Abstract

It is demonstrated that in Fourier domain the Navier-Stokes equations for an incompressible fluid can be reduced to a single complex scalar equation. An advantage of this equation is that any solution of this equation (approximate or exact) automatically represents real incompressible velocity field. An attempt was undertaken to check that in the absence of viscosity this equation represents a Hamiltonian system expressed in non-canonical variables. However, only one of two necessary conditions was shown to hold; the question of fulfillment of the second necessary condition (the Jacobi identity) remains open. Using the new representation it was demostrated that in the inviscous case there are only two translationally-invariant, second-order integrals of motion which correspond to conservation of energy and helicity.

A complex scalar form of the incompressible Navier-Stokes equations.

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1 Introduction

The Navier-Stokes equations for an incompressible fluid consist of four equations. In some cases, it may be desirable to have a more compact representation of this set. This paper demonstrates that in Fourier domain, the Navier-Stokes equations can be reduced to a single complex scalar equation, which is, however, non-local. A corresponding equation for the probability density functional which may be used for studying hydrodynamic turbulence was obtained. An attempt was also made to check that in the absence of viscosity the derived equation represents a Hamiltonian system expressed in non-canonical variables. However, only one of two necessary conditions was demonstrated to hold; the question of fulfillment of the second necessary condition (the Jacobi identity) was unanswered. Using the new representation it was demonstrated that in the inviscid case there are only two translationally-invariant, second-order in $\vec{v}$ integrals of motion which correspond to conservation of energy and helicity.

2 Derivation

The Navier-Stokes equations for an incompressible fluid in usual notations read:

$$\partial_t \vec{v} + (\vec{v}\nabla)\,\vec{v} - \upsilon \nabla^2 \vec{v} = -\nabla p + \vec{F} \tag{1}$$

$$\nabla \vec{v} = 0 \tag{2}$$

where p is a pressure normalized by a constant fluid density and $\vec{F}$ is an external force per unit mass. Let us consider these equations in Fourier domain:

$$\vec{v}_{\vec{r}} = \int \vec{v}_{\vec{k}} e^{i\vec{k}\vec{r}} d\vec{k} \tag{3}$$

where due to the velocity field being real

$$\vec{v}_{\vec{k}} = \vec{v}^{*}_{-\vec{k}} \tag{4}$$

(equations similar to (3),(4) hold for the external force F as well). The continuity equation (2) reads:

$$\left(\vec{k}, \vec{v}_{\vec{k}}\right) = 0 \tag{5}$$

where from now on $(,)$ stands for the dot product. Let us introduce two unit real vectors $\vec{e}_{\vec{k}}^{(1)}$ and $\vec{e}_{\vec{k}}^{(2)}$ which are orthogonal to $\vec{k}$ and to each other. Let vectors $\vec{e}_{\vec{k}}^{(1,2)}$ obey the following conditions:

$$\vec{e}_{-\vec{k}}^{(1)} = \vec{e}_{\vec{k}}^{(1)} \tag{6}$$

and

$$\vec{e}_{-\vec{k}}^{(2)} = -\vec{e}_{\vec{k}}^{(2)} \tag{7}$$

Vectors $\vec{e}_{\vec{k}}^{(1,2)}$ can be selected, for example, as vectors tangential to meridians and parallels on a sphere in the $\vec{k}$-space:

$$\vec{e}_{\vec{k}}^{(1)} = \frac{k^2\vec{N} - \left(\vec{k}, \vec{N}\right)\vec{k}}{k\sqrt{k^2 - \left(\vec{k}, \vec{N}\right)^2}} \tag{8}$$

$$\vec{e}_{\vec{k}}^{(2)} = \frac{\left[\vec{k}, \vec{N}\right]}{\left|\left[\vec{k}, \vec{N}\right]\right|} = \frac{\left[\vec{k}, \vec{N}\right]}{\sqrt{k^2 - \left(\vec{k}, \vec{N}\right)^2}} \tag{9}$$

where $[,]$ stands for the cross product, $\vec{N}$ is an arbitrary constant unit real vector, and $k = \left|\vec{k}\right|$. One can easily see that $\vec{k} = k\left[\vec{e}_{\vec{k}}^{(1)}, \vec{e}_{\vec{k}}^{(2)}\right]$ and that all conditions imposed on $\vec{e}_{\vec{k}}^{(1,2)}$ are satisfied. Instead of using the vectors $\vec{e}_{\vec{k}}^{(1,2)}$ it will be convenient to use the vectors $\vec{g}_{\vec{k}}^{(1,2)}$ which follow from $\vec{e}_{\vec{k}}^{(1,2)}$ after rotation by $\pi/4$ and reflection:

$$\vec{g}_{\vec{k}}^{(1)} = \frac{\vec{e}_{\vec{k}}^{(1)} + \vec{e}_{\vec{k}}^{(2)}}{\sqrt{2}}, \quad \vec{g}_{\vec{k}}^{(2)} = \frac{\vec{e}_{\vec{k}}^{(1)} - \vec{e}_{\vec{k}}^{(2)}}{\sqrt{2}} \tag{10}$$

The real unit vectors $\vec{g}_{\vec{k}}^{(1,2)}$ are also orthogonal to each other and to $\vec{k}$:

$$\left(g_{\vec{k}}^{(1,2)}, \vec{k}\right) = 0 \tag{11}$$

The set of two basis vectors $\vec{g}_{\vec{k}}^{(1,2)}$ considered for all $\vec{k}$ can be expressed in terms of a single real vector field $\vec{g}_{\vec{k}}$:

$$\vec{g}_{\vec{k}} = \vec{g}_{\vec{k}}^{(1)} = \frac{\vec{e}_{\vec{k}}^{(1)} + \vec{e}_{\vec{k}}^{(2)}}{\sqrt{2}} \tag{12}$$

since due to Eq. (7)

$$\vec{g}_{\vec{k}}^{(2)} = \vec{g}_{-\vec{k}} \tag{13}$$

The explicit expression for $\vec{g}_{\vec{k}}$ reads:

$$\vec{g}_{\vec{k}} = \frac{k^2\vec{N} - \left(\vec{k},\vec{N}\right)\vec{k} + k\left[\vec{k},\vec{N}\right]}{2^{1/2}k\sqrt{k^2 - \left(\vec{k},\vec{N}\right)^2}} \tag{14}$$

Note that unit vector $\vec{g}_{\vec{k}}$ in Eq. (14) is discontinuous at $\vec{k} = 0$ since $\vec{g}_{\vec{k}\to 0}$ depends on the direction along which $\vec{k}$ tends to zero; $\vec{g}_{\vec{k}}$ is also discontinuous when $\vec{k}$ becomes parallel to $\vec{N}$ (i.e. at $\left[\vec{k},\vec{N}\right] \to 0$).

Any vector field satisfying the condition of Eq. (5) can be represented as follows:

$$\vec{v}_{\vec{k}} = \left(\vec{v}_{\vec{k}},\vec{g}_{\vec{k}}^{(1)}\right)\vec{g}_{\vec{k}}^{(1)} + \left(\vec{v}_{\vec{k}},\vec{g}_{\vec{k}}^{(2)}\right)\vec{g}_{\vec{k}}^{(2)} = \left(\vec{v}_{\vec{k}},\vec{g}_{\vec{k}}\right)\vec{g}_{\vec{k}} + \left(\vec{v}_{\vec{k}},\vec{g}_{-\vec{k}}\right)\vec{g}_{-\vec{k}} \tag{15}$$

Let us introduce a complex scalar field

$$a_{\vec{k}} = \left(\vec{v}_{\vec{k}},\vec{g}_{\vec{k}}\right) \tag{16}$$

Then due to Eqs. (4)

$$a^*_{-\vec{k}} = \left(\vec{v}_{\vec{k}},\vec{g}_{-\vec{k}}\right) \tag{17}$$

and Eq. (15) becomes:

$$\vec{v}_{\vec{k}} = a_{\vec{k}}\vec{g}_{\vec{k}} + a^*_{-\vec{k}}\vec{g}_{-\vec{k}} \tag{18}$$

On the other hand, if $\vec{v}_{\vec{k}}$ is defined by Eq. (18) where $a_{\vec{k}}$ is an arbitrary complex scalar field (no relation between $a_{\vec{k}}$ and $a_{-\vec{k}}$ is assumed), then the condition Eq. (4) is apparently satisfied and as a result the corresponding velocity field in the spatial domain $\vec{v}_{\vec{r}}$ will be real; the incompressibility condition Eq. (5) is also satisfied. Thus, any real solenoidal vector field in 3D can be represented in terms of a scalar complex field and vice versa with the help of Eqs. (16), (18).

Let us introduce the following notation:

$$a^s_{\vec{k}} = \begin{cases} a_{\vec{k}}, & s = 1 \\ a^*_{\vec{k}}, & s = -1 \end{cases} \tag{19}$$

Then Eq. (18) can be rewritten as

$$\vec{v}_{\vec{k}} = \sum_{s=\pm 1} a^s_{s\vec{k}}\vec{g}_{s\vec{k}} \tag{20}$$

with the inverse transformation being

$$a^s_{\vec{k}} = \left(\vec{v}_{s\vec{k}},\vec{g}_{\vec{k}}\right) \tag{21}$$

In Fourier domain Eq. (1) reads:

$$\partial_t \vec{v}_{\vec{k}} + \int \left(\vec{v}_{\vec{k}_2}, i\vec{k}_1 \right) \vec{v}_{\vec{k}_1} \delta_{\vec{k}_1+\vec{k}_2-\vec{k}} d\vec{k}_1 d\vec{k}_2 + \upsilon k^2 \vec{v}_{\vec{k}} + i\vec{k} p_{\vec{k}} - \vec{F}_{\vec{k}} = 0 \tag{22}$$

Differentiating Eq. (16) with respect to time and using Eqs. (22), (20) one obtains an evolution equation for $a_{\vec{k}}$:

$$\dot{a}_{\vec{k}} = \left(\partial_t \vec{v}_{\vec{k}}, \vec{g}_{\vec{k}}\right) =$$

$$= -\left(\int \left(\vec{v}_{\vec{k}_2}, i\vec{k}_1 \right) \vec{v}_{\vec{k}_1} \delta_{\vec{k}_1+\vec{k}_2-\vec{k}} d\vec{k}_1 d\vec{k}_2 + \upsilon k^2 \vec{v}_{\vec{k}} - \vec{F}_{\vec{k}}, \vec{g}_{\vec{k}} \right) =$$

$$= -\upsilon k^2 a_{\vec{k}} + \left(\vec{F}_{\vec{k}}, \vec{g}_{\vec{k}}\right) - i \sum_{s_{1,2}=\pm 1} \int \left(a^{s_2}_{s_2\vec{k}_2} \vec{g}_{s_2\vec{k}_2}, \vec{k}_1 \right) \left(a^{s_1}_{s_1\vec{k}_1} \vec{g}_{s_1\vec{k}_1}, \vec{g}_{\vec{k}} \right) \delta_{\vec{k}_1+\vec{k}_2-\vec{k}} d\vec{k}_{1,2} \tag{23}$$

Taking into account that $\vec{k}_1 = \vec{k} - \vec{k}_2$ and that $\left(\vec{g}_{\vec{k}'}, \vec{k}'\right) = 0$ for any $\vec{k}'$ and replacing the integration variables $\vec{k}_1 \to s_1\vec{k}_1$, $\vec{k}_2 \to s_2\vec{k}_2$, we can represent Eq. (23) in the following form:

$$\dot{a}^s_{\vec{k}} = -\upsilon k^2 a^s_{\vec{k}} + \left(\vec{F}_{s\vec{k}}, \vec{g}_{\vec{k}}\right) - is \sum_{s_{1,2}=\pm 1} \int \left(\vec{g}_{\vec{k}}, \vec{g}_{\vec{k}_1} \right) \left(\vec{g}_{\vec{k}_2}, \vec{k} \right) a^{s_1}_{\vec{k}_1} a^{s_2}_{\vec{k}_2} \delta_{s_1\vec{k}_1+s_2\vec{k}_2-s\vec{k}} d\vec{k}_{1,2} \tag{24}$$

Symmetrizing the integrand in this equation with respect to 1, 2 indices we can also represent the evolution equation as follows:

$$\dot{a}^s_{\vec{k}} = -\upsilon k^2 a^s_{\vec{k}} + \left(\vec{F}_{s\vec{k}}, \vec{g}_{\vec{k}}\right) - is \sum_{s_{1,2}=\pm 1} \int V_{\vec{k}\vec{k}_1\vec{k}_2} a^{s_1}_{\vec{k}_1} a^{s_2}_{\vec{k}_2} \delta_{s_1\vec{k}_1+s_2\vec{k}_2-s\vec{k}} d\vec{k}_{1,2} \tag{25}$$

where

$$V_{\vec{k}\vec{k}_1\vec{k}_2} = \frac{1}{2} \left[\left(\vec{g}_{\vec{k}}, \vec{g}_{\vec{k}_1}\right) \left(\vec{g}_{\vec{k}_2}, \vec{k}\right) + \left(\vec{g}_{\vec{k}}, \vec{g}_{\vec{k}_2}\right) \left(\vec{g}_{\vec{k}_1}, \vec{k}\right) \right] \tag{26}$$

Note that if $\vec{k} \to 0$, then according to Eq. (26) $V^{ss_1s_2}_{\vec{k}\vec{k}_1\vec{k}_2} \to 0$. For this reason if in the initial moment of time $a_{\vec{k}=0}\,(t=0) = 0$, then in the absence of the spatially uniform external forces $\vec{F}_{\vec{k}=0} = 0$ it will always be $a_{\vec{k}=0}\,(t) = 0$.

The kernel (vertex) $V_{\vec{k}\vec{k}_1\vec{k}_2}$ is a real function symmetric with respect to permutation of the two last indices: $\vec{k}_1 \leftrightarrow \vec{k}_2$. Although $V_{\vec{k}\vec{k}_1\vec{k}_2}$ is defined for any $\vec{k}$, $\vec{k}_1$, $\vec{k}_2$ it has a meaning only at the subspaces $s_1\vec{k}_1 + s_2\vec{k}_2 - s\vec{k} = 0$. Just as $\vec{g}_{\vec{k}}$ is a discontinues function of $\vec{N}$ at $\left[\vec{k}, \vec{N}\right] \to 0$, $V_{\vec{k}\vec{k}_1\vec{k}_2}$ is also discontinues when any of its arguments become parallel to $\vec{N}$. Nevertheless, it is obvious that this function is limited:

$$\left| V_{\vec{k}\vec{k}_1\vec{k}_2} \right| \le \left| \vec{k} \right| = k \tag{27}$$

Note that the pressure field p does not enter Eq. (23); its effect is implicit by ensuring that in the course of evolution the velocity field is kept solenoidal. The price is the equation in Fourier domain being non-local. An explicit expression for pressure $p_{\vec{k}}$ follows after projecting Eq. (22) on vector $\vec{k}$.

From Eq. (18) one finds

$$\left|\vec{v}_{\vec{k}}\right|^2 = \left(\vec{v}_{\vec{k}}, \vec{v}_{-\vec{k}}\right) = \left|a_{\vec{k}}\right|^2 + \left|a_{-\vec{k}}\right|^2 \tag{28}$$

so that the expression for the kinetic energy reads:

$$\frac{1}{2}\int \left|\vec{v}_{\vec{k}}\right|^2 d\vec{k} = \int \left|a_{\vec{k}}\right|^2 d\vec{k} \tag{29}$$

Let us neglect in Eq. (25) the viscosity and the external force terms. Setting in Eq. (25) $a^s_{\vec{k}} = \operatorname{Re} a_{\vec{k}} + is \operatorname{Im} a_{\vec{k}}$ and separating the real and imaginary part one finds:

$$\partial_t \operatorname{Re} a_{\vec{k}} = 2 \sum_{s_{1,2}=\pm 1} \int s_2 V_{\vec{k}\vec{k}_1\vec{k}_2} \operatorname{Re} a_{\vec{k}_1} \operatorname{Im} a_{\vec{k}_2} \delta_{s_1\vec{k}_1 + s_2\vec{k}_2 - s\vec{k}} d\vec{k}_{1,2} \tag{30}$$

$$\partial_t \operatorname{Im} a_{\vec{k}} = \sum_{s_{1,2}=\pm 1} \int V_{\vec{k}\vec{k}_1\vec{k}_2} \left(s_1 s_2 \operatorname{Im} a_{\vec{k}_1} \operatorname{Im} a_{\vec{k}_2} - \operatorname{Re} a_{\vec{k}_1} \operatorname{Re} a_{\vec{k}_2}\right) \delta_{s_1\vec{k}_1 + s_2\vec{k}_2 - s\vec{k}} d\vec{k}_{1,2} \tag{31}$$

whence

$$\partial_t \left(\frac{\delta \operatorname{Re} a_{\vec{k}}}{\delta \operatorname{Re} a_{\vec{k}_1}} + \frac{\delta \operatorname{Im} a_{\vec{k}}}{\delta \operatorname{Im} a_{\vec{k}_1}}\right)_{\vec{k}_1 = \vec{k}} =$$

$$= 2 \sum_{s_{1,2}=\pm 1} \int V_{\vec{k}\vec{k}\vec{k}_2} s_2 \left(s_1 + 1\right) \operatorname{Im} a_{\vec{k}_2} \delta_{s_1\vec{k} + s_2\vec{k}_2 - \vec{k}} d\vec{k}_2 =$$

$$= 4 \sum_{s_2=\pm 1} \int V_{\vec{k}\vec{k}\vec{k}_2} s_2 \operatorname{Im} a_{\vec{k}_2} \delta_{s_2\vec{k}_2} d\vec{k}_2 = 2\left(\vec{g}_{\vec{k}_2=0}, \vec{k}\right) \operatorname{Im} a_{\vec{k}_2=0} \sum_{s_2=\pm 1} s_2 = 0 \tag{32}$$

Thus, the phase volume defined as

$$d\Gamma = \prod_{\vec{k}} d\left(\operatorname{Re} a_{\vec{k}}\right) d\left(\operatorname{Im} a_{\vec{k}}\right) \tag{33}$$

in the course of evolution in the absence of viscosity is conserved.

Let us write down the equation for the probability density function P:

$$dW = P\left(a_{\vec{k}}, a^*_{\vec{k}}\right) d\Gamma \tag{34}$$

where dW is a probability of the system to belong to $d\Gamma$. Derivation of corresponding (Liouville's) equation takes one line and for convenience it is reproduced below. If equation of motion reads $\dot{u} = f(u)$ one has:

$$\partial_t \delta\left(U - u(t)\right) =$$

$$= -\delta'\left(U - u(t)\right)\dot{u}(t) = -\partial_U\left[f(u)\,\delta(U-u)\right] = -\partial_U\left[f(U)\,\delta(U-u)\right] \tag{35}$$

where in Eq. (35) U is a scalar parameter. Generalization of this equation for the case of u being a vector is straightforward. Averaging this equation with respect to a statistical ensemble of initial conditions and returning to the original notation: $U \to u$ one obtains the continuity equation for P:

$$\partial_t P + \sum_k \partial_{u_k} \left(\dot{u}_k P \right) = 0 \tag{36}$$

If u_k is split into two components: $u_k = (x_k, y_k)$, where $x_k = \operatorname{Re} a_k$, $y_k = \operatorname{Im} a_k$, Eq. (36) becomes

$$\partial_t P + 2 \operatorname{Re} \sum_k \partial_{a_k} \left(\dot{a}_k P \right) = 0 \tag{37}$$

With respect to Eq. (25) with $\vec{F}_{\vec{k}} = 0$, Eq. (37) reads:

$$\partial_t P + 2 \operatorname{Re} \int \left(-\upsilon k^2 P + \dot{a}_{\vec{k}} \frac{\delta P}{\delta a_k} \right) d\vec{k} = 0$$

or

$$\partial_t P - 2\upsilon \int k^2 P d\vec{k} + \operatorname{Im} \sum_{s_{1,2}=\pm 1} \int V_{\vec{k}\vec{k}_1\vec{k}_2} a^{s_1}_{\vec{k}_1} a^{s_2}_{\vec{k}_2} \delta_{s_1\vec{k}_1+s_2\vec{k}_2-\vec{k}} \frac{\delta P}{\delta a_k} d\vec{k} d\vec{k}_{1,2} = 0 \tag{38}$$

If one assumes that the external forces $\vec{F}_{\vec{k}}$ in Eq. (25) are δ-correlated in time and possess spatially homogeneous statistics, Eq. (38) acquires in the RHS a corresponding diffusion term:

$$\int \frac{\delta^2}{\delta a_{\vec{k}} \delta a^*_{\vec{k}}} \sigma_{\vec{k}} P d\vec{k} \tag{39}$$

(the Fokker-Plank equation) where the diffusion coefficient $\sigma_{\vec{k}}$ is proportional to the intensity of the random force $\vec{F}_{\vec{k}}$.

Note also, that in contrast to the well-known Hopf equation [3] no extra conditions on the amplitudes $a^s_{\vec{k}}$ which ensure incompressibility are imposed here. This pertains also to the basic Eq. (25) any solution of which (either exact or approximate) represents real incompressible flow.

According to the theory of the linear, first-order partial differential equations (PDE) any function of the integrals of corresponding characteristic equations is a solution of the PDE. Thus, in the inviscid case any function of the integrals of Eq. (25) is a solution of Eq. (38). In particular, any function of kinetic energy given by Eq. (29) will be a solution; such solution was written down in [3] and it was checked there that it does satisfy to the Hopf equation. This solution, however, has no relation to real turbulent flows (as it was acknowledged in [3] as well).

Let us mention also that if one considers a two-dimensional motion of an incompressible fluid and introduces a stream function ψ so that velocity is given by $\vec{v} = \left[\nabla\psi, \vec{N}\right]$, then using Eq. (16) one finds that in the Fourier domain $a_{\vec{k}}$ and $\psi_{\vec{k}}$ are proportional to each other:

$$a_{\vec{k}} = \frac{i}{\sqrt{2}}\sqrt{k^2 - \left(\vec{k},\vec{N}\right)^2}\,\psi_{\vec{k}} \tag{40}$$

In the spatial domain relation between a and ψ will be, however, non-local.

3 Conservation laws

Let us consider translationally-invariant, second-order in $a^s_{\vec{k}}$ integral of motion:

$$I = \frac{1}{2}\sum_{s_1,s_2=\pm1}\int F^{s_1s_2}_{\vec{k}_1\vec{k}_2} a^{s_1}_{\vec{k}_1} a^{s_2}_{\vec{k}_2}\delta_{s_1\vec{k}_1+s_2\vec{k}_2} d\vec{k}_1 d\vec{k}_2 \tag{41}$$

where $F^{s_1s_2}_{\vec{k}_1\vec{k}_2} = F^{s_2s_1}_{\vec{k}_2\vec{k}_1}$. Horizontal translation of the velocity field: $\vec{v}_{\vec{r}} \to \vec{v}_{\vec{r}+\vec{d}}$ apparently leads to the following transformation of the corresponding Fourier components $\vec{v}_{\vec{k}}$: $\vec{v}_{\vec{k}} \to \vec{v}_{\vec{k}} e^{i\vec{k}\vec{d}}$. According to Eq. (21) the amplitudes $a^s_{\vec{k}}$ transform as follows: $a^s_{\vec{k}} \to a^s_{\vec{k}} e^{is\vec{k}\vec{d}}$. Thus, the δ-function in Eq. (41) ensures that the integral I is invariant with respect to the translation of the velocity field. Then one obtains:

$$I = \frac{1}{2}\sum_{s=\pm1}\int F^{s,-s}_{\vec{k},\vec{k}} a^s_{\vec{k}} a^{-s}_{\vec{k}} d\vec{k} + \frac{1}{2}\sum_{s=\pm1}\int F^{s,s}_{\vec{k},-\vec{k}} a^s_{\vec{k}} a^s_{-\vec{k}} d\vec{k} =$$

$$= \frac{1}{2}\sum_{s=\pm1}\int \left(A_{\vec{k}} a^{-s}_{\vec{k}} + B^s_{\vec{k}} a^s_{-\vec{k}}\right) a^s_{\vec{k}} d\vec{k} \tag{42}$$

where $A_{\vec{k}} = \left(F^{1,-1}_{\vec{k},\vec{k}} + F^{-1,-1}_{\vec{k},\vec{k}}\right)/2 = F^{1,-1}_{\vec{k},\vec{k}}$ and $B^s_{\vec{k}} = \left(F^{s,s}_{\vec{k},-\vec{k}} + F^{s,s}_{-\vec{k},\vec{k}}\right)/2 = F^{s,s}_{\vec{k},-\vec{k}}$. Apparently, function $A_{\vec{k}}$ in Eq. (42) does not depend on s and function $B^s_{\vec{k}}$ is an even function of $\vec{k}$:

$$B^s_{\vec{k}} = B^s_{-\vec{k}}. \tag{43}$$

Differentiating Eq. (42) with respect to time in the case of inviscid flow without external forces one finds:

$$\frac{dI}{dt} = \sum_{s=\pm1}\int \left(A_{\vec{k}} a^{-s}_{\vec{k}} + B^s_{\vec{k}} a^s_{-\vec{k}}\right)\dot{a}^s_{\vec{k}} d\vec{k} = -i\sum_{s=\pm1} s\int \left(A_{\vec{k}} a^{-s}_{\vec{k}} + B^s_{\vec{k}} a^s_{-\vec{k}}\right)\sum_{s_{1,2}=\pm1}$$

$$\int V_{\vec{k}\vec{k}_1\vec{k}_2} a^{s_1}_{\vec{k}_1} a^{s_2}_{\vec{k}_2}\delta_{s_1\vec{k}_1+s_2\vec{k}_2-s\vec{k}} d\vec{k}_{1,2} d\vec{k} = \frac{i}{3}\sum_{s,s_1,s_2=\pm1}\int \Big(sA_{\vec{k}} V_{\vec{k}\vec{k}_1\vec{k}_2} + s_1 A_{\vec{k}_1} V_{\vec{k}_1\vec{k}\vec{k}_2} +$$

$$+ s_2 A_{\vec{k}_2} V_{\vec{k}_2\vec{k}_1\vec{k}}\Big) a^s_{\vec{k}} a^{s_1}_{\vec{k}_1} a^{s_2}_{\vec{k}_2}\delta_{s\vec{k}+s_1\vec{k}_1+s_2\vec{k}_2} d\vec{k} d\vec{k}_1 d\vec{k}_2 - \frac{i}{3}\sum_{s,s_1,s_2=\pm1}\int \Big(sB^s_{\vec{k}} V_{-\vec{k}\vec{k}_1\vec{k}_2} +$$

$$+s_1 B^{s_1}_{\vec{k}_1} V_{-\vec{k}_1\vec{k}\vec{k}_2} + s_2 B^{s_2}_{\vec{k}_2} V_{-\vec{k}_2\vec{k}_1\vec{k}}\Big) a^s_{\vec{k}} a^{s_1}_{\vec{k}_1} a^{s_2}_{\vec{k}_2} \delta_{s\vec{k}+s_1\vec{k}_1+s_2\vec{k}_2} d\vec{k} d\vec{k}_1 d\vec{k}_2 \qquad (44)$$

Since $a^s_{\vec{k}}$ are arbitrary, to have $dI/dt = 0$ one has to have

$$sA_{\vec{k}} V_{\vec{k}\vec{k}_1\vec{k}_2} + s_1 A_{\vec{k}_1} V_{\vec{k}_1\vec{k}\vec{k}_2} + s_2 A_{\vec{k}_2} V_{\vec{k}_2\vec{k}_1\vec{k}} = 0 \qquad (45)$$

or

$$sB^s_{\vec{k}} V_{-\vec{k}\vec{k}_1\vec{k}_2} + s_1 B^{s_1}_{\vec{k}_1} V_{-\vec{k}_1\vec{k}\vec{k}_2} + s_2 B^{s_2}_{\vec{k}_2} V_{-\vec{k}_2\vec{k}_1\vec{k}} = 0 \qquad (46)$$

where in both cases

$$s\vec{k} + s_1\vec{k}_1 + s_2\vec{k}_2 = 0 \qquad (47)$$

One can easily check that the condition Eq. (45) is satisfied for $A_{\vec{k}} = 1$:

$$2\left(sV_{\vec{k}\vec{k}_1\vec{k}_2} + s_1 V_{\vec{k}_1\vec{k}\vec{k}_2} + s_2 V_{\vec{k}_2\vec{k}_1\vec{k}}\right) = \left(\vec{g}_{\vec{k}}, \vec{g}_{\vec{k}_1}\right)\left(\vec{g}_{\vec{k}_2}, s\vec{k} + s_1\vec{k}_1\right) +$$

$$+\left(\vec{g}_{\vec{k}}, \vec{g}_{\vec{k}_2}\right)\left(\vec{g}_{\vec{k}_2}, s\vec{k} + s_2\vec{k}_2\right) + \left(\vec{g}_{\vec{k}_1}, \vec{g}_{\vec{k}_2}\right)\left(\vec{g}_{\vec{k}}, s_1\vec{k}_1 + s_2\vec{k}_2\right) =$$

$$= -s_2\left(\vec{g}_{\vec{k}}, \vec{g}_{\vec{k}_1}\right)\left(\vec{g}_{\vec{k}_2}, \vec{k}_2\right) - s_1\left(\vec{g}_{\vec{k}}, \vec{g}_{\vec{k}_2}\right)\left(\vec{g}_{\vec{k}_1}, \vec{k}_1\right) - s\left(\vec{g}_{\vec{k}_1}, \vec{g}_{\vec{k}_2}\right)\left(\vec{g}_{\vec{k}}, \vec{k}\right) = 0 \qquad (48)$$

since $\left(\vec{g}_{\vec{k}}, \vec{k}\right) = 0$ always. The second condition Eq. (46) is satisfied for

$$B^s_{\vec{k}} = s\left|\vec{k}\right| = sk. \qquad (49)$$

In this case Eq. (46) reads:

$$kV_{-\vec{k}\vec{k}_1\vec{k}_2} + k_1 V_{-\vec{k}_1\vec{k}\vec{k}_2} + k_2 V_{-\vec{k}_2\vec{k}_1\vec{k}} = 0 \qquad (50)$$

Using the representation Eq. (20) one can check that the related conservation law in spatial domain corresponds to conservation of helicity:

$$I = \mathrm{Im}\int k a_{\vec{k}} a_{-\vec{k}} d\vec{k} = \frac{1}{16\pi^3}\int \left(\vec{v}, \nabla\times\vec{v}\right) d^3\vec{r} \qquad (51)$$

To make sure that the condition Eq. (50) holds as compared to transformation Eq. (48) requires more involved algebra. Those who would like to check Eq. (50) would better do this numerically, by choosing three arbitrary 3D vectors $\vec{k}_1$, $\vec{k}_2$, and $\vec{N}$, normalizing $\vec{N}$, calculating $\vec{k}$ according to Eq. (47) for an arbitrary set of s, s_1, s_2 indices, and then calculating the LHS of Eq. (50) according to Eqs. (26) and (14). The result will turn to zero to the machine precision for all eight different combinations of s-indices.

Such course of action allows also to obtain another result. Eq. (46) can be considered as a linear equation with respect to $B^s_{\vec{k}}$. Depending on selection of s-indices this equations generally links such values as $B^{\pm 1}_{\vec{k}_1}$, $B^{\pm 1}_{\vec{k}_2}$, $B^{\pm 1}_{\vec{k}_1+\vec{k}_2}$, $B^{\pm 1}_{\vec{k}_1-\vec{k}_2}$: eight variables altogether (note the condition Eq. (43)). Eq. (46) for all different combination of s-indices provides eight equations. The replacement of $\left(\vec{k}_1, \vec{k}_2\right)$

by $\left(\pm\vec{k}_1, \pm\vec{k}_2\right)$ which will be linking the same B-values, increases total number of equations to 32. Thus, one obtains 32 homogeneous linear equations with respect to 8 unknowns. Calculating numerically rank of this linear system using a standard function, one finds that the rank equals to 7. Hence, there might be no more than one nontrivial solution of these 32 equations. The conclusion is that there exist only one solution of functional equation (46) which is given by Eq. (49).

Similar analysis applied to condition Eq. (45) also shows that the solution $A_{\vec{k}} = 1$ is the only non-trivial one. Thus, there exist only two translationally-invariant, second-order conservation laws for the Euler equations.

4 Does the scalar equation represent a Hamiltonian system?

The question which might be asked, if Eq. (25) with $\nu = 0$ and $F_{\vec{k}} = 0$ represents a Hamiltonian system expressed in non-canonical variables. One would expect the answer to be affirmative, since the Euler equations for a compressible fluid are Hamiltonian [1], and for an incompressible fluid they are explicitly Hamiltonian if the velocity field is represented in terms of the Clebsch variables [2]. In this section an attempt is made to check directly if Eq. (25) is Hamiltonian.

The canonical, finite-dimensional Hamiltonian equations can be cast into a complex form as follows:

$$\dot{b}_k^s = -is\frac{\partial H}{\partial b_k^{-s}} \tag{52}$$

where $b_k = (q_k + ip_k)/\sqrt{2}$ and index $s = \pm 1$ is defined in Eq. (19). In arbitrary complex coordinates $a_k = a_k(b_1, b_1^*, b_2, b_2^*, ...)$ Eq. (52) reads:

$$\dot{a}_k^s = -i\left\{a_k^s, H\right\} \tag{53}$$

where the Poisson bracket is defined as follows:

$$\{f, g\} = \sum_{k'}\sum_{s'=\pm 1} s'\frac{\partial f}{\partial b_{k'}^{s'}}\frac{\partial g}{\partial b_{k'}^{-s'}} \tag{54}$$

where f, g are arbitrary smooth complex functions. It is obvious that $\{f, g\} = -\{g, f\}$ and one can easily check that the Jacobi identity with respect to this Poisson bracket is also satisfied. Substituting into Eq. (53) the equation

$$\frac{\partial H}{\partial b_{k'}^{-s'}} = \sum_{k'',s''}\frac{\partial H}{\partial a_{k''}^{s''}}\frac{\partial a_{k''}^{s''}}{\partial b_{k'}^{-s'}} \tag{55}$$

one obtains:

$$\dot{a}_k^s = -i\sum_{s',k'} J_{kk'}^{ss'}\frac{\partial H}{\partial a_k^{s'}} \tag{56}$$

where

$$J_{kk'}^{ss'} = \left\{ a_k^s, a_k^{s'} \right\} \tag{57}$$

Apparently, J is skew-symmetric:

$$J_{kk'}^{ss'} = -J_{k'k}^{s's} \tag{58}$$

and

$$\left(J_{kk'}^{ss'} \right)^* = -J_{k,k'}^{-s,-s'} \tag{59}$$

Let us consider the case where (cf. Eq. (29)):

$$H = \frac{1}{2} \sum_{s,k} a_k^s a_k^{-s} \tag{60}$$

and

$$J_{kk_1}^{ss_1} = \sum_{s_2,k_2} C_{kk_1k_2}^{ss_1s_2} a_{k_2}^{-s_2} \tag{61}$$

where

$$C_{kk_1k_2}^{ss_1s_2} = -C_{k_1kk_2}^{s_1ss_2} \tag{62}$$

(Lie-Poisson bracket [1]). Then Eq. (56) reads:

$$\dot{a}_k^s = -i \sum_{s_{1,2},k_{1,2}} C_{kk_1k_2}^{ss_1s_2} a_{k_1}^{-s_1} a_{k_2}^{-s_2} \tag{63}$$

Comparing Eqs. (63) and (24) in which we replace summation variables $s_{1,2} \rightarrow -s_{1,2}$ one finds:

$$C_{\vec{k}\vec{k}_1\vec{k}_2}^{ss_1s_2} = s\widetilde{C}_{\vec{k}\vec{k}_1\vec{k}_2} \delta_{s\vec{k}+s_1\vec{k}_1+s_2\vec{k}_2} \tag{64}$$

where

$$\widetilde{C}_{\vec{k}\vec{k}_1\vec{k}_2} = \left(\vec{g}_{\vec{k}}, \vec{g}_{\vec{k}_1} \right) \left(\vec{g}_{\vec{k}_2}, \vec{k} \right) \tag{65}$$

Let us check that the skew-symmetry condition Eq. (62) holds:

$$C_{\vec{k}_1\vec{k}\vec{k}_2}^{s_1ss_2} = s_1 \left(\vec{g}_{\vec{k}_1}, \vec{g}_{\vec{k}} \right) \left(\vec{g}_{\vec{k}_2}, \vec{k}_1 \right) \delta_{s\vec{k}+s_1\vec{k}_1+s_2\vec{k}_2} = \left(\vec{g}_{\vec{k}_1}, \vec{g}_{\vec{k}} \right) \left(\vec{g}_{\vec{k}_2}, -s\vec{k} - s_2\vec{k}_2 \right) \times$$

$$\times \delta_{s\vec{k}+s_1\vec{k}_1+s_2\vec{k}_2} = \left(\vec{g}_{\vec{k}_1}, \vec{g}_{\vec{k}} \right) \left(\vec{g}_{\vec{k}_2}, -s\vec{k} \right) \delta_{s\vec{k}+s_1\vec{k}_1+s_2\vec{k}_2} = -C_{\vec{k}\vec{k}_1\vec{k}_2}^{ss_1s_2} \tag{66}$$

Now let us consider the Jacobi identity:

$$\left\{ a_{\vec{k}_1}^{s_1}, \left\{ a_{\vec{k}_2}^{s_2}, a_{\vec{k}_3}^{s_3} \right\} \right\} + \left\{ a_{\vec{k}_2}^{s_2}, \left\{ a_{\vec{k}_3}^{s_3}, a_{\vec{k}_1}^{s_1} \right\} \right\} + \left\{ a_{\vec{k}_3}^{s_3}, \left\{ a_{\vec{k}_1}^{s_1}, a_{\vec{k}_2}^{s_2} \right\} \right\} =$$

$$= \left\{ a_{\vec{k}_1}^{s_1}, J_{\vec{k}_2\vec{k}_3}^{s_2s_3} \right\} + \left\{ a_{\vec{k}_2}^{s_2}, J_{\vec{k}_3\vec{k}_1}^{s_3s_1} \right\} + \left\{ a_{\vec{k}_3}^{s_3}, J_{\vec{k}_1\vec{k}_2}^{s_1s_2} \right\} =$$

$$= \sum_{s'} \int d\vec{k}' \left(\frac{\delta J_{\vec{k}_2\vec{k}_3}^{s_2s_3}}{\delta a_{\vec{k}'}^{s'}} J_{\vec{k}_1\vec{k}'}^{s_1s'} + \frac{\delta J_{\vec{k}_3\vec{k}_1}^{s_3s_1}}{\delta a_{\vec{k}'}^{s'}} J_{\vec{k}_2\vec{k}'}^{s_2s'} + \frac{\delta J_{\vec{k}_1\vec{k}_2}^{s_1s_2}}{\delta a_{\vec{k}'}^{s'}} J_{\vec{k}_3\vec{k}'}^{s_3s'} \right) = 0 \tag{67}$$

Using in this equation the representation Eq. (61) one obtains:

$$a_{\vec{k}}^{-s}\sum_{s'}\int d\vec{k}'\left(C_{\vec{k}_1,\vec{k}_2,\vec{k}'}^{s_1,s_2,-s'}C_{\vec{k}',\vec{k}_3,\vec{k}}^{s',s_3,s}+C_{\vec{k}_2,\vec{k}_3,\vec{k}'}^{s_2,s_3,-s'}C_{\vec{k}',\vec{k}_1,\vec{k}}^{s',s_1,s}+C_{\vec{k}_3,\vec{k}_1,\vec{k}'}^{s_3,s_1,-s'}C_{\vec{k}',\vec{k}_2,\vec{k}}^{s',s_2,s}\right)=0 \tag{68}$$

Since $a_{\vec{k}}^{-s}$ is arbitrary, the whole factor multiplying $a_{\vec{k}}^{-s}$ should turn to zero. Substituting into this equation Eq. (64) one obtains:

$$\sum_{s'=\pm 1}s'\left(s_1\widetilde{C}_{\vec{k}_1,\vec{k}_2,s'\left(s_1\vec{k}_1+s_2\vec{k}_2\right)}\widetilde{C}^{s',s_3,s}_{s'\left(s_1\vec{k}_1+s_2\vec{k}_2\right),\vec{k}_3,\vec{k}}+s_2\widetilde{C}_{\vec{k}_2,\vec{k}_3,s'\left(s_2\vec{k}_2+s_3\vec{k}_3\right)}\times\right.$$

$$\left.\times\widetilde{C}_{s'\left(s_2\vec{k}_2+s_3\vec{k}_3\right),\vec{k}_1,\vec{k}}+s_3\widetilde{C}_{\vec{k}_3,\vec{k}_1,s'\left(s_3\vec{k}_3+s_1\vec{k}_1\right)}\widetilde{C}_{s'\left(s_3\vec{k}_3+s_1\vec{k}_1\right),\vec{k}_2,\vec{k}}\right)\delta_{s_1\vec{k}_1+s_2\vec{k}_2+s_3\vec{k}_3+s\vec{k}}=0 \tag{69}$$

Direct numerical check of Eq. (69) for $\widetilde{C}$ defined by Eq. (65) for an arbitrary set of vectors $\vec{k}_{1,2,3}$ and $\vec{k}$ satisfying to the relation $s_1\vec{k}_1+s_2\vec{k}_2+s_3\vec{k}_3+s\vec{k}=0$ shows that Eq. (69) in fact does not hold. This, however, does not necessarily mean that Eq. (24) and equivalent to it Eq. (25) are not Hamiltonian. The reason is that $C_{kk_1k_2}^{ss_1s_2}$ in Eq. (63) is not supposed to be symmetric with respect to permutation of the second and third columns of indices: $\left(s_1,\vec{k}_1\right)\leftrightarrow\left(s_2,\vec{k}_2\right)$. As a result comparison of Eqs. (63) and (24) does not define $C_{kk_1k_2}^{ss_1s_2}$ uniquely: one can add to $C_{kk_1k_2}^{ss_1s_2}$ any function $E_{kk_1k_2}^{ss_1s_2}$ which is skew-symmetric with respect to permutations of all three columns of indices. Such addition will not affect the expression for $V_{\vec{k}\vec{k}_1\vec{k}_2}$ in Eq. (25) which due to its symmetry with respect to permutation of the second and third indices is defined uniquely. Thus, if Eq. (69) with $\widetilde{C}_{kk_1k_2}$ replaced by $\widetilde{C}_{kk_1k_2}+\widetilde{E}_{kk_1k_2}^{ss_1s_2}$ (with $\widetilde{E}_{kk_1k_2}^{ss_1s_2}$ being skew-symmetric with respect to permutations of all three columns of indices) has a solution with respect to $\widetilde{E}_{kk_1k_2}^{ss_1s_2}$, Eq. (25) is Hamiltonian; otherwise it is not. The author has not found such a solution, and the issue of a direct check of Eq. (25) being Hamiltonian remains unsolved.

5 Summary

It was demonstrated that the Navier-Stokes equations for an incompressible fluid can be represented in terms of a single complex scalar equation (25). The equation is formulated in Fourier domain and is non-local. Due to the former reason, this equation could be most useful for studying motions of unbounded fluid, or (after proper discretization of the $\vec{k}$-index) of motions within a rectangular parallelepiped finite along some (or all) coordinate axis. Numerical solution of this equation is also feasible. The advantage of doing this is twofold. The first is saving memory: instead of four variables (three components of velocity and pressure) one has only two (real and imaginary parts of amplitudes a). The second is that the resulting solution automatically represents incompressible motion. The disadvantage is that the equation in k-space is non-local and calculation of its RHS will be more time consuming.

A corresponding linear Liouville equation Eq. (38) for the probability density function corresponding to Eq. (25) was derived as well. Since using a-variables automatically ensures incompressibility, no external restrictions on solutions of Eq. (38) are imposed. In author's opinion, an attempt to build approximate solutions of this equation is potentially the main advantage of the suggested form of the Navier-Stokes equation.

The translationally-invariant, second-order in $\vec{v}_{\vec{k}}$ integrals of motion are considered in the inviscid case. There are two types of conservation laws possible. Each of two possibilities is realized; they correspond to energy- and helicity conservation. It was demonstrated that these two integrals exhaust translationally-invariant, second-order integrals.

A direct check was attempted to demonstrate that the derived scalar evolution equation Eq. (25) in a non-viscous case represents a Hamiltonian system. It was demonstrated that one of the two necessary conditions (the skew-symmetry of an appropriate coefficient function) does hold. However, check of the second necessary condition (the Jacobi identity) requires finding a solution of a set of second-order algebraic equations Eq. (69) (or a proof of existence of such solution), which was not accomplished.

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