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DEPARTMENT OF COMMERCE
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ABSTRACT

It is shown that a pulse undergoes a frequency shift while propagating through a medium whose losses depend on frequency. The amount of the frequency shift is proportional to the imaginary part of the complex group refractive index and the distance traveled in the medium. No new frequencies are generated that were not in the original signal; the signal energy is merely "redistributed" by selective absorption. In an inhomogeneous medium, the frequency shift is proportional to the imaginary part of the complex group path. Furthermore, a beam propagating in a medium where the losses depend on the direction of propagation will be bent toward the direction of least attenuation. Again, no new components in the angular spectrum are generated that were not in the original beam. The signal energy is merely redistributed by selective absorption.

1. SUMMARY

When a wave pulse propagates through a medium in which the attenuation depends on the frequency, the pulse spectrum is attenuated faster on one side than on the other. The farther the pulse propagates into the medium, the more the frequency is shifted.

This selective absorption might be interpreted as an effective Doppler shift of the pulse, even though the medium is time-independent. The effect depends partly on the time-dependence of the pulse; the effect vanishes for a monochromatic wave.

The imaginary part of the complex group refractive index

$$n' = \frac{d}{d\omega}(n\omega) = \frac{d}{d\omega}((\mu - i\chi)\omega)$$

is proportional to the frequency derivative of the absorption coefficient $\frac{\omega}{c}\chi$, and is thus a quantitative measure of this effect.

In an inhomogeneous medium, the imaginary part of the complex group path

$$P' = \frac{d}{d\omega}(\omega \int n \cdot ds)$$

measures the cumulative effect of the frequency shift just as the real part of P' gives the time of travel of the pulse.

Analogously, a beam that propagates in a medium that has angle-dependent losses will have its wave-normal direction shifted. This would be interpreted as an effective bending of the beam, even though the medium is homogeneous. The effect depends partly on the spatial inhomogeneity of the beam; the effect vanishes for a plane wave.

As a particular example, the frequency shift of a time-harmonic wave modulated by a Gaussian-shaped envelope is inversely proportional to the square of the pulse length in addition to being proportional to the imaginary part of the complex group refractive index and the distance traveled through the medium.

It is important to notice that all of the above effects are linear. That is, no <u>new</u> frequencies are generated that were not in the original signal. The signal energy is merely "redistributed" in frequency by selective absorption.

John Bennett (1974, page 1583) announced these results as a private communication from M. Jones. Here I document the results in full.

2. BACKGROUND

The propagation of pulses in lossless media is well understood (e.g., Stratton, 1941; Hines, 1951a; Panofsky and Phillips, 1955; Jeffreys and Jeffreys, 1956; Brillouin, 1960; Whitham, 1960; Budden, 1961; Jackson, 1962; Lighthill, 1965), including both isotropic and anisotropic media, in cases of anomalous dispersion where the group velocity no longer gives the velocity of energy flow, and the behavior of the precursor. For cases of normal dispersion, the velocity of energy flow is the same as the group velocity,

$$v_g = \frac{\partial \omega}{\partial \vec{k}} \quad . \qquad (1)$$

(Here, ω is the radian frequency, $|\vec{k}| = 2\pi/\lambda$ is the wave number, and $\vec{k}$ is normal to the phase fronts.)

In general, a pulse changes shape as it propagates. There is, then, no single velocity that describes its motion. Under conditions of normal dispersion, the group velocity (1) gives the velocity of propagation of the pulse peak. However, the complete calculation of the propagation of a pulse requires a calculation of the change in shape as well. The method is straightforward, although often difficult in practice. A Fourier transform is performed on the initial pulse, the propagation of each Fourier component is found, and an inverse Fourier transform is taken to find the amplitude and phase of the signal as a function of time and position. Under conditions of normal dispersion, the inverse transform can often be evaluated by asymptotic methods (Jeffreys and Jeffreys, 1956; Price, 1968; Felsen, 1969). With care, the results are unambiguous.

Many investigators have calculated the propagation of pulses of various shapes under various situations and calculated various aspects of the resulting distortion to the pulse (e.g., Felsen, 1969; Nicolis, 1967 (for a bell-shaped pulse); Wait, 1969a (exponential and bell-shaped pulse); Wait, 1969b (an example with a bell-shaped pulse that can be evaluated exactly); Vogler, 1969 (Gaussian-shaped pulse); Wait, 1970a (Gaussian-shaped pulse); Wait, 1970b (propagation of a Gaussian-shaped pulse over flat, imperfectly conducting earth); Wait, 1970c (Gaussian-shaped pulse); Vogler, 1970a, 1970b (Gaussian-shaped pulse)).

The propagation of pulses in lossy media is less well understood. The main difficulty arises from trying to interpret the significance of the group velocity in (1) when it takes on complex values. Booker (1939) suggested that, since a pulse is made up of a spectrum of frequencies, a complex frequency might be found in the spectrum that would make the group velocity real. Hines (1951 a,b) showed that the velocity of the spatial maximum and the time maximum of a pulse do not usually coincide in a lossy medium. He argued that the velocity of the time maximum is more closely related to what is actually measured. He showed that

$$u = \frac{c}{\mathrm{real}(n')} \tag{2}$$

gives the velocity of the time maximum of a pulse, where

$$n' = \frac{d}{d\omega}(n\omega) = \frac{d}{d\omega}((\mu - i\chi)\omega) \tag{3}$$

is the complex group refractive index (Budden, 1961). (Here n is the complex phase refractive index [with real part μ, and imaginary part $-\chi$], and c is the free-space speed of light.)

Furutsu (1952), apparently unaware of Hines' work, argued that the concept of group velocity and wave path do not exist in a lossy medium.

Suchy (1972 a,b,c, 1974) uses different arguments from Hines' to advocate real $(\partial\omega/\partial\vec{k})$ as a more appropriate group velocity. Suchy (1972a) argues that imaginary $(\partial\omega/\partial\vec{k})$ has no apparent physical meaning. Suchy (1974) interprets it in terms of a directional derivative of the real part of the wave number.

The difficulty with interpreting a complex group velocity is similar to that of interpreting the complex ray direction that occurs for ray tracing in complex space (Poeverlein, 1962; Budden and Jull, 1964; Jones, 1970; Budden and Terry, 1971; Keller and Streifer, 1971; Bertoni et al., 1971; Deschamps, 1972; Kratsov et al., 1974; Wang and Deschamps, 1974; Bennett, 1974; Connor and Felsen, 1974).

The seemingly peculiar behavior of wave propagation in dissipative media has led various investigations into various aspects. For example, Hines (1951 c,d) and Arsaev and Kinber (1968) consider the direction of energy flux in dissipative media. Poeverlein (1962) pointed out that absorption of waves can be represented by complex propagation vectors. Storey and Roehner (1970) and Roehner (1971) consider the direction of stationary phase for a beam of waves in an absorbing medium. Bertoni et al., (1971) consider the nonlocal nature of propagation in lossy media. Batorsky and Felsen (1971) consider complex waveguide modes. Denman and Buch (1973) derive a Hamiltonian for dissipative systems.

The practical calculation of the propagation of pulses in lossy media is not hindered by difficulties in interpreting a group velocity that takes on complex values. The method is the same as for lossless media, and the results for the amplitude and phase of the resulting signal as a function of time and position are just as unambiguous as for the lossless case.

Several investigators have calculated various aspects of pulse distortion in lossy media. Vogler (1969) calculated envelope shape distortion of a pulse propagating in a lossy troposphere. Vogler (1970a and b) considered pulse distortion further. Wait (1970b) considered pulse distortion for propagation over a flat, imperfectly-conducting earth.

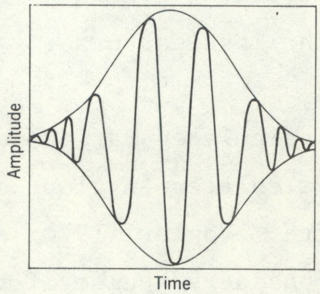
One particular aspect of the distortion of a pulse as it propagates through a lossy medium, that of the frequency shift of the carrier and the relation to the frequency dependence of losses and to the imaginary part of the complex group refractive index, does not seem to have received attention. John Bennett (1974), however, reported the result in the special issue of Proceedings IEEE on rays and beams, referring to some of my unpublished work. In the same issue, Connor and Felsen (1974), apparently independently, derive the same result for the case of a Gaussian-shaped pulse, but do not mention the relation to the imaginary part of the group refractive index.

As we shall see, the above frequency shift is proportional to the product of the imaginary part of the group refractive index and the distance traveled in the medium. This discovery complements the already-known relation between the time of travel of a pulse and the product of the real part of the group refractive index and the distance traveled in the medium.

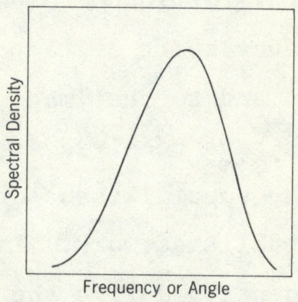
Section 3 illustrates the main results in a qualitative way. Section 4 establishes a quantitative relation between the imaginary part of n' and the frequency dependence of losses, and Section 5 derives a relation between the frequency dependence of losses and a frequency shift in the spectrum of a pulse. Section 6 applies the results to the case of a Gaussian-shaped pulse, Section 7 generalizes to inhomogeneous media, and Section 8 discusses the propagation of beams.

3. QUALITATIVE SUMMARY OF RESULTS

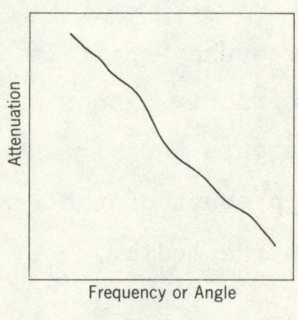
Suppose a wave pulse



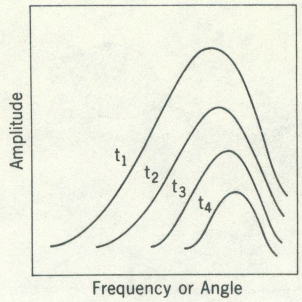
has a certain frequency and angular spectrum.



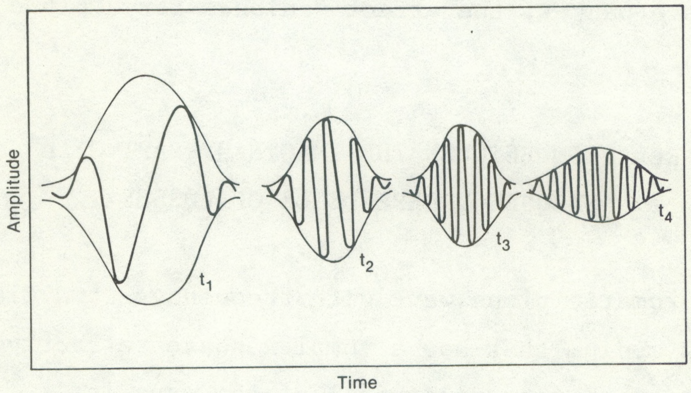
When such a pulse propagates through a medium in which the attenuation is a function of frequency and/or angle



the pulse spectrum will get attenuated (or "worn away") faster on one side than the other.



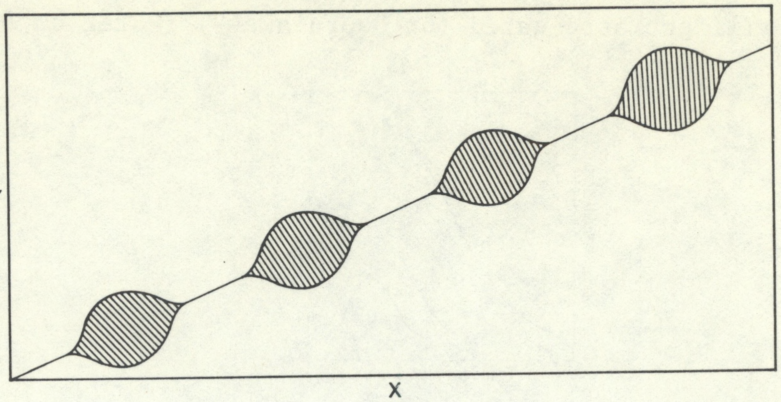
The result of frequency-dependent losses is that the farther the pulse propagates into the medium, the more the frequency gets shifted.



This selective absorption might be interpreted as an effective Doppler shift of the pulse, even though the medium is time-independent. The effect depends partly on the time dependence of the pulse; the effect vanishes for a monochromatic wave.

Such a "single pulse" Doppler shift has nothing to do with the Doppler shift measured by the pulse-to-pulse phase shift in a Doppler radar. This effect might influence range measurements in an FM-CW radar, however.

The result of angle-dependent losses is that the farther the wave packet propagates into the medium, the more the wave-normal direction gets shifted.



This could be interpreted as an effective bending of the wave packet, even though the medium is homogeneous. The effect depends partly on the spatial inhomogeneity of the wave packet; the effect vanishes for a plane wave.

4. THE RELATION BETWEEN THE IMAGINARY PART OF n' AND THE FREQUENCY DEPENDENCE OF LOSSES

Suppose a monochromatic plane wave with frequency ω travels in the z direction through a homogeneous medium that has a complex phase refractive index n. The time and space variation of the amplitude A of the wave is

$$A\, e^{i\phi} = A_o \exp(i\phi - \alpha z) = A_o \exp(i\omega t - i\,\frac{\omega n}{c}\, z) \tag{4}$$

where ϕ is the phase of the wave and α is the absorption coefficient.

To obtain an explicit expression for the frequency dependence of the phase and amplitude, let us take the derivative of the phase with respect to frequency,

$$\frac{d\phi}{d\omega} = t - \frac{z}{c}\,\mathrm{real}\,\frac{d}{d\omega}\,(n\omega) = t - \frac{z}{c}\,\mathrm{real}\,(n'), \tag{5}$$

and the logarithmic derivative of the amplitude with respect to frequency,

$$\frac{d}{d\omega}\,(\mathrm{Log}\,A) = -\,z\,\frac{d\alpha}{d\omega} = \frac{z}{c}\,\mathrm{imag}\,\frac{d}{d\omega}\,(n\omega) = \frac{z}{c}\,\mathrm{imag}\,(n'), \tag{6}$$

where we have used (3) for n'. Equation (5) leads to the well-known result (2) for the group velocity, while (6) gives the new result,

$$-\frac{d\alpha}{d\omega} = \frac{\mathrm{imag}\ (n')}{c} \quad . \tag{7}$$

This means that the imaginary part of the complex group refractive index is proportional to the frequency derivative of the absorption coefficient. Evidently, the group refractive index is real if and only if the absorption coefficient does not depend on frequency (at least at a given frequency).

5. THE FREQUENCY SHIFT OF A PULSE

Suppose we have a time-harmonic wave of frequency ω_o modulated by an envelope having one peak at $t = 0$:

$$m(t)\ \exp(i\omega_o t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(i\omega t)\ A(\omega)\ \exp(B(\omega))d\omega \tag{8}$$

where

$$A(\omega)\ \exp(B(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-i\omega t)\ m(t)\ \exp(i\omega_o t)dt \tag{9}$$

is the Fourier transform of $m(t)\ \exp(i\omega_o t)$ and has been divided into a slowly varying part $A(\omega)$ and a quickly varying part $\exp(B(\omega))$ to show its asymptotic behavior. Let the pulse propagate through a homogeneous medium that multiplies a monochromatic wave by

$$\exp(-\ i\ \frac{\omega}{c}\ \mu z)\ \exp(-\alpha z) \tag{10}$$

where μ is the real part of the phase refractive index, α is the absorption coefficient, and z is the distance traveled (along the wave normal direction) through the medium. The spectrum of the pulse at a point z in the medium is then

$$A(\omega)\ \exp(B(\omega) - i\,\frac{\omega}{c}\,\mu(\omega)\,z - \alpha(\omega)\,z)\ . \tag{11}$$

To find the dominant frequency ω_p in this spectrum, we set

$$\frac{d}{d\omega}\left|A(\omega)\ \exp(B(\omega) - \alpha(\omega)\,z)\right|_{\omega=\omega_p} = 0\ . \tag{12}$$

When the slowly varying term $A(\omega)$ is neglected, (12) gives

$$\text{real}\ \frac{d}{d\omega_p}\,B(\omega_p) - z\,\frac{d\alpha(\omega_p)}{d\omega_p} = 0\ . \tag{13}$$

We now expand $B(\omega)$ about $\omega = \omega_1$, where ω_1 is the dominant frequency of the transmitted pulse. That is, we expand about the point where

$$\text{real}\ \frac{d\,B(\omega_1)}{d\omega_1} = \text{real}\ B'(\omega_1) = 0\ . \tag{14}$$

(Notice that if B is an analytic function, the imaginary of the above is also zero, so that ω_1 is then a saddlepoint of the transmitted pulse.) (In many cases, $\omega_1 = \omega_o$.) The Taylor expansion gives

$$B(\omega) = B(\omega_1) + B'(\omega_1)(\omega-\omega_1) + \frac{1}{2}\,B''(\omega_1)(\omega-\omega_1)^2 + \ldots\ , \tag{15}$$

where primes indicate derivatives with respect to the argument. Substituting (15) into (13) gives

$$\text{real}\ B''(\omega_1)(\omega_p - \omega_1) - z\,\frac{d\,\alpha(\omega_p)}{d\,\omega_p} = 0\ . \tag{16}$$

We have neglected second and higher powers of $(\omega_p - \omega_1)$ to give the asymptotic high-frequency behavior. Solving (16) for the frequency shift, $(\omega_p - \omega_1)$, gives

$$\omega_p - \omega_1 = \frac{z}{\text{real}\ B''(\omega_1)}\ \frac{d\,\alpha(\omega_p)}{d\,\omega_p} = -\,\frac{z}{c}\,\frac{\text{imag}\ n'(\omega_p)}{\text{real}\ B''(\omega_1)}\ . \tag{17}$$

We have used (7) to combine with the result of the previous section. Thus, the medium shifts the frequency of the pulse by an amount proportional to the distance traveled in the medium and proportional to the frequency derivative of the absorption coefficient (and thus proportional to the imaginary part of the complex group refractive index). (It is probably accurate enough in most cases to use $n'(\omega_1)$ for $n'(\omega_p)$ in (17).)

The physical explanation for this frequency shift is clear. If the absorption depends on frequency then the medium will selectively absorb either the higher or the lower frequencies in the pulse spectrum, thus shifting the dominant frequency in the spectrum. It is also clear that this frequency shift should be proportional to z, the distance traveled in the medium. It is also reasonable that the frequency shift is larger, the larger the spectral width of the pulse, which is inversely proportional to the real part of B", the curvature of the spectrum peak.

6. EXAMPLE -- A GAUSSIAN-SHAPED PULSE

For a Gaussian-shaped pulse, let

$$m(t) = \exp(-\,(t/\tau)^2) \tag{18}$$

where τ is a measure of the pulse length.

(Although the Gaussian-shaped pulse is not realistic because it has no beginning, the propagation of the peak of a Gaussian-shaped pulse should be similar to that of realistic pulses with rounded peaks.)

Then from (9), the pulse shape in the frequency domain is

$$A(\omega)\,\exp(B(\omega)) = \frac{\tau}{\sqrt{2}}\,\exp\!\left(-\,\frac{\tau^2}{4}\,(\omega-\omega_o)^2\right) , \tag{19}$$

and thus

$$B(\omega) = -\,\frac{\tau^2}{4}\,(\omega-\omega_o)^2 \tag{20}$$

and

$$\omega_1 = \omega_o\,. \tag{21}$$

Substituting the above into (17) gives the frequency shift of the spectral peak

$$\omega_p - \omega_o = \frac{z}{c} \frac{\text{imag } n'(\omega_p)}{\tau^2/2} \quad . \tag{22}$$

Equation (22) shows that the frequency shift is larger for shorter pulse lengths (i.e., a wider pulse spectrum), as we expected.

7. INHOMOGENEOUS MEDIA

For inhomogeneous media, (10) can be replaced by the transfer function

$$a(\omega) \exp(- i \frac{\omega}{c} P(\omega)) \; , \tag{23}$$

where a is a slowly varying function of ω, and P is the phase path or phase integral (Budden, 1961, page 136). Thus, neglecting the slowly varying term $a(\omega)$, (16) would be replaced by

$$\text{real } B''(\omega_1)(\omega_p - \omega_1) + \frac{1}{c} \text{ imag } P'(\omega_p) = 0 \; , \tag{24}$$

where

$$P'(\omega) = \frac{d}{d\omega} (\omega \, P(\omega)) \tag{25}$$

is the group path or equivalent path (Budden, 1961, page 280). Solving (24) gives the frequency shift

$$\omega_p - \omega_1 = - \frac{\text{imag } P'(\omega_p)}{c \text{ real } B''(\omega_1)} \; . \tag{26}$$

Equation (26) is a general relation that applies to propagation through any medium that has a transfer function of the form (23).

For example, it would apply in the ray-theory approximation to waves propagating through an inhomogeneous anisotropic medium if we use

$$P = \int_{\substack{\text{ray} \\ \text{path}}} \underline{n} \cdot \underline{ds} \; . \tag{27}$$

P can be called the generalized or complex phase path (Budden, 1961, pages 173, 197, 507), $\underline{n}$ is a vector pointing in the wave-normal direction and having a value equal to the complex phase refractive index (Budden, 1961, page 531), s is the distance along the ray path, and $\underline{ds}$ is a vector pointing in the ray direction.

As another example, a full-wave reflection or transmission coefficient could be put into the form of (23). Then applying (26) would give the frequency shift of a pulse having a spectrum given by (9).

8. BENDING OF A BEAM IN A LOSSY HOMOGENEOUS MEDIUM

The results just presented show that a pulse undergoes a frequency shift while propagating through a medium whose losses depend on frequency. We ordinarily expect frequency shifts only in time-varying media, so obtaining a frequency shift in a stationary medium is surprising. Two conditions are necessary for this result:

1) A wave has a finite frequency spectrum.

2) The medium has frequency-dependent losses.

It is interesting to note that no <u>new</u> frequencies are generated that were not in the original signal. The signal energy is merely "redistributed" by selective absorption.

It is tempting to ask whether these results, derived for the frequency spectrum of a pulse, apply to the angular spectrum of a beam. More explicitly, will a beam bend in a homogeneous medium having losses that depend on the propagation direction of the beam? The improbable answer is yes, because components of the beam on one side of the angular spectrum will be attenuated faster than those on the other side as the beam travels through the medium, giving a curvature to the beam. However, the beam does not actually bend to directions where there was no wave energy to begin with.

Thus, in analogy to the above results, if the group velocity

$$\underline{u} = \frac{d\omega}{d\underline{k}} \tag{28}$$

is complex, then the medium has losses that depend on the direction of propagation, and a beam propagated in such a homogeneous medium will be bent toward the direction of lowest attenuation.

9. ACKNOWLEDGMENT

I thank T.M. Georges for helping me prepare this report.

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