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Development and Comparison of Bayesian and Classical Statistical Methods as Applied to Randomized Weather Modification Experiments

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WEATHER MODIFICATION PROGRAM OFFICE

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DEVELOPMENT AND COMPARISON OF BAYESIAN AND CLASSICAL STATISTICAL METHODS AS APPLIED TO RANDOMIZED WEATHER MODIFICATION EXPERIMENTS

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The Experimental Meteorology Laboratory has been conducting randomized dynamic cumulus seeding programs in south Florida to investigate the possible effects of cloud seeding on rainfall. There have been numerous statistical analyses performed under varying assumptions, including non parametric techniques, transformations of data for application of normal theory and the application of Bayesian techniques. However, none of the analyses have presented both a Bayesian, which requires numerous assumptions, and a classical analysis utilizing the same assumptions.

This report develops the Bayesian and Classical statistical analyses under similar assumptions concerning the underlying distributions. It is assumed that the basic distribution is a highly skewed gamma distribution and that the treatment effect is multiplicative. Within this framework, two situations are studied. The first assumes that the control, or natural, distribution is completely known. The second assumes only that the shape parameter of the gamma distribution is known. For each of these, Bayesian and Classical statistical methods are developed and compared. It is shown that the methods give the same numerical results when an improper diffuse prior $1/\theta$ is used for the seeding effect parameter in the first situation and, similarly, when the additional improper prior $1/\beta$ is used for the scale parameter in the control distribution for the second situation.

Key words: Weather Modification
Bayesian Analysis
Classical Analysis
Gamma Distribution

1. INTRODUCTION

In the past several years the Experimental Meteorology Laboratory (EML) has been involved in weather modification in the form of augmenting rainfall from cumulus clouds. The experiments conducted in south Florida have two common features. The first concerns the approach used in the actual seeding of cumuli, while the second concerns the statistical, or experimental, design. The dynamic seeding with silver iodide of cumulus clouds requires that an airplane actually enter the cumulus cloud to perform the seeding. While this allows an investigation of the seeding effect on individual cumuli, its use in an area experiment implies that not all clouds in the target area will actually be seeded. The statistical design may be termed a randomized treatment-control experiment. That is, an experimental unit is defined upon which a randomized decision is made whether to leave the unit alone as a control or to apply the treatment of seeding with silver iodide. It is this common feature and the resulting methods of analysis that is the subject in this investigation.

The statistical methodology associated with both the single cloud and multiple cloud experiments has the same basic structure. While using the actual experiments as examples, the emphasis will be on presenting several alternate analyses, classical and Bayesian, with their assumptions clearly stated. Some of the methods have been used in other publications analyzing the EML experiments. Most, however, are presentations of new statistical procedures based upon very explicit assumptions concerning the properties of the underlying statistical distributions. In addition to the development of the procedures, emphasis is given to the comparison of the Bayesian and classical procedures derived under the same assumptions. Because of the above, many of the tests used in analyzing weather modification data will not be presented. In particular, non-parametric procedures will not be discussed as well as optimal $C(\alpha)$ test procedures.

Section 2 will introduce the examples and data from the seeding experiments that will be used for illustrative purposes. Sections 3 and 4 will present the analyses possible, depending upon the parameters assumed to be known in the underlying gamma distribution. Within each section both Bayesian and classical results are given and compared.

2. BASIC MODEL AND EXPERIMENTAL CONDITIONS

Before introducing the experiments conducted by the EML, it is helpful to present the abstracted model that is common to them. It is this basic model upon which the various analyses are based. Hence it is essential that it be clearly stated and its applicability to the seeding experiments be studied.

In a number of experimental situations two independent data sets are obtained, one set giving independent replications on the measurement of a variable associated with a control and the other measurements of the same variable associated with the application of some treatment to the experimental unit. The data are to be analyzed to investigate for a possible treatment effect on the variable measured. In essence this is the usual two-sample problem for the detection of a treatment effect and, under the assumptions of an additive treatment effect and approximate normality of the variable, could be analyzed with the one-way classification experimental design (equivalent in this case to the two-sample t-test). In some situations these assumptions are clearly not justified, so that a correct analysis, either classical or Bayesian, based on them is not possible. In some instances an appropriate transformation may be made to the data so that the transformed data satisfy the assumptions, at least approximately. The approach considered here is to assume the basic structure given above, but then assume that the postulated effect of the treatment is multiplicative and that the distribution of the variable of interest is a highly skewed gamma distribution.

Needless to say, if an experiment is to be analyzed on the basis of these assumptions, their applicability to the experiment must be studied. The basic model results from the design of the experiment, so that once the design has been chosen and properly executed, the model will be satisfied. This is very much a part of the scientific process and is subject to all the various pitfalls associated with designing an experiment. From the statistical viewpoint, especially the classical, the proper application of the randomization is imperative. Although consideration of the validity of the basic model is important, this will not be examined for the examples presented. The additional assumptions of a multiplicative effect and an underlying gamma distribution will be discussed.

In 1968 and 1970, EML conducted massive airborne pyrotechnic AgI seeding experiments upon single isolated clouds over south Florida. This dynamic cumulus seeding aims to invigorate the cloud growth by means of fusion heat release. The purpose of the experiment was to examine the relationship between dynamic seeding and rainfall. In this series a total of 52 clouds, i.e. experimental units, was selected that met the criteria for seeding. The randomization procedure selected 26 clouds that were seeded, or treated, and 26 that were investigated as controls. The randomization was in large blocks and usually several seeded and control clouds were obtained in close succession on a single day. Further details of the design of the experiment have been reported by Simpson et al. (1970, 1971) and Simpson and Woodley (1971). It is sufficient for the present purpose to note that the experiment fits the basic model and that the variable measured for analysis is the amount of total rainfall from the time of the seeding pass until the cloud dissipated or merged with another cloud.

For the analyses to be presented here it is necessary to make additional assumptions that the total cloud lifetime rainfall has a gamma distribution and that the effect of seeding is multiplicative. That is, let θ be a parameter which is the factor by which, hypothetically, the treatment multiplies the average rainfall that would have occurred naturally. Extensive meteorological literature has shown that when a large enough sample is available, the gamma distribution fits a large class of rainfall data (Thom, 1947, 1951, 1957, 1958, 1968; Thom and Vestal, 1968; Mooley and Crutcher, 1968; Mooley, 1972; Barger, Shaw and Dale, 1959). The single cloud data were fit by the principle of maximum entropy to find the best fit gamma distributions and to compare the fits of other postulated distributions (Simpson, 1972). It was found that both the seeded and control sample could be described by a gamma distribution. Moreover, it was determined that the shape parameters were essentially the same so that the assumption of a multiplicative effect seemed realistic.

Before concluding that the assumptions are all satisfied, a few comments are in order. First, although the gamma distribution seemed to fit the data, there were other distributions that also gave adequate fits to the samples. Second, the sample sizes (26) are so small that it is difficult to determine what distribution the data follows. Similar comments hold for the assumption of a multiplicative effect. It is possible to analyze the experiment without making these assumptions, as done by Simpson et al. (1971). Because of the wider applicability of the procedures not requiring the above assumptions, they necessarily are not as powerful in detecting a treatment effect. By making the additional assumptions, however, it is possible to reach wrong conclusions if, in fact, the assumptions are invalid. Therefore, it is necessary to examine these assumptions carefully. Some investigators may not be willing to accept them, and in that case, they must seek alternate analyses.

During the single cloud experiments physical studies demonstrated that seeded rainfall was increased because of the larger volume and longer life of the seeded clouds. In the course of this work the natural cumulus merger process was documented. Consequently, the next and on-going phase of dynamic cumulus modification consists of multiple cloud seeding in a 4000 sq mi fixed target area, with the deliberate intention of promoting mergers. The experimental design is summarized in table 1. The use of a Daily Suitability Criterion, S-N_e, is an attempt to screen out objectively the disturbed, naturally rainy days. S is the maximum seedability in km (for a range of cloud diameters) and N_e is the number of hours between 10 a.m. and noon local time that one or more radar echoes are observed in the target. The experiment is randomized by days that meet the requirements in table 1. The experiment was conducted from mid-April to mid-September during 1970, 1971 and 1972. In addition to the randomized selection of seeded versus control days, a radar control program was operated as nearly identical to the randomized program as possible, with the exception that a single light aircraft was used so that actual seeding could not be carried out.

Although the radar control program departs from the basic model in that a randomized selection could not be made, the data resulting from the program will be used in the sequel. The importance of the departure lies in

DESIGN OF FLORIDA MULTIPLE CLOUD SEEDING EXPERIMENT

FIXED TARGET AREA — RANDOMIZATION BY DAY

CONTINUOUS UM/10-cm RADAR SURVEILLANCE OF TARGET

DAILY SUITABILITY CRITERION: $S-N_e \geq 1.50$? NO → NO OPERATIONS

YES

TARGET SUITABILITY: SUITABLE CLOUDS IN TARGET ? NO → RETURN TO BASE

YES

RANDOMIZER DETERMINES SEEDING DECISION

CARRYOUT SEEDING INSTRUCTION
(AVOID MATURE Cb's)

ACCEPTABILITY CRITERION: 6 SEEDED CLOUDS OR EXPENDITURE OF 60 FLARES ? NO → NO AREA ANALYSIS

YES

DO AREA ANALYSIS

Table 1. Decision procedure used in the design and operation of the multiple cloud experiments.

the possible introduction of bias to the experiment. The scientist aboard the light aircraft knows that the clouds selected cannot be seeded. Hence, even though he attempts to make an objective selection, as in the case of the randomized program where he does not know whether the clouds selected will be seeded or not, there is the possibility of a bias subconsciously being introduced. Further details of the randomized and radar control program are given by Simpson, Woodley, Cotton and Eden (1973).

Although the randomization was by days, there were two different experimental units, or targets, defined over the 4000 sq mi target area. The variable of interest was rain volume calculated for two types of targets, namely:

1. total target, comprising the rain falling in the entire 4000 sq mi target from the time of the first seeding to six hours after the first seeding,
2. floating target, comprising the rain from all radar echoes undergoing a "seeding" pass and all other echoes merging therewith, so long as they remain within the total target.

For the analysis presented, there were a total of seven random seeded days, four random control and five radar control. This is an extremely small sample for any assessment to be made of the validity of the assumptions of a gamma distribution and a multiplicative effect. A discussion of these assumptions is given by Simpson, Woodley, Olsen and Eden (1973) and Simpson, Woodley, Cotton and Eden (1973) along with further details concerning the multiple cloud area experiment. The main justifications for the assumptions are based upon analyzing the available data and on relying upon the transferability of the single cloud results to the multiple cloud experiment. Again it is emphasized that these assumptions must be verified, or the effect of departures from them studied.

The complete data sets used for the single cloud, floating target, and total target experiments are given in tables 2 and 3.

Table 2. Single Cloud Data for 1968 and 1970

Rain Volume for Total Cloud Lifetime (acre-ft)			
Seeded Clouds		Control Clouds	
129.6	7.7	26.1	28.6
31.4	1656.0	26.3	830.1
2745.6	978.0	87.0	345.5
489.1	198.6	95.0	1202.6
430.0	703.4	372.4	36.6
302.8	1697.8	0.0	4.9
119.0	334.1	17.3	4.9
4.1	118.3	24.4	41.1
92.4	255.0	11.5	29.0
17.5	115.3	321.2	163.0
200.7	242.5	68.5	244.3
274.7	32.7	81.2	147.8
274.7	40.6	47.3	21.7

Table 3. Floating Target and Total Target Rain Volume for Cloud Seeding Experiments 1970, 1971, 1972 (Values are in acre-ft x 10^4)

Floating Target Rainfall		Total Target Rainfall	
Seeded	Control	Seeded	Control
0.16	0.78	3.23	7.53
1.11	0.26	1.94	1.58
8.96	0.35	11.90	7.54
4.56	0.96	8.42	1.88
0.23	1.00	0.25	7.00
1.58	0.10	2.99	0.22
1.67	0.23	4.90	0.26
	2.15		3.04
	1.15		2.71

Since the basic model and distributional assumptions are the same for the three data sets, it is possible to present a single abstract model for them. This not only facilitates the analysis, but also explicitly exposes the assumptions that are made. Briefly, the situation is as follows: once an experimental unit is determined to be eligible for the experiment, a randomized decision is made to determine if the unit is to be subjected to the treatment or left as a control. In either case, the appropriate observation is taken. Let the random variable X represent the response measured on an experimental unit designated to receive the control, and let the random variable Y represent the response measured on an experimental unit designated to receive the treatment. Then let $P_X(\alpha, \beta_C)$ be the distribution on the positive real line with density

$$p(x \mid \alpha, \beta_C) = \frac{\beta_C^{\alpha}}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\beta_C x}$$

where $\alpha > 0$ and $\beta_C > 0$. Similarly, for the random variable Y, associated with the treatment, the distribution is designated by $P_Y(\alpha, \beta_T)$. Under these conditions, the postulated effect of the treatment is a change in scale for the distribution.

If the expected values of X and Y are given by $\mu_C = \alpha/\beta_C$ and $\mu_T = \alpha/\beta_T$, then the parameter of interest to the meteorologist is defined by $\theta = \mu_T/\mu_C$. In terms of the scale parameters of the distributions $\beta_T = \beta_C/\theta$ so that θ indicates a scale change. Moreover, the distribution of θX is the same as the distribution of Y. Under these conditions θ is the factor by which hypothetically, the treatment multiplies the average response that would occur if the treatment had not been applied to the experimental unit. Thus, $\theta = 1$ means no treatment effect, while $\theta = 1.4$ means a 40 percent increase due to the treatment and $\theta = 0.6$ means a 40 percent decrease.

At the completion of an experiment, it is assumed there are m independent observations on X and n observations on Y upon which the investigation of the treatment effect θ is to be based.

3. ANALYSES ASSUMING α AND β_C ARE KNOWN

In this section both Bayesian and classical analyses will be given when it is assumed that the control random variable X has a known distribution. This is equivalent under the assumptions of the previous section to knowing the parameters α and β_C for the control distribution. For the three situations considered as examples, this assumption is probably not true. However, some meaningful results may be possible by using the control sample to provide estimates for the parameters, even though strictly speaking the parameters should not be estimated in this manner in the context of the model.

3.1 Bayesian Approach

Since α and β_C are assumed to be known, only the distribution of the treated sample Y has an unknown parameter, β_T. Then θ may be introduced as the unknown parameter since $\beta_T = \beta_C/\theta = \alpha/(\mu_C\theta)$ and hence

$$p(y \mid \theta) = \left(\frac{\alpha}{\mu_C\theta}\right)^{\alpha} \frac{y^{\alpha - 1}}{\Gamma(\alpha)} \exp\left\{-\frac{\alpha y}{\mu_C\theta}\right\}, \quad y > 0.$$

Since α and μ_C are known with this formulation, it is now possible to assign a prior distribution to θ and obtain the posterior distribution given the n observations on Y. Having obtained the posterior, summary statistics such as the posterior mean, mode, standard deviation and Bayes posterior intervals on θ for a specified content can be derived. Both equal tail and shortest intervals are given.

In order to proceed with the development it is necessary to assign a prior distribution to θ. This assignment is not to be taken lightly and considerable forethought should be given in selecting it. Because of the expository nature of this report in this and the following sections various families of prior distributions will be presented to illustrate the prior's effect on the posterior distribution.

A natural family of prior distributions to consider is the family of conjugate priors (see Raiffa and Schlaiffer, 1961). In this case, the inverse gamma family, given by

$$p_1(\theta) = \frac{K_2^{K_1 + 1}}{\Gamma(K_1 + 1)} \theta^{-K_1 - 2} e^{-K_2/\theta}, \qquad \theta > 0$$

$K_1 > 0$ and $K_2 > 0$, is the conjugate family. By applying Bayes rule, the posterior distribution of θ given the n observations on the treatment random variable Y is

$$p_1(\theta/\underset{\sim}{y}) = \frac{(K_2 + \Delta)^{n\alpha + K_1 + 1}}{\Gamma(n\alpha + K_1 + 1)} \theta^{-n\alpha - K_1 - 2} e^{-(K_2 + \Delta)/\theta}, \quad \theta > 0$$

where $\Delta = n\alpha\bar{y}/\mu_C$, $\bar{y} = \sum_{i=1}^{n} y_i/n$ and $\underset{\sim}{y}$ denotes the sample. Note that the posterior is a member of the same family, i.e., inverse gamma, as the prior and that the parameters of the prior are changed by adding information gained from the sample. The posterior mean, mode and variance are given respectively by

$$\mu_1 = \frac{K_2 + \Delta}{K_1 + n\alpha}, \quad \eta_1 = \frac{K_2 + \Delta}{K_1 + 2 + n\alpha}, \quad \sigma_1^2 = \frac{(K_2 + \Delta)^2}{(K_1 + n\alpha)^2 (K_1 + n\alpha - 1)}.$$

There are two types of posterior intervals on θ that are of interest. The first is the Bayes equal tail posterior interval on the parameter θ with a prescribed content. If an interval containing 95 percent of the posterior distribution is desired, then the endpoints of the equal tail interval $[\ell, u]$ are determined by

$$\int_0^{\ell} p_1(\theta \mid \underset{\sim}{y})\,d\theta = 0.025 \quad \text{and} \quad \int_u^{\infty} p_1(\theta \mid \underset{\sim}{y})\,d\theta = 0.025.$$

The solution of the equations may be obtained numerically by using Newton's method and numerical integration, such as gaussian quadrature. This is the procedure used by EML and the program is available. An alternate procedure would be to make a change of variable in the integral so that the resulting integrand would be a Chi-square density. The Chi-square tables could then be utilized, with the probable need to interpolate for non integer degrees of freedom.

The Bayes shortest posterior interval θ is the shortest interval with prescribed content in the domain of the posterior density of θ. If the endpoints of the interval are $[L, U]$, then L and U would be selected such that the length $D=U-L$ was minimized for all endpoints that satisfy

$$\int_L^U p_1(\theta \mid \underset{\sim}{y})\,d\theta = 0.95,$$

where the prescribed content is 95 percent. This is accomplished by differentiating D with respect to L giving

$$\frac{dD}{dL} = \frac{dU}{dL} - 1$$

and similarly differentiating the side condition to give

$$p_1(U \mid \underset{\sim}{y}) \frac{dU}{dL} - p_1(L \mid \underset{\sim}{y}) = 0.$$

Hence

$$\frac{dD}{dL} = \frac{p_1(L \mid \underset{\sim}{y})}{p_1(U \mid \underset{\sim}{y})} - 1.$$

and the critical value, which yields a minimum, occurs when

$$p_1(L \mid \underset{\sim}{y}) = p_1(U \mid \underset{\sim}{y}).$$

Since $p_1(\theta \mid \underset{\sim}{y})$ is continuous, unimodal and monotone decreasing to zero on both sides of the mode, the endpoints of the shortest interval are on opposite sides of the mode and must satisfy

$$p_1(L \mid \underset{\sim}{y}) = p_1(U \mid \underset{\sim}{y}) \quad \text{and} \quad \int_L^U p_1(\theta \mid \underset{\sim}{y})\,d\theta = 0.95.$$

This may be solved numerically by using Newton's method for two equations and two unknowns. Starting values for the iteration may be obtained by using the endpoints of the equal tail interval.

A prior distribution that will be of interest in the comparison with the classical procedures is the improper prior with density proportional to $1/\theta$. It is easily seen that the resulting posterior distribution is a special case of the inverse gamma posteriors just presented. In particular the correct posterior is given when the parameters for the inverse gamma prior are $K_1 = -1$ and $K_2 = 0$. As a result, the summary statistics are given by the same equations. Note, however, that it is necessary for $n\alpha$ to be greater than 1 for the posterior mean to exist and greater than 2 for the posterior variance to exist. For any reasonable sample size and α these restrictions are easily met.

By following the basic procedures described above for the inverse gamma prior, it is in principle possible to obtain the same posterior information for any other selection of a prior on θ. The only limitation is the ability to perform the necessary operations. Since θ corresponds to a multiplicative effect, the choice of a prior distribution should make use of this information. That is, if the prior is to be unprejudiced toward an increase or decrease to be ascribed to the treatment, then the prior should assign 1/2 of its probability less than 1 and 1/2 greater than 1. One particular family that achieves this is based on a uniform distribution for θ on the interval $[\ell, b]$ and is given by

$$p_2(\theta) = \begin{cases} \dfrac{1}{2(b-1)\theta^2} & 1/b \leq \theta \leq 1 \\[2ex] \dfrac{1}{2(b-1)} & 1 \leq \theta \leq b \end{cases}$$

where $b > 1$. The resulting posterior distribution is

$$p_2(\theta/\underset{\sim}{y}) = \begin{cases} \dfrac{\Delta^{n\alpha+1}}{A(n\alpha+1,\ n\alpha-1)} \; \theta^{-n\alpha-2} \; e^{-\Delta/\theta} & 1/b \leq \theta \leq 1 \\[2ex] \dfrac{\Delta^{n\alpha+1}}{A(n\alpha+1,\ n\alpha-1)} \; \theta^{-n\alpha} \; e^{-\Delta/\theta} & 1 \leq \theta \leq b, \end{cases}$$

where $A(c, d) = \gamma(c, \Delta b) - \gamma(c, \Delta) + \Delta^2 [\gamma(d, \Delta) - \gamma(d, \Delta/b)]$ and γ is the unnormalized incomplete gamma function. The posterior mean and variance are determined from

$$\mu_2 = \Delta \frac{A(n\alpha,\ n\alpha-2)}{A(n\alpha+1,\ n\alpha-1)}$$

and

$$\sigma_2^2 = \frac{\Delta^2}{A(n\alpha + 1, n\alpha - 1)} \left[A(n\alpha - 1, n\alpha - 3) - \frac{A^2(n\alpha, n\alpha - 2)}{A(n\alpha + 1, n\alpha - 1)} \right].$$

The determination of the posterior mode is complicated by the dependency of the density form on the range of θ. In any case the mode is given by the member of the set $\{1/b, \Delta/(n\alpha + 2), 1, \Delta/(n\alpha), b\}$ that maximizes $p_2(\theta \mid \underset{\sim}{y})$. The Bayes equal tail and shortest posterior intervals are determined numerically in the same manner as before, and are denoted by $[\ell_2, u_2]$ and $[L_2, U_2]$ respectively. However, since the range of θ is finite, it may be possible that the shortest interval will have one of its end-points as $1/b$ or b instead of satisfying the equation $p_2(L_2 \mid \underset{\sim}{y}) = p_2(U_2 \mid \underset{\sim}{y})$. This is a result of the density not approaching zero within the range of θ.

The final family of prior distributions to be considered is uniform on an interval $[a, b]$, i.e.,

$$p_3(\theta) = \frac{1}{b - a}, \qquad a \leq \theta \leq b,$$

so that the posterior density is

$$p_3(\theta \mid \underset{\sim}{y}) = C_3\, \theta^{-n\alpha}\, e^{-\Delta/\theta}, \quad a \leq \theta \leq b$$

and zero elsewhere where

$$C_3 = \frac{\Delta^{n\alpha - 1}}{[\gamma(n\alpha - 1, \Delta/a) - \gamma(n\alpha - 1, \Delta/b)]}.$$

The posterior mean and variance are determined by

$$\mu_3 = \Delta \frac{\gamma(n\alpha - 2, \Delta/a) - \gamma(n\alpha - 2, \Delta/b)}{\gamma(n\alpha - 1, \Delta/a) - \gamma(n\alpha - 1, \Delta/b)},$$

$$\sigma_3^2 = \Delta^2 \frac{\gamma(n\alpha - 3, \Delta/a) - \gamma(n\alpha - 3, \Delta/b)}{\gamma(n\alpha - 1, \Delta/a) - \gamma(n\alpha - 1, \Delta/b)} - \mu_3^2$$

and the posterior mode is the member of the set $\{a, \Delta/(n\alpha), b\}$ that maximizes $p_3(\theta \mid \underset{\sim}{y})$. The Bayes posterior intervals are determined and denoted in a manner similar to the previous family of priors.

To illustrate the procedures of this section the three data sets given in tables 2 and 3 of section 2 are used for selected members of each of the three families of prior distributions. More detailed descriptions of the implications with regard to the cloud seeding can be found in Simpson et al., 1973a, 1973b, and 1973c. Although the parameters α and μ_c necessary for the specification of the posterior distributions are not actually known, as assumed in the development, the control sample is used to help specify their values. Table 4 gives the values used for the three data sets.

Table 4. Values Used for the Specification of the Posterior Distributions (excluding the prior distribution).

Parameter	Single Cloud	Floating Target	Total Target
α	0.6	1.0	1.0
n	26	7	7
μ_c	164.588	0.775	3.529
$\overline{y}$	441.985	2.610	4.800
Δ	41.892	23.574	9.524

The results of the examples are presented in tables 5, 6 and 7 in the form of the summary statistics for the posterior distributions resulting from selected prior distributions. In all of the examples the mode is smaller than the posterior mean indicating that the posterior is skewed with the long tail toward higher seeding effects. This is also implied by the Bayes shortest posterior sets being shifted to the left of the equal tail posterior sets. For the single cloud example the selection of the prior does not greatly affect the posterior. The relatively large sample sizes have the most effect on the posterior and minimizes the contribution of the prior. This is also reflected in the length of the Bayes posterior sets of 95 percent. Since none of the posterior sets include a seeding effect of $\theta = 1$, it can be concluded that there is a positive seeding effect. A point estimate of the size depends upon whether the mean or mode is the criterion used and upon the specific prior selected. In general, a seeding effect of 2.5 for the single cloud data would be reasonable.

For the floating and total target examples the effect of the prior is more pronounced due to smaller data sample size. There does seem to be a positive seeding effect. However, its actual magnitude is more dependent upon the prior selected and the criterion used. Also, the posterior distributions are more skewed than the single cloud example. The total target example is inconclusive concerning the seeding effect because virtually all the Bayes posterior sets include one. Although the posterior distributions are not as skewed as the floating target, this is more a function of the distribution being centered on smaller seeding effects than anything else.

As a final comment, the validity of using the control sample to specify α and μ_C deserves careful examination. This has been investigated by Simpson, Woodley, Cotton and Eden (1973) and Simpson, Woodley, Olsen and Eden (1973) in the context of the sensitivity of the posterior to changes in α and μ_C. Their results show that reasonable changes in α do not significantly affect the results, but that changes in μ_C are important.

Table 5. Single Cloud Example. Posterior Summary Statistics for Selected Prior Distributions on θ.

Prior Distribution	Mean	Mode	Std. Dev.	Equal Tail ℓ	Equal Tail u	Shortest L	Shortest U
Inverse Gamma							
1. K_1=2.25 K_2=6.75	2.73	2.45	0.66	1.72	4.30	1.59	4.05
2. K_1=1 K_2=3	2.70	2.41	0.68	1.68	4.33	1.55	4.07
3. K_1=1 K_2=1	2.58	2.31	0.65	1.60	4.14	1.48	3.89
4. K_1=1.0 K_2=0.5	2.55	2.28	0.65	1.59	4.09	1.46	3.85
5. K_1=10 K_2=20	2.42	2.24	0.49	1.65	3.54	1.56	3.39
6. K_1=10 K_2=5	1.83	1.70	0.37	1.25	2.69	1.18	2.57
7. K_1=0.5 K_2=0.5	2.63	2.34	0.68	1.63	4.25	1.49	3.99
8. K_1=-1 K_2=0	2.87	2.52	0.78	1.73	4.74	1.57	4.42
Modified Uniform							
9. b = 5	2.99	2.69	0.72	1.82	4.61	1.73	4.49
10. b = 10	3.08	2.69	0.87	1.82	5.17	1.65	4.81
Uniform							
11. a=0.5 b=10	3.08	2.69	0.87	1.82	5.17	1.65	4.81
12. a=0.8 b= 5	2.99	2.69	0.72	1.82	4.61	1.73	4.49

Table 6. Floating Target Example. Posterior Summary Statistics for Selected Prior Distributions on θ.

Prior Distribution		Mean	Mode	Std. Dev.	Equal Tail ℓ	Equal Tail u	Shortest L	Shortest U
Inverse Gamma								
1. K_1=2.25	K_2=6.75	3.28	2.70	1.14	1.74	6.10	1.51	5.53
2. K_1= 1	K_2= 3	3.32	2.66	1.25	1.69	6.46	1.43	5.78
3. K_1= 1	K_2= 1	3.07	2.46	1.16	1.56	5.97	1.33	5.35
4. K_1=1.0	K_2=0.5	3.01	2.51	1.14	1.53	5.85	1.30	5.24
5. K_1=10	K_2=20	2.56	2.29	0.64	1.60	4.08	1.48	3.84
6. K_1=10	K_2= 5	1.68	1.50	0.42	1.05	2.68	0.97	2.52
7. K_1=0.5	K_2=0.5	3.21	2.53	1.26	1.59	6.36	1.34	5.67
8. K_1=-1	K_2=0	3.92	2.95	1.73	1.81	8.36	1.47	7.30
Modified Uniform								
9. b = 5		3.48	3.37	0.83	1.91	4.90	2.11	5.00
10. b =10		4.43	3.37	1.72	2.01	8.71	1.71	8.10
Uniform								
11. a=0.5	b=10	4.43	3.37	1.72	2.01	8.71	1.71	8.10
12. a=0.8	b= 5	3.48	3.37	0.83	1.91	4.90	2.11	5.00

Table 7. Total Target Example. Posterior Summary Statistics for Selected Prior Distributions on θ.

Prior Distribution		Mean	Mode	Std. Dev.	Equal Tail ℓ	Equal Tail u	Shortest L	Shortest U
Inverse Gamma								
1. K_1=2.25	K_2=6.75	1.76	1.45	0.61	0.93	3.27	0.81	2.97
2. K_1= 1	K_2= 3	1.56	1.25	0.59	0.79	3.05	0.67	2.73
3. K_1= 1	K_2= 1	1.32	1.05	0.50	0.67	2.55	0.57	2.29
4. K_1=1.0	K_2=0.5	1.25	1.00	0.47	0.64	2.42	0.54	2.18
5. K_1=10	K_2=20	1.74	1.55	0.43	1.08	2.77	1.00	2.60
6. K_1=10	K_2= 5	0.86	0.76	0.21	0.53	1.29	0.50	1.24
7. K_1=0.5	K_2=0.5	1.33	1.05	0.52	0.66	2.64	0.56	2.36
8. K_1=-1	K_2= 0	1.59	1.19	0.71	0.73	3.38	0.59	2.96
Modified Uniform								
9. b = 5		1.81	1.36	0.80	0.73	3.87	0.58	3.47
10. b = 10		1.86	1.36	0.93	0.74	4.28	0.57	3.66
Uniform								
11. a=0.5	b=10	1.90	1.36	0.92	0.82	4.31	0.64	3.69
12. a=0.8	b= 5	1.87	1.36	0.77	0.89	3.92	0.80	3.45

3.2 Classical Approach

The analyses presented in this subsection follow the classical statistics approach for the same problem considered in section 3.1. That is, it is assumed that the underlying probability distribution is a gamma distribution and the parameters α and μ_c (or β_c) are assumed to be known. In this situation the only unknown parameter is θ and the only probability density of interest is that of the seeded sample, i.e.

$$p(y \mid \theta) = \left(\frac{\alpha}{\mu_c \theta}\right)^{\alpha} \frac{y^{\alpha - 1}}{\Gamma(\alpha)} \exp\left\{- \frac{\alpha}{\mu_c \theta}\, y\right\}, \quad y > 0.$$

Moreover, a random sample of size n on the random variable Y is available for analysis.

In classical statistics there are two main areas of inference - estimation and hypothesis testing. Under estimation interest is centered on point estimation or interval estimation for a parameter. Although there are many possible procedures for obtaining point estimators for θ, only the method of maximum likelihood will be used. This method consists of forming the likelihood function,

$$\ell(\theta \mid \underset{\sim}{y}) = \prod_{i=1}^{n} p(y_i \mid \theta),$$

as a function of θ and then finding the value of θ that maximizes $\ell(\theta \mid \underset{\sim}{y})$. By taking the natural log and differentiating it is easily seen that

$$\hat{\theta} = \overline{y}/\mu_c$$

is the maximum likelihood estimator. Moreover, the expected value of $\hat{\theta}$ is θ so that the estimator is also unbiased. By noting that $\overline{y}$ is a complete sufficient statistic for θ and appealing to the Rao-Blackwell Theorem (see Ferguson, 1967) it can be concluded that $\hat{\theta}$ is a minimum variance unbiased estimator for θ.

Since the confidence intervals are derived from the hypothesis testing problem, it is convenient to consider the testing problem first. There are two type of hypotheses concerning the parameter θ that may be of interest in this particular situation. The first is to test the null hypothesis H_o: $\theta = \theta_o$, θ_o specified, versus the alternate hypothesis H_A: $\theta \neq \theta_o$. In particular, $\theta_o = 1$ tests whether seeding has an effect, either positive or negative, on the amount of rainfall. The second hypothesis is similar, except that the null hypothesis specifies that θ is contained in the interval $[\theta_o, \theta_1]$ while the alternative states that $\theta < \theta_o$ or $\theta > \theta_1$. At this stage in the investigation the first hypothesis is more relevant.

It was previously noted that $\overline{y}$ is a sufficient statistic for θ so that the hypothesis test may be based on it and its distribution, gamma with shape parameter $n\alpha$ and scale parameter $n\alpha/(\mu_c\theta)$. A reasonable test statistic is

$$Z = \frac{2n\alpha\overline{Y}}{\mu_c\theta_o}$$

where the null hypothesis is rejected if $Z < Z_1$ or $Z > Z_2$. Z_1 and Z_2 are to be selected based on the significance level chosen for the test and the criteria used to specify the type of test desired. For instance, one test would be an equal tail test where 1/2 of the significance probability is to be in each tail.

This is easily accomplished by noting that under the null hypothesis Z has a Chi-square distribution with $2n\alpha$ degrees of freedom. Then if a 5 percent significance level is selected, Z_1 and Z_2 are determined such that

$$\Pr\{Z < Z_1 \mid \theta = \theta_0\} = 0.025 \quad \text{and} \quad \Pr\{Z > Z_2 \mid \theta = \theta_0\} = 0.025 .$$

The actual values are determined by consulting a set of Chi-square tables using $2n\alpha$ degrees of freedom. It may be necessary to interpolate on the degrees of freedom. Alternately, since

$$\Pr\{Z < Z_1 \mid \theta = \theta_0\} = \int_0^{Z_1} \frac{z^{n\alpha - 1} e^{-z/2}}{2^{n\alpha}\,\Gamma(n\alpha)}\,dz$$

and

$$\Pr\{Z > Z_2 \mid \theta = \theta_0\} = 1 - \int_0^{Z_2} \frac{z^{n\alpha - 1} e^{-z/2}}{\Gamma(n\alpha)\,2^{n\alpha}}\,dz ,$$

Z_1 and Z_2 may be determined numerically.

Based on this equal tail test is an equal tail confidence interval. This is derived by noting that the acceptance region of the test statistic is the interval $Z_1 < Z = \dfrac{2n\alpha\overline{Y}}{\mu_c\theta} < Z_2$. By inverting the interval so that an interval on θ results, the confidence interval is

$$\frac{2n\alpha\overline{Y}}{\mu_c Z_2} < \theta < \frac{2n\alpha\overline{Y}}{\mu_c Z_1}$$

with confidence level equal to one minus the significance level of the test.

Although the above equal tail test and its associated confidence interval is very reasonable, a "better" test can be derived by using the theory associated with hypothesis testing. In particular a uniformly most powerful unbiased test can be derived following the procedures given by Ferguson (1967). The terminology can be simply explained in terms of the power function of a test $\Phi(\bar{y})$. Given the significance level, or size of the test desired, consideration is limited to all tests based on $\bar{Y}$ that have the desired size and are such that their power is always greater than or equal to the size. Note that this refers to the hypothesis H_0: $\theta = \theta_0$ versus H_A: $\theta \neq \theta_0$. That is, the test must have a probability of rejecting the null hypothesis when it is false at least as great as the significance level. This is the property of unbiasedness. A uniformly most powerful unbiased test is then a test of the appropriate size that is unbiased and has a power function that is uniformly, over $\theta \neq \theta_0$, higher than any other test of the same size that is unbiased.

If ξ denotes the size of the test desired and $E_\theta(\,-\,)$ denotes the expectation of a function given the parameter value θ, then the uniformly most powerful unbiased test is of the form

$$\Phi(\bar{y}) = \begin{cases} 1 & \bar{y} < c_1 \text{ or } \bar{y} > c_2 \\ 0 & c_1 \leq \bar{y} \leq c_2 \end{cases}$$

where $\Phi(\bar{y}) = 1$ indicates the null hypothesis is to be rejected and c_1 and c_2 are chosen so that

$$E_{\theta_0}[\Phi(\bar{Y})] = \xi$$

and

$$E_{\theta_0}[\bar{Y}\Phi(\bar{Y})] = \xi E_{\theta_0}[\bar{Y}].$$

The test and above conditions can be restated in terms of the test statistic used in the equal tail hypothesis test as follows: If $Z = 2n\alpha Y/(\mu_c\theta_0)$, then

$$\Phi(Z) = \begin{cases} 1 & Z < Z_1 \text{ or } Z > Z_2 \\ 0 & Z_1 \leq Z \leq Z_2 \end{cases}$$

where Z_1 and Z_2 are determined from

$$\int_{Z_1}^{Z_2} \frac{z^{n\alpha - 1} e^{-z/2}}{2^{n\alpha} \Gamma(n\alpha)} dz = 1 - \xi$$

and

$$\int_{Z_1}^{Z_2} \frac{z^{n\alpha} e^{-z/2}}{2^{n\alpha} \Gamma(n\alpha)} dz = 2n\alpha(1 - \xi) .$$

These equations may be solved numerically to find Z_1 and Z_2. If the Chi-square density of Z is denoted by $g_\nu(z)$ where $\nu = 2n\alpha$ and if $\lambda = \theta/\theta_0$, then the two conditions can be rewritten in the form

$$P(1) = 1 - \xi$$

and

$$Z_1 g_\nu(Z_1) = Z_2 g_\nu(Z_2)$$

where

$$P(\lambda) = \int_{Z_1\lambda}^{Z_2\lambda} g_\nu(z) dz .$$

The advantage of stating the conditions in this form is that Z_1 and Z_2 have been tabulated subject to them. In particular, Tate and Klett (1959) have given Z_1 and Z_2 to four decimal places for $\nu = 2(1)29$ and $\xi = .001, .005, .01, .05, .10$. Lindley, East and Hamilton (1960) tabulated Z_1 and Z_2 to five significant figures for $\nu = 1(1)100$, $\xi = .001, .01, .05$. Moreover, Guenther, 1972 gives the framework for deriving the unbiased confidence interval for θ with confidence coefficient $1 - \xi$. In this case the interval is

$$\frac{2n\alpha\bar{Y}}{\mu_c Z_2} \leq \theta \leq \frac{2n\alpha\bar{Y}}{\mu_c Z_1}$$

where Z_1 and Z_2 are determined from the above conditions, either numerically or through the use of the tables.

In addition to the equal tail and unbiased confidence intervals arising from the hypothesis tests, a shortest confidence interval with a specified confidence coefficient can be obtained in a manner similar to that used in the Bayesian shortest posterior set. The procedure is outlined by Guenther (1969). That is, the length of the interval

$$L = \frac{2n\alpha Y}{\mu_C}\left[\frac{1}{Z_1} - \frac{1}{Z_2}\right]$$

is minimized with respect to Z_1 and Z_2 subject to the condition

$$\Pr[Z_1 \leq Z \leq Z_2 \mid \theta = 1] = 1 - \xi.$$

The minimum length is given by Z_1 and Z_2 when they satisfy the relationships

$$P(1) = 1 - \xi$$

and

$$Z_1^2\, g_\nu(Z_1) = Z_2^2\, g_\nu(Z_2).$$

These procedures were applied to the three sample data sets with the pertinent results presented in table 8. An inspection of the results reveals the overall conclusions to be very similar to those given in section 3.1. This is particularly true for the diffuse or flat priors used for θ. In fact in the case of some of the classical statistical results, it can be shown that there exists a prior distribution which will give the same numerical answers from the Bayesian approach. For the maximum likelihood estimator the numerical agreement occurs when a uniform prior is used on θ such that the mode of the posterior does not occur at the endpoints of the posterior range. Similarly, the equal tail and shortest confidence intervals are numerically the same as their counterpart Bayes posterior sets when the improper prior $1/\theta$ is used.

The numerical equivalence of the intervals is easily checked theoretically for both the equal tail and shortest case. The lower limit of the classical equal tail interval is given by $L = 2n\alpha\bar{y}/(\mu_C Z_2) = 2\Delta/Z_2$ where Z_2 is selected to satisfy

$$\int_{Z_2}^{\infty} g_\nu(x)\,dx = \xi/2 .$$

In terms of L the integration limits become

$$\int_{2\Delta/L}^{\infty} g_\nu(x)\,dx = \xi/2.$$

For the lower limit L of the Bayesian equal tail interval L is selected such that

$$\int_{0}^{L} \frac{\Delta^{n\alpha}}{\Gamma(n\alpha)} \theta^{-(n\alpha+1)} e^{-\Delta/\theta} \, d\theta = \xi/2$$

where the improper prior $1/\theta$ is used. By making the change of variable $x = 2\Delta/\theta$ the above intergral becomes

$$\int_{2\Delta/L}^{\infty} g_\nu(x)\,dx = \xi/2.$$

Hence, the two lower limits agree. A similar procedure shows the upper limits also agree.

For the shortest Bayesian and classical intervals the same type of procedure as above will yield the desired agreement. It is most readily established by using the conditions

$$Z_1^2\, g_\nu(Z_1) = Z_2^2\, g_\nu(Z_2)$$

and

$$\int_{Z_1}^{Z_2} g_\nu(x)\,dx = 1 - \xi$$

for the classical interval and the conditions

$$p_1(L \mid \underset{\sim}{y}) = p_1(U \mid \underset{\sim}{y})$$

and

$$\int_{L}^{U} p_1(\theta \mid \underset{\sim}{y})\,d\theta = 1 - \xi$$

for the Bayesian interval with the improper prior $1/\theta$ being utilized.

Table 8. Summary of Sample Classical Statistical Analyses for the Detection of a Seeding Effect Assuming the Control Distribution is Known.

Summary Statistics	Single Cloud		Floating Target		Total Target	
Maximum Likelihood Estimate of Seeding Effect θ	2.69		3.37		1.36	
Observed Value of Test Statistics	83.79		47.15		19.04	
95% Confidence Limits						
	Lower	Upper	Lower	Upper	Lower	Upper
Equal Tail	1.73	4.74	1.81	8.38	0.73	3.38
Unbiased	1.69	4.63	1.73	7.93	0.70	3.20
Shortest	1.57	4.42	1.47	7.31	0.59	2.95
5% Hypothesis Test Critical Limits						
	Lower	Upper	Lower	Upper	Lower	Upper
Equal Tail	17.69	48.48	5.63	26.12	5.63	26.12
Unbiased	18.11	49.45	5.95	27.26	5.95	27.26

The importance of having the classical results and Bayesian results agree is not necessarily in the agreement itself, but in pointing out that the choice of the methods must be based on the methodology involved. That is to say that Bayesian statistics is not necessarily more powerful, or able to reach a conclusion faster, than classical statistics when the same assumptions are used.

4. ANALYSES ASSUMING SHAPE PARAMETER α IS KNOWN

The analysis of section 3 assumed that the control random variable X had a known distribution. For most experiments this assumption is not valid so that the analyses presented may not be used. However, it may be reasonable, expecially with a highly skewed gamma distribution, to assume that the common shape parameter α is known. This section will present Bayesian and classical statistical analyses based upon this assumption. These procedures make direct use of the observations on the control and treated random variables.

4.1 Bayesian Approach

Under the above assumptions the unknown parameters are $\beta \equiv \beta_C$ and θ so that it is necessary to assign a joint prior distribution to them. Moreover, the data now consists of both samples $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$

(designated by $\underset{\sim}{x}$ and $\underset{\sim}{y}$ respectively) with the associated joint conditional density

$$p(\underset{\sim}{x}, \underset{\sim}{y} \mid \beta, \theta) = \prod_{i=1}^{m} \left[\frac{\beta^{\alpha} x_i^{\alpha - 1}}{\Gamma(\alpha)} e^{-\beta x_i} \right] \prod_{j=1}^{n} \left[\frac{\beta^{\alpha} y_i^{\alpha - 1}}{\Gamma(\alpha)\theta^{\alpha}} e^{-\beta y_i/\theta} \right].$$

The same procedure as before is used to obtain the joint posterior distribution of β and θ. In this case, however, since only information concerning the seeding effect θ is of interest, an additional step is taken to derive the marginal posterior distribution of θ. The summary statistics can then be derived.

In the assessment of the joint prior on β and θ it is assumed that they are independent. Under this assumption there are numerous choices available for the selection of the two marginal prior distributions. The remarks concerning the selection of a prior made in section 3.1 apply equally well here. The three families of prior distributions previously introduced for θ are used for illustrative purposes. The families of priors selected for the scale parameter β are the gamma family and the uniform family on the positive real axis.

Under the restriction that the other parameter is known, both β and θ have an associated conjugate prior distribution, a gamma distribution for β and an inverse gamma distribution for θ, i.e.,

$$f_4(\beta) = \frac{k_2^{k_1}}{\Gamma(k_1)} \beta^{k_1 - 1} e^{-k_2 \beta} \qquad \beta > 0$$

and

$$p_1(\theta) = \frac{K_2^{K_1 + 1}}{\Gamma(K_1 + 1)} \theta^{-(K_1 + 2)} e^{-K_2/\theta} \qquad \theta > 0.$$

For these prior distributions the joint posterior probability density is easily determined as proportional to

$$\theta^{-(n\alpha + K_1 + 2)} e^{-K_2/\theta} \beta^{n\alpha + m\alpha + k_1 - 1} e^{-(k_2 + m\bar{x} + n\bar{y}/\theta)\beta}$$

where $\theta > 0$, $\beta > 0$. The relevant marginal posterior density for θ is determined by a simple integration and is

$$p_4(\theta \mid \underset{\sim}{x}, \underset{\sim}{y}) \approx \frac{\theta^{-(n\alpha + K_1 + 2)} e^{-K_2/\theta}}{(k_2 + m\bar{x} + n\bar{y}/\theta)^{n\alpha + m\alpha + k_1}}, \quad \theta > 0.$$

For any given set of values the normalizing constant as well as the posterior mean, mode and variance of θ can be determined numerically. The posterior mode η_4 is the single positive root of the quadratic in θ

$$(k_2 + m\bar{x})(n\alpha + K_1 + 2)\theta^2 + [n\bar{y}(K_1 + 2 - m\alpha - k_1) - K_2(k_2 + m\bar{x})]\theta - n\bar{y}K_2 .$$

It is of interest to note that the sample observations enter into the posterior only in terms of the sample average. This is a consequence of assuming that the shape parameter α is known. The Bayes shortest and equal tail posterior intervals of a prescribed content are computed numerically as in section 3.1.

Using the inverse gamma prior for θ and a uniform prior for β the joint posterior density is

$$p_5(\theta, \beta \mid \underset{\sim}{x}, \underset{\sim}{y}) \approx \theta^{-(n\alpha + K_1 + 2)} e^{-K_2/\theta} \beta^{n\alpha + m\alpha} e^{-(m\bar{x} + n\bar{y}/\theta)\beta}$$

where $\theta > 0$ and $k_1 \leq \beta \leq k_2$, and the marginal posterior density for θ becomes

$$p_5(\theta \mid \underset{\sim}{x}, \underset{\sim}{y}) \approx \frac{\Gamma[n\alpha + m\alpha + 1, (m\bar{x} + n\bar{y}/\theta)k_2] - \Gamma[n\alpha + m\alpha + 1, (m\bar{x} + n\bar{y}/\theta)k_1]}{\theta^{n\alpha + K_1 + 2} e^{-K_2/\theta} (m\bar{x} + n\bar{y}/\theta)^{n\alpha + m\alpha + 1}}$$

where $\theta > 0$ and $\Gamma(a, b)$ is the normalized incomplete gamma function with the shape parameter designated by a. The posterior summary statistics must be found numerically as before. If the improper uniform prior on the positive real line is used for β, then the marginal posterior for θ is given by the density $p_4(\theta \mid \underset{\sim}{x}, \underset{\sim}{y})$ where $k_1 = 1$ and $k_2 = 0$.

The final combination of prior distributions to be presented assumes the gamma prior for β and the modified uniform prior $p_2(\theta)$ introduced in section 3.1 for θ. The joint posterior density then becomes

$$p_6(\theta, \beta \mid \underset{\sim}{x}, \underset{\sim}{y}) \approx \begin{cases} \theta^{-(n\alpha+2)} \beta^{n\alpha + m\alpha + k_1 - 1} \exp\{-(k_2 + m\bar{x} + n\bar{y}/\theta)\beta\}, & 1/b \leq \theta \leq 1 \\ \theta^{-n\alpha} \beta^{n\alpha + m\alpha + k_1 - 1} \exp\{-(k_2 + m\bar{x} + n\bar{y}/\theta)\beta\}, & 1 \leq \theta \leq b \end{cases}$$

and the posterior marginal density for θ is

$$p_6(\theta \mid \underset{\sim}{x}, \underset{\sim}{y}) \approx \begin{cases} \dfrac{\theta^{-(n\alpha+2)}}{(k_2 + m\bar{x} + n\bar{y}/\theta)^{n\alpha + m\alpha + k_1}} & 1/b \leq \theta \leq 1 \\ \dfrac{\theta^{-n\alpha}}{(k_2 + m\bar{x} + n\bar{y}/\theta)^{n\alpha + m\alpha + k_1}} & 1 \leq \theta \leq b \end{cases}$$

The posterior mode η_6 for this marginal density form is the member of the set

$$\left\{ 1/b,\ 1,\ b,\ \frac{n\bar{y}\,(m\alpha + k_1 - 2)}{(k_2 + m\bar{x})\,(n\alpha + 2)},\ \frac{n\bar{y}\,(m\alpha + k_1)}{n\alpha(k_2 + m\bar{x})} \right\}$$

that maximizes $p_6(\theta \mid \underset{\sim}{x}, \underset{\sim}{y})$. As in the previous cases the proportionally constant and the remainder of the summary statistics can be obtained numerically.

For the later comparison of the statistical methods under the present assumptions it is convenient to consider the use of improper priors on both θ and β and to present the form of the posterior in a slightly different form. In section 3.2 it was shown that the improper prior $1/\theta$ gave results that agreed numerically with the classical approach. Under the present assumptions a reasonable procedure would be to assume the same improper prior for θ, and additionally the same form of improper prior for β; i.e., $1/\beta$ would give the agreement with the classical procedures. As before the necessary posterior marginal density for θ is given by the selection of particular parameter values for the gamma and inverse gamma priors. In particular $K_1 = -1$ and $K_2 = 0$ for the inverse gamma prior on θ and $k_1 = k_2 = 0$ for the gamma prior on β. Under these assumptions the posterior is

$$p_4(\theta \mid \underset{\sim}{x}, \underset{\sim}{y}) = \frac{\Gamma(n\alpha + m\alpha)}{\Gamma(n\alpha)\Gamma(m\alpha)} \left(\frac{n\bar{y}}{m\bar{x}}\right)^{n\alpha} \frac{\theta^{-(n\alpha+1)}}{\left(1 + \frac{n\bar{y}}{m\bar{x}\theta}\right)^{n\alpha + m\alpha}}, \quad \theta > 0.$$

It is possible to show that the rth moments of θ are given by

$$\left(\frac{n\bar{y}}{m\bar{x}}\right)^r \frac{(m\alpha + r - 1)\ . \ . \ . \ (m\alpha)}{(n\alpha - 1)\ . \ . \ . \ (n\alpha - r)}$$

so that the posterior expected value and variance are

$$\frac{n\alpha}{n\alpha - 1}\ \frac{\bar{y}}{\bar{x}} \quad \text{and} \quad \frac{\bar{y}^2 n^2 \alpha(n\alpha + m\alpha - 1)}{\bar{x}^2 m(n\alpha - 1)^2(n\alpha - 2)}$$

respectively. Note that it is necessary for $n\alpha$ to be greater than 1 for the mean to exist and greater than 2 for the variance to exist.

Tables 9, 10, and 11 present the summary statistics for the posterior distribution of θ for the same three examples utilized in section 3. In making a general comparison with the results of section 3, where it was assumed that the scale parameter β was known, the shapes of the posterior distributions are very similar. However, the spread of the distributions is larger, as indicated by the increase in the standard deviations and the wider confidence sets. The introduction of β as an unknown parameter should lead to this increase as more uncertainty is being introduced into the analysis.

The selection of a prior distribution for β is more difficult than the choice of a prior for θ. Although β is a scale parameter with some possible physical interpretation, its value, or possible range in a particular situation is usually not known prior to an examination of the data. This excludes the cases where sufficient previous experimentation has been performed to form a prior distribution. There is also an indication from tables 10 and 11 that the marginal posterior distribution of θ is sensitive to the choice of prior for β. This is in agreement with the sensitivity of the posterior of θ to percentage changes in μ_c given in section 3.1. It should be noted that the data samples used for tables 10 and 11 were very small so that the posterior is sensitive to the choice of priors.

Table 9. Single Cloud Example. Posterior Summary Statistics on θ for Selected Prior Distributions on θ and β.

Prior Distributions	Mean	Mode	Std. Dev.	Equal Tail ℓ	u	Shortest L	U
θ Prior Inv. Gamma K_1=1.0 K_2=0.5							
β Prior Gamma							
1. k_1 = 2 k_2 = 500	2.32	1.95	0.83	1.11	4.33	0.94	3.99
2. k_1 = 2 k_2 = 200	2.48	2.07	0.89	1.18	4.61	1.00	4.25
3. k_1 = 12 k_2 = 3300	2.39	2.05	0.77	1.25	4.23	1.10	3.93
4. k_1 = 12 k_2 = 5000	1.96	1.68	0.63	1.03	3.47	0.90	3.22
5. Uniform [0, + ∞]	2.43	2.03	0.89	1.14	4.56	0.96	4.19
θ Prior Mod. Unif. b = 5							
β Prior Gamma							
6. k_1 = 2 k_2 = 500	3.03	2.71	0.87	1.52	4.77	1.55	4.79
7. k_1 = 2 k_2 = 200	3.16	2.89	0.86	1.62	4.82	1.72	4.90
8. k_1 = 12 k_2 = 3300	3.00	2.68	0.82	1.60	4.72	1.58	4.69
9. k_1 = 12 k_2 = 5000	2.55	2.19	0.78	1.32	4.38	1.19	4.18
θ Prior Unif. [.8, 5]							
β Prior Gamma							
10. k_1 = 2 k_2 = 500	3.03	2.71	0.87	1.53	4.77	1.55	4.79
11. k_1 = 2 k_2 = 200	3.16	2.89	0.86	1.62	4.82	1.72	4.90
12. k_1 = 12 k_2 = 3300	3.00	2.68	0.82	1.60	4.72	1.58	4.69
13. k_1 = 12 k_2 = 5000	2.55	2.19	0.78	1.32	4.38	1.19	4.17
θ Prior 1/θ and β Prior 1/β							
14.	2.87	2.36	1.08	1.31	5.49	1.10	5.02

Table 10. Floating Target Example. Posterior Summary Statistics on θ for Selected Prior Distribution on θ and β.

Prior Distributions		Mean	Mode	Std. Dev.	Equal Tail ℓ	Equal Tail u	Shortest L	Shortest U
	θ Prior Inv. Gamma $K_1=1.0$ $K_2=0.5$							
β Prior Gamma								
1. $k_1=$ 2	$k_2=$ 8	1.50	1.08	0.75	0.55	3.38	0.40	2.96
2. $k_1=$ 2	$k_2=$ 4	2.00	1.44	1.01	0.72	4.54	0.53	3.97
3. $k_1=$ 12	$k_2=$ 50	0.85	0.65	0.37	0.37	1.80	0.29	1.62
4. $k_1=$ 12	$k_2=$ 100	0.50	0.38	0.21	0.22	0.82	0.20	0.79
5. Uniform	$[0, +\infty]$	2.76	1.95	1.43	0.95	6.38	0.67	5.55
	θ Prior Mod. Unif. b = 5							
β Prior Gamma								
6. $k_1=$ 2	$k_2=$ 8	2.47	1.92	1.02	0.81	4.67	0.72	4.56
7. $k_1=$ 2	$k_2=$ 4	2.99	2.62	1.01	1.21	4.84	1.41	5.00
8. $k_1=$ 12	$k_2=$ 50	1.19	0.68	0.70	0.39	3.05	0.28	2.60
9. $k_1=$ 12	$k_2=$ 100	0.52	0.36	0.29	0.23	1.28	0.20	1.03
	θ Prior Unif. [.8, 5]							
β Prior Gamma								
10. $k_1=$ 2	$k_2=$ 8	2.52	1.92	0.99	1.01	4.68	0.86	4.47
11. $k_1=$ 2	$k_2=$ 4	3.00	2.62	0.99	1.26	4.84	1.42	4.97
12. $k_1=$ 12	$k_2=$ 50	1.54	0.96	0.67	0.83	3.39	0.80	2.91
13. $k_1=$ 12	$k_2=$ 100	1.18	0.80	0.42	0.81	2.33	0.80	1.99
	θ Prior $1/\theta$ and β Prior $1/\beta$							
14.		3.91	2.62	2.19	1.25	9.63	0.83	8.24

Table 11. Total Target Example. Posterior Summary Statistics on θ for Selected Prior Distributions on θ and β.

Prior Distributions		Mean	Mode	Std. Dev.	Equal Tail ℓ	u	Shortest L	U
θ Prior Inv. Gamma K_1=1.0 K_2=0.5								
β Prior Gamma								
1. k_1= 2	k_2= 4	1.18	0.86	0.59	0.44	2.66	0.32	2.33
2. k_1= 2	k_2= 2	1.24	0.90	0.62	0.46	2.80	0.34	2.45
3. k_1= 12	k_2= 24	1.52	1.16	0.67	0.65	3.21	0.52	2.84
4. k_1= 12	k_2= 36	1.27	0.97	0.56	0.55	2.67	0.43	2.36
5. Uniform [0, + ∞]		1.19	0.85	0.60	0.43	2.71	0.31	2.37
θ Prior Mod. Unif. b = 5								
β Prior Gamma								
6. k_1= 2	k_2= 4	1.98	1.48	0.98	0.56	4.36	0.40	3.99
7. k_1= 2	k_2= 2	2.09	1.56	0.99	0.61	4.45	0.45	4.13
8. k_1= 12	k_2= 24	2.36	1.81	0.96	0.88	4.56	0.75	4.36
9. k_1= 12	k_2= 36	1.99	1.49	0.91	0.68	4.27	0.52	3.91
θ Prior Unif. [.8, 5]								
β Prior Gamma								
10. k_1= 2	k_2= 4	2.13	1.48	0.92	0.90	4.40	0.80	4.01
11. k_1= 2	k_2= 2	2.21	1.56	0.94	0.91	4.47	0.80	4.11
12. k_1= 12	k_2= 24	2.39	1.81	0.94	1.01	4.57	0.85	4.29
13. k_1= 12	k_2= 36	2.08	1.49	0.88	0.91	4.30	0.80	3.88
θ Prior $1/\theta$ and β Prior $1/\beta$								
14.		1.59	1.06	0.92	0.50	3.92	0.33	3.34

4.2 Classical Approach

When α is the only parameter assumed to be known, it is possible to directly incorporate both the control and treated sample data into the analysis. In this case a random sample of size m is available from the control random variable X with its associated density function

$$p(x \mid \beta) = \frac{\beta^{\alpha} \; x^{\alpha - 1}}{\Gamma(\alpha)} \quad e^{-\beta x} \quad , \quad x > 0$$

and a random sample of size n on the treated random variable Y is available with the density function

$$p(y \mid \beta, \theta) = \frac{\beta^{\alpha} \; y^{\alpha - 1}}{\Gamma(\alpha) \; \theta^{\alpha}} \quad e^{-\beta y/\theta} \quad , \quad y > 0 .$$

Based upon these two data samples this section will derive point estimates, confidence intervals and hypothesis tests for the seeding effect θ. It will be assumed that there is no direct interest in estimating β and, additionally, that the procedures used in making inferences concerning θ are not dependent upon the value assumed by β. That is, procedures are desired that are invariant to scale changes.

A point estimator for θ can be obtained by forming the log likelihood function and finding the parameter value that maximizes it. The log likelihood function, as a function of the unknown parameters, is

$$LN = (n\alpha + m\alpha) \log\beta - \beta m\bar{x} - \beta n\bar{y}/\theta - n\alpha\log\theta + \text{constant}.$$

By differentiating with respect to β and θ and then simplifying, the maximum likelihood estimates for β and θ are

$$\tilde{\beta} = \alpha/\bar{x} \qquad\qquad \tilde{\theta} = \bar{y}/\bar{x} .$$

The estimator for θ is the same form as derived when β was assumed known. The difference is that for this case $\overline{x}$ is a sample estimator for μ_c. In the present case, however, the maximum likelihood estimator for θ is positively biased, i.e.,

$$E[\tilde{\theta}] = \frac{m\alpha}{m\alpha - 1} \quad \theta.$$

Since α is known, an unbiased estimator is easily determined to be

$$\hat{\theta} = \frac{m\alpha - 1}{m\alpha} \quad \tilde{\theta}.$$

By noting that $(\overline{X}, \overline{Y})$ forms a complete sufficient statistic for the parameter (β, θ), $\hat{\theta}$ is recognized as the minimum variance unbiased estimator for θ (Ferguson, 1967). It is also informative to note that the estimator for θ is invariant under scale changes.

Since $(\overline{X}, \overline{Y})$ forms a complete sufficient statistic for (β, θ), attention may be restricted to estimators and tests based upon it. Furthermore, the invariance to scale changes can and will be formally introduced into the analysis. This will reduce the statistical problem further without the loss of any essential information regarding the seeding effect θ. This is accomplished by defining the group of transformations

$$\mathcal{G} = \{g_\delta : g_\delta\ (\overline{X}, \overline{Y}) = (\delta\overline{X}, \delta\overline{Y}) \quad \delta > 0\}$$

and showing the family of distributions for $(\overline{X}, \overline{Y})$ is invariant under this group (Ferguson, 1967). A maximal invariant statistic is $W = \overline{Y}/\overline{X}$, i.e., W remains constant under scale changes and maximal in the sense that

$W_1 = W_2$ implies

$$\bar{x}_1 = \delta \bar{x}_2 \text{ and } \bar{y}_1 = \delta \bar{y}_2 \text{ for some } \delta > 0.$$

More importantly the probability distribution of the maximal invariant W depends only upon the seeding effect θ. This is easily verified by considering that $2n\beta\bar{y}/\theta$ and $2m\beta\bar{x}$ have independent Chi-squared distributions with $2n\alpha$ and $2m\alpha$ degrees of freedom respectively. Hence

$$T(\theta) = \frac{2n\beta\bar{y}/(2n\alpha\theta)}{2m\beta\bar{x}/(2m\alpha)} = \frac{W}{\theta}$$

has an F-distribution with $2n\alpha$ and $2m\alpha$ degrees of freedom. A simple change of variable then shows the distribution of W to depend only upon θ.

Since the variable $T(\theta)$ has a well-known distribution that is independent of θ, and since it is a function of the complete sufficient statistics, it is reasonable to derive confidence intervals and hypothesis tests based upon it. Moreover, in comparing the definition of $T(\theta)$ to the statistic utilized in section 3.2, the effect of assuming β to be unknown becomes clear.

Based on $T(\theta)$ it is a simple matter to derive equal tail, shortest and unbiased confidence intervals for θ. Corresponding to the equal tail and unbiased confidence intervals are an equal tail and an unbiased hypothesis test of $\theta = 1$ versus $\theta \neq 1$. Under the null hypothesis $T(1) = T$ has an F-distribution and an equal tail test is constructed by selecting T_1 and T_2 such that

$$\Pr[T \leq T_1 \mid \theta = 1] = \xi/2$$

and

$$\Pr[T \geq T_2 \mid \theta = 1] = \xi/2.$$

The critical region of the test is then $T \leq T_1$ or $T \geq T_2$. Moreover, a $1 - \xi$ equal tail confidence interval is given, in the usual manner, by

$$T/T_2 \leq \theta \leq T/T_1 .$$

Also, the power function of the test is easily computed since

$$\begin{aligned} \text{Power} = K(\theta) &= \Pr\,[T(\theta) \leq T_1 \text{ or } T \geq T_2 \mid \theta] \\ &= 1 - \Pr\,[T_1 \leq T \leq T_2 \mid \theta] \\ &= 1 - \Pr\,[T_1/\theta \leq T(\theta) \leq T_2/\theta] \end{aligned}$$

and $T(\theta)$ has an F-distribution $2n\alpha$, $2m\alpha$. Hence the power can be determined from either tables of the F-distribution and incomplete Beta distribution or the appropriate integration can be performed numerically.

An unbiased test based on $T(\theta)$ can be derived in a manner similar to section 3.2 by selecting T_1 and T_2 to satisfy the equations

$$\Pr[T_1 \leq T(\theta) \leq T_2 \mid \theta = 1] = 1 - \xi$$

and

$$T_1\, f_{\nu_1,\,\nu_2}(T_1) = T_2\, f_{\nu_1,\,\nu_2}(T_2)$$

where $f_{\nu_1,\,\nu_2}(T)$ is an F-distribution with $\nu_1 = 2n\alpha$ and $\nu_2 = 2m\alpha$ degrees of freedom. Once T_1 and T_2 are determined the power of the test and an unbiased confidence interval are given by the same procedures described for the equal tail case.

It is also possible to show with a little algebraic manipulation that the likelihood ratio test derived from the original sample is equivalent to the unbiased test.

A final statistic that may be of interest is a shortest confidence interval based on $T(\theta)$. The length of a confidence interval is given by $L = T\ (T_1^{-1} - T_2^{-1})$ and is minimized subject to the constraint $\Pr\ [T_1 \leq T(\theta) \leq T_2 \mid \theta = 1] = 1 - \xi$. This minimization leads to the selection of T_1 and T_2 satisfying the simultaneous equations

$$\Pr\ [T_1 \leq T(\theta) \leq T_2 \mid \theta = 1] = 1 - \xi$$

and

$$T_1^2 f_{\nu_1, \nu_2}(T_1) = T_2^2 f_{\nu_1, \nu_2}(T_2) .$$

The application of the above procedures to the three data sets is presented in summary form in table 12. The interpretation of the results leads to the same conclusions as those previously presented. It is noted that the effect of assuming the scale parameter β to be unknown is to introduce a bias to the maximum likelihood estimate of the seeding effect θ, and to increase the length of the confidence intervals compared to the analysis of section 3.2. This is precisely the same effect as noted for the Bayesian analysis case.

By examining tables 9, 10, 11 and 12 regarding interval estimates for θ it is suggested that the improper priors $1/\theta$ and $1/\beta$ lead to the same numerical intervals for the equal tail and shortest intervals. This is, in fact, the case and can be shown theoretically in a manner similar to that in section 3.2. The classical equal tail interval has the lower limit $L = T/T_2$ where

$T = \overline{y}/\overline{x}$ and T_2 is such that

$$\int_{T_2}^{\infty} f_{\nu_1, \nu_2}(t)\,dt = \xi/2$$

and hence in terms of L

$$\int_{T/L}^{\infty} f_{\nu_1, \nu_2}(t)\,dt = \xi/2 .$$

The Bayesian equal tail lower limit is determined from

$$\int_{0}^{L} \frac{\Gamma(n\alpha + m\alpha)}{\Gamma(n\alpha)\Gamma(m\alpha)} \left(\frac{n\overline{y}}{m\overline{x}}\right)^{n\alpha} \frac{\theta^{-(n\alpha+1)}}{\left(1 + \frac{ny}{mx\theta}\right)^{n\alpha + m\alpha}}\, d\theta = \xi/2$$

and by making the change of variable $t = \overline{y}/(\overline{x}\theta)$ it is easily verified that the integrand is the same as in the classical case. The upper limit for the equal tail intervals and the shortest intervals are shown to be equivalent in a similar manner.

Because of the simple manner in which the test statistic can be altered to determine its distribution for various alternative hypotheses, the power functions for the tests presented in this section and section 3.2 are easily derived. That is,a scaler multiple of the appropriate test statistic has the same distribution under an alternate hypothesis. The power functions for the unbiased test under the assumptions of both α and β known and only α known were computed for the two different situations presented by the single cloud and floating (total) target data. Figure 1 presents the comparisons. The upper curves give the floating and total target power functions. The large effect of assuming β is known is clearly visualized; similarly, for the single cloud situation in the lower curves. For example, at the alternate

Table 12. Summary of Classical Statistical Analysis for the Detection of a Seeding Effect Assuming the Shape Parameter α is known.

Summary Statistics	Single Cloud		Floating Target		Total Target	
Point Estimates						
Max. Likelihood $\tilde{\theta}$	2.69		3.37		1.36	
Unbiased $\tilde{\theta}$	2.51		2.99		1.21	
Max. Likelihood $\tilde{\beta}$	.0036		1.29		0.28	
Test Statistic T	2.69		3.37		1.36	
95% Confidence Limits						
	Lower	Upper	Lower	Upper	Lower	Upper
Equal Tail	1.31	5.49	1.25	9.70	0.50	3.92
Unbiased	1.31	5.49	1.24	9.59	0.50	3.87
Shortest	1.10	5.02	0.83	8.27	0.34	3.34
5% Hypothesis Test Critical Limits						
	Lower	Upper	Lower	Upper	Lower	Upper
Equal Tail	0.49	2.04	0.35	2.70	0.35	2.70
Unbiased	0.49	2.04	0.35	2.73	0.35	2.73

hypothesis value of $\theta = 2$, approximately a 50 percent increase in power results by assuming that β is known.

The power function has as parameters the known shape parameter α and the sample sizes n (and m if β unknown). For both tests the parameters enter in the form $2n\alpha$ (and $2m\alpha$) as the degrees of freedom associated with the Chi-squared distribution and Snedecor's F-distribution. By increasing the degrees of freedom, the performance characteristics of the power functions can be improved. This is illustrated by the power functions for the same test in the upper and lower panels of figure 1.

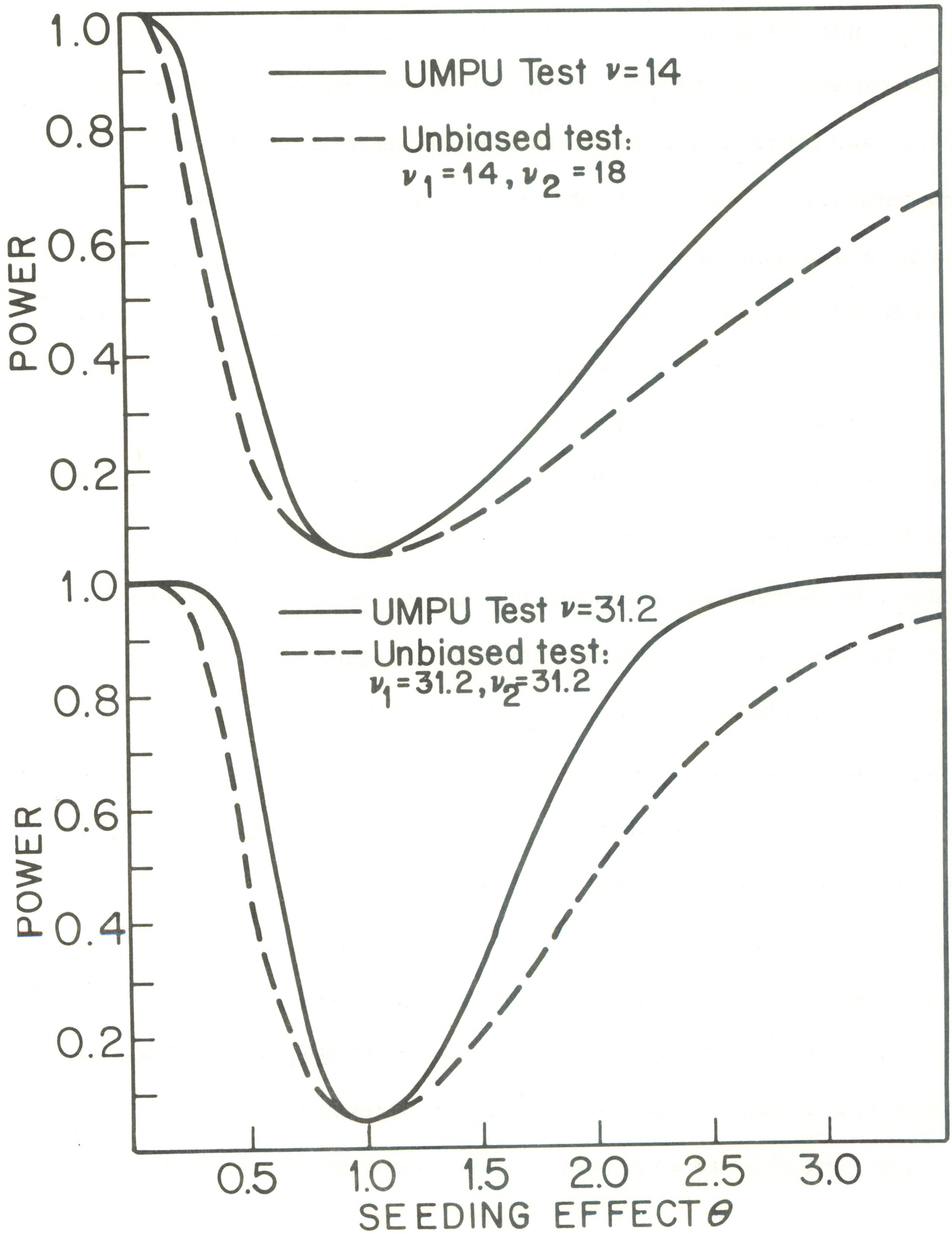

Figure 1. Power functions of the optimal statistical tests for single cloud sample size (lower panel) and multiple cloud sample size (upper panel).

5. SUMMARY

The preceding sections have presented a comparison of the possible analyses under the assumptions of an underlying gamma distribution for the random variable observed and a multiplication treatment effect from both the Bayesian and classical viewpoints. In all of the analyses it was assumed the shape parameter α was known. Different procedures were developed depending upon whether the scale parameter β of the control distribution was known or unknown. The effect of this assumption on all cases was to increase the dispersion of the appropriate test statistics or posterior distribution.

In addition to deriving useful statistical procedures for use under the present nonstandard assumptions, one of the more important aspects of the work is the comparison of the Bayesian and classical approaches to the problem. In section 3.2 it was shown that some of the classical procedures lead to the same numerical results as the Bayesian approach when the appropriate prior distribution is chosen for the seeding effect θ. In this case it is the improper prior $1/\theta$. The motivation for presenting these comparisons was to show explicitly that the two approaches, under the same assumptions, give similar results when a noninformative or diffuse prior is selected. Hence a choice between the procedures must be made on their philosophical difference and not on the belief that one procedure is inherently more powerful than the other. This is exclusive of the situation when enough prior evidence is available to formulate a nondiffuse prior for the Bayesian procedures. Similar results were obtained when both α and β were unknown in section 4.2.

An extension of the procedures to the case of an unknown scale parameter α was not presented. Although there are asymptotic results in the form of likelihood ratio and optimal $C(\alpha)$ tests available, they were not presented since the Bayesian development under the same assumptions could not be derived. It is anticipated that further work will be completed in this area.

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