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Wave Propagation Laboratory
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ABSTRACT

Transforming the linearized inviscid Navier-Stokes equations to a symmetric hyperbolic system of partial differential equations leads to a unique dispersion relation for the propagation of acoustic-gravity waves in a baroclinic atmosphere. The dispersion relation differs from the usual one because of the presence of a baroclinic term. In addition, $C(\gamma-1)^{-1/2} \nabla \ell n\, \theta_o$ replaces the Brunt-Väisälä frequency, where C is sound speed, γ is the usual ratio of specific heats, and θ_o is potential temperature, and $C^2((\gamma-1)(\nabla \ell n\, \theta_o)^2 + (1/\gamma - 0.5) \vec{\nabla} \ell n\, p_o + 0.5 \vec{\nabla} \ell n\, C)^2)$ replaces the square of the acoustic cut-off frequency where p_o is atmospheric pressure. It is argued that this dispersion relation be accepted as the standard for acoustic-gravity waves in either a baroclinic or a barotropic atmosphere.

1. INTRODUCTION

1.1 Ambiguities in Acoustic-Gravity Wave Dispersion Relations for a Non-Isothermal Atmosphere

The dispersion relation for acoustic-gravity waves in an isothermal atmosphere is unambiguous. In the presence of temperature gradients, however, the differential equation for each wave variable (pressure, temperature, density, etc.) is different, so that there is a different dispersion relation for each wave variable. Einaudi and Hines (1970) give the differential equations for several of the wave variables and the corresponding dispersion relations.

There is no inconsistency from a practical viewpoint in that the solutions to the various wave equations are consistent among themselves, and it is possible to calculate the solutions for all of the wave variables from that for any one of them. From the point of view of wave propagation theory, however, one ought to expect there to be a single dispersion relation that describes the propagation of the whole wave. In particular, for use in an acoustic-gravity ray tracing program such as that by Georges (1971), one needs an appropriate dispersion relation. We might expect that such a dispersion relation would somehow involve several wave variables as a system. The purpose of the present work is to derive just such a dispersion relation.

1.2 Transforming the Linearized Inviscid Navier-Stokes Equations to a Symmetric Hyperbolic System of First-Order Partial Differential Equations

There are two methods to obtain a dispersion relation for linear, second-order partial differential equations. In the first, one substitutes $i\omega$ for $\partial/\partial t$, $-ik_x$ for $\partial/\partial x$, $-ik_y$ for $\partial/\partial y$, and $-ik_z$ for $\partial/\partial z$ in a single second-order differential equation for one wave variable. In the second, one makes the same substitution in a system of first-order partial differential equations, and sets the determinant of the coefficients of the wave variables to zero. If we are looking for a dispersion relation that involves several wave variables, then the second of the two methods discussed above seems more appropriate.

It is straightforward to write the linearized inviscid Navier-Stokes equations as a system of first-order partial differential equations. In fact, this can be done in many ways, depending on which wave variables one chooses and which combinations of differential equations one uses for the system. In two dimensions, one requires four wave variables. Adding the third dimension requires one more wave variable, but adds no new physical effects. The choice of wave variables and the choice of combinations of differential equations to be used in the system determine the dispersion relation.

It is possible to change wave variables and combinations of equations in a systematic way. One does this by first arbitrarily choosing an independent set of wave variables and an independent set of the partial differential equations that describe the system. One then writes that system of equations as a matrix equation in which the wave variables appear as a column vector multiplied on the left by a square matrix some of whose elements involve differential operators. Each row in the matrix corresponds to one of the differential equations. To change to a new set of wave variables, one can replace the original column vector by a column vector of new variables multiplied on the left by a transformation matrix. This corresponds to making a change in basis functions, and is equivalent to making a similarity transformation on the original matrix. To change to a new combination of differential equations, one simply multiplies the original matrix on the left by some matrix.

Thus, by making similarity transformations on the original matrix that represents the system of partial differential equations, and by multiplying that matrix on the left by various matrices (whose elements, like those for the transformation matrix, may depend on position or time), one can obtain a representation of the original system of partial differential equations for any set of wave variables and any combinations of the original set of equations. All of these representations are equivalent, in that, from the solution in one representation, one can obtain the solution in any other representation by multiplying the column vector of wave variables for which the solution is known by the appropriate transformation matrix. However, the dispersion relation will in general depend on the representation.

One wonders whether there is a representation in which the dispersion relation is "special" in that it somehow describes the whole wave. What kinds of criteria could we apply to the matrix representing the system in a given representation to decide that it is special? To begin, because there are no losses, the resulting dispersion relation must be real. A sufficient condition for obtaining a real dispersion relation is that the matrix representing the system be Hermitian (the transpose equals the complex conjugate). As it turns out, the Hermitian requirement is sufficient to yield a unique dispersion relation.

There are other reasons for finding a representation in which the matrix is Hermitian. In such a representation, the differential equations form a symmetric hyperbolic system. The advantage of that representation is that Cauchy's initial value problem is well posed for symmetric hyperbolic systems (Courant and Hilbert, 1962; Garabedian, 1964; Friedrichs and Lax, 1971). That is, specifying the solution on a space-like hypersurface determines a unique solution, at least in a neighborhood of the initial surface. There is also a relationahip between conservation equations and the symmetric form for a system of partial differential equations (Friedrichs and Lax, 1971). Further, expressing the system of equations in symmetric form simplifies the calculation of wave amplitudes with the eikonal method (Weinberg, 1962). I had hoped to show that there would be no "additional memory" contributions to the phase (Budden and Smith, 1976) when the system is expressed in symmetric form, but I have not been able to show that.

1.3 Summary of Present Acoustic-Gravity Wave Dispersion Relations

Dispersion relations for acoustic-gravity waves in the atmosphere have the form (e.g., Hines, 1960)

$$\frac{\Omega^2}{C^2} - k^2 - \frac{\omega_2^2}{C^2} + \frac{\omega_1^2}{\Omega^2} k_x^2 = 0 , \tag{1.1}$$

where

$$C^2 = \gamma\, p_o/\rho_o \tag{1.2}$$

is the square of the sound speed, γ is the ratio of specific heat at constant pressure to that at constant volume, p_o is the background pressure, ρ_o is the background density,

$$\Omega = \omega - \vec{k} \cdot \vec{U}_o \tag{1.3}$$

is the intrinsic frequency, ω is the wave frequency, $\vec{k}$ is the wave vector, k_x is the horizontal component of $\vec{k}$, and $\vec{U}_o$ is the background wind velocity. (Appendix G defines the notation more completely.)

The variables ω_1 and ω_2 depend on local properties of the atmosphere. For an isothermal atmosphere, they are unambiguously given by (Gossard and Hooke, 1975, p. 114)

$$\omega_1^2 = \omega_g^2 \equiv (\gamma-1)g^2/C^2 = \frac{(\gamma-1)g}{\gamma H} = -\frac{(\gamma-1)g}{\gamma\, p_o}\frac{\partial p_o}{\partial z} \tag{1.4}$$

and

$$\omega_2^2 = \omega_a^2 \equiv \frac{g^2\gamma^2}{4C^2} = \frac{C^2}{4H^2} = \frac{\gamma g}{4H}\,. \tag{1.5}$$

The variable ω_g is sometimes called the "isothermal" Brunt-Väisälä frequency, and ω_a is called the acoustic cutoff frequency. H is the scale height and g is the acceleration due to gravity.

When the atmosphere is not isothermal, however, one arrives at various dispersion relations having the form (1.1), but with different forms for ω_1 and ω_2 depending on which wave variable (pressure, density, temperature, etc.) is used as the dependent variable in the wave equation. Einaudi and Hines (1970) have presented some of the wave equations and the resulting forms for ω_1 and ω_2.

Usually, for a non-isothermal atmosphere, one uses (Gossard and Hooke, 1975, p. 73)

$$\omega_1^2 = \omega_B^2 = \frac{g}{\theta_o}\frac{d\,\theta_o}{dz} \equiv g\left(\frac{d\,p_o/dz}{\gamma p_o} - \frac{d\,\rho_o/dz}{\rho_o}\right), \tag{1.6}$$

where ω_B is variously called the Brunt-Väisälä or Väisälä-Brunt frequency and θ_o is potential temperature. Sometimes, for a non-isothermal atmosphere, one uses (Gossard and Hooke, 1975, p. 114)

$$\omega_2^2 = \omega_B^2 + C^2\,\Gamma^2 = 1/4\; C^2((d\rho_o/dz)/\rho_o)^2\,, \tag{1.7}$$

where

$$\Gamma = \frac{g}{2C^2}(2-\gamma) - \frac{1}{C}\frac{dC}{dz} \tag{1.8}$$

is Eckart's coefficient (Eckart, 1960; Gossard and Hooke, 1975, p. 92).

Tolstoy (1963) suggests another form for ω_2 for a non-isothermal atmosphere, and considers situations where ω_2 may be less than ω_1, as do Maeda and Young (1966) and Johnston (1967). Johnston (1967) suggests that situations in which ω_2 is less than ω_1 may lead to a gravity wave instability. Einaudi and Hines (1970) show that no such stability exists, and recommend using the isothermal form (1.5) for ω_2 (even though the situation where ω_2 is less than ω_1 may sometimes occur) because non-isothermal forms for ω_2 differ significantly from the isothermal form only where the W.K.B. approximation is invalid. Notice that ω_2 can never be less than ω_1 if (1.6) and (1.7) are used.

As pointed out by Hines (1971), when the background flow accelerates,

$$g = |\vec{g}| = |-\vec{g}| \tag{1.9}$$

should be replaced in (1.6) by

$$|D\,\vec{U}_o/Dt - \vec{g}| \tag{1.10}$$

as an effective acceleration due to gravity because an acceleration and a gravitational field act in identical ways on the fluid. The operator

$$D/Dt \equiv \partial/\partial t + \vec{U}_o \cdot \vec{\nabla} \tag{1.11}$$

is the Eulerian or substantial derivative. The impossibility of telling the difference between acceleration and a gravitational field follows from Einstein's equivalence principle in general relativity. In fluid flow we can see that this is true because $\vec{g}$ and $D\,\vec{U}_o/Dt$ appear only in the combination indicated inside the absolute-value lines in (1.10). For example, the momentum equation for the background flow is

$$D\,\vec{U}_o/Dt - \vec{g} = -\frac{\vec{\nabla}p_o}{\rho_o} \quad . \tag{1.12}$$

Wherever the left side of (1.12) appears in the inviscid Navier-Stokes equations, it can be replaced by the right side of (1.12). Thus,

$$\frac{\vec{\nabla} p_o}{\rho_o} = \frac{p_o}{\rho_o}\frac{\vec{\nabla} p_o}{p_o} = \frac{C^2}{\gamma}\frac{\vec{\nabla} p_o}{p_o} \tag{1.13}$$

can be considered to be an effective value for $\vec{g}$. The effective $\vec{g}$ may not always point vertically, so that some concepts (such as horizontal and vertical wave numbers) will have to be generalized.

An appropriate generalization for the Brunt-Väisälä frequency in (1.6) is

$$\omega_B^2 = -\frac{C^2}{\gamma}\frac{\vec{\nabla} p_o}{p_o} \cdot \frac{\vec{\nabla}\theta_o}{\theta_o}, \tag{1.14}$$

at least for the barotropic case where $\vec{\nabla} p_o$ is parallel to $\vec{\nabla}\theta_o$. For the present work, I will use (1.14) for the Brunt-Väisälä frequency.

For a short time during this study, I thought that bicharacteristic rays and the usual geometrical optical rays were the same because for many of the examples given in textbooks, they are the same. To help others avoid this misconception, I describe the difference in Appendix F.

1.4 Why Generalize to a Baroclinic Atmosphere?

The reader might wonder why I have generalized my treatment to a baroclinic atmosphere (in which $\vec{\nabla} p_o$ is not parallel to $\vec{\nabla}\theta_o$). Although Hess (1959, p. 193) argues that the atmosphere is normally significantly baroclinic in that the wind shear depends mainly on the horizontal rather than the vertical temperature gradient, baroclinity in the atmosphere is small. For example, an estimate of baroclinity from latitude effects (Dutton, 1976, Figs. 4.2 and 4.11, pp. 82 and 92) gives only about 0.2° between contours of constant pressure and potential temperature. Although baroclinity may be larger in weather fronts and sea breezes (Hess, 1959, pp. 244-247), it may still be too small to have significant effect on acoustic-gravity wave propagation.

The real reason for including baroclinity is that a barotropic atmosphere seems to be a special case for acoustic-gravity wave propagation. In transforming to a symmetric system of equations, I found a special solution for that transformation valid only when the whole atmosphere is exactly barotropic. It led to a dispersion relation of the form (1.1) with ω_1 given by the generalized Brunt-Väisälä frequency (1.14).

For an atmosphere that is only slightly baroclinic (or barotropic only in some regions), however, that solution does not apply. For a baroclinic atmosphere it is also possible to transform to a symmetric system of equations. The resulting dispersion relation is a generalization of (1.1) in that a baroclinic term is added. In addition, ω_1 is not equal to ω_B, *even in barotropic regions.* Thus, a baroclinic calculation has some effect even for a barotropic atmosphere. This result suggests that even in a barotropic atmosphere, the baroclinic dispersion relation (specialized to the barotropic case) be used rather than the usual barotropic dispersion relation.

2. SUMMARY

Transforming the linearized inviscid Navier-Stokes equations to a symmetric hyperbolic system (3.1) leads to a unique dispersion relation (4.4). Except for a baroclinic term (4.9), the dispersion relation (4.4) has the same form as (1.1) with

$$\omega_1 = C k_g , \tag{2.1}$$

$$\omega_2 = C k_a , \tag{2.2}$$

and $k_\perp$ replacing k_x (where k_g is given by (4.5), k_a is given by (4.8), and $k_\perp$ is the component of the wave vector $\vec{k}$ perpendicular to the gradient of potential temperature). Ck_g differs from the Brunt-Väisälä frequency ω_B except for an isothermal atmosphere, and Ck_a differs from the acoustic cutoff frequency except for an isothermal atmosphere, but the differences are not great for normal atmospheric temperature gradients.

Equation (4.10) shows that

$$\omega_2^2 = \omega_1^2 + C^2 k_c^2 \qquad (2.3)$$

where k_c is given by (4.6). Thus

$$\omega_2 \geq \omega_1 , \qquad (2.4)$$

so that locally the acoustic regime never overlaps the gravity wave regime, and when k_c is not zero, there is a frequency gap separating the two regimes (as in Gossard and Hooke, 1975, Fig. 23-1, p. 116).

There is a special case for an exactly barotropic atmosphere for which the dispersion relation has the form (1.1) with ω_1 equal to the Brunt-Väisälä frequency ω_B. However, this particular solution cannot be generalized to a slightly baroclinic atmosphere, and thus seems to be extraneous.

That (4.4) is the unique dispersion relation for the linearized baroclinic inviscid Navier-Stokes equations when expressed as a symmetric hyperbolic system suggests that it be adopted as the standard dispersion relation for acoustic-gravity waves in the atmosphere.

After completing this study, I became aware of similar work by Turkel (1973) and Abarbanel and Gottlieb (1981) on symmetrizing the Navier-Stokes equations. Both Turkel (1973) and Abarbanel and Gottlieb (1981) included nonlinearities in the Navier-Stokes equations. Abarbanel and Gottlieb included viscosity, but Turkel did not. Neither Turkel nor Abarbanel and Gottlieb included the effect of spatial derivatives of the transformation matrix (whose effect required the main effort in the present study).

That they neglected the spatial derivatives of their transformation matrices implies that their symmetric equations were only approximations to the Navier-Stokes equations, or that their equations would be only approximately symmetric when the spatial variation of the transformation matrix is taken into account.

In spite of these differences, the symmetric representation derived here is very similar to the parabolic symmetrization of Abarbanel and Gottlieb (1981). In fact, if I take

$$\frac{B_{22}}{B_{11}} = \frac{\sin(2\delta + 2\varepsilon + 2b)}{\sin(2\delta + 2\varepsilon)} \quad , \tag{2.5}$$

then the transformation derived here (when extended in an obvious way to three dimensions) would symmetrize all nine of their matrices (in their sense, not including the effects of spatial variation of the transformation), although it would not diagonalize their parabolic coefficient matrices C, D, and K.

In the present study, the transformation matrix depended only on background values of the variables, so that spatial derivatives of the transformation matrix contribute to the matrix B that multiplies the wave variables. In the procedure used by Abarbanel and Gottlieb (1981), in which no background component is identified or separated, spatial derivatives of the transformation matrix contribute to various terms in the equations. It would probably still be possible to symmetrize those contributions by carefully choosing the transformation matrix. I estimate that for the inviscid case the amount of work required would be about the same as that for the present study, and would probably be worth doing.

3. INVISCID NAVIER-STOKES EQUATIONS AS A SYMMETRIC HYPERBOLIC SYSTEM

Appendix A writes the linearized inviscid Navier-Stokes equations as a first-order system of equations expressed in matrix form. Appendix B transforms the equations to a symmetric hyperbolic system by using a new set of wave variables and using equations that are a linear combination of the original equations. The symmetry conditions put various constraints on the elements of the transformation matrix, some of the constraints being differential constraints.

Appendix E argues that the elements of the transformation matrix should be a function of only local properties of the medium (including local derivatives), and should not be a function of the route by which the local point was reached. This reasoning leads to a requirement that the differential constraints be represented as total differentials. The total differential constraint leads to a unique determination of the transformation matrix (except for some constants of integration) for a specified multiplier matrix (that determines the linear combination of the original equations). Appendix E calculates the transformation matrix for a constant multiplier matrix. The result is that the inviscid Navier-Stokes equations can be written as

$$B_t\, D\phi/Dt + B_z\,\partial\phi/\partial z + B_x\,\partial\phi/\partial x + B\phi = 0 \ , \tag{3.1}$$

where

$$D/Dt = \partial/\partial t + \vec{U}_o \cdot \vec{\nabla} \tag{3.2}$$

is the Eulerian or substantial derivative, ϕ is the column vector

$$\phi = \begin{pmatrix} \left(\dfrac{p-p_o}{p_o}\cos(\delta+\varepsilon) - \dfrac{\rho-\rho_o}{\rho_o}\dfrac{\cos\delta}{\cos\varepsilon}\right) \Big/ \left(\sin\varepsilon \ \sin b\right) \\ \left(-\dfrac{p-p_o}{p_o}\cos(\delta+b+\varepsilon) + \dfrac{\rho-\rho_o}{\rho_o}\dfrac{\cos(\delta+b)}{\cos\varepsilon}\right) \Big/ \left(\sin\varepsilon \ \sin b\right) \\ \dfrac{U_z - U_{oz}}{C}\,\gamma \\ \dfrac{U_x - U_{ox}}{C}\,\gamma \end{pmatrix} \left(\frac{p_o \ C}{\gamma p_{oo} \ C_{oo}}\right)^{1/2} \tag{3.3}$$

γ is the usual ratio of specific heats, C is sound speed,

$$\cos^2\varepsilon = 1/\gamma \ , \tag{3.4}$$

$$\delta = \frac{2}{\sqrt{\gamma-1}}\,\ell n(C/C_o) \ , \tag{3.5}$$

C_o, C_{oo}, and p_{oo} are constants chosen to make dimensionless the expressions in which they appear, and the coefficients in (3.1) are matrices. The constant b is defined by

$$B_t = \begin{pmatrix} 1 & \cos b & 0 & 0 \\ \cos b & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.6}$$

and can be chosen arbitrarily with the possible exceptions of integer multiples of $\pi/2$.

$$B_z = C \begin{pmatrix} 0 & 0 & \cos(\delta+b) & 0 \\ 0 & 0 & \cos\delta & 0 \\ \cos(\delta+b) & \cos\delta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.7}$$

$$B_x = C \begin{pmatrix} 0 & 0 & 0 & \cos(\delta+b) \\ 0 & 0 & 0 & \cos\delta \\ 0 & 0 & 0 & 0 \\ \cos(\delta+b) & \cos\delta & 0 & 0 \end{pmatrix} \tag{3.8}$$

$$B = C \begin{pmatrix} 0 & E^T \\ -E & 0 \end{pmatrix}, \tag{3.9}$$

where

$$E = \begin{pmatrix} -k_{cz} \cos(\delta+b) + \dfrac{\sin(\delta+b)}{\sqrt{\gamma-1}} \dfrac{1}{\theta_o} \dfrac{\partial\theta_o}{\partial z} & -k_{cz} \cos\delta + \dfrac{\sin\delta}{\sqrt{\gamma-1}} \dfrac{1}{\theta_o} \dfrac{\partial\theta_o}{\partial z} \\ -k_{cx} \cos(\delta+b) + \dfrac{\sin(\delta+b)}{\sqrt{\gamma-1}} \dfrac{1}{\theta_o} \dfrac{\partial\theta_o}{\partial x} & -k_{cx} \cos\delta + \dfrac{\sin\delta}{\sqrt{\gamma-1}} \dfrac{1}{\theta_o} \dfrac{\partial\theta_o}{\partial x} \end{pmatrix}, \tag{3.10}$$

$C^2 = \gamma\, p_o/\rho_o$ is the square of the sound speed, θ_o is the potential temperature defined by

$$\frac{\vec{\nabla}\theta_o}{\theta_o} = \frac{1}{\gamma}\frac{\vec{\nabla}p_o}{p_o} - \frac{\vec{\nabla}\rho_o}{\rho_o} , \tag{3.11}$$

k_{cx} and k_{cz} are components of a vector

$$\vec{k}_c = - \frac{2-\gamma}{2\gamma}\vec{\nabla}\,\ell n\, p_o - \frac{1}{2}\vec{\nabla}\,\ell n\, C . \tag{3.12}$$

The background pressure and density are restricted to satisfy

$$Dp_o/Dt = D\rho_o/Dt = 0 . \tag{3.13}$$

When (3.13) is not satisfied, there is the possibility of exchange of energy between the background and the wave, for which the equations here are not adequate.

Although (3.1) is written for only two spatial dimensions, the extension to include the third dimension is clear.

4. DISPERSION RELATIONS

A dispersion relation is found by letting

$$\begin{aligned} &- i\, \partial/\partial t \rightarrow \omega && \text{(a)} \\ &i\, \partial/\partial z \rightarrow k_z && \text{(b)} \\ &i\, \partial/\partial x \rightarrow k_x && \text{(c)} \\ &- i\, D/Dt \rightarrow \Omega = \omega - \vec{U}_o \cdot \vec{k} && \text{(d)} \end{aligned} \tag{4.1}$$

in (3.1) and setting the determinant of the matrix coefficient of ϕ to zero. That is,

$$|B_t\,\Omega - B_z\,k_z - B_x\,k_x - i\,B| = 0 . \tag{4.2}$$

Appendix C calculates the dispersion relation for the matrices given in Sec. 3. The result is

$$\frac{\Omega^2}{C^2} - k^2 - k_a^2 + \frac{C^2}{\Omega^2}\left((\vec{k}_g \times \vec{k})^2 + (\vec{k}_g \times \vec{k}_c)^2\right) = 0 \ , \tag{4.3}$$

which is equivalent to

$$\frac{\Omega^2}{C^2} - k^2 - k_a^2 + \frac{C^2\, k_g^2}{\Omega^2}\,(k_\perp^2 + k_{c\perp}^2) = 0 \ , \tag{4.4}$$

where

$$\vec{k}_g = (\gamma-1)^{-1/2}\, \vec{\nabla}\, \ell n\, \theta_o \ , \tag{4.5}$$

and

$$\vec{k}_c = \vec{\Gamma} + \frac{1}{2}\vec{\nabla}\, \ell n\, C = -\frac{2-\gamma}{2\gamma}\vec{\nabla}\, \ell n\, p_o - \frac{1}{2}\vec{\nabla}\, \ell n\, C \ , \tag{4.6}$$

where

$$\vec{\Gamma} = -\frac{2-\gamma}{2\gamma}\vec{\nabla}\, \ell n\, p_o - \vec{\nabla}\, \ell n\, C \tag{4.7}$$

is the obvious vector generalization (using (1.13) as an effective value of $\vec{g}$) of Eckart's coefficient (1.8),

$$k_a^2 = k_g^2 + k_c^2 \ , \tag{4.8}$$

and $\vec{k}_\perp$ and $\vec{k}_{c\perp}$ are components of $\vec{k}$ and $\vec{k}_c$ perpendicular to $\vec{k}_g$ (and thus also perpendicular of $\vec{\nabla}\,\theta_o$).

The term

$$C^2\, k_g^2\, k_{c\perp}^2/\Omega^2 = C^2(\vec{k}_g \times \vec{k}_c)^2/\Omega^2 \tag{4.9}$$

has no counterpart in the usual dispersion relation for a barotropic atmosphere. It is zero for a barotropic atmosphere because $\vec{k}_c$ is then parallel to $\vec{\nabla}\,\theta_o$, as can be seen from (4.6). It seems appropriate to call that term the baroclinic term.

Except for the baroclinic term, (4.4) has the same form as (1.1) with k replacing k_x,

$$\omega_2^2 = C^2 k_a^2 = C^2(k_g^2 + k_c^2) \tag{4.10}$$

(which differs from either of the usual formulas (1.5) or (1.7) for the acoustic cutoff frequency except for an isothermal atmosphere), and

$$\omega_1^2 = C^2 k_g^2 \tag{4.11}$$

(which differs from the usual formula (1.6) for the Brunt-Väisälä frequency except for an isothermal atmosphere).

Appendix D considers a special case in which the atmosphere is exactly barotropic everywhere. In that case, there is no baroclinic term so that the dispersion relation has the form (1.1). For a barotropic atmosphere, it is possible (though not necessary) to choose a particular solution to the symmetry conditions in which

$$\omega_1^2 = \omega_B^2 = -\frac{C^2}{\gamma}\frac{\vec{\nabla}p_o}{p_o}\cdot\frac{\vec{\nabla}\,\theta_o}{\theta_o} , \tag{4.12}$$

equal to the generalized formula (1.14) for the Brunt-Väisälä frequency. The additional condition that the matrix B_t in (3.1) and (B.15) is constant leads to

$$\omega_2^2 = C^2(\omega_1^2 + (\vec{\Gamma} + \frac{1}{2}\vec{\nabla}\,\ell n\ C)^2) \ . \tag{4.13}$$

For an even slightly baroclinic atmosphere, however, even if it contains barotropic regions, the solutions to the symmetry conditions in (4.12) and (4.13) are not valid. This result from Appendix E follows because some of the symmetry conditions are differential equations. This, combined with the requirement that the dispersion relation should depend on only local properties of the atmosphere (including derivatives) and not on the route by which the wave arrived at some particular point in the atmosphere, leads uniquely to (4.4) for the dispersion relation. Although it would be possible in a baroclinic atmosphere to choose (4.12) to hold at some particular barotropic point, one of the differential symmetry conditions would require ω_1 to change with position in such a way that there would be no guarantee that (4.12) would still hold at some other barotropic point (or even the same barotropic point reached by some circuitous route).

Equation (4.5) follows from transforming the inviscid Navier-Stokes equations to a symmetric hyperbolic system. Equation (4.6) required the additional assumption that the matrix B_t was constant. Thus, it is possible to alter (4.6) by relaxing that assumption. In fact, once that assumption is relaxed, it would be possible to specify $\vec{k}_c$ nearly arbitrarily. However, no matter what the value of $\vec{k}_c$, we will always have

$$\omega_2 \geq \omega_1 \tag{4.14}$$

because

$$\omega_2^2 = \omega_1^2 + C^2 k_c^2 . \tag{4.15}$$

Thus, the acoustic regime and the gravity-wave regime never overlap, and when $\vec{k}_c$ is not zero, there is a frequency gap separating them.

It is interesting to notice that for a baroclinic atmosphere it is the component of $\vec{k}$ perpendicular to $\vec{\nabla}\theta_o$ that enters the dispersion relation. This takes the place of the k_x in the usual barotropic dispersion relation. It is not at all obvious why the component of $\vec{k}$ perpendicular to $\vec{\nabla}\theta_o$ is important rather than the component perpendicular to $\vec{\nabla}p_o$, say.

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APPENDIX A.--Linearized Inviscid Navier-Stokes Equations in Matrix Form

As pointed out in the introduction, there are advantages to writing the inviscid Navier-Stokes equations as a symmetric hyperbolic system. It is easier to express the symmetry condition if the equations are first written in matrix form. The transformation to a symmetric system is also more systematic when the equations are in matrix form. Let us define the following perturbation or wave-associated quantities.

$$p_1 \equiv (p - p_o)/p_o \qquad \text{(a)}$$

$$\rho_1 \equiv (\rho - \rho_o)/\rho_o \qquad \text{(b)} \qquad \text{(A.1)}$$

$$\vec{u} \equiv (\vec{U} - \vec{U}_o)/C \quad , \qquad \text{(c)}$$

where C is the sound speed, p is the pressure, ρ is the density, and $\vec{U}$ is the flow velocity. Then in the usual way, we can write the linearized Navier-Stokes equations for the wave-associated quantities neglecting dissipation, Coriolis force, and Reynolds stress.

The continuity equation combined with the equation of adiabatic state is

$$\partial p_1/\partial t + \vec{U}_o \cdot \vec{\nabla} p_1 + C\gamma \vec{\nabla} \cdot \vec{u} + \gamma \vec{u} \cdot \vec{\nabla} C + C\vec{\nabla} p_o \cdot \vec{u}/p_o = 0 \ . \qquad \text{(A.2)}$$

The continuity equation is

$$\partial \rho_1/\partial t + \vec{U}_o \cdot \vec{\nabla} \rho_1 + C\vec{\nabla} \cdot \vec{u} + \vec{u} \cdot \vec{\nabla} C + C\vec{\nabla} \rho_o \cdot \vec{u}/\rho_o = 0 \ . \qquad \text{(A.3)}$$

The momentum equation is

$$\partial \vec{u}/\partial t + \frac{\vec{u}}{C}\frac{\partial C}{\partial t} + (\vec{U}_o \cdot \vec{\nabla})\vec{u} + (\vec{U}_o \cdot \vec{\nabla} C)\vec{u}/C$$

$$+ \frac{1}{C}(p_o/\rho_o)\vec{\nabla} p_1 + \frac{1}{C}(\vec{\nabla} p_o/\rho_o)p_1 - \frac{1}{C}(\vec{\nabla} p_o/\rho_o)\rho_1 = 0 \ . \qquad \text{(A.4)}$$

The background flow also satisfies continuity,

$$D\rho_o/Dt + \rho_o(\vec{\nabla} \cdot \vec{U}_o) = 0 \ , \tag{A.5}$$

and the momentum equation

$$D\vec{U}_o/Dt + \vec{\nabla}p_o/\rho_o - \vec{g} = 0 \ , \tag{A.6}$$

where

$$\vec{g} = (g_x,\ g_y,\ g_z) = (0,\ 0,\ -\ g) \tag{A.7}$$

is the vector acceleration due to the Earth's gravitational field, and

$$D/Dt \equiv \partial/\partial t + \vec{U}_o \cdot \vec{\nabla} \tag{A.8}$$

is the Eulerian or substantial derivative. Equations (A.2) through (A.4) can be written in the condensed form

$$\partial\psi/\partial t + \vec{U}_o \cdot \vec{\nabla}\psi + A_z\partial\psi/\partial z + A_x\partial\psi/\partial x + A_y\partial\psi/\partial y + A\psi = 0 \ , \tag{A.9}$$

where ψ is a column vector made from the wave-associated quantities in (A.1), and A_z, A_x, A_y, and A are matrices.

If we include all three spatial dimensions, then ψ will have five elements and the matrices will be five-by-five matrices. However, including the third spatial dimension adds nothing significant, but neglecting the y direction simplifies the algebra. The generalization to the third dimension is easily made at the end. Neglecting the y direction allows us to set $A_y = 0$, and use four-by-four matrices. Thus, we have (using (1.2))

$$\psi = \begin{pmatrix} p_1 \\ \rho_1 \\ u_z \\ u_x \end{pmatrix} , \tag{A.10}$$

$$A_z = C \begin{pmatrix} 0 & 0 & \gamma & 0 \\ 0 & 0 & 1 & 0 \\ 1/\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (A.11)$$

$$A_x = C \begin{pmatrix} 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1/\gamma & 0 & 0 & 0 \end{pmatrix}, \qquad (A.12)$$

and

$$A = C \begin{pmatrix} 0 & 0 & \dfrac{\partial p_o/\partial z}{p_o} + \dfrac{\gamma \partial C/\partial z}{C} & \dfrac{\partial p_o/\partial x}{p_o} + \dfrac{\gamma \partial C/\partial x}{C} \\ 0 & 0 & \dfrac{\partial \rho_o/\partial z}{\rho_o} + \dfrac{\partial C/\partial z}{C} & \dfrac{\partial \rho_o/\partial x}{\rho_o} + \dfrac{\partial C/\partial x}{C} \\ \dfrac{\partial p_o/\partial z}{\gamma p_o} & -\dfrac{\partial p_o/\partial z}{\gamma p_o} & \dfrac{1}{C^2}\dfrac{DC}{Dt} & 0 \\ \dfrac{\partial p_o/\partial x}{\gamma p_o} & -\dfrac{\partial p_o/\partial x}{\gamma p_o} & 0 & \dfrac{1}{C^2}\dfrac{DC}{Dt} \end{pmatrix}. \qquad (A.13)$$

It is interesting to notice that even for a time-varying background, there is no explicit time dependence of the background in (A.11), (A.12), or (A.13). The only explicit dependence on background gradients occurs in (A.13) through the terms containing $\vec{\nabla}C$, $\vec{\nabla}p_o$, or $\vec{\nabla}\rho_o$. Also, the equations (A.9), (A.11), (A.12), and (A.13) are symmetric with respect to x and z. The asymmetry of the Earth's gravitational field in (A.7) enters only indirectly through the background momentum equation (A.6). The (3,2) and (4,2) elements in (A.13) arise from the effective $\vec{g}$ that includes the inertial effect of the acceleration (if present) of the background flow.

APPENDIX B.--Symmetrizing the Matrices

The next step in transforming the inviscid Navier-Stokes equations into a symmetric hyperbolic system is to symmetrize the matrices in (A.11), (A.12), and (A.13). We define the following operators:

$$
\begin{aligned}
\hat{\omega} &= - i\, \partial/\partial t && \text{(a)} \\
\hat{k}_z &= i\, \partial/\partial z && \text{(b)} \\
\hat{k}_x &= i\, \partial/\partial x && \text{(c)} \\
\hat{\Omega} &= - i\, D/Dt = \hat{\omega} - \vec{U}_o \cdot \hat{k} && \text{(d)}
\end{aligned}
\qquad \text{(B.1)}
$$

The choice of signs in (B.1) implies an exp(+iωt) time convention for time-harmonic waves. We now assume a generalization of (A.9) in which the first two terms also have a matrix coefficient (but we neglect the y dependence). Using the operator notation in (B.1) (and multiplying by -i), we have

$$(B_t \hat{\Omega} - B_z \hat{k}_z - B_x \hat{k}_x - i\, B)\phi = 0 \ . \qquad \text{(B.2)}$$

The dispersion relation is found by letting $\hat{\omega} \rightarrow \omega$, $\hat{k}_z \rightarrow k_z$, $\hat{k}_x \rightarrow k_x$, $\hat{\Omega} \rightarrow \Omega = \omega - \vec{U}_o \cdot k$, where ω, k_z, k_x, and Ω are no longer operators, but are frequencies and wave numbers (Ω is called the intrinsic frequency, the frequency seen by an observer moving with the mean flow), and by setting the determinant

$$\left| B_t\, \Omega - B_z\, k_z - B_x\, k_x - i\, B \right| = 0 \ . \qquad \text{(B.3)}$$

For the dispersion relation in (B.3) to represent a wave that neither grows nor decays, it must be possible to find solutions of (B.3) for which Ω, k_z, and k_x are all real. A sufficient condition is that the matrix in (B.3) be Hermitian. The matrix is Hermitian for real B_t, B_z, B_x, B, Ω, k_z, and k_x when B_t, B_z, and B_x are symmetric about the main diagonal and B is antisymmetric about the main diagonal.

To transform from the system of equations (A.9) to the symmetric system (B.2) in which B_t, B_z, and B_x are symmetric and B is antisymmetric requires changing to a new set of wave variables and multiplying the set of equations on the left by some matrix B_t. Changing to a new set of wave variables can be done by a linear transformation matrix S.

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = S^{-1} \psi \quad . \tag{B.4}$$

The transformation (B.4) has two purposes. First, it changes to new wave variables that are linear combinations of the old wave variables. Second, it scales the wave variables by position-dependent coefficients. The scaling is known to be necessary because in an isothermal atmosphere, for example, the wave variables in (A.1) and (A.10) grow exponentially with height without energy exchange between the wave and the background. The effect is well understood, and scaling by $p_o^{1/2}$ gives wave variables whose amplitudes more nearly represent wave energy even when the background temperature varies with height. For a baroclinic atmosphere, the scaling may be more complicated.

Without going into the details, I can give a general outline of the process of converting from the system (A.9) to the symmetric system (B.2). We begin by writing the inverse of (B.4):

$$\psi = S\,\phi \quad . \tag{B.5}$$

We substitute (B.5) into (A.9) and multiply on the left by S^{-1}. This gives

$$S^{-1}\, D(S\phi)/Dt + S^{-1}\, A_z\, \partial(S\phi)/\partial z + A^{-1}\, A_x\, \partial(S\phi)/\partial x + S^{-1}\, A\, S\phi = 0 \ , \tag{B.6}$$

where (A.8) is used to shorten the equation. Several terms in (B.6) involve derivatives of the product $S\phi$. Expanding out those derivatives into two terms gives

$$S^{-1} S \, D\phi/Dt + S^{-1} A_z S \, \partial\phi/\partial z + S^{-1} A_x \, \partial\phi/\partial x +$$

$$(S^{-1} A S + S^{-1} DS/Dt + S^{-1} A_z \, \partial S/\partial z + S^{-1} A_x \, \partial S/\partial x)\phi = 0 \quad . \tag{B.7}$$

Multiplying (B.7) on the left by B_t gives

$$B_t \, D\phi/Dt + B_z \, \partial\phi/\partial z + B_x \, \partial\phi/\partial x + B\phi = 0 \quad , \tag{B.8}$$

where

$$B_z = B_t S^{-1} A_z S \, , \tag{B.9}$$

$$B_x = B_t S^{-1} A_x S \, , \tag{B.10}$$

and

$$B = B_t (S^{-1} A S + S^{-1} DS/Dt + S^{-1} A_z \, \partial S/\partial z + S^{-1} A_x \, \partial S/\partial x) \quad . \tag{B.11}$$

It turns out to be sufficient to consider transformations of the form

$$S = \begin{pmatrix} S_{11} & S_{12} & 0 & 0 \\ S_{21} & S_{22} & 0 & 0 \\ 0 & 0 & S_{33} & 0 \\ 0 & 0 & 0 & S_{44} \end{pmatrix} \tag{B.12}$$

to make B_t, B_z, and B_x symmetric and B antisymmetric. The transformation (B.12) converts the wave variables in (A.1) to a set in which the pressure and density variables are replaced by two linear combinations of density and pressure. The velocity variables are merely scaled. They are not mixed with each other or with density and pressure.

The inverse of (B.12) is

$$S^{-1} = \begin{pmatrix} S_{22}/\Delta & -S_{12}/\Delta & 0 & 0 \\ -S_{21}/\Delta & S_{11}/\Delta & 0 & 0 \\ 0 & 0 & 1/S_{33} & 0 \\ 0 & 0 & 0 & 1/S_{44} \end{pmatrix} , \tag{B.13}$$

where

$$\Delta = \begin{vmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{vmatrix} = S_{11}\ S_{22} - S_{21}\ S_{12} \ . \tag{B.14}$$

With the transformation (B.12), symmetry of B_t, B_z, and B_x requires

$$B_t = \begin{pmatrix} B_{11} & B_{12} & 0 & 0 \\ B_{21} & B_{22} & 0 & 0 \\ 0 & 0 & B_{33} & 0 \\ 0 & 0 & 0 & B_{44} \end{pmatrix} , \tag{B.15}$$

$$B_{12} = B_{21} = (S_{12}\ B_{11}(\gamma S_{22} - S_{12}) - S_{11}\ B_{22}(S_{11} - \gamma S_{21}))/d \ , \tag{B.16}$$

and

$$\begin{aligned} \frac{B_{33}}{S_{33}^2\ \gamma} = \frac{B_{44}}{S_{44}^2\ \gamma} \\ &= (B_{11}(\gamma\ S_{22} - S_{12}) + B_{12}(S_{11} - \gamma S_{21}))/(S_{11}\ \Delta) \\ &= (B_{21}(\gamma\ S_{22} - S_{12}) + B_{22}(S_{11} - \gamma S_{21}))/(S_{12}\ \Delta) \\ &= C_1 \ , \end{aligned} \tag{B.17}$$

where

$$d \equiv S_{11}(\gamma S_{22} - S_{12}) - S_{12}(S_{11} - \gamma S_{21})$$

$$= \gamma S_{11} S_{22} + \gamma S_{12} S_{21} - 2S_{11} S_{12} , \qquad (B.18)$$

and

$$C_1 \equiv (B_{11}(\gamma S_{22} - S_{12})^2 - B_{22}(S_{11} - \gamma S_{21})^2)/(d\Delta) \quad . \qquad (B.19)$$

The other two symmetric matrices are

$$B_z = B_t S^{-1} A_z S = C_1 S_{33} C \begin{pmatrix} 0 & 0 & S_{11} & 0 \\ 0 & 0 & S_{12} & 0 \\ S_{11} & S_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (B.20)$$

and

$$B_x = B_t S^{-1} A_x S = C_1 S_{44} C \begin{pmatrix} 0 & 0 & 0 & S_{11} \\ 0 & 0 & 0 & S_{12} \\ 0 & 0 & 0 & 0 \\ S_{11} & S_{12} & 0 & 0 \end{pmatrix} \quad . \qquad (B.21)$$

(I have purposely left out the intermediate algebra because it is straightforward and tedious. However, the correctness of the above equations can be checked by direct substitution.)

It is not possible to make the matrix B antisymmetric under all conditions, because under some conditions there is energy exchange between the wave and the background mean flow. This energy exchange would be manifested in complex eigenvalues for the wave, which would be seen in non-Hermitian matrices in the wave equation. Because the present equations are not sufficiently general to treat correctly energy exchange between the wave and the mean flow (e.g., they neglect Reynolds stress), we need to restrict the mean flow so that energy exchange does not appear.

A sufficient condition for energy exchange not to occur is for the background flow to satisfy

$$\dot{p}_o \equiv D\, p_o/Dt \equiv \partial p_o/\partial t + \vec{U}_o \cdot \vec{\nabla} p_o = 0 \qquad \text{(a)}$$

and

$$\dot{\rho}_o \equiv D\, \rho_o/Dt \equiv \partial \rho_o/\partial t + \vec{U}_o \cdot \vec{\nabla} \rho_o = 0 \qquad \text{(b)} \qquad \text{(B.22)}$$

which implies

$$\dot{C} \equiv DC/Dt \equiv \partial C/\partial t + \vec{U}_o \cdot \vec{\nabla} C = 0 \quad . \qquad \text{(B.23)}$$

The conditions (B.22) and (B.23) turn out to be sufficient to allow the requirement

$$\dot{S} \equiv DS/Dt \equiv \partial S/\partial t + \vec{U}_o \cdot \vec{\nabla} S = 0 \qquad \text{(B.24)}$$

on the transformation matrix S. Notice that (B.24) requires each element of S to satisfy (B.24).

After considerable algebra, it is possible to show that

$$B = C_1\, S_{33}\, C \begin{pmatrix} 0 & 0 & -S_{11}\,\alpha_z & -S_{11}\,\alpha_x \\ 0 & 0 & -S_{12}\,\beta_z & -S_{12}\,\beta_x \\ S_{11}\,\alpha_z & S_{12}\,\beta_z & 0 & 0 \\ S_{11}\,\alpha_x & S_{12}\,\beta_x & 0 & 0 \end{pmatrix}, \qquad \text{(B.25)}$$

where α_x and α_z are components of a vector

$$\vec{\alpha} = (1 - S_{21}/S_{11})\, \vec{\nabla} p_o/p_o + \vec{\nabla} S_{11}/S_{11}$$

$$= \vec{k}_c - (\gamma S_{21}/S_{11} - 1)\, C_2/C_1\, \vec{\nabla}\theta_o/\theta_o \quad , \qquad \text{(B.26)}$$

and β_x and β_z are components of a vector

$$\vec{\beta} = (1 - S_{22}/S_{12})\ \vec{\nabla}p_o/p_o + \vec{\nabla}S_{12}/S_{12}$$

$$= \vec{k}_c - (\gamma S_{22}/S_{12} - 1)\ C_2/C_1\ \vec{\nabla}\theta_o/\theta_o \quad , \qquad \text{(B.27)}$$

where

$$\vec{k}_c = - \vec{\nabla}p_o/\gamma p_o - \vec{\nabla}C/C - \vec{\nabla}S_{33}/S_{33} \quad , \qquad \text{(B.28)}$$

$$C_2 = (B_{11}\ S_{12}^2 - B_{22}\ S_{11}^2)/(d\Delta) \ , \qquad \text{(B.29)}$$

and

$$\vec{\nabla}\theta_o/\theta_o = \vec{\nabla}p_o/\gamma p_o - \vec{\nabla}\rho_o/\rho_o \qquad \text{(B.30)}$$

is the generalization of the definition for potential temperature implicitly implied in (1.6).

There are conditions on the elements of the matrices B_t and S. These are

$$S_{44} = S_{33} \quad , \qquad \text{(B.31)}$$

$$B_{44} = B_{33} \quad , \qquad \text{(B.32)}$$

$$\frac{\vec{\nabla}(S_{11}\ S_{33})}{S_{11}\ S_{33}} = - \frac{\vec{\nabla}p_o}{p_o} - \frac{\vec{\nabla}C}{C} + \left(\frac{S_{21}}{S_{11}} - \frac{1}{\gamma}\right)\left(\frac{\vec{\nabla}p_o}{p_o} - \gamma\ \frac{C_2}{C_1}\ \frac{\vec{\nabla}\theta_o}{\theta_o}\right) \quad , \qquad \text{(B.33)}$$

and

$$\frac{\vec{\nabla}(S_{12}\ S_{33})}{S_{12}\ S_{13}} = - \frac{\vec{\nabla}p_o}{p_o} - \frac{\vec{\nabla}C}{C} + \left(\frac{S_{22}}{S_{12}} - \frac{1}{\gamma}\right)\left(\frac{\vec{\nabla}p_o}{p_o} - \gamma\ \frac{C_2}{C_1}\ \frac{\vec{\nabla}\theta_o}{\theta_o}\right) \quad . \qquad \text{(B.34)}$$

The sum of (B.33) and (B.34),

$$\frac{\vec{\nabla}(S_{11}\ S_{12}\ S_{33}^2)}{S_{11}\ S_{12}\ S_{33}^2} = -\,2\,\frac{\vec{\nabla}p_o}{p_o} - 2\,\frac{\vec{\nabla}C}{C} + \frac{d}{\gamma S_{11}\ S_{12}}\left(\frac{\vec{\nabla}p_o}{p_o} - \gamma\,\frac{C_2}{C_1}\,\frac{\vec{\nabla}\theta_o}{\theta_o}\right) , \qquad \text{(B.35)}$$

gives approximately the overall scaling of the wave variables. The main contribution is the $-2\ \vec{\nabla}p_o/p_o$ term. It leads to the $p_o^{1/2}$ scaling mentioned earlier. The other terms give corrections to that scaling. The difference between (B.33) and (B.34),

$$\frac{\nabla(S_{11}/S_{12})}{S_{11}/S_{12}} = -\,\frac{\Delta}{S_{11}\ S_{12}}\left(\frac{\vec{\nabla}p_o}{p_o} - \gamma\,\frac{C_2}{C_1}\,\frac{\vec{\nabla}\theta_o}{\theta_o}\right) , \qquad \text{(B.36)}$$

plays a role in calculating the dispersion relation in Appendix C.

At this stage, we can consider the matrix B_t to be arbitrarily specifiable except that it must be symmetric. S_{44} equals S_{33}, and (B.16), (B.17), (B.31), (B.33), and (B.34) give five conditions on the elements of S. That leaves one free condition on the elements of S still to specify.

We also have from (1.2) and (B.30)

$$2\,\frac{\vec{\nabla}C}{C} = \frac{\vec{\nabla}p_o}{p_o} - \frac{\vec{\nabla}\rho_o}{\rho_o} = (1 - 1/\gamma)\,\frac{\vec{\nabla}p_o}{p_o} + \frac{\vec{\nabla}\theta_o}{\theta_o} . \qquad \text{(B.37)}$$

APPENDIX C.--Dispersion Relations

Substituting (B.15), (B.20), (B.21) and (B.25) into (B.3) leads to the dispersion relation. The substitution is straightforward but tedious. An intermediate form of the dispersion relation is

$$0 = |B_t| \, \Omega^4 + C^4 \, C_1^4 \, S_{33}^4 \, S_{11}^2 \, S_{12}^2 \, [(\vec{\alpha} \times \vec{\beta})^2 + (\vec{k} \times (\vec{\alpha} - \vec{\beta}))^2]$$

$$- C^2 \, C_1^2 \, S_{33}^2 \, B_{33} \, k^2 \, \Omega^2 \, (B_{11} \, S_{12}^2 - 2 \, S_{12} \, S_{11} \, B_{12} + B_{22} \, S_{11}^2)$$

$$- C^2 \, C_1^2 \, S_{33}^2 \, B_{33} \, \Omega^2 \, (B_{11} \, S_{12}^2 \, \beta^2 + B_{22} \, S_{11}^2 \, \alpha^2 - 2 \, B_{12} \, S_{11} \, S_{12} \, \vec{\alpha} \cdot \vec{\beta}) \qquad (C.1)$$

where

$$|B_t| = (B_{11} \, B_{22} - B_{12} \, B_{21}) \, B_{33} \, B_{44} = - \, C_1 \, C_2 \, B_{33}^2 \, \Delta^2 \quad , \qquad (C.2)$$

$$\alpha^2 = \vec{\alpha} \cdot \vec{\alpha} \quad , \qquad (C.3)$$

$$\beta^2 = \vec{\beta} \cdot \vec{\beta} \quad , \qquad (C.4)$$

and (C.1) clearly generalizes to three dimensions if we let

$$k^2 = k_z^2 + k_x^2 \rightarrow k_z^2 + k_x^2 + k_y^2 \; . \qquad (C.5)$$

Some of the quantities in (C.1) are

$$\vec{\alpha} \times \vec{\beta} = \frac{\Delta \, \gamma \, C_2}{C_1 \, S_{11} \, S_{12}} \; \frac{\vec{\nabla}\theta_o}{\theta_o} \times \vec{k}_c \qquad (C.6)$$

$$\vec{\alpha} - \vec{\beta} = - \frac{\Delta \, \gamma \, C_2}{C_1 \, S_{11} \, S_{12}} \; \frac{\vec{\nabla}\theta_o}{\theta_o} \quad , \qquad (C.7)$$

$$B_{11} \, S_{12}^2 - 2 \, S_{12} \, S_{11} \, B_{12} + B_{22} \, S_{11}^2 = - \, \gamma C_2 \, \Delta^2 \quad , \qquad (C.8)$$

and

$$B_{11} S_{12}^2 \beta^2 - 2 B_{12} S_{11} S_{12} \alpha \cdot \beta + B_{22} S_{11}^2 \alpha^2 = - \gamma C_2 \Delta^2 \left(k_c^2 - \frac{C_2}{C_1} \left(\frac{\nabla \theta_o}{\theta_o} \right)^2 \right) . \quad (C.9)$$

Substituting (C.2) and (C.6) through (C.9) in (C.1) gives

$$0 = \frac{\Omega^2}{C^2} - k^2 - k_c^2 + \frac{C_2}{C_1} \left(\frac{\nabla \theta_o}{\theta_o} \right)^2 - \frac{C_2}{C_1} \frac{C^2}{\Omega^2} \left[\left(\vec{k} \times \frac{\vec{\nabla} \theta_o}{\theta_o} \right)^2 + \left(\vec{k}_c \times \frac{\vec{\nabla} \theta_o}{\theta_o} \right)^2 \right] , \quad (C.10)$$

where $\vec{k}_c$ is given by (B.28), C_1 is given by (B.19), and C_2 is given by (B.29). Equation (C.10) can be rewritten

$$0 = \frac{\Omega^2}{C^2} - k^2 - k_c^2 - k_g^2 + \frac{C^2 k_g^2}{\Omega^2} \left(k_\perp^2 + k_{c\perp}^2 \right) , \quad (C.11)$$

where $\vec{k}_\perp$ and $\vec{k}_{c\perp}$ are components of $\vec{k}$ and $\vec{k}_c$ perpendicular to $\vec{\nabla}\theta_o$, and

$$k_g^2 = - \frac{C_2}{C_1} \left(\frac{\nabla \theta_o}{\theta_o} \right)^2 . \quad (C.12)$$

The variables C_1, C_2, and $\vec{k}_c$ so far have some arbitrariness. However, Appendix E applies the differential symmetry conditions for a baroclinic atmosphere to determine them. Equation (E.33) gives

$$C_2/C_1 = - 1/(\gamma-1) , \quad (C.13)$$

and (E.61) gives

$$\vec{k}_c = - \frac{2-\gamma}{2\gamma} \vec{\nabla} \ell n \, p_o - \frac{1}{2} \vec{\nabla} \, \ell n \, C . \quad (C.14)$$

Substituting (C.13) into (C.12) gives

$$k_g^2 = \frac{1}{\gamma-1} (\vec{\nabla} \, \ell n \, \theta_o)^2 . \quad (C.15)$$

Thus, from (C.15), we may write

$$\vec{k}_g = (\gamma-1)^{-1/2} \vec{\nabla} \, \ell n \, \theta_o . \quad (C.16)$$

APPENDIX D.--Optional Dispersion Relation for a Barotropic Atmosphere

Equation (B.36) contains the factor

$$\frac{\vec{\nabla} p_o}{p_o} - \gamma \frac{C_2}{C_1} \frac{\vec{\nabla} \theta_o}{\theta_o} \quad . \tag{D.1}$$

In a barotropic atmosphere, the two terms in (D.1) are parallel, so that it is possible to choose C_2/C_1 to make (D.1) zero. Because of (B.36), that would make S_{11}/S_{12} constant, resulting in a simplification. Setting (D.1) to zero,

$$\frac{\vec{\nabla} p_o}{p_o} - \gamma \frac{C_2}{C_1} \frac{\vec{\nabla} \theta_o}{\theta_o} = 0 \quad , \tag{D.2}$$

gives

$$\frac{C_2}{C_1} = \frac{1}{\gamma} \frac{\nabla p_o / p_o}{\nabla \theta_o / {}_o} \quad . \tag{D.3}$$

Substituting (D.3) into (C.12) gives

$$k_g^2 = - (1/\gamma)(\nabla p_o / p_o)(\nabla \theta_o / \theta_o) \quad . \tag{D.4}$$

If we assume the elements of the matrix B_t are constant, and we use the above result that S_{11}/S_{12} is constant, then D defined in (E.13) is constant and W defined in (E.5) is constant. Thus, from (E.15),

$$S_{33}/S_{11} = \text{constant.} \tag{D.5}$$

Substituting (D.2) into (B.33) gives

$$\frac{\vec{\nabla}(S_{11}\ S_{33})}{S_{11}\ S_{33}} = - \frac{\vec{\nabla} p_o}{p_o} - \frac{\vec{\nabla} C}{C} \quad . \tag{D.6}$$

Combining (D.5) with (D.6) gives

$$\frac{\vec{\nabla} S_{11}}{S_{11}} = \frac{\vec{\nabla} S_{33}}{S_{33}} = -\frac{1}{2}\frac{\vec{\nabla} p_o}{p_o} - \frac{1}{2}\frac{\vec{\nabla} C}{C} . \tag{D.7}$$

Substituting (D.7) into (B.28) gives

$$\vec{k}_c = -\frac{2-\gamma}{2\gamma}\frac{\vec{\nabla} p_o}{p_o} - \frac{1}{2}\frac{\vec{\nabla} C}{C} = \vec{\Gamma} + \frac{1}{2}\frac{\vec{\nabla} C}{C} , \tag{D.8}$$

where

$$\vec{\Gamma} = -\frac{2-\gamma}{2\gamma}\frac{\vec{\nabla} p_o}{p_o} - \frac{\vec{\nabla} C}{C} \tag{D.9}$$

is the obvious vector generalization (using (1.13) as an effective $\vec{g}$) of Eckart's coefficient (1.8).

For a barotropic atmosphere, the baroclinic term $C^2 k_g^2 k_{c\perp}^2/\Omega^2$ in (C.11) is zero, so that the dispersion relation (C.11) has the barotropic form (1.1). Using (D.4) and (D.8) in (C.11) gives (1.1), with

$$C_1^2 = C^2 k_g^2 = \omega_B^2 = -\frac{C^2}{\gamma}\frac{\vec{\nabla} p_o}{p_o} \cdot \frac{\vec{\nabla} \theta_o}{\theta_o} , \tag{D.10}$$

equal to the square of the generalized Brunt-Väisälä frequency (1.14), and

$$C_2^2 = C^2(k_g^2 + k_c^2) = \omega_B^2 + C^2\left(\vec{\Gamma} + \frac{1}{2}\frac{\vec{\nabla} C}{C}\right)^2 \tag{D.11}$$

for the square of an effective acoustic cutoff frequency.

The dispersion relation derived above for a barotropic atmosphere is optional. It results from choosing the condition (D.2) as an assumption. It is not necessary to choose that condition for a barotropic atmosphere. The more general procedure followed in Appendix E for a baroclinic atmosphere is applicable to a barotropic atmosphere also, and gives a dispersion relation different from that derived here, even for a barotropic atmosphere.

For a baroclinic atmosphere, however, there is no generalization of the procedure in this appendix, because when the two vectors in (D.1) are not parallel, there is no value for the scalar function C_2/C_1 that will make (D.1) zero. Because the real atmosphere is never exactly barotropic, the dispersion relation derived in this appendix under the assumption (D.2) has no practical significance.

This example illustrates, however, a situation in which results derived for a special case are not generalizable to broader situations. Previous calculations of the dispersion relation for acoustic-gravity waves have been for propagation in a barotropic atmosphere. Perhaps there is a relationship between that reliance on barotropic calculations and the result of getting the Brunt-Väisälä frequency for ω_1 in the dispersion relation (1.1).

Notice that whereas (D.4) follows from transforming the inviscid Navier-Stokes equations to a symmetric hyperbolic system plus the assumption (D.2), (D.8) required the additional assumption that the matrix B_t was constant. Thus, it is possible to alter (D.8) by relaxing that assumption. In fact, once that assumption is relaxed, it would be possible to specify $\vec{k}_c$ nearly arbitrarily. However, no matter what the value of $\vec{k}_c$, we will always have

$$\omega_2 \geq \omega_1 \tag{D.12}$$

because

$$\omega_2^2 = \omega_1^2 + C^2 k_c^2 \quad . \tag{D.13}$$

Thus, the acoustic regime and the gravity-wave regime never overlap, and when $\vec{k}_c$ is not zero, there is a frequency gap separating them.

APPENDIX E.--Satisfying the Symmetry Conditions for a Baroclinic Atmosphere

Appendix B contains several symmetry conditions on the elements of the matrix S, namely, (B.16), (B.17), (B.31), (B.33), and (B.34). [Or, (B.35) and (B.36) instead of the last two.] To see the full effect of these conditions, it is useful to define some new variables.

Let

$$F \equiv \gamma \frac{S_{22}}{S_{12}} - 1 \tag{E.1}$$

and

$$G \equiv 1 - \gamma \frac{S_{21}}{S_{11}} \ . \tag{E.2}$$

Then from (B.14),

$$\Delta = S_{11} S_{12} (F + G)/\gamma \ , \tag{E.3}$$

and from (B.18),

$$d = S_{11} S_{12} (F - G) \ . \tag{E.4}$$

If we let

$$W^2 \equiv \frac{B_{22}}{B_{11}} \left(\frac{S_{11}}{S_{12}} \right)^2 \ , \tag{E.5}$$

then from (B.19),

$$C_1 = \gamma \frac{B_{11}}{S_{11}^2} \frac{F^2 - W^2 G^2}{F^2 - G^2} \tag{E.6}$$

and from (B.29),

$$C_2 = \gamma \frac{B_{11}}{S_{11}^2} \frac{1 - W^2}{F^2 - G^2} \ , \tag{E.7}$$

so that

$$\frac{C_2}{C_1} = \frac{1 - W^2}{F^2 - W^2 G^2} \quad . \tag{E.8}$$

From (B.16), we now have

$$\frac{B_{12}}{B_{11}} = \frac{S_{12}}{S_{11}} \frac{F - W^2 G}{F - G} \quad . \tag{E.9}$$

From (B.17), we have

$$\frac{B_{33}}{B_{11}} = \gamma^2 \frac{S_{33}^2}{S_{11}^2} \frac{F^2 - W^2 G^2}{F^2 - G^2} \, . \tag{E.10}$$

We can solve (E.8) and (E.9) simultaneously for F and G in terms of the other variables. The result is

$$F = \frac{D\,W - 1}{R\sqrt{D^2 - 1}} \tag{E.11}$$

and

$$G = \frac{D - W}{\sqrt{WR\,D^2 - 1}} \, , \tag{E.12}$$

where

$$D^2 \equiv \frac{B_{11}\;B_{22}}{B_{12}\;B_{21}} = B_{11}\;B_{22}/B_{12}^2 \tag{E.13}$$

and

$$R^2 \equiv - C_2/C_1 \quad , \tag{E.14}$$

and (E.5) is used to eliminate S_{11}/S_{12}. We can substitute (E.11) and (E.12) into (E.10) to get

$$\frac{S_{33}^2}{S_{11}^2} = \frac{1}{\gamma^2} \frac{B_{33}}{B_{11}} \frac{D(D\,W^2 + D - 2W)}{(D^2 - 1)\,W} \quad . \tag{E.15}$$

We also get

$$F + G = \frac{D\,W^2 + D - 2W}{R\sqrt{D^2 - 1}\,W} \tag{E.16}$$

and

$$F - G = \frac{D(W^2 - 1)}{RW\sqrt{D^2 - 1}} \quad . \tag{E.17}$$

Substituting (E.3), (E.4), and (E.14) into (B.36) gives

$$\frac{\vec{\nabla}(S_{11}/S_{12})}{(S_{11}/S_{12})} = -\frac{F + G}{\gamma}\left(\frac{\vec{\nabla}p_o}{p_o} + \gamma R^2 \frac{\vec{\nabla}\theta_o}{\theta_o}\right) \quad . \tag{E.18}$$

If we now assume that B_{22}/B_{11} is constant, then

$$\frac{\vec{\nabla}(S_{11}/S_{12})}{(S_{11}/S_{12})} = \frac{\vec{\nabla}(\sqrt{B_{22}/B_{11}}\; S_{11}/S_{12})}{\sqrt{B_{22}/B_{11}}\; S_{11}/S_{12}} = \frac{\vec{\nabla}W}{W} \quad . \tag{E.19}$$

Substituting (E.16) and (E.19) into (E.18) gives

$$\frac{\vec{\nabla}W}{W} = -\frac{D\,W^2 - 2W + D}{\gamma RW \quad D^2 - 1}\left(\frac{\vec{\nabla}p_o}{p_o} + \gamma R^2 \frac{\vec{\nabla}\theta_o}{\theta_o}\right) \quad . \tag{E.20}$$

For the dispersion relation (C.11) to be useful, the terms that depend on the medium need to be determined completely by the local properties of the medium (including local gradients); these terms should not depend on properties of the medium some distance away. Thus, we must have C_2/C_1 (and therefore R) depend only on local properties. In addition, the elements of S must depend only on local properties.

If we consider only the simplest scalar functions of the gradient, then we may write

$$R = R(p_o, \theta_o, \vec{\nabla} p_o \cdot \vec{\nabla} p_o, \vec{\nabla} p_o \cdot \vec{\nabla}\theta_o, \vec{\nabla}\theta_o \cdot \vec{\nabla}\theta_o) \tag{E.21}$$

and

$$W = W(p_o, \theta_o, \vec{\nabla} p_o \cdot \vec{\nabla} p_o, \vec{\nabla} p_o \cdot \vec{\nabla}\theta_o, \vec{\nabla}\theta_o \cdot \vec{\nabla}\theta_o) . \tag{E.22}$$

There are other gradient terms we may consider, but let us consider the above first. In (E.20), we need the gradient of W. Using (E.22), we obtain

$$\vec{\nabla} W = \frac{\partial W}{\partial p_o} \vec{\nabla} p_o + \frac{\partial W}{\partial \theta_o} \vec{\nabla}\theta_o + \frac{\partial W}{\partial \vec{\nabla} p_o \cdot \vec{\nabla} p_o} \vec{\nabla}(\vec{\nabla} p_o \cdot \vec{\nabla} p_o)$$

$$+ \frac{\partial W}{\partial \vec{\nabla} p_o \cdot \vec{\nabla}\theta_o} \vec{\nabla}(\vec{\nabla} p_o \cdot \vec{\nabla}\theta_o) + \frac{\partial W}{\partial \vec{\nabla}\theta_o \cdot \vec{\nabla}\theta_o} \vec{\nabla}(\vec{\nabla}\theta_o \cdot \vec{\nabla}\theta_o) . \tag{E.23}$$

Let us consider the fourth term in (E.23). Using the formula for the gradient of a dot product (Whitmer, 1952, p. 257) gives

$$\vec{\nabla}(\vec{\nabla} p_o \cdot \vec{\nabla}\theta_o) = (\vec{\nabla} p_o \cdot \vec{\nabla})\vec{\nabla}\theta_o + (\vec{\nabla}\theta_o \cdot \vec{\nabla})\vec{\nabla} p_o + \vec{\nabla} p_o \times (\nabla \times \vec{\nabla}\theta_o)$$

$$+ \vec{\nabla}\theta_o \times (\vec{\nabla} \times \vec{\nabla} p_o)$$

$$= (\vec{\nabla} p_o \cdot \vec{\nabla})\vec{\nabla}\theta_o + (\vec{\nabla}\theta_o \cdot \vec{\nabla})\vec{\nabla} p_o , \tag{E.24}$$

where the final form follows because the curl of a gradient is zero. If we were to substitute (E.24) into (E.23) and then into (E.20), we would get terms on the left of (E.20) proportional to the two terms on the right of (E.24). These are essentially directional derivatives of a gradient, and there is no term like that on the right of (E.20), nor could such a term arise from the scalar factor in front. Therefore, W cannot depend directly on $(\vec{\nabla} p_o \cdot \vec{\nabla}\theta_o)$.

The same result holds for the fourth and fifth arguments in (E.22). In addition, it is difficult to imagine any way W could depend on a scalar function of derivatives that would not give the same result. Therefore, we conclude that W is a function of p_o and θ_o only.

$$W = W(\ell n\, p_o, \ell n\, \theta_o) , \tag{E.25}$$

where (E.25) expresses that dependence in terms of logarithms for later convenience. Therefore (E.23) becomes

$$\vec{\nabla} W = \frac{\partial W}{\partial \ell n\, p_o} \frac{\vec{\nabla} p_o}{p_o} + \frac{\partial W}{\partial \ell n\, \theta_o} \frac{\vec{\nabla} \theta_o}{\theta_o} . \tag{E.26}$$

Substituting (E.26) into (E.20) and equating the coefficients of the two gradient terms gives

$$\frac{\partial W}{\partial \ell n\, p_o} = - \frac{D\, W^2 - 2W + D}{\gamma\, R \sqrt{D^2 - 1}} \tag{E.27}$$

and

$$\frac{\partial W}{\partial \ell n\, \theta_o} = - \frac{D\, W^2 - 2W + D}{\sqrt{D^2 - 1}} R . \tag{E.28}$$

Taking the ratio of (E.28) and (E.27) gives

$$\frac{\partial W / \partial \ell n\, \theta_o}{\partial W / \partial \ell n\, p_o} = \gamma R^2 . \tag{E.29}$$

Because W does not depend on gradients of p_o or θ_o, neither can the derivatives on the left of (E.29). Therefore, the right side of (E.29) cannot depend on gradients of p_o or θ_o either. Therefore, (E.21) becomes

$$R = R(p_o, \theta_o) . \tag{E.30}$$

To determine the functional dependence in (E.30), we consider an isothermal atmosphere. In that case, (D.4) gives the correct formula for the square of the Brunt-Väisälä frequency with

$$\frac{\vec{\nabla}\theta_o}{\theta_o} = -\frac{\gamma-1}{\gamma}\frac{\vec{\nabla}P_o}{\theta_o} \tag{E.31}$$

from (B.30) because

$$\frac{\vec{\nabla}\rho_o}{\rho_o} = \frac{\vec{\nabla}p_o}{p_o} \tag{E.32}$$

for an isothermal atmosphere. Comparing (C.12) with (D.4) gives

$$\frac{C_2}{C_1} = -R^2 = -\frac{1}{\gamma-1} \tag{E.33}$$

for an isothermal atmosphere, independent of the local values of p_o and θ_o. Therefore, R must be a constant, so that (E.33) holds in general, not just for an isothermal atmosphere.

We substitute (E.33) into (E.20) and use (B.37) to obtain

$$\vec{\nabla}W = -\frac{D\,W^2 - 2W + D}{\gamma\sqrt{D^2 - 1}}\sqrt{\gamma-1}\left(\frac{\vec{\nabla}p_o}{p_o} + \frac{\gamma}{\gamma-1}\frac{\vec{\nabla}\theta_o}{\theta_o}\right)$$

$$= -\frac{D\,W^2 - 2W + D}{\gamma\sqrt{D^2 - 1}}\sqrt{\gamma-1}\,\frac{2\gamma}{\gamma-1}\frac{\vec{\nabla}C}{C}\ . \tag{E.34}$$

Clearly, from (E.34), W depends on only the local sound velocity. Therefore, (E.25) becomes

$$W = W(C) \tag{E.35}$$

and (E.26) becomes

$$\vec{\nabla} W = \frac{dW}{dC} \vec{\nabla} C \quad . \tag{E.36}$$

Substituting (E.36) into (E.34) gives

$$\frac{dW}{dC} = - \frac{D W^2 - 2W + D}{\sqrt{\gamma-1} \sqrt{D^2 - 1}} \frac{2}{C} \tag{E.37}$$

which can be written

$$\sqrt{\gamma-1} \sqrt{D^2 - 1} \frac{dW}{D W^2 - 2W + D} = - 2 \frac{dC}{C} \quad . \tag{E.38}$$

Integrating (E.38) gives

$$W = \frac{1}{D} - \frac{\sqrt{D^2 - 1}}{D} \tan\delta \ , \tag{E.39}$$

where

$$\delta = \frac{2}{\sqrt{\gamma-1}} \ln (C/C_o) \ , \tag{E.40}$$

where C_o is a constant of integration, and I have assumed D to be constant. We can rewrite (E.39) as

$$W = \frac{\cos(\delta+b)}{\cos\delta} = \left(\frac{B_{22}}{B_{11}}\right)^{1/2} \frac{S_{11}}{S_{12}} \tag{E.41}$$

where

$$\cos b = \frac{1}{D} \quad . \tag{E.42}$$

From (E.41) and (E.42), we have

$$D W^2 - 2W + D = \sin b \tan b/\cos^2 \delta \ , \tag{E.43}$$

$$D\,W - 1 = -\tan b \tan\delta \quad , \tag{E.44}$$

and

$$D - W = \tan b \sin(\delta + b)/\cos\delta \ . \tag{E.45}$$

Substituting (E.33), (E.42), and (E.44) into (E.11) gives

$$F = -\sqrt{\gamma - 1}\,\tan\delta \ . \tag{E.46}$$

Substituting (E.33), (E.42), (E.41), and (E.45) into (E.12) gives

$$G = \sqrt{\gamma - 1}\,\tan(\delta + b) \ . \tag{E.47}$$

Substituting (E.33), (E.41), (E.42), and (E.43) into (E.16) gives

$$F + G = \sqrt{\gamma - 1}\,\frac{\sin b}{\cos\delta\,\cos(\delta + b)} \ . \tag{E.48}$$

Substituting (E.33), (E.41), and (E.42) into (E.17) gives

$$F - G = \sqrt{\gamma - 1}\,\frac{\cos^2(\delta + b) - \cos^2\delta}{\cos\delta\,\cos(\delta + b)} \ . \tag{E.49}$$

Substituting (E.41), (E.46), and (E.47) into (E.10) gives

$$S_{33} = \frac{S_{11}}{\gamma}\left(\frac{B_{33}}{B_{11}}\right)^{1/2}\frac{1}{\cos(\delta + b)} \ . \tag{E.50}$$

Using (E.33) gives

$$\frac{\vec{\nabla} p_o}{p_o} - \gamma\,\frac{C_2}{C_1}\,\frac{\vec{\nabla}\theta_o}{\theta_o} = \frac{\vec{\nabla} p_o}{p_o} + \frac{\gamma}{\gamma - 1}\,\frac{\vec{\nabla}\theta_o}{\theta_o} = \frac{2\gamma}{\gamma - 1}\,\frac{\vec{\nabla} C}{C} \ . \tag{E.51}$$

Using (E.51) and (E.2) in (B.33) gives

$$\frac{\vec{\nabla}(S_{11}\,S_{33})}{S_{11}\,S_{33}} = -\frac{\vec{\nabla} p_o}{p_o} - \frac{\vec{\nabla} C}{C} - \frac{2G}{\gamma - 1}\,\frac{\vec{\nabla} C}{C} \ . \tag{E.52}$$

Taking the derivative of (E.40) gives

$$d\,\delta = \frac{2}{\sqrt{\gamma-1}}\frac{dC}{C} \quad . \tag{E.53}$$

Substituting (E.47) and (E.53) into (E.52) gives

$$\frac{\vec{\nabla}(S_{11}\ S_{33})}{S_{11}\ S_{33}} = -\frac{\vec{\nabla}p_o}{p_o} - \frac{\vec{\nabla}C}{C} - \tan(\delta+b)\ \vec{\nabla}\ \delta \quad . \tag{E.54}$$

Equation (E.54) can be written as a perfect differential:

$$\vec{\nabla}\ell n(S_{11}\ S_{33}) = -\vec{\nabla}\ell n(p_o/p_{oo}) - \vec{\nabla}\ell n(C/C_{oo}) + \vec{\nabla}\ell n\ \cos(\delta+b) \ . \tag{E.55}$$

We can take

$$\ell n(S_{11}\ S_{33}) = -\ell n(p_o/p_{oo}) - \ell n(C/C_{oo}) + \ell n\ \cos(\delta+b) \ , \tag{E.56}$$

where p_{oo} and C_{oo} are constants of integration. Equation (E.56) is equivalent to

$$S_{11}\ S_{33} = \frac{\cos(\delta+b)}{(p_o/p_{oo})(C/C_{oo})} \ . \tag{E.57}$$

Combining (E.57) and (E.50) gives

$$S_{11} = \gamma^{1/2}(B_{11}/B_{33})^{1/4}\frac{\cos(\delta+b)}{(p_o/p_{oo})^{1/2}\ (C/C_{oo})^{1/2}} \tag{E.58}$$

and

$$S_{33} = (B_{33}/B_{11})^{1/4}\frac{1}{\gamma^{1/2}(p_o/p_{oo})^{1/2}\ (C/C_{oo})^{1/2}} \ . \tag{E.59}$$

We have therefore

$$\frac{\vec{\nabla}S_{33}}{S_{33}} = -\frac{1}{2}\frac{\vec{\nabla}p_o}{p_o} - \frac{1}{2}\frac{\vec{\nabla}C}{C} \tag{E.60}$$

so that (B.28) becomes

$$\vec{k}_c = - \frac{2-\gamma}{2\gamma} \frac{\vec{\nabla} p_o}{p_o} - \frac{1}{2} \frac{\vec{\nabla} C}{C} \quad . \tag{E.61}$$

Substituting (E.41), (E.46), and (E.47) into (E.6) gives

$$C_1 = \gamma \frac{B_{11}}{S_{11}^{\;2}} \cos^2(\delta+b) \quad . \tag{E.62}$$

From (E.41), we have

$$S_{12} = S_{11}(B_{22}/B_{11})^{1/2} \cos\delta/\cos(\delta+b) \; . \tag{E.63}$$

Substituting (E.50), (E.62), and (E.63) into (B.20) gives

$$B_z = B_{33}^{\;1/2} \; C \begin{pmatrix} 0 & 0 & B_{11}^{\;1/2}\cos(\delta+b) & 0 \\ 0 & 0 & B_{22}^{\;1/2}\cos\delta & 0 \\ B_{11}^{\;1/2}\cos(\delta+b) & B_{22}^{\;1/2}\cos\delta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{E.64}$$

Similarly, from (B.21),

$$B_x = B_{33}^{\;1/2} \; C \begin{pmatrix} 0 & 0 & 0 & B_{11}^{\;1/2}\cos(\delta+b) \\ 0 & 0 & 0 & B_{22}^{\;1/2}\cos\delta \\ 0 & 0 & 0 & 0 \\ B_{11}^{\;1/2}\cos(\delta+b) & B_{22}^{\;1/2}\cos\delta & 0 & 0 \end{pmatrix} . \tag{E.65}$$

Using (E.2), (E.47), and (E.33) in (B.26) gives

$$\vec{\alpha} = \vec{k}_c - \frac{\tan(\delta+b)}{\sqrt{\gamma-1}} \frac{\vec{\nabla}\theta_o}{\theta_o} \quad . \tag{E.66}$$

Using (E.1), (E.46), and (E.33) in (B.27) gives

$$\vec{\beta} = \vec{k}_c - \frac{\tan\delta}{\sqrt{\gamma-1}} \frac{\vec{\nabla}\theta_o}{\theta_o} \quad . \tag{E.67}$$

Substituting (E.50), (E.62), (E.63), (E.66), and (E.67) into (B.25) gives

$$B = B_{33}^{1/2} \, C \begin{pmatrix} 0 & E^T \\ -E & 0 \end{pmatrix} , \tag{E.68}$$

where

$$E \equiv \begin{pmatrix} B_{11}^{1/2}\left(-k_{cz} \cos(\delta+b) + \frac{\sin(\delta+b)}{\sqrt{\gamma-1}} \frac{\partial\theta_o/\partial z}{\theta_o}\right) & B_{22}^{1/2}\left(-k_{cz} \cos\delta + \frac{\sin\delta}{\sqrt{\gamma-1}} \frac{1}{\theta_o} \frac{\partial\theta_o}{\partial z}\right) \\ B_{11}^{1/2}\left(-k_{cx} \cos(\delta+b) + \frac{\sin(\delta+b)}{\sqrt{\gamma-1}} \frac{1}{\theta_o} \frac{\partial\theta_o}{\partial x}\right) & B_{22}^{1/2}\left(-k_{cx} \cos\delta + \frac{\sin\delta}{\sqrt{\gamma-1}} \frac{1}{\theta_o} \frac{\partial\theta_o}{\partial x}\right) \end{pmatrix} . \tag{E.69}$$

From (E.13) and (E.42), we have

$$B_{12} = B_{21} = (B_{11} B_{22})^{1/2} \cos b \; . \tag{E.70}$$

Substituting (E.70) and (B.32) into (B.15) gives

$$B_t = \begin{pmatrix} B_{11} & (B_{11} B_{22})^{1/2}\cos b & 0 & 0 \\ (B_{11} B_{22})^{1/2}\cos b & B_{22} & 0 & 0 \\ 0 & 0 & B_{33} & 0 \\ 0 & 0 & 0 & B_{33} \end{pmatrix} . \tag{E.71}$$

There is no loss in generality (and no change in the dispersion relation) if we take

$$B_{11} = B_{22} = B_{33} = 1 \; , \tag{E.72}$$

and this leads to some simplification in (E.64), (E.65), (E.68), (E.69), and (E.71).

From (E.1) and (E.46), we have

$$\frac{S_{22}}{S_{12}} = \frac{F+1}{\gamma} = \frac{1 - \sqrt{\gamma-1}\,\tan\delta}{\gamma} = \cos(\delta+\varepsilon)\cos\varepsilon/\cos\delta \tag{E.73}$$

where ε is a constant defined by

$$\cos^2\varepsilon \equiv 1/\gamma \; . \tag{E.74}$$

From (E.2), (E.47), and (E.74) we have

$$\frac{S_{21}}{S_{11}} = \frac{1-G}{\gamma} = \frac{1 - \sqrt{\gamma-1}\,\tan(\delta+b)}{\gamma} = \cos(\delta+b+\varepsilon)\cos\varepsilon/\cos(\delta+b) \; . \tag{E.75}$$

From (E.3), (E.48), and (E.74) we have

$$\Delta = \frac{S_{11}\, S_{12}\, \sin\varepsilon\, \cos\varepsilon\, \sin b}{\cos\delta\, \cos(\delta+b)} = (B_{22}/B_{11})^{1/2}\, S_{11}^{\;2}\, \frac{\sin\varepsilon\, \cos\varepsilon\, \sin b}{\cos^2(\delta+b)} \; . \tag{E.76}$$

Substituting (B.13), (B.31), (E.73), (E.75), (E.76), (E.63), (E.50), and (E.58) and (A.10) into (B.4) gives

$$\phi = \begin{pmatrix} (p_1\, \cos(\delta+\varepsilon) - \rho_1\, \cos\delta/\cos\varepsilon)/(B_{11}^{\;1/2}\, \sin\varepsilon\, \sin b) \\ (-p_1\, \cos(\delta+b+\varepsilon) + \rho_1\, \cos(\delta+b)/\cos\varepsilon)\, \dfrac{B_{22}^{\;-1/2}}{\sin\varepsilon\, \sin b} \\ \dfrac{u_z}{C}\, B_{33}^{\;-1/2}\, \gamma \\ \dfrac{u_x}{C}\, B_{33}^{\;-1/2}\, \gamma \end{pmatrix} B_{11}^{\;1/2}\, \frac{\cos(\delta+b)}{S_{11}} \tag{E.77}$$

where

$$\frac{\cos(\delta+b)}{S_{11}} = \gamma^{-1/2}(B_{33}/B_{11})^{1/4}(p_o/p_{oo})^{1/2}(C/C_{oo})^{1/2} \; . \tag{E.78}$$

APPENDIX F.--The Difference Between Bicharacteristic Rays and Geometrical Acoustic-Gravity Rays

Sometimes there is confusion between the bicharacteristic rays constructed to solve partial differential equations by the method of characteristics and ordinary geometrical rays calculated from a dispersion relation such as that for acoustic-gravity waves. I have not found the difference explicitly explained in the literature, and because much of the same terminology and methods are used for both, I think it is useful to explain the difference here. In working on the present problem in finding a unique dispersion relation for acoustic-gravity waves, I had the impression for some time that the two were the same and I think possibly others may have the same confusion. Part of the confusion may arise because there is a class of partial differential equations for which the two are the same.

Definitions and uses of characteristics and bicharacteristic rays may be found in standard textbooks (e.g., Courant and Hilbert, 1962; Garabedian, 1964). Here I give only brief definitions and try to emphasize how they differ from the usual geometrical rays.

A characteristic surface for a partial differential equation or for a system of partial differential equations is a surface where solutions of the differential equation can be discontinuous. Because of that property, characteristic surfaces cannot be used as surfaces for specifying initial-value data to generate a solution of the differential equation. Physically, a characteristic surface is the surface of a wave front in space-time. A wave front is not in general the same as a surface of constant phase in a finite-frequency wave except for a particular class of partial differential equations. At a wave front, the field can be discontinuous, and the discontinuities propagate with the wave front.

A bicharacteristic ray is a path within the characteristic surface along which properties of the initial-value data propagate. A bicharacteristic ray is analogous to the usual kind of geometrical ray. The same terminology is used to describe them, and the method for calculating them is the same. However only for a particular class of partial differential equations are they the same.

Bicharacteristic rays and the usual geometrical rays are both calculated using Hamilton's equations, but the difference is in the Hamiltonian used. For a first-order system of partial differential equations such as (3.1), only the terms involving derivatives of the wave variables are considered in calculating bicharacteristic rays, not terms that are simply proportional to wave variables. Thus, the final term on the left of (3.1) is not considered when calculating bicharacteristic rays, although it is considered when calculating the usual geometrical rays.

Thus, to calculate the usual geometrical rays for acoustic-gravity waves, one uses Hamilton's equations with the dispersion relation (4.2) for a Hamiltonian

$$H = \left| B_t \Omega - B_z k_z - B_x k_x - i B \right| , \tag{F.1}$$

whereas to calculate bicharacteristic rays, one uses Hamilton's equations with

$$H = \left| B_t \Omega - B_z k_z - B_x k_x \right| \tag{F.2}$$

for a Hamiltonian. Except for some factors, (F.2) is equivalent to

$$H = \Omega^2/C^2 - k^2 , \tag{F.3}$$

which can be verified by substituting (3.6) through (3.8) [or (B.15), (B.20), and (B.21)] into (F.2). Equation (F.3) implies that information about initial data travels along bicharacteristic rays at the local sound speed. Bicharacteristic rays are independent of the wave variables used to express the differential equation under consideration, which is not true of the rays calculated from (F.1) as a Hamiltonian.

The dispersion relation for characteristics [setting (F.3) to zero] corresponds to that for acoustic-gravity waves (4.3) in the limit of infinite frequency and infinite wave number, as would be expected for the propagation of discontinuities.

Clearly, the special class of differential equations for which bicharacteristic rays are the same as the usual geometrical rays are those (in the case of systems of first-order equations) in which there are no terms that contain the wave variables undifferentiated. The first-order system

$$B_t \, D\phi/Dt + B_z \, \partial\phi/\partial z + B_x \, \partial\phi/\partial x = 0 \tag{F.4}$$

is in this special class, as are pure acoustic waves (without an acoustic cutoff frequency). In general, waves without dispersion (phase velocity independent of frequency) are in this class.

APPENDIX G.--Notation

A	the matrix defined in (A.13)
A_x	the matrix defined in (A.12)
A_z	the matrix defined in (A.11)
B	the matrix defined in (B.11), calculated as (B.25), then (3.9)
B_{ij}	components of B_t
B_t	the matrix defined in (B.15), then calculated as (E.71) and finally as (3.6)
B_x	the matrix defined in (B.21), then calculated as (E.65), and finally as (3.8)
B_z	the matrix defined in (B.20), then calculated as (E.64), and finally as (3.7)
b	defined in (E.42)
C	$= (\gamma p_o/\rho_o)^{1/2}$, sound speed
C_o	constant with the units of speed
C_{oo}	constant with the units of speed
C_1	defined in (B.19), then calculated as (E.6), and finally as (E.62)
C_2	defined in (B.29), then calculated as (E.7)
D	defined in (E.13)
D/Dt	substantial derivative defined in (3.2)
d	defined in (B.18), and calculated as (E.4)
E	defined in (E.69), then simplified following (3.9)
F	defined in (E.1), then calculated as (E.46)
G	defined in (E.2), then calculated as (E.47)
$\vec{g}$	vector acceleration due to gravity
H	pressure scale height $\left(1/H = -\frac{1}{p_o}\frac{\partial p_o}{\partial z}\right)$ (in Appendix F, the Hamiltonian)
$\vec{k}$	wave vector ($\|k\|$ = the wave number = $2\pi/\lambda$, $\vec{k}$ points in the wave normal direction)
k_a	parameter having the dimensions of inverse length appearing in the dispersion relation for acoustic-gravity waves (4.4), and defined in (4.8)
$\vec{k}_c$	vector having the dimensions of inverse length appearing in the dispersion relation for acoustic-gravity waves (4.4), and given in (4.6), originally defined in (B.28)

$\vec{k}_g$	vector having the dimensions of inverse length appearing in the dispersion relation for acoustic-gravity waves (4.4), and defined in (4.5)
k_x	x component of $\vec{k}$
$\hat{k}_x$	the operator defined in (B.1c)
k_y	y component of $\vec{k}$
k_z	z component of $\vec{k}$
$\hat{k}_z$	the operator defined in (B.1b)
p	pressure
p_o	background pressure
p_{oo}	constant with the units of pressure
p_1	dimensionless pressure wave variable defined in (A.1a)
R	defined in (E.14)
S	transformation matrix defined in (B.4), and given by (B.12)
S_{ij}	components of S
t	time
$\vec{U}$	wind velocity
$\vec{U}_o$	background wind (of the mean flow)
$\vec{u}$	dimensionless wind wave variable defined in (A.1c)
W	defined in (E.5), then calculated as (E.41)
x,y,z	position variables
$\vec{\alpha}$	defined in (B.26), then calculated as (E.66)
α_x	x component of $\vec{\alpha}$
α_z	z component of $\vec{\alpha}$
$\vec{\beta}$	defined in (B.27), then calculated as (E.67)
β_x	x component of $\vec{\beta}$
β_z	z component of $\vec{\beta}$
Γ	Eckart's coefficient, defined in (1.8)
$\vec{\Gamma}$	vector generalization of Eckart's coefficient, defined in (4.7)
γ	ratio of specific heat at constant pressure to that at constant density
Δ	defined in (B.14), then calculated as (E.3), and finally as (E.76)
δ	defined in (3.5)
ε	defined in (3.4)
λ	wavelength

ϕ — the wave variable column vector defined in (B.4), then calculated as (E.77), and finally as (3.3)

ψ — the wave variable column vector defined in (A.10)

ρ — density

ρ_o — background density

ρ_1 — dimensionless density wave variable defined in (A.1b)

θ — potential temperature

θ_o — potential temperature of the background

Ω — intrinsic frequency, $= \omega - \vec{k} \cdot \vec{U}_o$

$\hat{\Omega}$ — the operator defined in (B.1d)

ω — radian frequency

$\hat{\omega}$ — the operator defined in (B.1a)

ω_a — acoustic cutoff frequency, defined in (1.5)

ω_B — Brunt-Väisälä frequency, defined in (1.6) and generalized in (1.14)

ω_1 — parameter having the dimensions of frequency appearing in the barotropic dispersion relation for acoustic-gravity waves (1.1)

ω_2 — parameter having the dimensions of frequency appearing in the barotropic dispersion relation for acoustic-gravity waves (1.1)