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THE LONGITUDINAL-TRANSVERSE SPATIAL COHERENCE FUNCTION FOR A SPHERICAL WAVE PROPAGATING THROUGH HOMOGENEOUS ATMOSPHERIC TURBULENCE: IMPLICATIONS FOR RASS

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Richard J. Lataitis

ABSTRACT

The parabolic wave equation and the Bourret approximation are used to derive an expression for the second-order spatial coherence function of a spherical wave propagating through homogeneous atmospheric turbulence. Both the longitudinal and transverse coherence of the wave field are considered. In contrast with existing plane-wave results, the spherical-wave coherence normalized by its value in the absence of inhomogeneities depends not only on the transverse separation of the observation points but also on their mean transverse position. This inhomogeneity of the normalized second-order statistics disappears as the longitudinal separation between the observation planes tends to zero. The normalized coherence for a spherical wave can be expressed as the exponential of a complex factor, as in the plane wave case. The imaginary component of the exponent describes the turbulence induced excess phase-path difference between the two observation planes. The modulus of the normalized coherence can be used to estimate the longitudinal and transverse coherence lengths ρ_l and ρ_t, respectively. The longitudinal spherical-wave coherence length is found to be essentially identical to the

plane-wave result. For uniform isotropic turbulence described by a von Kármán-type refractive index spectrum, it can be approximated by $\rho_l \approx \rho_1 \rho_2/(\rho_1 + \rho_2)$, where $\rho_1 = (1.82 \times 10^{-2} k^2 L_o^{5/3} C_n^2)^{-1}$, $\rho_2 = (0.632\, k^{7/6} L\, C_n^2)^{-6/5}$, $k = 2\pi/\lambda$, λ is the wavelength, C_n^2 is the refractive index structure parameter, L is the propagation pathlength, and L_o is the turbulence outer scale. The transverse coherence length for a spherical wave is given by the well-known result $\rho_t = (0.546 k^2 L C_n^2)^{-3/5}$. For a nonuniform distribution of turbulence strength along the propagation path, C_n^2 in ρ_1 corresponds to a local value at a path position $z = L$; C_n^2 in ρ_2 represents a uniformly-weighted, path-integrated value; and C_n^2 in ρ_t corresponds to a path-integrated value with a $z^{5/3}$ weighting. For typical profiles of acoustic C_n^2 with height, the echo power associated with the radio acoustic sounding of temperature does not appear to be significantly affected by the reduction in longitudinal coherence of the acoustic wave.

1. INTRODUCTION

In the radio-acoustic sounding of temperature, a pulsed Doppler radar can be used to infer the speed of a vertically propagating acoustic wave, from which the virtual temperature at a given altitude can be obtained (e.g., May et al., 1990). Atmospheric turbulence distorts the acoustic wave, which can have a significant impact on the returned power. In theories describing the operation of a radio-acoustic sounding system (RASS), the effects of turbulence are usually described in terms of the second-order spatial coherence function (i.e. the second moment) of the acoustic field at the range of interest (Nalbandyan, 1976, 1977; Clifford and Wang, 1977, 1978; Clifford et al., 1978; Kon, 1984a,b). In the most general case this function describes both the longitudinal and transverse degradation of the acoustic wave, which are characterized in terms of a longitudinal and transverse coherence lengths ρ_l and ρ_t, respectively. In general, the relative size of ρ_l to the range resolution Δ, and ρ_t to the width W of the radar beam, at the range of interest determines the extent to which the returned power is affected by turbulence.

Many formulations for the transverse coherence function exist (e.g., Tatarskii, 1971) and the form for the transverse coherence length ρ_t is well known. Only two formulations, however, exist for the full second-order longitudinal-transverse coherence function. The earliest was due to Klyatskin (1970), who expressed his result as a convolution involving the transverse coherence function at a single range. The functional form of the transverse coherence function is such that this convolution is impossible to evaluate analytically, making it difficult to estimate ρ_l. The second formulation is due to Nalbandyan and Tatarskii (1977) who, using a different approach, arrived at a relatively simple expression for the full coherence function, from which they were able to obtain a reasonable estimate for ρ_l. It was their expression for the second-order coherence function that was used in earlier investigations of the effects of turbulence on a RASS.

There are two potential difficulties with applying the results of Nalbandyan and Tatarskii (1977) to the RASS problem. The first is that their formulation is limited to plane waves, whereas the acoustic wave in most RASS configurations approximates a spherical wave. The second is their assumption of a uniform distribution of turbulence strength along the propagation path. This is particularly inappropriate for vertical sounding of temperature because the turbulence strength is known to decrease dramatically with height, especially above the boundary layer. For this situation it is critical to know how the contributions from turbulence at different heights are weighted in the final expression for the coherence lengths.

Consider, for example, the expression for ρ_l implied in the result of Nalbandyan and Tatarskii (1977), that is, $\rho_l = (0.632\, k^{7/6} L\, C_n^{\,2})^{-6/5}$, where, for the RASS problem, $k = 2\pi/\lambda$ is the acoustic wavenumber, λ the corresponding wavelength, L the sounding range, and $C_n^{\,2}$ the acoustic refractive index structure parameter. If we assume that the turbulence is uniform (i.e. $C_n^{\,2}$ is constant with height), we need to decide on an appropriate value of $C_n^{\,2}$. Brown and Clifford [1976, Eq. (17)] present a typical acoustic $C_n^{\,2}$ profile for convective midday conditions. If we choose $C_n^{\,2} = 5.7 \times 10^{-6}\, m^{-2/3}$, which is characteristic of conditions roughly 2m above the surface, and use $\lambda = 37.5$ cm (corresponding to a radar frequency of 400 MHz) and $L = 1$ km, we obtain $\rho_l \approx 17m$. On the other hand if we choose $C_n^{\,2} = 3.8 \times 10^{-7}\, m^{-2/3}$, which is more representative of values in the upper boundary layer at heights of approximately 1 km, we obtain $\rho_l \approx 430$ m. Assuming that the range resolution $\Delta \approx 150$ m, it is clear that any conclusion regarding the impact of the turbulence-induced longitudinal degradation of the acoustic wave on the echo power depends critically on the choice of $C_n^{\,2}$. A more careful examination of the development of Nalbandyan and Tatarskii (1977) indicates that $C_n^{\,2}$ in their expression for ρ_l represents a uniformly weighted path-integrated value. Using the same parameters as above, and the $C_n^{\,2}$ profile presented by Brown and Clifford (1976), we obtain $\rho_l \approx 370$ m $> \Delta$, indicating that, for this example, the reduction in the longitudinal coherence of the acoustic wave has little effect on the RASS echo power. The corresponding path weighting function for a spherical wave is not easy to glean from the work of Nalbandyan and Tatarskii (1977), although physical reasoning suggests that it is also uniform.

In this report we extend the plane wave formulation for the longitudinal-transverse coherence function presented by Nalbandyan and Tatarskii (1977) to include spherical waves. Particular emphasis is placed on determining the appropriate path-weighting of C_n^2 in the expression for ρ_l. The development is based on the scalar parabolic approximation to Maxwell's equations, and is therefore appropriate only for electromagnetic waves in the quasi-optical limit. We note, however, that results derived from this starting point are often a good approximation to those based on the corresponding hydrodynamic equations describing the propagation of acoustic waves through a turbulent medium, provided the appropriate form of the refractive index spectrum is used in the final result (Clifford and Brown, 1970; Tatarskii, 1971). We therefore assume that the form of the coherence function obtained by this approach is valid not only for electromagnetic waves, but is reasonably accurate for acoustic waves as well.

2. THEORY

We want to obtain an expression for the second-order spherical-wave coherence function

$$\Gamma_2(\vec{\rho}_1, z_1; \vec{\rho}_2 z_2) = \langle \gamma_2(\vec{\rho}_1, z_1; \vec{\rho}_2, z_2) \rangle = \langle U(\vec{\rho}_1, z_1)\, U^*(\vec{\rho}_2, z_2) \rangle \,, \tag{1}$$

where $U(\vec{\rho}_i, z_i)$ describes the field at a range z_i and transverse position $\vec{\rho}_i$, and the angle brackets identify an ensemble average. Using the scalar parabolic approximation for the wave equation and the Bourret approximation (Bourret, 1962), we follow the development of Nalbandyan and Tatarskii (1977) to obtain the following result for the governing integro-differential equation of the coherence function Γ_2:

$$2ik \frac{\partial \Gamma_2}{\partial z} + 2 \vec{\nabla}_{\vec{R}} \cdot \vec{\nabla}_{\vec{\rho}} \Gamma_2 + 2k^2 \langle \delta n \, \gamma_2 \rangle = 0, \tag{2a}$$

where $\Gamma_2 = \Gamma_2(\vec{\rho}, \xi, \vec{R}, z)$, $\vec{\rho} = \vec{\rho}_1 - \vec{\rho}_2$, $\xi = z_1 - z_2$, $\vec{R} = (\vec{\rho}_1 + \vec{\rho}_2)/2$, $z = (z_1 + z_2)/2$,

$$\begin{aligned} \delta n &= n(\vec{\rho}_1, z_1) - n(\vec{\rho}_2, z_2) \\ &= n\left(\vec{R} + \frac{\vec{\rho}}{2}, z + \frac{\xi}{2}\right) - n\left(\vec{R} - \frac{\vec{\rho}}{2}, z - \frac{\xi}{2}\right), \end{aligned} \tag{2b}$$

$\gamma_2 = \gamma_2(\vec{\rho}, \xi, \vec{R}, z)$,

$$\begin{aligned} \langle \delta n \, \gamma_2 \rangle = ik \int_{\xi/2}^{z} dz' \int d^2\vec{\rho}\,' \int d^2\vec{R}' \; G(\vec{\rho} - \vec{\rho}\,', \vec{R} - \vec{R}', z - z') \\ \times \langle \delta n(\vec{\rho}\,', \xi, \vec{R}', z') \, \delta n(\vec{\rho}, \xi, \vec{R}, z) \rangle \, \Gamma_2(\vec{\rho}\,', \xi, \vec{R}', z'), \end{aligned} \tag{2c}$$

and G is the Green's function for (2a) defined by

$$G(\vec{\rho}, \vec{R}, z) = \frac{k^2}{4\pi^2 z^2} \exp(ik\vec{\rho} \cdot \vec{R}/z). \tag{2d}$$

The parabolic approximation used in deriving (2) requires that several conditons be satisfied. These were discussed by Tatarskii (1971) and summarized by Strohbehn (1978).

The most fundamental is that the propagation wavelength be sufficiently short so that the scattering is confined to a narrow angle about the propagation direction (i.e. $\lambda << l_o$, where l_o is the inner scale of turbulence). In addition, use of the Bourret approximation requires that $(k\,L_c\,\sigma_n)^2 << 1$, where L_c is the correlation radius of the refractive index fluctuations and $\sigma_n{}^2$ is the variance of the refractive index fluctuations about a mean value of unity (Rytov et al., 1989). This restriction can be rewritten by using the approximate relations $L_c \sim L_o$ and $\sigma_n{}^2 \approx C_n{}^2 L_o^{2/3}$ (Strohbehn, 1978), which yields $k^2 L_o^{8/3} C_n{}^2 << 1$.

To evaluate (2a) we assume that the function Γ_2 in (2c) is sufficiently smooth relative to the other terms in the integrand that it can be evaluated at the coordinates $\vec{\rho}' = \vec{\rho}, \vec{R}' = \vec{R}$, and $z' = z$ and factored out of the integral. In this limit we obtain the following expression for the governing differential equation:

$$\frac{\partial \Gamma_2}{\partial z} - \frac{i}{k}\,\vec{\nabla}_{\vec{R}} \cdot \vec{\nabla}_{\vec{\rho}}\,\Gamma_2 + k^2 H\,\Gamma_2 = 0\,, \tag{3a}$$

where

$$H = H(\vec{\rho}, \xi, \vec{R}, z) = \int_{\xi/2}^{z} dz' \int d^2\vec{\rho}\,' \int d^2\vec{R}'\, G(\vec{\rho} - \vec{\rho}\,', \vec{R} - \vec{R}', z - z')$$
$$\times \langle\, \delta n(\vec{\rho}\,', \xi, \vec{R}', z')\, \delta n(\vec{\rho}, \xi, \vec{R}, z) \rangle\,. \tag{3b}$$

Nalbandyan and Tatarskii (1977) suggest that this is a valid procedure provided

$0.5\, k^{7/6} L_0^{11/6} C_n{}^2 << 1$, that is provided the intensity fluctuations of a wave propagating a distance on the order of the outer-scale are small. Equation (3b) has been evaluated in the Appendix. The result is given by

$$H(\vec{\rho}, \xi, z) \approx 2\pi \int d^2\vec{K}\, \Phi_n(\vec{K}, z) \left\{ 1 - \exp\left[-i\left(\vec{K}\cdot\vec{\rho} + K^2 \frac{\xi}{2k} \right) \right] \right\}, \tag{4}$$

where $\Phi_n(\vec{K}, z)$ is the three-dimensional refractive index spectrum evaluated at the transverse wavenumber $\vec{K}$ and mean path position z. The integrals in (3b) were evaluated by using the Markov approximation, that is, by assuming the fluctuations in the refractive index were δ-correlated in the direction of propagation. The impact of this assumption on the solution for the second-order coherence function was evaluated by Tatarskii (1971) and was summarized by Strohbehn (1978). We note only that use of the Markov approximation in the calculation of the second-order coherence is generally valid provided $\lambda << \rho_t, l_o$.

We will solve (3a) subject to the boundary condition

$$\begin{aligned} \Gamma_2\left(\vec{\rho}, \xi, \vec{R}, \frac{\xi}{2}\right) &= \langle U(\vec{\rho}_1, \xi)\, U^*(\vec{\rho}_2, 0)\rangle \\ &= U^*(\vec{\rho}_2, 0)\, \langle U(\vec{\rho}_1, \xi)\rangle \\ &= U^*\left(\vec{R} - \frac{\vec{\rho}}{2}, 0\right) \left\langle U\left(\vec{R} + \frac{\vec{\rho}}{2}, \xi\right)\right\rangle \end{aligned} \tag{5}$$

suggested by Nalbandyan and Tatarskii (1977). Since the function H defined in (4) is independent of $\vec{R}$, (3a) can be solved in a manner analogous to that used by Tatarskii (1971) to evaluate the transverse second-order coherence. He obtained his solution by substituting for Γ_2 its Fourier integral representation with respect to $\vec{R}$, and solving the resulting equation for the Fourier transform of Γ_2 using the method of characteristics. An inverse transform was then applied to obtain the final result. The details of this method are straightforward and will not be repeated here. (An alternative approach is described by Rytov et al., 1989). The solution to (3a) is given by

$$\Gamma_2(\vec{\rho},\xi,\vec{R},L) = \frac{1}{4\pi^2}\int d^2\vec{R}'\int d^2\vec{K}\; U^*\left(\vec{R}' - \frac{\vec{\rho}}{2} + \frac{\vec{K}(L-\xi/2)}{2k},0\right)\left\langle U\left(\vec{R}' + \frac{\vec{\rho}}{2} - \frac{\vec{K}(L-\xi/2)}{2k},\xi\right)\right\rangle$$

$$\times \exp\left\{i\vec{K}\cdot(\vec{R}-\vec{R}') - k^2\int_{\xi/2}^{L} dz'\, H\left[\vec{\rho}-\frac{\vec{K}}{k}(L-z'),\xi,z'\right]\right\}, \tag{6}$$

where we have set $z = L$. Assuming a spherical wave field of the form

$$U(\vec{\rho},z) = \frac{A_o}{z}\exp\left[ik\,\rho^2/(2z)\right],$$

we have

$$\left\langle U\left(\vec{R}' + \frac{\vec{\rho}}{2} - \frac{\vec{K}(L-\xi/2)}{2k},\xi\right)\right\rangle = \frac{A_o}{\xi}\exp\left\{ik\left[\vec{R}' + \frac{\vec{\rho}}{2} - \frac{\vec{K}(L-\xi/2)}{2k}\right]^2/(2\xi)\right\}F(\xi)\,, \tag{7a}$$

where F is the mean field $<U>$ normalized by its value U_o in the absence of fluctuations. In the limit of the Markov approximation, $F(\xi)$ can be expressed as (Tatarskii, 1969)

$$F(\xi) \approx \exp\left\{-\pi k^2\int_{L-\xi/2}^{L+\xi/2} dz'\int d^2\vec{K}\,\Phi_n(\vec{K},z')\right\}. \tag{7b}$$

The integration over $\vec{R}'$ in (6) can be evaluated by assuming the following Gaussian form for the source field given by $U(\vec{\rho},0) = \exp\{-[\rho^2/(2a^2)]\}$, where a is the effective radius of the transmitting aperture, and taking the limit as a tends to zero, which yields a delta function in $\vec{R}'$. Noting that the Huygens-Fresnel integral (Goodman, 1968) gives $A_o = -ika^2$, we have

$$\Gamma_2(\vec{\rho},\xi,\vec{R},L) = \frac{I_o}{2\pi i k\xi} F(\xi) \exp\left\{ i\left[\frac{k\rho^2}{2\xi} - \frac{k\xi(\vec{R}-\vec{\rho}L/\xi)}{2(L^2-\xi^2/4)}\right]\right\}$$
$$\times \int d^2\vec{K} \exp\left\{ i\,\frac{(L^2-\xi^2/4)}{2k\xi}\left[\vec{K} + \frac{k\xi(\vec{R}-\vec{\rho}L/\xi)}{L^2-\xi^2/4}\right]^2\right\}$$
$$\times \exp\left\{-k^2 \int_{\xi/2}^{L} dz' H\left[\vec{\rho} - \frac{\vec{K}}{k}(L-z'),\, \xi,\, z'\right]\right\}, \tag{8}$$

where $I_o = |A_o|^2$. The second term in the integrand of (8) has the form of a coherence function. The transverse scale of this term is given approximately by the transverse coherence length ρ_t. If we require that $\rho_t >> \sqrt{\lambda\xi}$, the first exponential within the integrand is much narrower than the second, effectively sampling the second exponential at the wavenumber $\vec{K} = \vec{K}_o = -k\xi(\vec{R}-\vec{\rho}L/\xi)/(L^2-\xi^2/4)$. Therefore, in this limit, the integration over $\vec{K}$ can be evaluated by replacing $\vec{K}$ in the second exponential by $\vec{K}_o$, removing it from the integrand and evaluating the remaining integral. Noting that the vacuum coherence

$$\Gamma_o(\vec{\rho},\xi,\vec{R},L) = \frac{I_o}{z_1 z_2} \exp\left[i\frac{k}{2}\left(\rho_1^2/z_1 - \rho_2^2/z_2\right)\right]$$
$$= \frac{I_o}{L^2-\xi^2/4} \exp\left\{ i\frac{k}{2}\left[2L\vec{\rho}\cdot\vec{R} - \xi(R^2-\rho^2/4)\right]/(L^2-\xi^2/4)\right\}, \tag{9}$$

we have for the normalized coherence $M_2 = \Gamma_2/\Gamma_o$

$$M_2(\vec{\rho},\xi,\vec{R},L) = F(\xi)\exp\left[-k^2\int_{\xi/2}^{L} dz'\, H\left\{\vec{\rho}\left[1-\frac{(L-z')L}{L^2-\xi^2/4}\right]+\vec{R}\,\frac{(L-z')\xi}{L^2-\xi^2/4},\xi,z'\right\}\right]$$

$$\approx F(\xi)\exp\left\{-k^2\int_{o}^{L} dz'\, H\left[\vec{\rho}\,\frac{z'}{L}+\vec{R}\left(1-\frac{z'}{L}\right)\frac{\xi}{L},\,\xi,z'\right]\right\},\ \xi<<L, \tag{10}$$

In (10) we have replaced ξ by 0 in the lower limit of the integration and excluded terms of order $(\xi/L)^2$ in the argument of H, which introduces a negligible error provided $\xi<<L$.

We note that (10) has essentially the same form as the plane wave result of Nalbandyan and Tatarskii (1977), which can be recovered by setting $z'/L=1$. It is the product of two terms. The first term, $F(\xi)$, describes the decay of the mean field in propagating from one observation plane to the other, which depends only on the local value of C_n^2. The second term describes the contribution to the coherence from the entire propagation path. It is easy to show that the transverse argument of the function H in (10) is simply the difference between the transverse vectors in the planes at $z'-\xi/2$ and $z'+\xi/2$ locating the two rays connecting the source to the observation points at $L-\xi/2$ and $L+\xi/2$. For a plane wave source these two rays are parallel and have a constant separation $\vec{\rho}$. For a spherical wave we obtain the more complicated dependence shown in (10).

3. DISCUSSION

Equations (4), (7b) and (10) describe the longitudinal-transverse, spherical-wave, spatial coherence function for an arbitrary refractive index spectrum $\Phi_n(\vec{K},z)$. The explicit dependence of the spectrum on the path position z allows for a nonuniform distribution of turbulence strength along the propagation path. We can simplify the result somewhat by writing the function H in (4) as the sum of the following two terms;

$$H(\vec{\rho}, \xi, z) = 2\pi \int d^2\vec{K}\, \Phi_n(\vec{K}, z) \{ 1 - \exp[-i K^2 \xi/(2k)]\}$$
$$+ 2\pi \int d^2\vec{K}\, \Phi_n(\vec{K}, z) \exp[-i K^2 \xi/(2k)] [1 - \exp(-i\vec{K}\cdot\vec{\rho})], \tag{11}$$

Consider the second term of this expression. If we assume that the spectrum $\Phi_n(\vec{K}, z)$ has an inner-scale cutoff described by $\exp\{-K^2/K_m^2\}$, where $K_m = 5.92/l_o$, the exponential term within the integrand containing ξ can be considered as part of the spectrum if we define an effective inner-scale $l_{eff} = \left(l_o^2 + i\,17.5\,\xi/k\right)^{1/2}$ as suggested by Nalbandyan and Tatarskii (1977). If we require that $\rho >> l_{eff}$ this inner-scale dependence can be ignored. In this limit the first term in (11) depends only on ξ and the second only on $\vec{\rho}$. If in addition we assume $\rho << L_o$ and $l_o << \sqrt{\lambda\,\xi} << L_o$, the Kolmogorov spectrum

$$\Phi_n(\vec{K}, z) = 0.033\, C_n^2(z)\, K^{-11/3}$$

can be used in (11) to obtain

$$H(\vec{\rho}, \xi, z) = 2.44\, C_n^2(z)\, k^{-5/6} (i\,\xi)^{5/6} + 1.46\, C_n^2(z)\, \rho^{5/3}\,. \tag{12}$$

Substituting this result into (10) and assuming a uniform distribution of turbulence along the propagation path gives

$$M_2(\vec{\rho},\xi,L) = F(\xi)\exp\Big[-\Big(0.632\,k^{7/6}\,L\,C_n^2\,\xi^{5/6}$$

$$+\,i\,2.36k^{7/6}\,L\,C_n^2\,\xi^{5/6} + 0.546k^2\,L\,C_n^2\,\rho^{5/3}\Big)\Big]. \tag{13}$$

We now need to evaluate the function $F(\xi)$ defined in (7b). Since this function clearly depends on the turbulence outer-scale L_o we will use the following uniform spectral form of the von Kármán type (Tatarskii, 1961);

$$\Phi_n(\vec{K}) = \frac{0.033\,C_n^2}{(K^2 + K_o^2)^{11/6}},$$

where $K_o = 2\pi/L_o$. Writing (7b) in the more recognizable form

$$F(\xi) = \exp\left[-\frac{k^2}{2}\,\xi\,A_n(0)\right], \tag{14a}$$

where

$$A_n(0) = 2\pi\int d^2\vec{K}\,\Phi_n(\vec{K}) = \sigma_n{}^2\,L_i, \tag{14b}$$

$\sigma_n{}^2 = C_n^2/(1.91\,K_o{}^{2/3})$ and $L_i = 0.237\,L_o$ is the corresponding integral scale, we have

$$F(\xi) = \exp(-0.0182\,k^2\,L_o{}^{5/3}\,C_n^2\,\xi) \tag{14c}$$

The combination of (13) and (14c) finally yields

$$M_2(\rho,\xi,L) = \exp\left\{-\left[\frac{\xi}{\rho_1} + \left(\frac{\xi}{\rho_2}\right)^{5/6} + i\left(\frac{\xi}{3.73\rho_2}\right)^{5/6} + \left(\frac{\rho}{\rho_t}\right)^{5/3}\right]\right\}, \tag{15}$$

$\rho_1 = (1.82 \times 10^{-2}\, k^2 L_o^{5/3} C_n^2)^{-1}$, $\rho_2 = (0.632\, k^{7/6} L\, C_n^2)^{-6/5}$, and $\rho_t = (0.546\, k^2 L\, C_n^2)^{-3/5}$. The longitudinal component of the normalized coherence defined in (1) is identical to the plane wave result obtained by Nalbandyan and Tatarskii (1977). Note that if we change the 5/6 power law of the second factor in (1) to 1, the longitudinal coherence length ρ_l, which can be estimated from the modulus of M_2, can be approximated by $\rho_l \approx \rho_1\rho_2/(\rho_1 + \rho_2)$. For a nonuniform distribution of turbulence strength along the propagation path, C_n^2 in ρ_1 corresponds to the local value at a path position $z \approx L$, C_n^2 in ρ_2 represents a uniformly weighted path-integrated value, and C_n^2 in ρ_t corresponds to a path-integrated value with a $z^{5/3}$ path weighting.

4. CONCLUSIONS

We have derived an expression for the longitudinal-transverse, second-order, spatial coherence function for a spherical wave propagating through homogeneous turbulence. Our development parallels that of Nalbandyan and Tatarskii (1977), who considered the same problem for plane waves, and is general enough to allow for a nonuniform distribution of turbulence along the propagation path. Our results indicate that the longitudinal coherence length for a spherical wave is identical to that for a plane wave. We also find that the path-weighting of turbulence strength for the longitudinal coherence length is the same for both plane and spherical waves, consisting of a uniformly weighted local component about the range of interest and a uniformly weighted path-integrated component.

Earlier calculations of the longitudinal coherence length with implications for RASS assumed uniform turbulence and used exceedingly small acoustic C_n^2 values. This procedure yielded long coherence lengths and led to the conclusion that the RASS echo power was not significantly affected by the degradation of the longitudinal coherence of the acoustic wave. We have reached the same conclusion using more appropriate values of C_n^2 and including its variability with height.

APPENDIX

We will evaluate (3b) using the assumption of δ-correlated refractive index fluctuations along the propagation path for which

$$\langle \delta n(\vec{\rho}', \xi, \vec{R}', z') \, \delta n(\vec{\rho}, \xi, \vec{R}, z) \rangle = \delta(z - z') A_n \left[\vec{R} - \vec{R}' + \frac{1}{2}(\vec{\rho} - \vec{\rho}'); z + \xi/2 \right]$$
$$+ \delta(z - z') A_n \left[\vec{R} - \vec{R}' - \frac{1}{2}(\vec{\rho} - \vec{\rho}'); z - \xi/2 \right]$$
$$- \delta(z - z' + \xi) A_n \left[\vec{R} - \vec{R}' + \frac{1}{2}(\vec{\rho} + \vec{\rho}'); z + \xi/2 \right]$$
$$- \delta(z - z' - \xi) A_n \left[\vec{R} - \vec{R}' + \frac{1}{2}(\vec{\rho} + \vec{\rho}'); z - \xi/2 \right], \tag{A1}$$

where δ is the Dirac delta function,

$$A_n(\vec{\rho}_o, z_o) = 2\pi \int d^2\vec{K} \, \Phi_n(\vec{K}, z_o) \exp(i\vec{K} \cdot \vec{\rho}_o), \tag{A2}$$

and $\Phi_n(\vec{K}, z_o)$ is the three-dimensional refractive index spectrum evaluated at the transverse wavenumber $\vec{K}$ and mean path position z_o. Substitution of (A1) into (3b) yields a sum of four terms, which we denote each by I_1, I_2, I_3, and I_4, respectively. The delta function associated with the first two terms in (A1) makes the evaluation of I_1 and I_2 trivial. We find that

$$I_1 + I_2 = [A_n(0; z + \xi/2) + A_n(0; z - \xi/2)]/2 \approx A_n(0; z),$$

provided the strength of the refractive index fluctuations does not change significantly over the distance ξ.

Evaluation of I_3 and I_4 is slightly more difficult. In our development we defined ξ as the separation between the observation planes, which is always positive. Technically, ξ

can be either positive or negative, although the distinction has not been important so far. If we assume that ξ can be either positive or negative but small compared to z, then the delta functions in the expressions for I_3 and I_4 are such that when $\xi > 0$, $I_3 = 0$, and when $\xi < 0$, $I_4 = 0$. We note, however, that the sum $I_3 + I_4$ does not depend on the sign of ξ. We therefore consider only $\xi > 0$ for which $I_3 = 0$, and

$$I_4 \approx \frac{k^2}{4\pi^2 \xi^2} \int d^2 \vec{\rho}^{\,\prime} \int d^2 \vec{R}^{\prime} \exp\left[ik(\vec{\rho} - \vec{\rho}^{\,\prime}) \cdot (\vec{R} - \vec{R}^{\prime})/\xi\right]$$

$$A_n\left[\vec{R} - \vec{R}^{\prime} - \frac{1}{2}(\vec{\rho} + \vec{\rho}^{\,\prime})\ ;\ z\right], \tag{A3}$$

where we replaced $z - \xi/2$ in the argument for A_n by z. Using (A2) in this expression yields

$$I_4 = 2\pi \int d^2\vec{K}\, \Phi_n(\vec{K}, z) \exp\left[-(\vec{K} \cdot \vec{\rho} + K^2 \xi/2k)\right], \tag{A5}$$

which leads directly to (4) in the text.

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