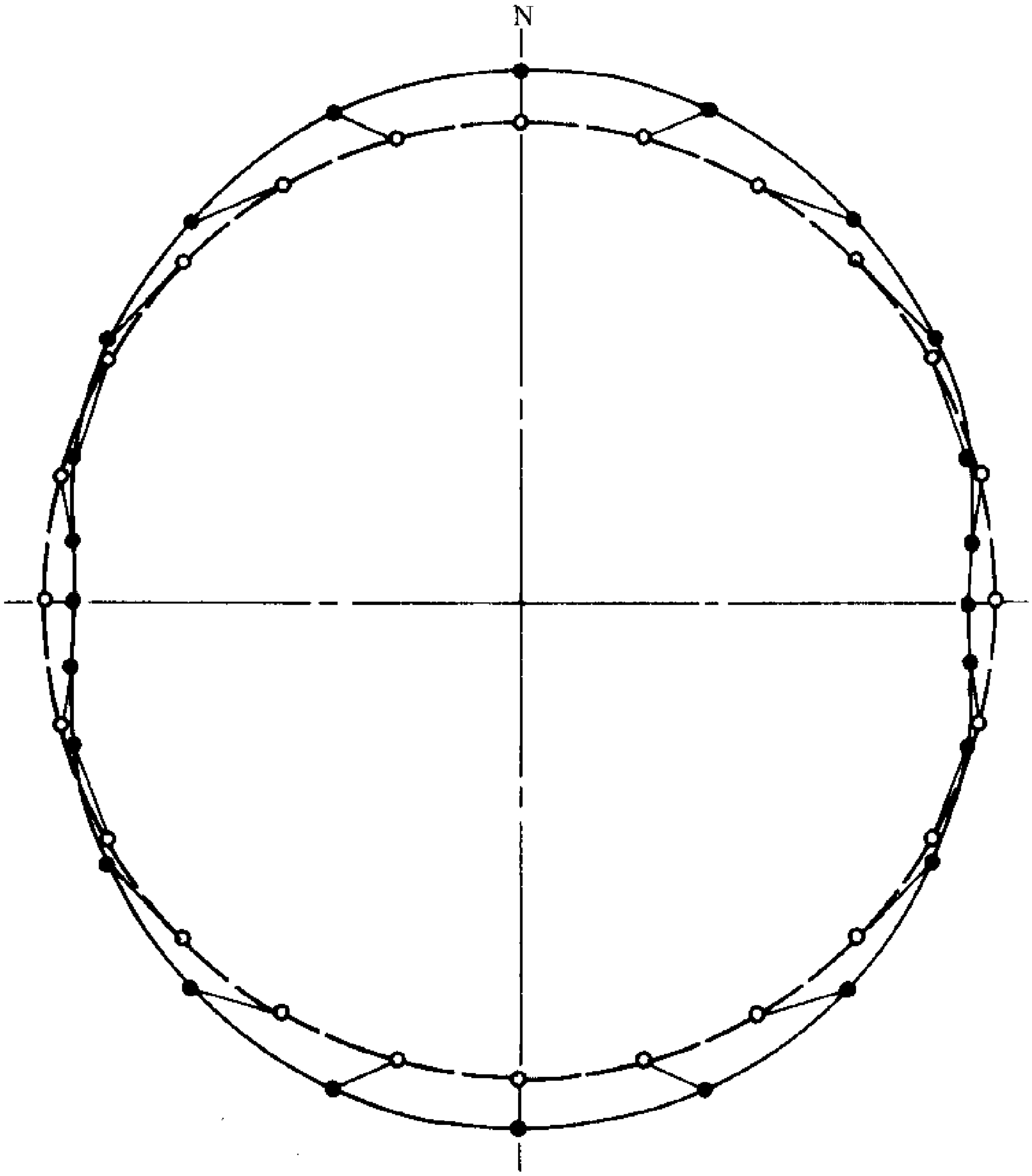

Fig. 8 The mode shape for n=2 of the upper branch

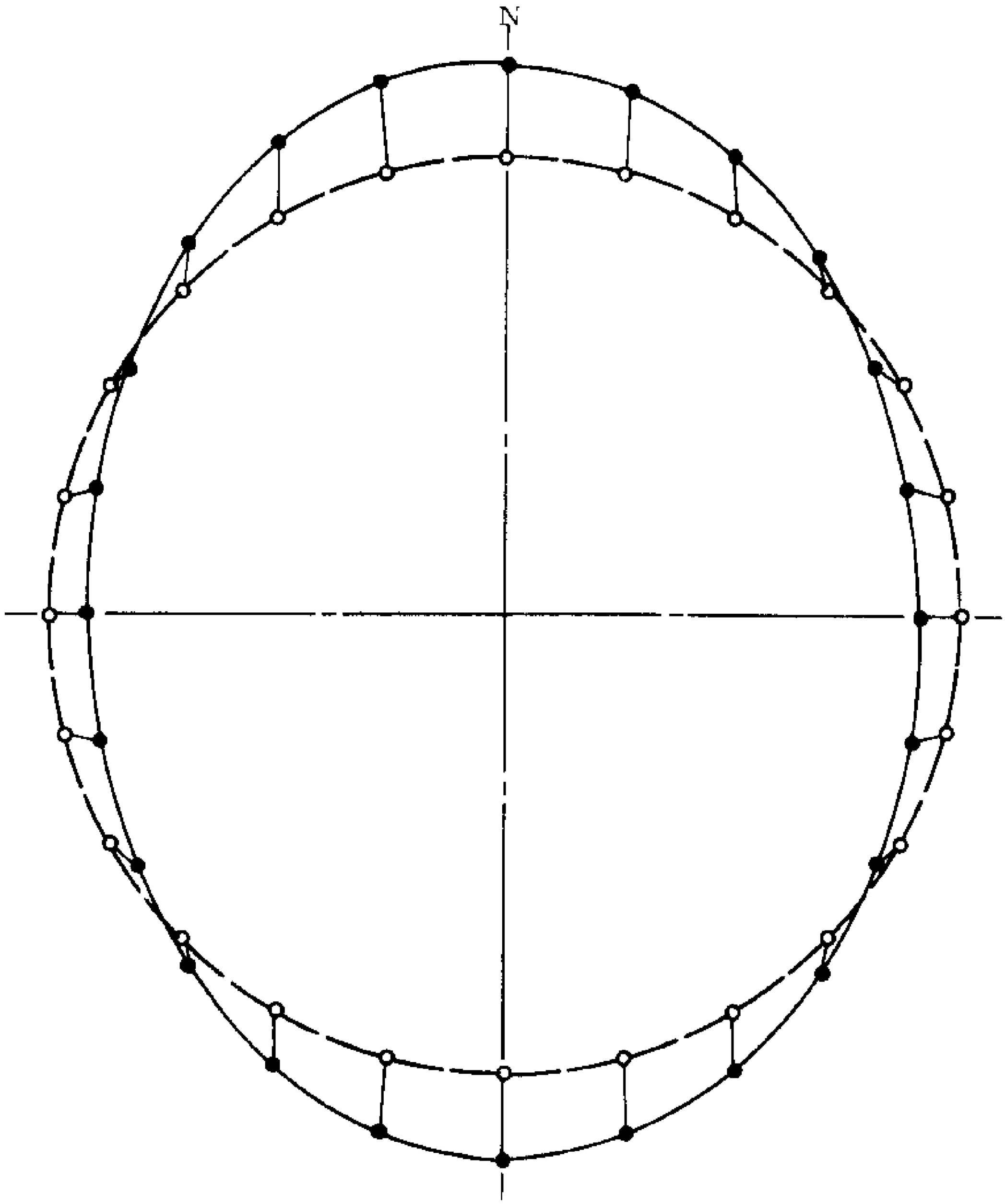

Fig. 9 The mode shape for n=2 of the lower branch

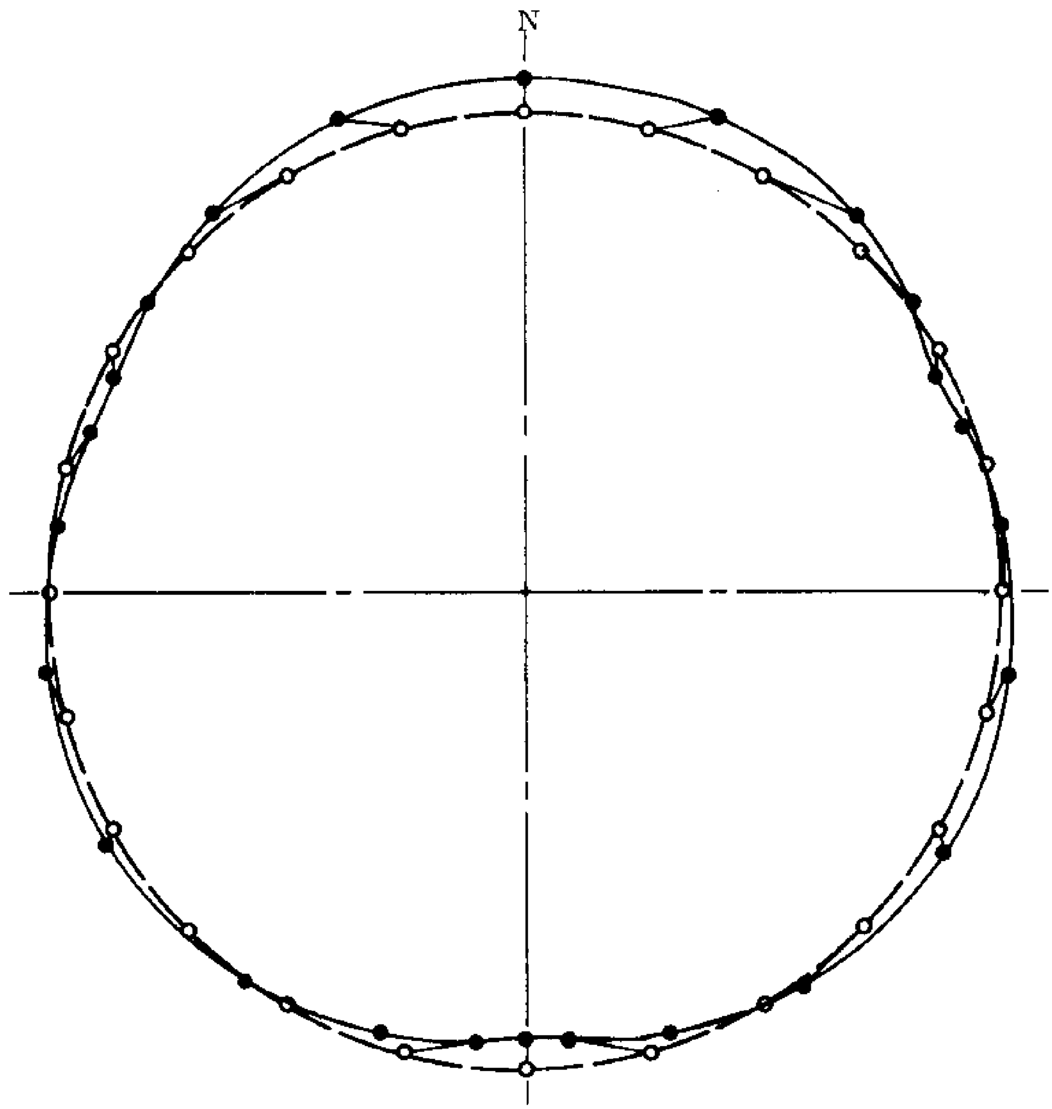

Fig. 10 The mode shape for n=3 of the upper branch

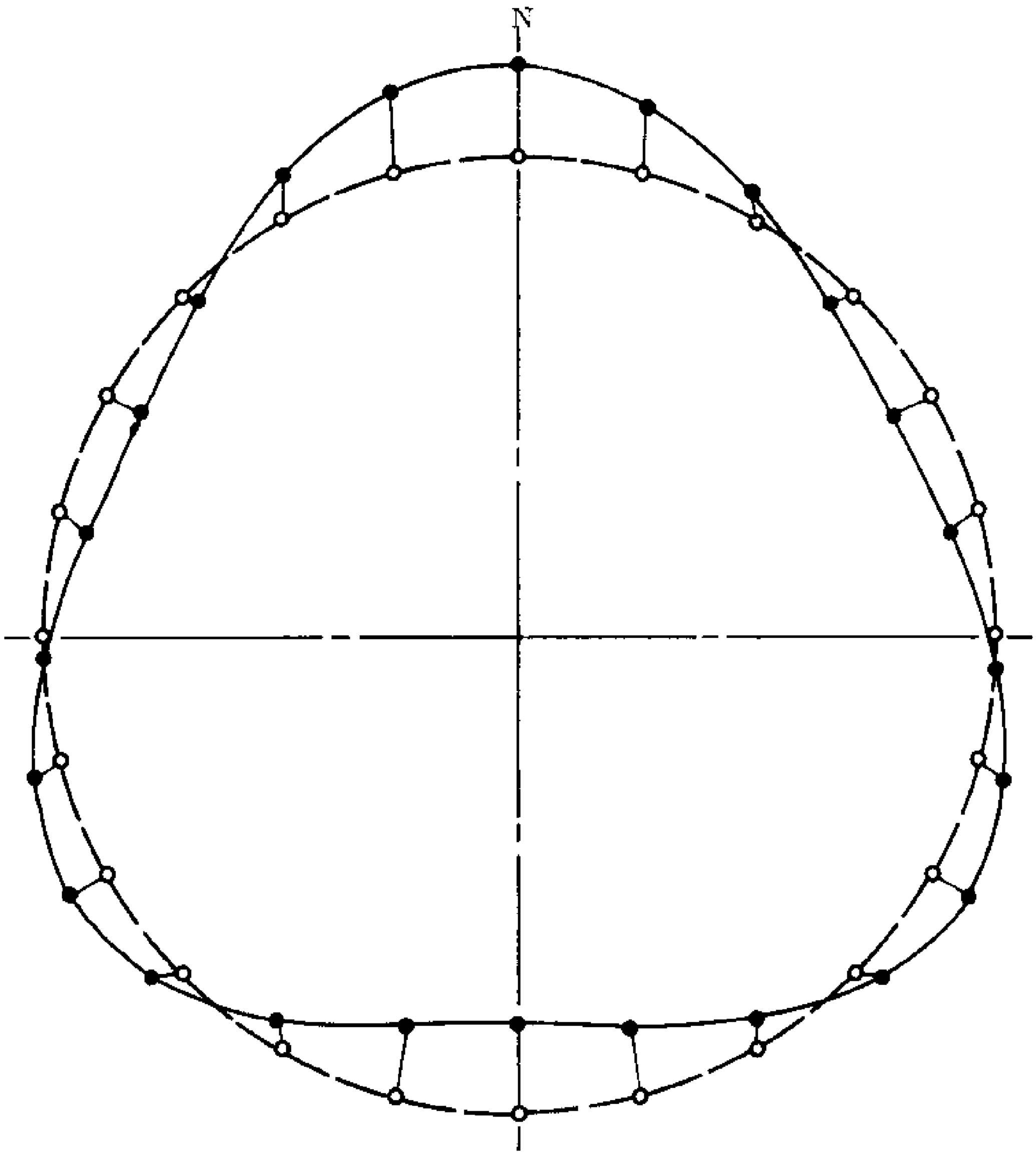

Fig. 11 The mode shape for n=3 of the lower branch

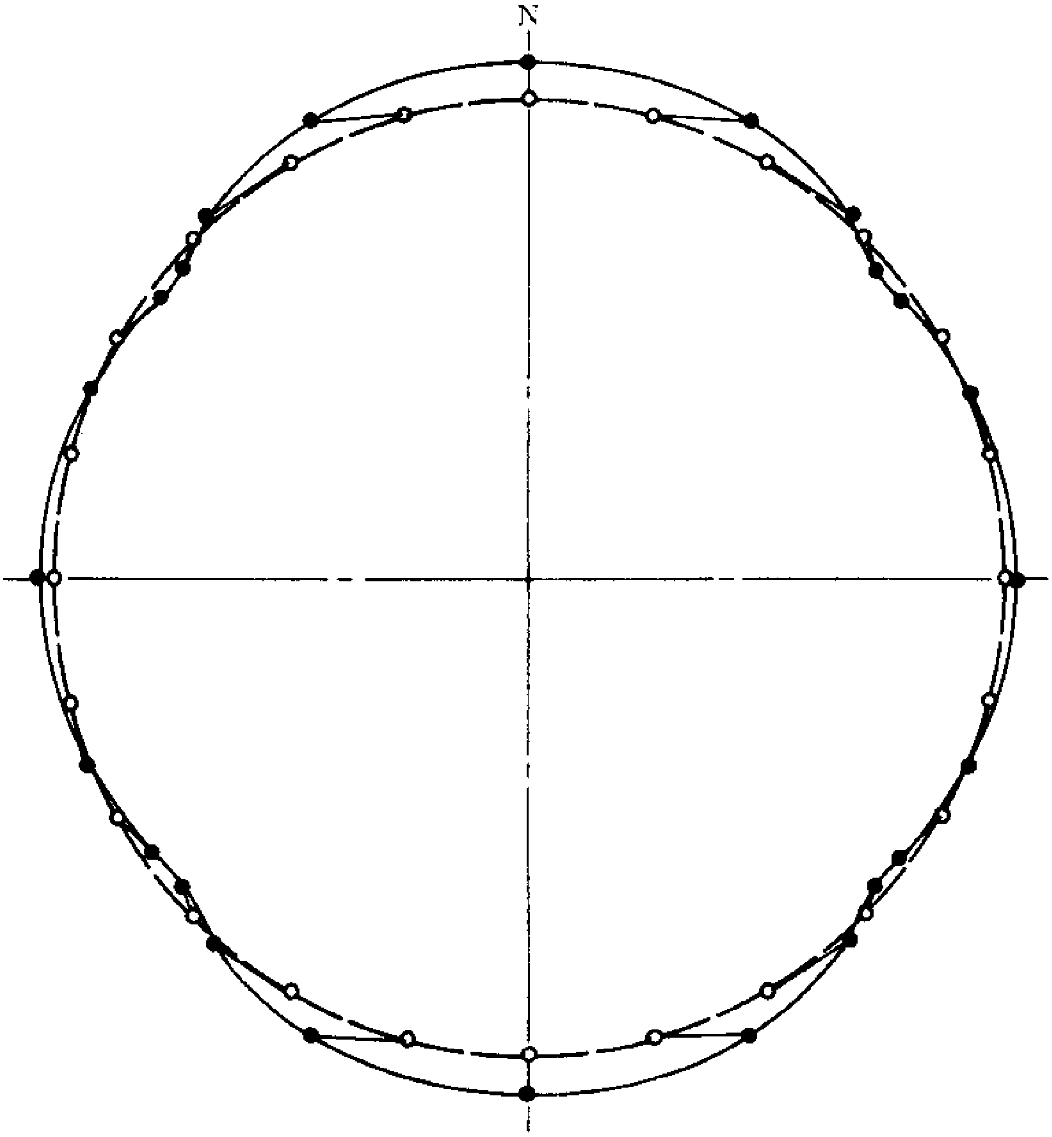

Fig. 12 The mode shape for n=4 of the upper branch

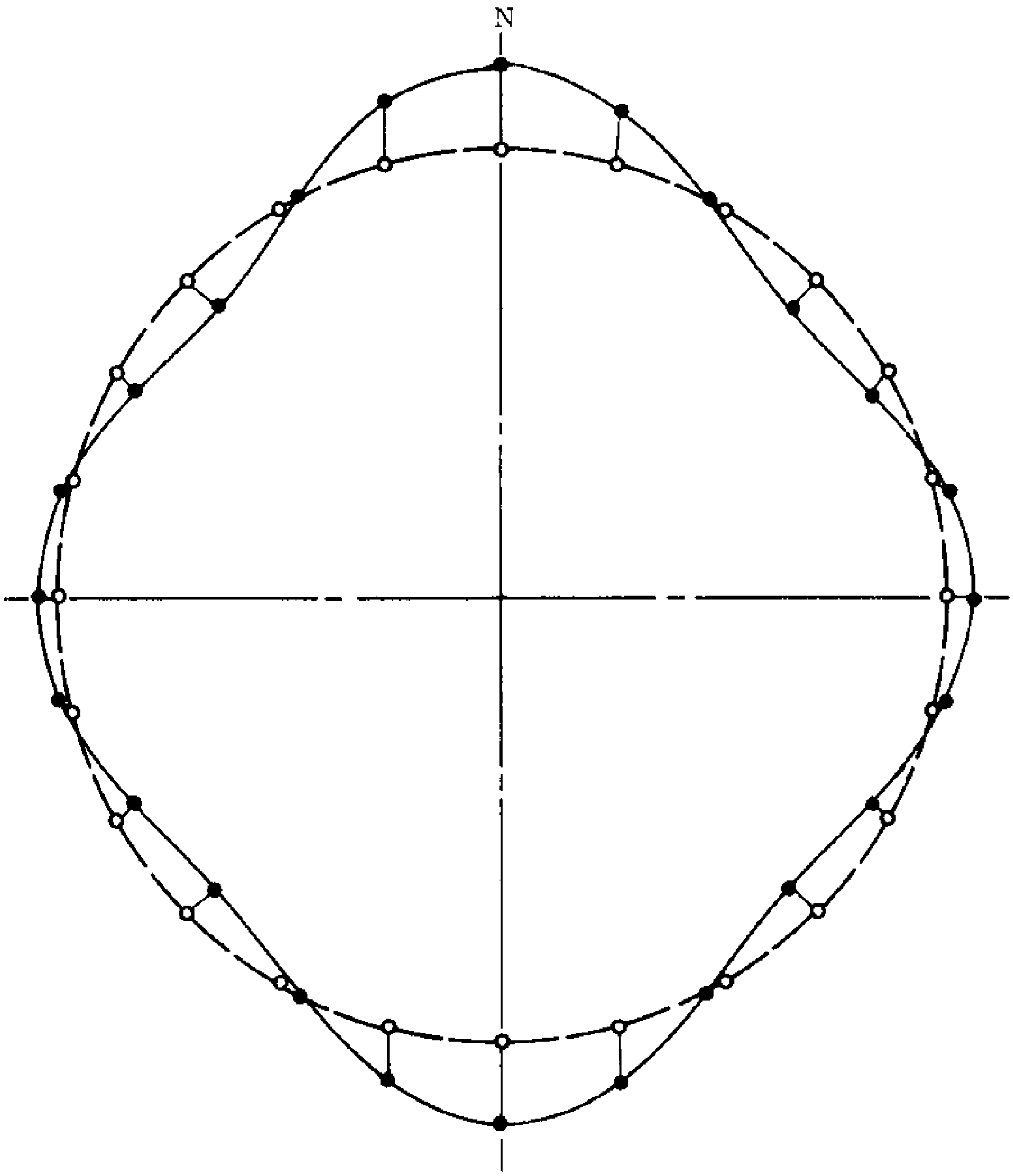

Fig. 13 The mode shape for n=4 of the lower branch

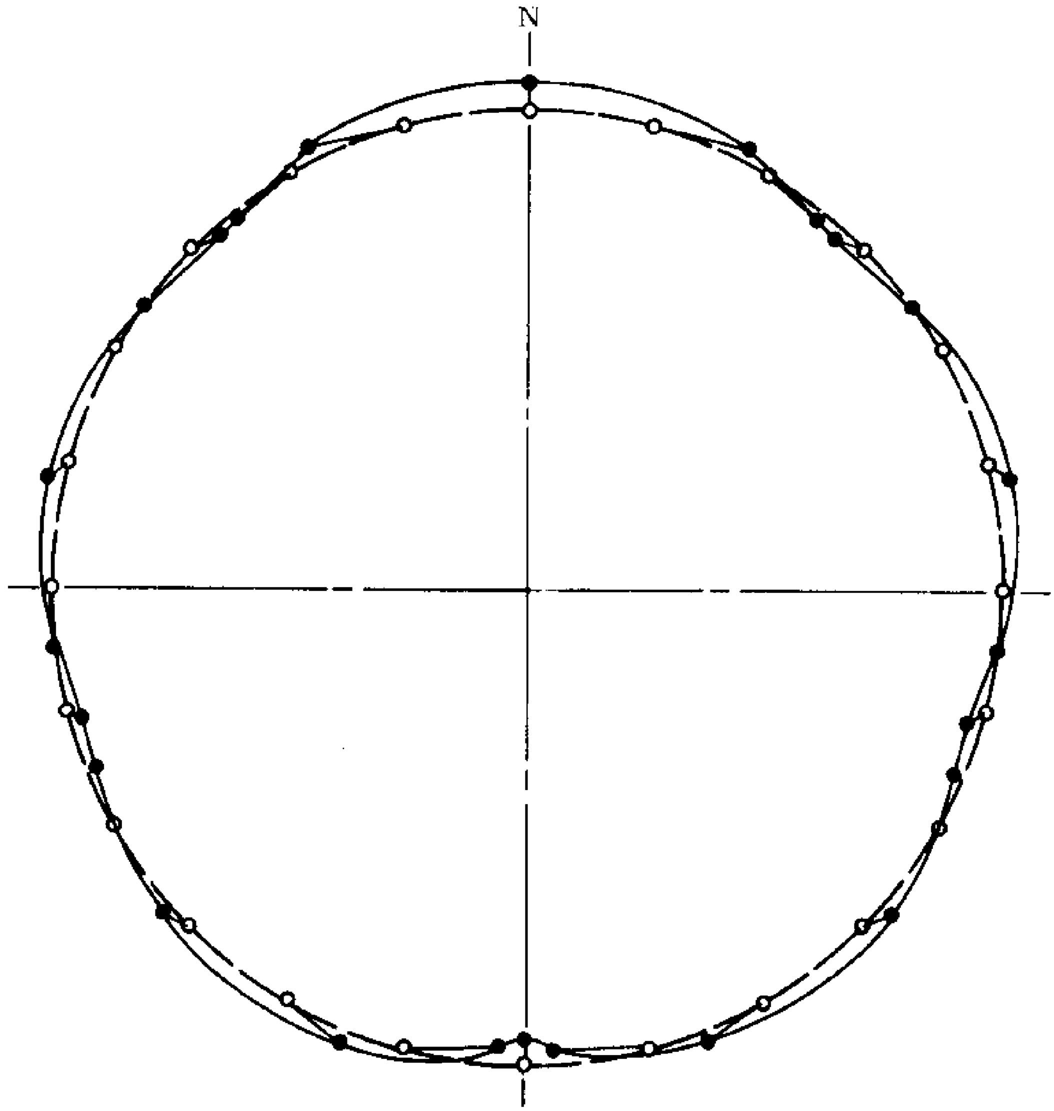

Fig. 14 The mode shape for n=5 of the upper branch

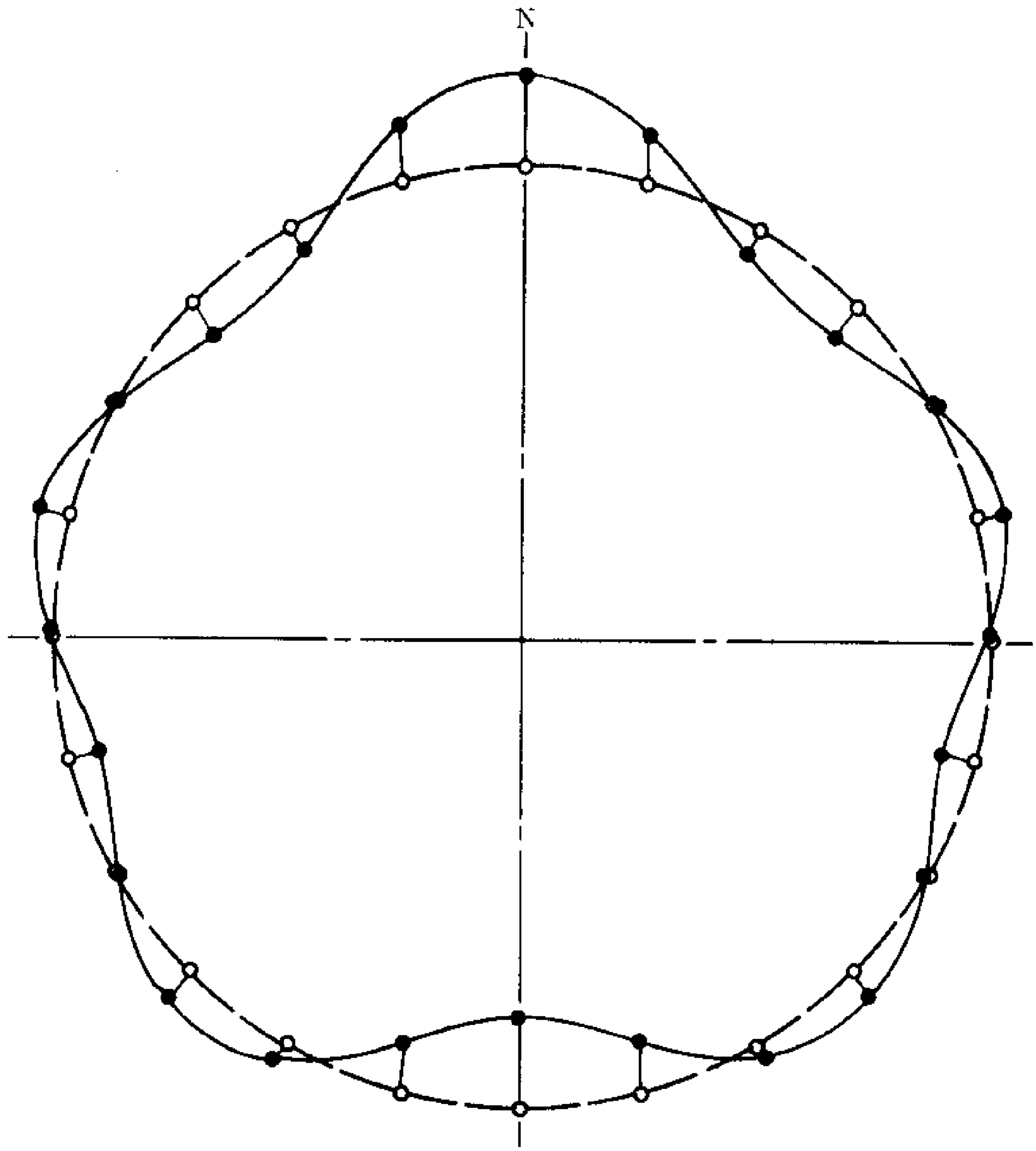

Fig. 15 The mode shape for n=5 of the lower branch